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ALGEBRAICALLY CONSTRUCTIBLE CHAINS

by Hélène PENNANEAC'H

1. Introduction.

In [KS], Kashiwara and Schapira define subanalytic chains on a real analytic manifold. Now let X be a real algebraic variety. We can define analogously semialgebraic chains on X: the group $\mathrm{C}_n(X)$ of n-chains is generated by symbols $[S]$ where $S \subset X$ is a n-dimensional oriented Nash manifold with the relations

1) $[S^a] = -[S]$ if S^a is S with the opposite orientation;

2) $[S \sqcup S'] = [S] + [S']$;

3) $[S] = [S']$ if S' is dense in S with the induced orientation.

There is a boundary for these chains. This leads on one hand to the notion of cycles which is useful for computing characteristic cycles of constructible functions as we shall see later, and on the other hand to a homology which is in fact the Borel-Moore homology of X.

Here is introduced a new definition of $C_*(X)$ by considering "constructible functions on the oriented real spectrum of the function fields of n-dimensional subvarieties of X". This approach through real spectrum is analogous to the one of Scheiderer in [Sch]. He shows that the Borel-Moore homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$ of a real algebraic variety X can be computed from the complex

$$\cdots \longrightarrow \bigoplus_{y \in X_{(2)}} H^0(y_r, \mathbb{Z}/2\mathbb{Z}) \longrightarrow \bigoplus_{y \in X_{(1)}} H^0(y_r, \mathbb{Z}/2\mathbb{Z}) \longrightarrow \bigoplus_{y \in X_{(0)}} H^0(y_r, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

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where $X_{(n)}$ is the set of n-dimensional points of X and y_r is the real spectrum of the residue field $\kappa(y)$ of y (*i.e.*, the space of its orderings). This complex can be identified with $C_*(X) \otimes \mathbb{Z}/2\mathbb{Z}$.

In order to deal with integral coefficients, we have to take into account the orientation. We introduce the oriented real spectrum $\hat{y}_r$ of the field $\kappa(y)$. If $\kappa(y)$ has transcendence degree n over the base field R, the points of $\hat{y}_r$ are orderings of $\kappa(y)$ equipped with an equivalence class $\overline{\omega}^\alpha$ of nonzero Kähler differentials of degree n of $\kappa(y)$ over R where $\overline{\omega}^\alpha = \overline{f\omega}^\alpha$ if and only if f is an element of $\kappa(y)$ positive for α. The oriented real spectrum $\hat{y}_r$ is a two-sheeted covering of the real spectrum y_r. In the complex above, we replace $H^0(y_r, \mathbb{Z}/2\mathbb{Z})$ by the group of constructible functions on $\hat{y}_r$, *i.e.* the continuous functions $\varphi : \hat{y}_r \longrightarrow \mathbb{Z}$ such that $\varphi(\alpha, \overline{-\omega}^\alpha) = -\varphi(\alpha, \overline{\omega}^\alpha)$ (this corresponds to condition 1) in the previous definition of $C_*(X)$). We can distinguish among those functions the "algebraically constructible" ones, *i.e.* those for which $\alpha \mapsto \varphi(\alpha, \overline{\omega}^\alpha)$ is a sum of signs of elements of $\kappa(y)$ (or equivalently the signature of a quadratic form on $\kappa(y)$). The notion of constructible functions on the oriented real spectrum is explained in Section 2, as far as most technical points concerning it.

We thus have a complex $C_*(X)$, whose boundary we can compute using the specialization in the real spectrum as done in [Sch], Proposition 2.6. We take into account the orientation *via* a Poincaré residue for Kähler differentials. Then half of the boundary of an algebraically constructible chain is still an algebraically constructible chain, and so we get a new complex $AC_*(X)$, of algebraically constructible chains, whose boundary is half of the previous one. This complex can be seen as the "signature" of the Witt complex introduced by Schmid in his thesis (*cf.* [S]): he tensorizes the Witt ring of $\kappa(y)$ by the Kähler differentials.

In Section 3, we are mostly interested in the case where X is a variety over a real closed field and we establish some relations between algebraically constructible homology and Borel-Moore homology. In Section 4, we generalize the complex to schemes essentially of finite type over a field of characteristic 0. We establish the functorial properties of the complex and compute the homologies of affine and projective spaces. These are similar to the results of [S].

As the Witt ring is associated to the graded Witt ring, it is natural to consider the filtration of the algebraically constructible functions of $\kappa(y)$ by the functions divisible by 2^ℓ with $0 \leq \ell \leq n$. This leads to many complexes $C_*^{k-\mathrm{ac}}(X)$: the k-algebraically constructible n-chains are the algebraically constructible functions on y_r which are divisible by 2^{k+n}, considered modulo 2^{k+n+1}. This construction is of the type of Rost's cycle modules (see [R]), and is explained in Section 5. If $k \geq 0$ we recover Scheiderer's complex. We obtain in some cases a filtration between usual algebraic homology (see [BCR]) and Borel-Moore homology with coefficients modulo 2.

Another interest of algebraically constructible chains is the following: if X is a smooth algebraic variety, the group of constructible functions on X is isomorphic via the characteristic cycle to the group of Lagrangian semialgebraic cycles of T^*X. On the other hand, McCrory and Parusiński introduced algebraically constructible functions which are sums of Euler characteristics of fibres of proper morphisms (*cf.* [MP]). We will prove that, in this isomorphism, the Lagrangian algebraically constructible cycles are exactly the images of algebraically constructible functions. This result also enables us to get a nice characterization of Lagrangian algebraically constructible cycles in terms of sums of pushforwards by proper morphisms of characteristic cycles of characteristic functions of algebraic smooth varieties. This is done in Section 6.

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2. Constructible functions on the real spectrum with coefficients.

In the following, R will denote a fixed field of characteristic 0.

For a field K define $K^\bullet = K \setminus \{0\}$; the real spectrum of K (*i.e.* the set of orderings on K, *cf.* [BCR]) will be denoted by Σ_K (or Σ if there is no possible confusion). An ordering $\alpha \in \Sigma_K$ is seen as a map $K^\bullet \to \{-1, 1\}$. For $f \in K$, $f(\alpha)$ is the image of f in the real closure of K for α. Then $f(\alpha) > 0 \Leftrightarrow \alpha(f) = 1$. The Harrison topology on Σ_K is the one such that the basic open sets are the $\{\alpha \in \Sigma_K \mid f_1(\alpha) > 0, \ldots, f_k(\alpha) > 0\}$ where $f_i \in K$. A constructible set of Σ_K is a finite union of basic sets.

The fields we consider are extensions of R.

2.1. Real spectrum with coefficients.

Let K be a field and H a 1-dimensional K-vector field. For each ordering α on K, there are two equivalence classes in $H^\bullet = H \setminus \{0\}$ for the relation "$h \sim_\alpha fh$ if $f \in K^\bullet$ is positive for α". For a given nonzero h, those classes are $\overline{h}^\alpha$ and $\overline{-h}^\alpha$. We shall denote

$$\Sigma^H = \{(\alpha, \overline{h}^\alpha), (\alpha, \overline{-h}^\alpha) \mid \alpha \in \Sigma\}.$$

There is a projection $p : \Sigma^H \longrightarrow \Sigma$ and for each $h \in H^\bullet$ a *distinguished* section $s_h : \Sigma \to \Sigma^H : \alpha \mapsto (\alpha, \overline{h}^\alpha)$. We put a boolean topology on Σ^H such that the basic open sets are the images by the s_h of the basic open sets of Σ where h ranges through $H^\bullet$. Thus p and s_h are continuous.

Note that there is a canonical homeomorphism $\Sigma_K^H \to \Sigma_K^{H^*}$. Indeed an isomorphism $H \to H'$ induces an homeomorphism $\Sigma_K^H \to \Sigma_K^{H'}$; the homeomorphism $d_H : \Sigma_K^H \to \Sigma_K^{H^*}$ induced by $H \longrightarrow H^* : h \mapsto \ell(h) \cdot \ell$ does not depend on the choice of $\ell \in H^*$. The application d_H preserves the distinguished sections. For an extension $K \hookrightarrow L$, there is also a base change $\Sigma_L^{H\otimes_K L} \longrightarrow \Sigma_K^H : (\alpha, \overline{h \otimes 1}^\alpha) \mapsto (\alpha_{|K}, \overline{h}^{\alpha|K})$.

If K is of transcendence degree n over R, the K-vector space of n-Kähler differentials $\Omega^n_{K/R} = \bigwedge^n \Omega_{K/R}$ is 1-dimensional (*cf.* [Ku]). We shall denote $\Sigma^\wedge$ or $\widehat{\Sigma} = \Sigma^{\Omega^n_{K/R}}$ and for each 1-dimensional H, $(\Sigma^H)^\wedge$ or $\widehat{\Sigma^H} = \Sigma^{\Omega^n_{K/R}\otimes_K H}$.

Let $\mathrm{Cons}(\Sigma, \mathbb{Z})$ be the set of continuous functions from Σ to $\mathbb{Z}$, *i.e.* the functions which can be written $\sum_{i\in I} m_i \mathbf{1}_{S_i}$ with finite I, $m_i \in \mathbb{Z}$ and S_i constructible subset of Σ (they can be chosen disjoint or basic). Let $\mathrm{AlgCons}(\Sigma, \mathbb{Z})$ be the set of algebraically constructible functions, *i.e.* those which can be written $\sum_{i=1}^m \mathrm{sign}(f_i)$ with $f_i \in K^\bullet$ and $\mathrm{sign}(f_i)(\alpha) = \alpha(f_i)$.

There is a continuous involution

$$i : \Sigma^H \longrightarrow \Sigma^H : (\alpha, \overline{h}^\alpha) \longmapsto (\alpha, \overline{-h}^\alpha).$$

We shall denote by

$$\mathrm{Cons}(\Sigma^H, \mathbb{Z}) = \{\varphi : \Sigma^H \to \mathbb{Z} \mid \varphi \text{ continuous and } \varphi \circ i = -\varphi\}.$$

For each $h \in H^\bullet$ and each $\varphi : \Sigma^H \to \mathbb{Z}$, we define $\varphi_h = \varphi \circ s_h$. Then the shrinking morphism $\varphi \mapsto \varphi_h$ is an isomorphism from the group $\mathrm{Cons}(\Sigma^H, \mathbb{Z})$ to $\mathrm{Cons}(\Sigma, \mathbb{Z})$.

$\mathrm{Cons}(\Sigma^H, \mathbb{Z})$ is a $\mathrm{Cons}(\Sigma, \mathbb{Z})$-module for the multiplication

$$(\psi \cdot \varphi)(\alpha, \overline{h}^\alpha) = \psi(\alpha)\varphi(\alpha, \overline{h}^\alpha).$$

DEFINITION 1. — $\mathrm{AlgCons}(\Sigma^H,\mathbb{Z})$ *is the subgroup of* $\mathrm{Cons}(\Sigma^H,\mathbb{Z})$ *of continuous functions* φ *such that* $\varphi\circ i=-\varphi$ *and there exists* $h\in H^\bullet$ *for which* $\varphi_h\in\mathrm{AlgCons}(\Sigma,\mathbb{Z})$.

Remarks.

- $\varphi\in\mathrm{AlgCons}(\Sigma^H,\mathbb{Z})\Leftrightarrow\forall h\in H^\bullet,\varphi_h\in\mathrm{AlgCons}(\Sigma,\mathbb{Z})$. Indeed if $\varphi_h=\mathrm{sign}(g)$, then $\varphi_{fh}=\mathrm{sign}(gf)$.
- $\mathrm{AlgCons}(\Sigma^H,\mathbb{Z})$ is a $\mathrm{AlgCons}(\Sigma,\mathbb{Z})$-module.

Fan criterion. — The representation theorem of Becker and Bröcker (*cf.* [BB]) implies that $\varphi\in\mathrm{AlgCons}(\Sigma,\mathbb{Z})$ if and only if for all F fan in Σ, $\sum_{x\in F}\varphi(x)=0\bmod|F|$. Thus $\varphi\in\mathrm{AlgCons}(\Sigma^H,\mathbb{Z})$ if and only if there exists h such that for all F fan in Σ,

$$\sum_{x\in s_h(F)}\varphi(x)=0\bmod|F|.$$

2.2. Restriction and corestriction.

Let K be a field of transcendence degree n over R. Let $K\to L$ be an extension of transcendence degree r. Define a restriction morphism $r_{L/K}:\mathrm{Cons}(\Sigma_K,\mathbb{Z})\to\mathrm{Cons}(\Sigma_L,\mathbb{Z})$ by $r_{L/K}\varphi(\alpha)=\varphi(\alpha_{|K})$.

Using the canonical isomorphism

$$\Omega^{n+r}_{L/R}\otimes_L(\Omega^r_{L/K})^*\xrightarrow{\sim}\Omega^n_{K/R}\otimes_K L,$$

we get for a 1-dimensional K-vector space H,

$$(\Sigma_L{}^{H\otimes_K\Omega^r_{L/K}})^\wedge\xrightarrow{\sim}\Sigma_L{}^{H\otimes_K\Omega^n_{K/R}\otimes_K L}.$$

Using base change, we get a morphism $(\Sigma_L{}^{H\otimes_K\Omega^r_{L/K}})^\wedge\to\widehat{\Sigma^H_K}$ and thus a restriction morphism

$$r_{L/K}:\mathrm{Cons}(\widehat{\Sigma^H_K},\mathbb{Z})\to\mathrm{Cons}\big((\Sigma_L{}^{H\otimes_K\Omega^r_{L/K}})^\wedge,\mathbb{Z}\big).$$

Clearly, $r_{L/K}$ induces a restriction morphism

$$r_{L/K}:\mathrm{AlgCons}(\widehat{\Sigma^H_K},\mathbb{Z})\to\mathrm{AlgCons}\big((\Sigma_L{}^{H\otimes_K\Omega^r_{L/K}})^\wedge,\mathbb{Z}\big).$$

If $L\to M$ is another extension, $r_{M/L}\circ r_{L/K}=r_{M/K}$.

Now we define corestriction morphisms for extensions of finite degree.

PROPOSITION-DEFINITION 2. — *Let $K \to L$ an algebraic extension of finite degree. The formula*

$$c_{L/K}(\varphi)(\alpha) = \sum_{\beta|K=\alpha} \varphi(\beta)$$

defines a morphism (called corestriction) $c_{L/K}\colon \mathrm{Cons}(\Sigma_L,\mathbb{Z}) \to \mathrm{Cons}(\Sigma_K,\mathbb{Z})$.

Proof. — It is sufficient to see that this is well defined for $\varphi = \mathbf{1}_S$ with S basic, *i.e.* $S = \{f_1 > 0, \ldots, f_k > 0\}$, $f_i \in L^\bullet$. If a is a primitive element of the extension, and P is its minimal polynomial, $f_i = F_i(a)$ with F_i a polynomial with coefficients in K; for $\alpha \in \Sigma_K$, each β extending α on L corresponds to a root b of P in the real closure R_α of K for α, and then $f_i(\beta) > 0 \Leftrightarrow F_i(b) > 0$ (*cf.* [BCR], Proposition 1.3.7); thus

$$c_{L/K}(\varphi)(\alpha) = \mathrm{card}\big\{b \in R_\alpha \mid P(b) = 0,\ F_1(b) > 0, \ldots, F_k(b) > 0\big\}\,;$$

there is a constructible set S_m such that

$$\alpha \in S_m \Leftrightarrow \mathrm{card}\big\{b \in R_\alpha \mid P(b) = 0,\ F_1(b) > 0, \ldots, F_k(b) > 0\big\} = m\,;$$

then $c_{L/K}(\varphi) = \sum_{m\in\mathbb{N}} m\mathbf{1}_{S_m}$ is constructible. □

PROPOSITION 3. — *The image by* $c_{L/K}$ *of* $\mathrm{AlgCons}(\Sigma_L,\mathbb{Z})$ *is contained in* $\mathrm{AlgCons}(\Sigma_K,\mathbb{Z})$. *Thus we define*

$$c_{L/K}\colon \mathrm{AlgCons}(\Sigma_L,\mathbb{Z}) \longrightarrow \mathrm{AlgCons}(\Sigma_K,\mathbb{Z}).$$

Proof. — If we assume that $\varphi = \mathrm{sign}(f)$ for $f = F(a) \in L^\bullet$,

$$\begin{aligned} c_{L/K}(\varphi)(\alpha) = \mathrm{card}\big\{b \in R_\alpha \mid P(b) = 0,\ F(b) > 0\big\} \\ - \mathrm{card}\big\{b \in R_\alpha \mid P(b) = 0,\ F(b) < 0\big\}. \end{aligned}$$

Then $c_{L/K}(\varphi)$ is the signature of the Hermite quadratic form associated to P and F (see for example [KnS]) and is algebraically constructible. □

Now, as $\Omega^n_{L/R} = \Omega^n_{K/R} \otimes_K L$, for a 1-dimensional K-vector space H we can define

$$c_{L/K} : \mathrm{Cons}(\widehat{\Sigma_L^{H\otimes_K L}}, \mathbb{Z}) \longrightarrow \mathrm{Cons}(\widehat{\Sigma_K^H}, \mathbb{Z})$$

and

$$c_{L/K} : \mathrm{AlgCons}(\widehat{\Sigma_L^{H\otimes_K L}}, \mathbb{Z}) \longrightarrow \mathrm{AlgCons}(\widehat{\Sigma_K^H}, \mathbb{Z})$$

in the following way:

$$c_{L/K}(\varphi)(\alpha, \overline{\omega \otimes h^\alpha}) = \sum_{\beta|K=\alpha} \varphi(\beta, \overline{\omega \otimes 1 \otimes h \otimes 1^\beta}).$$

Then $c_{L/K}(\varphi_{\omega\otimes h}) = (c_{L/K}\varphi)_{\omega\otimes h}$.

If $L \to M$ is another algebraic extension of finite degree,

$$c_{L/K} \circ c_{M/L} = c_{M/K}.$$

LEMMA 4. — *Let K be a field of transcendence degree n over R. Let $K \to L$ be an algebraic extension of finite degree, $K \to M$ an extension of transcendence degree r, H a 1-dimensional K-vector space, $N = M \otimes_K L$. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_m$ be the maximal ideals of N, and $N_i = N/\mathfrak{m}_i$. Then the following diagram (and the corresponding one with* AlgCons) *is commutative*:

$$\begin{array}{ccc}
\mathrm{Cons}\big(\widehat{\Sigma_L^{H\otimes_K L}}, \mathbb{Z}\big) & \xrightarrow{\oplus r_{N_i/L}} & \bigoplus_{i=1}^{m} \mathrm{Cons}\big((\Sigma_{N_i}{}^{H\otimes_K \Omega^r_{N_i/L}})^\wedge, \mathbb{Z}\big) \\
\Big\downarrow c_{L/K} & & \Big\downarrow \Sigma c_{N_i/M} \\
\mathrm{Cons}\big(\widehat{\Sigma_K^H}, \mathbb{Z}\big) & \xrightarrow{\quad r_{M/K} \quad} & \mathrm{Cons}\big((\Sigma_M{}^{H\otimes_K \Omega^r_{M/K}})^\wedge, \mathbb{Z}\big)
\end{array}$$

Proof. — Choosing $h \in H^\bullet$, $\omega'' \in \Omega^n_{K/R}{}^\bullet$, $\omega' \in \Omega^r_{M/K}{}^\bullet$ and using the shrinking isomorphisms, it is enough to show the commutativity of the diagram on the level of constructible functions on the real spectrum without coefficients. Then it is enough to show that for each $\alpha \in \Sigma_M$,

$$\{\beta \in \Sigma_L \mid \beta_{|K} = \alpha_{|K}\}$$

is the disjoint union for $i = 1, \ldots, m$ of the sets

$$\{\beta_{|L} \mid \beta \in \Sigma_{N_i} \text{ such that } \beta_{|M} = \alpha\}.$$

Let a be a primitive element of the extension $K \to L$, and P its minimal polynomial. Let $Q_1, \ldots, Q_m$ be the irreducible factors of P in $M[T]$. Then $\mathfrak{m}_i = (Q_i)$ and

$$N = M[T]/(P) \simeq M[T]/(Q_1) \times \cdots \times M[T]/(Q_m).$$

Let M_α be the real closure of M for α and K_α the real closure of K for $\alpha_{|K}$; M_α is an extension of K_α and a root of P in K_α is exactly a root of one and only one Q_i in M_α. □

2.3. The boundary map.

Let B be a valuation ring of K with residue field k_B. Denote by v_B, λ_B the valuation and place associated to B. If $x \in B$, put $\overline{x} = \lambda_B(x)$. If β is an ordering of K, B is said to be compatible with β if the maximal ideal $\mathfrak{m}_B$ of B is convex for β (*i.e.* if for $f \in B$ and $g \in \mathfrak{m}_B$, $0 < f(\beta) < g(\beta) \Rightarrow f \in \mathfrak{m}_B$). Then β induces on k_B an ordering $\overline{\beta}$ such that: for $u \in B \backslash \mathfrak{m}_B$, $\overline{\beta}(\overline{u}) = \beta(u)$. If B is a discrete valuation ring and α is an ordering of k_B, then there exists two orderings β of K and only two such that $\overline{\beta} = \alpha$. The two orderings differ by the sign they give to a uniformizing parameter of B (*cf.* [BCR], Chapter 10).

PROPOSITION-DEFINITION 5. — *Let B a discrete valuation ring of K, with residue field k_B; the formula*

$$\sigma_B(\varphi)(\alpha) = \sum_{\overline{\beta}=\alpha} \varphi(\beta)$$

gives a morphism $\sigma_B\colon \mathrm{Cons}(\Sigma_K,\mathbb{Z}) \to \mathrm{Cons}(\Sigma_{k_B},\mathbb{Z})$.

Proof. — Assume $\varphi = \mathbf{1}_S$ with $S = \{f_1 > 0, \ldots, f_k > 0\}$; multiplying the f_i's by squares if necessary, we can assume their valuation to be 0 or 1. We have

$$\sigma_B(\varphi)(\alpha) = \mathrm{card}\{\beta \in S \mid \overline{\beta} = \alpha\}.$$

If every f_i has valuation 0, then we put $g_i = \overline{f}_i$. Then $\sigma_B(\varphi) = 2 \times \mathbf{1}_{S'}$ with $S' = \{g_1 > 0, \ldots, g_k > 0\}$. If f_1 is of valuation 1 and the others are of valuation 0, we put $g_i = \overline{f}_i$ for $i > 1$, and then $\sigma_B(\varphi) = \mathbf{1}_{S'}$ with $S' = \{g_2 > 0, \ldots, g_k > 0\}$. At last, if $f_1, \ldots, f_\ell$ are of valuation 1 and the others are of valuation 0, we put for $i = 2, \ldots, \ell$, $f'_i = f_i/f_1$ which is of valuation 0. Then

$$S = \{f_1 > 0, f'_2 > 0, \ldots, f'_\ell > 0, f_{\ell+1} > 0, \ldots, f_k > 0\}$$

and this is the preceding case. □

PROPOSITION 6. — *If $\varphi \in \mathrm{AlgCons}(\Sigma_K,\mathbb{Z})$, then $\sigma_B(\varphi)$ is even and $\sigma_B(\varphi)/2 \in \mathrm{AlgCons}(\Sigma_{k_B},\mathbb{Z})$.*

Proof. — If $\varphi = \mathrm{sign}(f)$ with $f \in K^\bullet$ of valuation 0 or 1, then $\sigma_B(\varphi) = 2\,\mathrm{sign}(\lambda_B(f))$. □

Let K be a field of transcendence degree n over R and B a valuation ring of K. The ring B is said to be geometric if the restriction of the valuation v_B to R is trivial. All valuation rings are now assumed to be geometric.

Let t be an uniformizing parameter of B and $x_1, \dots, x_{n-1} \in B \setminus \mathfrak{m}_B$ such that $t, x_1, \dots, x_{n-1}$ make a transcendence basis of K over R. Each $\omega \in {\Omega^n_{K/R}}^{\bullet}$ can be written

$$t^{v_B(\omega)} u \, \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_{n-1} \wedge \mathrm{d}t$$

with $u \in B \setminus \mathfrak{m}_B$; $v_B(\omega)$ is independent of the choices of t and x_i, this is the valuation of ω. Let

$$\widetilde{\Omega} = \big\{\omega \in \Omega^n_{K/R} \mid v_B(\omega) = -1\big\}.$$

Then following for example [Ku], Chap. 17, exercise 1, we can construct a Poincaré-residue

$$\mathrm{Res}^*_B : \widetilde{\Omega} \longrightarrow {\Omega^{n-1}_{k_B/R}}^{\bullet}$$

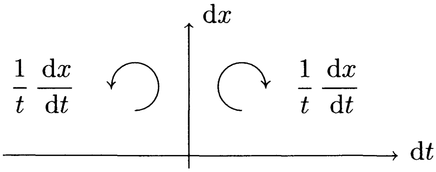
putting $\mathrm{Res}^*_B(\omega) = \overline{u} \, \mathrm{d}\overline{x}_1 \wedge \dots \wedge \mathrm{d}\overline{x}_{n-1}$ if ω is written as before. This is independent of the choices.

This residue is surjective: given a transcendence basis $\{y_1, \dots, y_{n-1}\}$ of k_B over R, let $x_1, \dots, x_{n-1}$ be the pullbacks of $y_1, \dots, y_{n-1}$ in $B \setminus \mathfrak{m}_B$; then $\{t, x_1, \dots, x_{n-1}\}$ is a transcendence basis of K over R. If $\tau = \overline{g} \, \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_{n-1}$ with $g \in B \setminus \mathfrak{m}_B$, then

$$\omega = (g/t) \, \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_{n-1} \wedge \mathrm{d}t$$

is such that $\mathrm{Res}^*_B(\omega) = \tau$.

Example. — If K is the function field of a real smooth variety V and B is the local ring of a 1-codimensional subvariety W, then τ gives in almost every point of W a local orientation and the constructed ω gives on V a local orientation which induces on W the orientation given by τ.



If $f \in K^\bullet$ and $\omega \in \widetilde{\Omega}$,

$$f\omega \in \widetilde{\Omega} \iff v_B(f) = 0 \quad \text{and} \quad \mathrm{Res}^*_B(f\omega) = \lambda_B(f)\,\mathrm{Res}^*_B(\omega).$$

Thus if β is an ordering of K compatible with B and α its specialization in k_B,

$$\mathrm{Res}^*_B(f\omega) \sim_\alpha \mathrm{Res}^*_B(\omega) \iff f\omega \sim_\beta \omega.$$

Then for $\alpha \in \Sigma_{k_B}$ and $\tau \in {\Omega^{n-1}_{k_B/R}}^\bullet$, the set $\{\omega \in \widetilde{\Omega} \mid \mathrm{Res}^*_B(\omega) \sim_\alpha \tau\}$ is contained in an equivalence class of ${\Omega^n_{K/R}}^\bullet$ for $\sim_\beta$.

Definition 7. — *Let H be a free B-module of rank 1. The formula*

$$\sigma_B(\varphi)\big(\alpha, \overline{\tau \otimes h \otimes \lambda_B(a)^\alpha}\big) = \sum_{\overline{\beta}=\alpha} \varphi\big(\beta, \overline{\omega \otimes h \otimes a^\beta}\big)$$

where $\mathrm{Res}^*_B(\omega) = \tau$ *defines a morphism (called boundary map)*

$$\sigma_B\colon \mathrm{Cons}\big(\Sigma_K^{\widehat{H\otimes_B K}}, \mathbb{Z}\big) \longrightarrow \mathrm{Cons}\big(\Sigma_{k_B}^{\widehat{H\otimes_B k_B}}, \mathbb{Z}\big).$$

We still have

$$\sigma'_B = \sigma_B/2 : \mathrm{AlgCons}\big(\Sigma_K^{\widehat{H\otimes_B K}}, \mathbb{Z}\big) \longrightarrow \mathrm{AlgCons}\big(\Sigma_{k_B}^{\widehat{H\otimes_B k_B}}, \mathbb{Z}\big).$$

Lemma 8. — *Let K be a field of transcendence degree n over R. Let $K \to L$ an algebraic extension of finite degree, B a discrete valuation ring of K, $B_1, \dots, B_m$ the valuation rings of L such that $B_\imath \cap K = B$, H a free B-module of rank 1. Then the following diagram (and the corresponding one with* AlgCons *and σ'_B) is commutative*:

$$\begin{tikzcd}
\mathrm{Cons}\big(\Sigma_L^{\widehat{H\otimes_B L}}, \mathbb{Z}\big) \arrow[r, "\oplus\sigma_{B_\imath}"] \arrow[d, "c_{L/K}"] & \bigoplus_{i=1}^{m} \mathrm{Cons}\big((\Sigma_{k_{B_\imath}}^{H\otimes_B k_{B_\imath}})^\wedge, \mathbb{Z}\big) \arrow[d, "\Sigma c_{k_{B_\imath}/k_B}"] \\
\mathrm{Cons}\big((\Sigma_{k_B}^{H\otimes_B k_B})^\wedge, \mathbb{Z}\big) \arrow[r, "\sigma_B"] & \mathrm{Cons}\big(\Sigma_K^{\widehat{H\otimes_B K}}, \mathbb{Z}\big)
\end{tikzcd}$$

Proof. — Let $h \in H^\bullet$, let $x_1, \dots, x_{n-1} \in B$ such that $\overline{x}_1, \dots, \overline{x}_{n-1}$ make a transcendence basis of k_B over R, and t an uniformizing parameter of B. Let

$$\omega = \mathrm{d}\overline{x}_1 \wedge \dots \wedge \mathrm{d}\overline{x}_{n-1} \in \Omega^{n-1}_{k_B/R};$$

then

$$\tau = (1/t)\,\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_{n-1} \wedge \mathrm{d}t \in \Omega^n_{K/R}$$

is such that $\mathrm{Res}^*_B(\tau) = \omega$. If $\tilde{\tau} = \tau \otimes 1 \in \Omega^n_{L/R}$ and, for each i,

$\widetilde{\omega}_i = \omega \otimes 1 \in \Omega^n_{k_{B_i}/R}$, then $\mathrm{Res}^*_{B_i}(\tilde{\tau}) = e_i\widetilde{\omega}_i$ where e_i is the ramification index of B_i over B: indeed there exists an uniformizing parameter t_i of B_i and a unit u_i of B_i such that $t = t_i^{e_i}u_i$; then $\mathrm{d}t/t = e_i\,\mathrm{d}t_i/t_i + \mathrm{d}u_i/u_i$ and $\tilde{\tau} = e_i(1/t_i)\,\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{n-1} \wedge \mathrm{d}t_i + \tau_i$ with $v_{B_i}(\tau_i) = 0$. Using shrinking isomorphisms it is equivalent to prove commutativity on the level of real spectrum without coefficients.

We have to show that for each $\alpha \in \Sigma_{k_B}$ and each $\gamma \in \Sigma_L$, $\gamma_{|K}$ is compatible with B and $\overline{(\gamma_{|K})} = \alpha$ if and only if there is a unique i in $\{1,\ldots,m\}$ such that γ is compatible with B_i and $(\overline{\gamma}^{\,i})_{|k_B} = \alpha$. If γ extends on L a pullback β of α in K, let B' be the convex hull for γ of the ring B; B' is one of the B_i since $B' \cap K$ is the convex hull of B for $\gamma_{|K} = \beta$ in K, *i.e.* B; it is the only one compatible with γ. Then the specialization $\overline{\gamma}^{\,i}$ of γ in $k_{B'}$ extends α.

On the other hand, if δ extends α to k_{B_i}, let t and t_i be uniformizing parameters of B and B_i such that $t = t_i^{e_i}u_i$; let γ_+, γ_- be the pullbacks of δ whose sign on t_i is respectively positive and negative, ε the sign of $\lambda_{B_i}(u_i)$ for δ, and β_+, β_- the pullbacks of α whose sign on t is respectively positive and negative. For a unit u of B, we have

$$\gamma_\pm(u) = \delta\big(\lambda_B(u)\big) = \alpha\big(\lambda_B(u)\big) = \beta_\pm(u).$$

We have $\varepsilon\gamma_+(t) = 1$ thus γ_+ extends β_ε; if e_i is odd, $\varepsilon\gamma_-(t) = -1$ thus γ_- extends $\beta_{-\varepsilon}$; if e_i is even, $\varepsilon\gamma_-(t) = 1$ thus γ_- extends β_ε. □

LEMMA 9. — *Let K be a field of transcendence degree n over R. Let $K \to L$ an extension of transcendence degree r, B a discrete valuation ring of K, B' a discrete valuation ring of L such that $B' \cap K = B$ and whose ramification index over B is* 1, *H a free B-module of rank* 1. *Then the following diagram (and the corresponding one with* AlgCons *and σ'_B) is commutative*:

$$\begin{array}{ccc}
\mathrm{Cons}\big((\Sigma_L^{H\otimes_B\Omega^r_{L/K}})^\wedge,\mathbb{Z}\big) & \xrightarrow{\ \sigma_{B'}\ } & \mathrm{Cons}\big((\Sigma_{k_{B'}}^{H\otimes_B\Omega^r_{k_{B'}/k_B}})^\wedge,\mathbb{Z}\big) \\
\Big\uparrow{\scriptstyle r_{L/K}} & & \Big\uparrow{\scriptstyle r_{k_{B'}/k_B}} \\
\mathrm{Cons}\big((\Sigma_{k_B}^{H\otimes_B k_B})^\wedge,\mathbb{Z}\big) & \xrightarrow{\ \sigma_B\ } & \mathrm{Cons}\big(\Sigma_K^{\widehat{H\otimes_B K}},\mathbb{Z}\big)
\end{array}$$

Proof. — First, we have to prove that the composition is well-defined: this means

$$\sigma_{B'} : \mathrm{Cons}\big(\Sigma_L^{\widehat{H\otimes_B\Omega^r_{L/K}}},\mathbb{Z}\big) \longrightarrow \mathrm{Cons}\big((\Sigma_{k_{B'}}^{H\otimes_B\Omega^r_{k_{B'}/k_B}})^\wedge,\mathbb{Z}\big).$$

First $\operatorname{tr}\deg_{k_B} k_{B'} = r$. As $e = 1$, $\mathfrak{m}_{B'} = \mathfrak{m}_{B'}^2 + \mathfrak{m}_B B'$ as they are both generated by an uniformizing parameter t of B and the exact sequence of [Ku], Corollary 6.5,

$$0 \to \mathfrak{m}_{B'}/(\mathfrak{m}_{B'}^2 + \mathfrak{m}_B B') \longrightarrow \Omega^1_{B'/B} \otimes_{B'} k_{B'} \longrightarrow \Omega^1_{k_{B'}/k_B} \to 0$$

gives an isomorphism

$$\Omega^1_{B'/B} \otimes_{B'} k_{B'} \xrightarrow{\sim} \Omega^1_{k_{B'}/k_B} : \ \mathrm{d}_{B'/B}x \otimes 1 \longmapsto \mathrm{d}_{k_{B'}/k_B}\overline{x}.$$

We also have $\Omega^1_{B'/B} \otimes_{B'} L \simeq \Omega^1_{L/K}$; $\Omega^1_{B'/B}$ is a free B'-module of rank r as if we choose $z_1, \dots, z_r \in B'$ such that $\overline{z}_1, \dots, \overline{z}_r$ make a transcendence basis of $k_{B'}$ over k_B, then $\mathrm{d}_{B'/B}z_1, \dots, \mathrm{d}_{B'/B}z_r$ is a B'-basis of $\Omega^1_{B'/B}$. If we put $H' = H \otimes_B \Omega^r_{B'/B}$, H' is a free B'-module of rank 1 and by definition $\sigma_{B'} : \operatorname{Cons}((\Sigma_L^{H' \otimes_{B'} L})^\wedge, \mathbb{Z}) \to \operatorname{Cons}((\Sigma_{k_{B'}}^{H' \otimes_{B'} k_{B'}})^\wedge, \mathbb{Z})$ which gives exactly what we want.

Choosing $h \in H^\bullet$, $x_1, \dots, x_{n-1} \in B$ such that $\overline{x}_1, \dots, \overline{x}_{n-1}$ make a transcendence basis of k_B over R, we define coefficients in $H \otimes_B \Omega^n_{K/R}$, $H \otimes_B \Omega^r_{L/K} \otimes_L \Omega^{n+r}_{L/R}$, $H \otimes_B \Omega^r_{k_{B'}/k_B} \otimes_{k_{B'}} \Omega^{n+r-1}_{k_{B'}/R}$ and $H \otimes_B \Omega^{n-1}_{k_B/R}$ such that using the shrinking isomorphisms, it is equivalent to show commutativity on the level of real spectrum without coefficients. It is enough to see that for each $\alpha \in \Sigma_{k_{B'}}$, if β^+ and β^- are the pullbacks of α in L, then $\beta^+{}_{|K}$ and $\beta^-{}_{|K}$ are the pullbacks of $\alpha_{|k_B}$ in K, which is true as on one hand for $u \in B \setminus \mathfrak{m}_B$, $u \in B' \setminus \mathfrak{m}_{B'}$ thus $\beta^\varepsilon{}_{|K}(u) = \alpha(u) = \alpha_{|k_B}(u)$, and on the other hand, $\beta^\varepsilon{}_{|K}(t) = \beta^\varepsilon(t) = \varepsilon$. $\square$

3. Semialgebraic chains on an algebraic variety.

For a scheme X, the set of n-dimensional (respectively n-codimensional) points of X will be denoted by $X_{(n)}$ (respectively $X^{(n)}$). The real spectrum of X obtained by glueing together the real spectra of its affine open subschemes will be denoted by X_r. For x a point of X, x_r will denote the real spectrum of its residual field $\kappa(x)$. If $\mathcal{L}$ is a line bundle over X, $\mathcal{L}(x) = \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is a 1-dimensional $\kappa(x)$-vector space. For another point y, we will write

$$x \succ y \quad \text{if } y \text{ is a specialization of } x,$$

i.e. $y \in \overline{\{x\}}$; if moreover y is 1-codimensional in $\overline{\{x\}}$, we will write $x \succ_1 y$.

3.1. Two complexes.

All the schemes we consider are essentially of finite type over the base field R. This means that they are localizations of schemes of finite type over R. Let X be such a scheme.

DEFINITION 10. — *We put:*

$$C_n(X) = \bigoplus_{x \in X_{(n)}} \mathrm{Cons}(\widehat{x}_r, \mathbb{Z}), \quad AC_n(X) = \bigoplus_{x \in X_{(n)}} \mathrm{AlgCons}(\widehat{x}_r, \mathbb{Z}).$$

For $\mathcal{L}$ a line bundle over X:

$$C_n(X, \mathcal{L}) = \bigoplus_{x \in X_{(n)}} \mathrm{Cons}(\widehat{x_r{}^{\mathcal{L}(x)}}, \mathbb{Z}),$$

$$AC_n(X, \mathcal{L}) = \bigoplus_{x \in X_{(n)}} \mathrm{AlgCons}(\widehat{x_r{}^{\mathcal{L}(x)}}, \mathbb{Z}).$$

Now we will define a boundary for those complexes. We will show that they are actually complexes in the next section.

For x and y points of X, let us define

$$(\partial_X)_y^x = \partial_y^x : \mathrm{Cons}(\widehat{x_r{}^{\mathcal{L}(x)}}, \mathbb{Z}) \longrightarrow \mathrm{Cons}(\widehat{y_r{}^{\mathcal{L}(y)}}, \mathbb{Z}).$$

Let $Z = \overline{\{x\}}$ and let $\pi : \widetilde{Z} \to Z$ be the normalization of Z: consider a covering of Z by open affine subschemes $\operatorname{Spec} A$. The normalization is obtained by glueing together the open affine schemes $\operatorname{Spec} \widetilde{A}$ where $\widetilde{A}$ is the integral closure of A; $\widetilde{A}$ is a finite A-algebra thus π is a finite morphism. If y is a specialization of x of codimension 1,

$$\partial_y^x = \sum_{\tilde{y} \in \pi^{-1}(y)} c_{\kappa(\tilde{y})/\kappa(y)} \circ \sigma_{\mathcal{O}_{\tilde{Z},\tilde{y}}}.$$

Else $\partial_y^x = 0$. If y is a specialization of x of codimension 1, let $U = \operatorname{Spec} A$ be an affine open subset of Z containing y such that A is noetherian. If $\widetilde{A}$ is the integral closure of A in $\kappa(x)$, $\pi^{-1}(U) = \operatorname{Spec} \widetilde{A}$. Let $\mathfrak{p}_y$ be the prime ideal of A corresponding to y, denote $\mathcal{B}_x(y)$ the set of discrete valuation rings of $\kappa(x)$ containing A and such that the center of the associated place in A is $\mathfrak{p}_y$. They are discrete valuation rings and in fact they are the $\widetilde{A}_{\tilde{\mathfrak{p}}}$ where the $\widetilde{\mathfrak{p}}$ are the prime ideals of $\widetilde{A}$ such that $\widetilde{\mathfrak{p}} \cap A = \mathfrak{p}_y$, *i.e.* the ideals corresponding to the $\tilde{y} \in \pi^{-1}(y)$ (indeed $\widetilde{A}$ is an integrally closed noetherian ring and thus a Krull ring, *cf.* [ZS], VI, §13). Thus we have $\partial_y^x = \sum_{B\in\mathcal{B}_x(y)} c_{k_B/\kappa(y)} \circ \sigma_B$.

Define still

$$\partial'^x_y = \partial_y^x/2 : \mathrm{AlgCons}(\widehat{x_r{}^{\mathcal{L}(x)}}, \mathbb{Z}) \to \mathrm{AlgCons}(\widehat{y_r{}^{\mathcal{L}(y)}}, \mathbb{Z}).$$

Sections 2.2 and 2.3 imply that this is well defined.

LEMMA 11. — *If* $\varphi \in \mathrm{Cons}(\widehat{x_r{}^{\mathcal{L}(x)}},\mathbb{Z})$, *then* $\partial_y^x\varphi = 0$ *for almost every* y (*this means*: *for all except a finite number of* y).

Proof. — It is sufficient to see that for Z normal, $\sigma_{\mathcal{O}_{Z,y}}\varphi = 0$ for almost every $y \in Z^{(1)}$. It is sufficient to see this for

$$\sigma_{\mathcal{O}_{Z,y}} : \mathrm{Cons}(\widehat{x_r}, \mathbb{Z}) \longrightarrow \mathrm{Cons}(\hat{y}_r, \mathbb{Z}).$$

Fix ω in $\Omega^n_{\kappa(x)/R}$ (where $n = \dim(x)$). We can assume $\varphi_\omega = \mathbf{1}_{\{f_1>0,\dots,f_k>0\}}$, $f_i \in \kappa(x)^\bullet$. Let v_y be the valuation associated to $\mathcal{O}_{Z,y}$. There is a finite number of y such that one of the $v_y(f_i)$ or $v_y(\omega)$ is nonzero (*cf.* [Ha], Lemma II 6.1). Let y be such that they are all zero, t an uniformizing parameter of $\mathcal{O}_{Z,y}$, $\omega = t\widetilde{\omega}$, $\tau = \mathrm{Res}^*_{\mathcal{O}_{Z,y}}(\widetilde{\omega})$. We have

$$\varphi_{\widetilde{\omega}} = \mathbf{1}_{\{t>0,f_1>0,\dots,f_k>0\}} - \mathbf{1}_{\{t<0,f_1>0,\dots,f_k>0\}}$$

and then from Proposition 5,

$$(\sigma_{\mathcal{O}_{Z,y}}\varphi)_\tau = \mathbf{1}_{\{g_1>0,\dots,g_k>0\}} - \mathbf{1}_{\{g_1>0,\dots,g_k>0\}} = 0$$

where $g_i = \lambda_{\mathcal{O}_{Z,y}}(f_i)$. □

Thus we can define a boundary map $\partial : C_{n+1}(X) \to C_n(X)$ by putting

$$\partial\Big(\sum_{x\in X_{(n+1)}} \varphi_x\Big) = \sum_{x\in X_{(n+1)}} \sum_y \partial_y^x \varphi_x$$

where $\varphi_x \in \mathrm{Cons}(\widehat{x_r{}^{\mathcal{L}(x)}}, \mathbb{Z})$ is zero except for a finite number of x. The same way we define a boundary $\partial' : AC_{n+1}(X) \to AC_n(X)$.

The boundary can be written directly on the real spectrum: for $\varphi \in C_n(X)$, $\alpha \in X_{r(n-1)}$ and $\omega \in \Omega^{n-1}_{\kappa(\mathrm{supp}(\alpha))/R}{}^\bullet$,

$$\partial\varphi(\alpha, \overline{\omega}^\alpha) = \sum_{\beta\in X_r|\beta\succ_1\alpha} \varphi(\beta, \overline{\tau}^\beta)$$

where if $Z = \overline{\{\mathrm{supp}(\beta)\}}$, Spec A is an affine open subscheme of Z containing $\mathrm{supp}(\alpha)$ and B_α is the convex hull of $A_{\mathrm{supp}(\alpha)}$ in $\mathrm{Frac}(A)$ for the ordering β, $\mathrm{Res}^*_{B_\alpha}(\tau) \sim_\alpha \omega$.

3.2. Semialgebraic chains.

Until the end of Section 3, R is a real closed field, X will denote an algebraic subset of R^p and $X_{(n)}$ denotes the set of n-dimensional irreducible algebraic subset of X. For such a $V \in X_{(n)}$, $\mathcal{R}(V)$ will denote the ring of regular functions on V, $\kappa(V)$ its function field, V_r the real spectrum of $\kappa(V)$. Then $C_n(X) = \bigoplus_{V\in X_{(n)}} \mathrm{Cons}(\widehat{V_r},\mathbb{Z})$ and $AC_n(X) = \bigoplus_{V\in X_{(n)}} \mathrm{AlgCons}(\widehat{V_r},\mathbb{Z})$ as the real spectrum of non real points is empty.

There is an injective map $j : V_r \longrightarrow \mathrm{Spec}_r\,\mathcal{R}(V)$ and an isomorphism $S \longmapsto \widetilde{S}$ of Boolean algebra between the set of semialgebraic subsets of V and the set of constructibles sets of $\mathrm{Spec}_r\,\mathcal{R}(V)$ (*cf.* [BCR], §7.2). We denote by $S^\sharp$ the constructible set $j^{-1}(\widetilde{S})$ in V_r. If C is a constructible set of V_r, denote by $\underline{C}$ the intersection of the closure of $j(C)$ in $\mathrm{Spec}_r\,\mathcal{R}(V)$ with V. Then $(\underline{C})^\sharp = C$. A *wall* of C is a $(n-1)$-dimensional irreducible component of the Zariski boundary of $\underline{C}$ (this means the Zariski closure of $\mathrm{Adh}(\underline{C}) \setminus \mathrm{Int}(\underline{C})$ where the adherence and interior are taken for the Euclidean topology). If $\varphi \in \mathrm{Cons}(V_r,\mathbb{Z})$, φ can be written as $\sum_{i\in\mathbb{Z}} i\mathbf{1}_{C_i}$ where $C_i = \varphi^{-1}(i)$ (almost all the C_i are empty); a *wall* of φ is a wall of one of the C_i and $\mathcal{M}(\varphi)$ is the set of walls of φ.

In [KS], Chapter 9, the group of semialgebraic n-chains is the group generated by symbols $[S]$, where S ranges through oriented Nash n-dimensional manifolds in X, with the relations:

- $[S^a] = -[S]$ if S^a is S with the opposite orientation;
- $[S \sqcup S'] = [S] + [S']$;
- $[S] = [S']$ if S' is dense in S with the induced orientation.

Using stratifications, it is easy to see that these two definitions are equivalent: for a function $\varphi \in \mathrm{Cons}(\widehat{V_r},\mathbb{Z})$, fix $\omega \in {\Omega^n_{\kappa(V)/R}}^\bullet$, and a Nash stratification compatible with the singular points of V, the set of zeroes and poles of ω, the walls of φ_ω; then the associated n-chain is $\sum m_s[S]$ where S are n-dimensional strata oriented by ω and m_S is the value of φ_ω on $S^\sharp$; this is independent of the choices of ω and the stratification. The converse application is done exactly the same way. Also, the boundaries are the same.

If we choose for the points the positive orientation (leaving the point is counted negatively) we have the following

PROPOSITION 12 (Reciprocity). — *Let X a compact curve. Then for each* 1*-chain C of X, the sum of the values of ∂C is zero.*

Proof. — If $C = [S]$ where $S \subset X$ is a 1-dimensional connected Nash manifold, then either S is homeomorphic to a circle and in this case $\partial[S] = 0$, or S is homeomorphic to a segment, and in this case $\partial[S]$ consists of two points (possibly coinciding), one with the value 1, the other one with the value -1, thus summing the values we obtain 0. □

PROPOSITION 13. — $\partial \circ \partial = 0$.

Proof. — Let first $[S]$ be a 2-chain of support Z, and z a point of Z. It is sufficient to show that the component of $\partial \circ \partial[S]$ in z is zero. Embed Z in an R^n and let $S(z,\varepsilon) = \{u \in R^n \mid d(u,z) = \varepsilon\}$ (d Euclidean distance) be the sphere, and $L = L(z,\varepsilon) = S(z,\varepsilon) \cap Z$ the link of z. For ε sufficiently small, $L(z,\varepsilon)$ intersects S and the structure of $L(z,\varepsilon)$ does not change taking smaller ε. Let $[S']$ be the chain given by $S = L \cap S$ and the orientation τ such that, if $\mathrm{d}r$ is the orientation to the exterior of the sphere and ω the orientation of S, we have $\mathrm{d}r \wedge \tau = \omega$. Then the value of $\partial \circ \partial[S]$ on z is exactly the sum of the values of the boundaries of $[S']$, which is zero using the property of reciprocity applied to L.

If now $[S]$ is a chain of dimension greater than 2 of support Z, we must see if $\partial \circ \partial[S]$ is zero on each 2-codimensional subvariety of Z. Let $\partial \circ \partial[S] = \sum m_i[S_i]$, and let z be a point of S_i. Let P be a 2-dimensional subvariety of Z transverse to S_i in z. Refining it if necessary, we may assume that the stratification of Z compatible with the S and S_i verifies Whitney-condition (b); then using [G], there exists for each i a neighborhood U_i of S_i in Z such that U_i is isomorphic as stratified space to $P \cap U_i \times S_i$. Let $[S']$ the 2-chain of P given by $S' = S \cap P$ and the orientation τ of S', such that, if τ_i is the orientation of S_i and ω the orientation of S, we have $\tau \wedge \tau_i = \omega$. Then the value of $\partial \circ \partial[S]$ on S_i is the same that the value of $\partial \circ \partial[S']$ on z, and thus is zero. □

Thus we get a homology denoted by $H_*(X)$.

PROPOSITION 14. — *$H_*(X)$ is isomorphic to the Borel-Moore homology of X* (*cf.* [BCR], 11.7.13).

Proof. — We use the algebraic Alexandrov compactification X' of X, and triangulations of subvarieties of X' compatible with $X' \setminus X$. It is quite easy this way to see that the computation is the same in both cases. □

As $\partial \circ \partial = 0$ then $\partial' \circ \partial' = 0$ and thus we get a homology $H_*^{\mathrm{ac}}(X)$.

Observe that $H_0(X) = H_0^{\mathrm{ac}}(X) = \mathbb{Z}^{\mathrm{bc}(X)}$ where $\mathrm{bc}(X)$ is the number of closed and bounded semialgebraically connected components of X.

Let us compare $H_*^{\mathrm{ac}}(X)$ with $H_*(X)$.

There is a first morphism

$$\psi_n : C_n(X) \longrightarrow AC_n(X)$$

given by $\psi_n(\varphi) = 2^n\varphi$. Indeed if V a n-dimensional irreducible algebraic subset of X, every constructible function on V becomes algebraically constructible after multiplication by 2^n. As

$$\partial' \circ \psi_n(\varphi) = 2^{n-1}\partial(\varphi) = \psi_{n-1} \circ \partial,$$

we obtain a morphism $H_n(X) \to H_n^{\mathrm{ac}}(X)$. The cokernel is of torsion 2^m with $m < n$. In every case ψ_n is injective at chain level. If $n = \dim(X)$ then ψ_n is injective at homology level.

Now we define a second morphism: put $Z_n := \operatorname{Ker}\partial$, $Z_n^{\mathrm{ac}} := \operatorname{Ker}\partial'$, $B_n = \operatorname{Im}\partial$ and $B_n^{\mathrm{ac}} = \operatorname{Im}\partial'$, we have $Z_n^{\mathrm{ac}} \hookrightarrow Z_n$ and $2B_n^{\mathrm{ac}} \hookrightarrow B_n$, thus there is a morphism

$$\theta_n : H_n^{\mathrm{ac}}(X) \longrightarrow H_n(X)/\big(H_n(X)\big)_2$$

where $(\mathrm{H}_n(X))_2$ is the 2-torsion of H_n. When is θ_n injective? This means: if we have $\varphi \in C_{n+1}(X)$ such that $\partial\varphi$ is even and $\partial\varphi/2 \in AC_n(X)$, when does there exist $\varphi' \in AC_{n+1}(X)$ such that $\partial\varphi' = \partial\varphi$? It is enough to see it on each irreducible component of X.

θ_0 is always an isomorphism and $\theta_{\dim}(X)$ is always injective.

If X is a smooth d-dimensional compact connected irreducible variety and $\varphi \in \mathrm{Cons}(\widehat{X}_r, \mathbb{Z})$ is such that $\partial\varphi$ is even, then φ equals an algebraically constructible chain modulo 4. First φ is constant modulo 2: choose $\omega \in \Omega^d_{\kappa(X)/R}{}^{\bullet}$; if V is a wall of φ_ω, then $\mathcal{B}_X(V) = \{B\}$ where B is the local ring of V in X; then if V is a wall of φ_ω mod 2, we have

$$\partial_V\varphi(\alpha, \tau) = \sum_{\overline{\beta}=\alpha} \varepsilon(\alpha)\varphi_\omega(\beta) \neq 0 \mod 2.$$

We can assume φ even and put $\psi = \frac{1}{2}\varphi$. Put

$$S = \big\{\psi \equiv 1 \mod 2\big\}.$$

S is principal: indeed if $V_1, \ldots, V_m$ are the walls of ψ mod 2, we have $\sum_{i=1}^{m}[V_i] = 0$ in $H_{n-1}(X, \mathbb{Z}/2\mathbb{Z})$ as this is the boundary of ψ mod $2 \in H_n(X, \mathbb{Z}/2\mathbb{Z})$. From Proposition 12.4.6 of [BCR] there exists $f \in \kappa(X)$ changing sign passing through the V_i's; we can assume $S = \{f > 0\}$, then $\psi \equiv (1 + \operatorname{sign} f)/2 \bmod 2$ thus $\varphi_\omega \equiv 1 + \operatorname{sign} f \bmod 4$. In particular if $d = 2$, φ is algebraically constructible and θ_1 is injective in the case of a smooth compact connected surface.

In either dimension following the proof of "generic criterion" of I. Bonnard [B], if the walls of φ_ω are non singular and normal crossing we can show that this is algebraically constructible .

The assumptions on the walls can be removed if dimension of X is 2 or 3. Also, if X is non necessarily connected, but orientable, we can do the same on each connected component and we can add $\xi_\omega = \mathbf{1}_{\{\varphi \equiv 1 \bmod 2\}}$ to φ_ω so that it is 0 mod 2 (this add no component to the boundary if we choose ω without zero nor pole). Finally we get:

Proposition 15. — *If X is a 2 or 3-dimensional smooth compact algebraic variety then θ_{d-1} is injective.*

Now we compare algebraically constructible homology with the usual algebraic homology (*cf.* [BCR], §11.3). Changing the coefficients of $AC_n(X)$ to $\mathbb{Z}/2\mathbb{Z}$, we obtain an homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$ that is not the algebraic homology $\mathrm{H}_*^{\mathrm{alg}}(X, \mathbb{Z}/2\mathbb{Z})$.

Example. — If $X = P^2(\mathbb{R})$, we have $\mathrm{H}_2^{\mathrm{ac}}(X, \mathbb{Z}/2\mathbb{Z}) = 0$, when $\mathrm{H}_2^{\mathrm{alg}}(X, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

Indeed, an algebraic 2-chain on $P^2(\mathbb{R})$ can be seen as a generically algebraically constructible function φ on $\mathbb{R}^2$ canonically oriented. The boundary on an oriented affine curve $\mathcal{C}$ is then computed doing half the difference between the value of φ on the side inducing the orientation of $\mathcal{C}$ and the value on the other side; the boundary on the line at infinity is computed doing half the sum of the values on both "sides".

If $\mathcal{C}$ is an affine curve, let $\partial_{\mathcal{C}}\varphi$ be half the difference of φ along $\mathcal{C}$ (it has integral values). If we want the corresponding element of

$$AC_1(X, \mathbb{Z}/2\mathbb{Z}) = AC_1(X) \otimes \mathbb{Z}/2\mathbb{Z} \simeq AC_1(X)/2AC_1(X)$$

to be zero, we need that $\partial_{\mathcal{C}}\varphi \in 2AC_1(X)$, *i.e.* $\partial_{\mathcal{C}}\varphi \equiv \mathrm{Cte} \bmod 4$ where $\mathrm{Cte} = 0$ or 2.

If φ is a cycle, two values of φ differ by 0 or 4 modulo 8; let a be a value of φ, f a polynomial positive where $\varphi \equiv a \bmod 8$, and negative where $\varphi \equiv a+4 \bmod 8$; then generically $\varphi = a + 2(1+\operatorname{sign} f) + 8\psi$ (ψ integer valued). Now the computation of the boundary of φ along the line at infinity gives a modulo 2; then $a \equiv 0 \bmod 2$ and φ is zero in $AC_2(X, \mathbb{Z}/2\mathbb{Z})$.

4. Generalizations. Functoriality.

In this section, R is not necessarily a real closed field. Remember that every scheme is essentially of finite type over R.

4.1. The complexes in the general case.

We prove that $(C(X,\mathcal{L}),\partial)$ and $(AC(X,\mathcal{L}),\partial')$ are actually complexes for general X. They also verify a reciprocity property.

PROPOSITION 16. — *One has $\partial \circ \partial = 0$. Hence we get two chain complexes $(C_*(X,\mathcal{L}),\partial)$ and $(AC_*(X,\mathcal{L}),\partial')$, giving two homologies $H_*(X,\mathcal{L})$ and $H_*^{\mathrm{ac}}(X,\mathcal{L})$.*

Proof. — We have to show that for R non necessarily real closed (this is true for real closed fields from Proposition 13), for X integral local of dimension 2, and x the generic point of X and $z \in X_{(0)}$, we have

$$\sum_{y \in X_{(1)}} \partial_z^y \partial_y^x = 0.$$

Let $\alpha \in \Sigma_R$. For $\varphi \in \mathrm{Cons}(\widehat{x_r}, \mathbb{Z})$, the function $\sum_{y\in X_{(1)}} \partial_z^y \partial_y^x \varphi$ is in $\mathrm{Cons}(\widehat{z_r}, \mathbb{Z})$. Let $m = \operatorname{tr\,deg}_R \kappa(z)$. Let $\alpha' \in z_r$ extending α and let ω be a nonzero element in $\Omega^m_{\kappa(z)/R}$. One has

$$\sum_{y\in X_{(1)}} \partial_z^y \partial_y^x \varphi(\alpha', \overline{\omega}^{\alpha'}) = \sum_{\gamma \in X_r^{(0)} | \gamma \rightsquigarrow \alpha'} \varphi(\gamma, \overline{\tau}^{\gamma})$$

where $\gamma \rightsquigarrow \alpha'$ signifies $\gamma \succ_1 \beta \succ_1 \alpha'$ and $\tau \in \Omega^{m+2}_{\kappa(x)/R}$ is such that $\mathrm{Res}^*_{B_\alpha}(\mathrm{Res}^*_{B_\beta} \tau) \sim_{\alpha'} \omega$ (for the meaning of notation B_α, see the end of Subsection 3.1).

Put $X_{\kappa(\alpha)} = X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, \kappa(\alpha)$. Let $x_\alpha^1, \dots, x_\alpha^{p_\alpha}$ be the points of $X^{(0)}_{\kappa(\alpha)}$ and $X_\alpha^i = \overline{\{x_\alpha^i\}}$. X_α^i is integral of dimension 2 on $\kappa(\alpha)$. We have

$$\kappa(x) \otimes_R \kappa(\alpha) = \prod_{i=1}^{p_\alpha} \kappa(x_\alpha^i), \quad x_r = \bigsqcup_{\alpha \in \Sigma_R} \bigsqcup_{i=1}^{p_\alpha} (x_\alpha^i)_r\,;$$

on the other hand

$$\Omega^{m+2}_{\kappa(x)/R} \otimes_R \kappa(\alpha) \hookrightarrow \prod_{i=1}^{p_\alpha} \Omega^1_{\kappa(x^i_\alpha)/R}$$

and if $\tau \in \Omega^{m+2}_{\kappa(x)/R}$, $\tau \otimes 1 = (\tau^1_\alpha, \ldots, \tau^{p_\alpha}_\alpha)$; thus we can define $\varphi^i_\alpha \in \mathrm{Cons}(\widehat{(x^i_\alpha)_r}, \mathbb{Z})$ by $(\varphi^i_\alpha)_{\tau^i_\alpha} = (\varphi_\tau)_{|(x^i_\alpha)_r}$.

Let z^i_α be the inverse image of z in X^i_α by $X_{\kappa(\alpha)} \to X$. Then

$$\prod_{i=1}^{p_\alpha} \kappa(z^i_\alpha) = \kappa(z) \otimes_R \kappa(\alpha), \quad \Omega^m_{\kappa(z)/R} \otimes_R \kappa(\alpha) \hookrightarrow \prod_{i=1}^{p_\alpha} \Omega^m_{\kappa(z^i_\alpha)/R}$$

then $\omega \otimes 1 = (\omega^1_\alpha, \ldots, \omega^{p_\alpha}_\alpha)$ and $z_r = \bigsqcup_{\alpha \in \Sigma_R} \bigsqcup_{i=1}^{p_\alpha} (z^i_\alpha)_r$.

Let i be such that $\alpha' \in (z^i_\alpha)_r$. Then

$$\sum_{y \in X_{(1)}} \partial^y_z \partial^x_y \varphi(\alpha', \overline{\omega}^{\alpha'}) = \sum_{y \in (X^i_\alpha)_{(1)}} \partial^y_{z^i_\alpha} \partial^{x^i_\alpha}_y \varphi^\alpha_i(\alpha', \overline{\omega}^{i\alpha'}_\alpha).$$

Applying the property for each X^i_α over $\kappa(\alpha)$ shows it for X. □

LEMMA 17. — *Let $f: X \to X'$ proper; if $x \in X$ and $x' = f(x)$ are such that* $\dim(x) = \dim(x')$, *let z' be a specialization of x' of codimension* 1. *Let for $B' \in \mathcal{B}_{f(x)}(z')$,*

$$\mathcal{B}_{B'} = \{B \textit{ real valuation ring of } \kappa(x) \mid B \cap \kappa(f(x)) = B'\}.$$

Then

$$\bigsqcup_{y \in f^{-1}(z')} \mathcal{B}_x(y) = \bigsqcup_{B' \in \mathcal{B}_{x'}(z')} \mathcal{B}_{B'}.$$

Proof. — Let $Z = \overline{\{x\}}$ and $Z' = \overline{\{x'\}}$, V' an affine open subset of Z' containing z', $A' = \mathcal{O}_{Z'}(V')$, $A = \mathcal{O}_Z(f^{-1}(V'))$; we shall denote by $\mathfrak{p}_y$ or $\mathfrak{p}'_y$ the points of $\operatorname{Spec} A$ or $\operatorname{Spec} A'$ corresponding to $y \in X$ or X'.

Let $y \in f^{-1}(z')$. Then $\mathfrak{p}_y \cap A' = \mathfrak{p}'_{z'}$; let $B \in \mathcal{B}_x(y)$, we have $\mathfrak{m}_B \cap A = \mathfrak{p}_y$; if $B' = B \cap \kappa(x')$ we then have $\mathfrak{m}_{B'} \cap A' = \mathfrak{p}'_{z'}$ and thus $B' \in \mathcal{B}_{x'}(z')$.

Conversely, for $B' \in \mathcal{B}_{x'}(z')$, let $B \in \mathcal{B}_{B'}$; we have a morphism $\operatorname{Spec} \kappa(x) \to Z$; the inclusion $\kappa(x') \subset \kappa(x)$ is such that B dominates $A'_{\mathfrak{p}'_{z'}}$ (indeed $B \cap A'_{\mathfrak{p}'_{z'}} = B' \cap A_{\mathfrak{p}'_{z'}} = \mathfrak{p}'_{z'}$ as $B' \in \mathcal{B}_{x'}(z')$) thus gives a morphism

$\operatorname{Spec} B \to Z'$ (*cf.* [Ha], Lemma II, 4.4) such that the diagram

$$\begin{array}{ccc} \operatorname{Spec}\kappa(x) & \longrightarrow & Z \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ \operatorname{Spec} B & \longrightarrow & Z' \end{array}$$

commutes. The valuative criterion for properness (see [Ha], Theorem II.4.7) gives a unique morphism $\operatorname{Spec} B \to Z$ such that the whole diagram commutes and then a unique point y, specialization of x, such that $f(y) = z'$ and that $B \in \mathcal{B}_x(y)$. □

PROPOSITION 18 (Reciprocity). — *Let X an integral proper curve over R. Let x be the generic point of X, we have*

$$\sum_{y \in X_{(0)}} c_{\kappa(y)/F} \circ \partial_y^x = 0.$$

Proof. — If R is real closed, this is Proposition 12. If R is not real closed, let $\alpha \in \Sigma_R$ and $\omega \in R \hookrightarrow \kappa(\alpha)$. For $\varphi \in \operatorname{Cons}(\widehat{x_r}, \mathbb{Z})$, $\sum_{y \in X_{(0)}} c_{\kappa(y)/R} \circ \partial_y^x \varphi$ is in $\operatorname{Cons}(\Sigma_R^R, \mathbb{Z})$. Then

$$\Big(\sum_{y \in X_{(0)}} c_{\kappa(y)/R} \circ \partial_y^x \varphi\Big)(\alpha, \overline{\omega}^{\alpha}) = \sum_{\beta \in X_r{}^{(0)} | \beta \succ \alpha} \varphi(\beta, \overline{\tau}^{\beta})$$

where $\beta \succ \alpha$ means $\beta \succ \gamma$ with γ of dimension 0 extending α in the extension $R \to \kappa(\operatorname{supp}(\gamma))$ and τ is such that $\operatorname{Res}^*_{B_\gamma}(\tau) = \omega$.

Let $X_{\kappa(\alpha)} = X \times_{\operatorname{Spec} R} \operatorname{Spec} \kappa(\alpha)$, $x_\alpha^1, \ldots, x_\alpha^{p_\alpha}$ the points of $X_{\kappa(\alpha)}^{(0)}$ and $X_\alpha^i = \overline{\{x_\alpha^i\}}$. X_α^i is an integral proper curve on $\kappa(\alpha)$ (*cf.* [Ha], II, Corollary 4.8).

Define φ_α^i as in proof of Proposition 16.

We have

$$\begin{aligned} \sum_{\beta \in X_r{}^{(0)} | \beta \succ \alpha} \varphi(\beta, \overline{\tau}^{\beta}) &= \sum_{i=1}^{p_\alpha} \sum_{\beta \in (x_\alpha^i)_r | \beta \succ \alpha} \varphi(\beta, \overline{\tau}^{\beta}) \\ &= \sum_{i=1}^{p_\alpha} \sum_{\beta \in (X_\alpha^i)_r{}^{(0)} | \beta \succ \alpha} \varphi_\alpha^i(\beta, \overline{\tau_\alpha^i}^{\beta}) \\ &= \sum_{i=1}^{p_\alpha} \sum_{y \in (X_\alpha^i)_{(0)}} c_{\kappa(y)/\kappa(\alpha)} \circ \partial_y^{x_\alpha^i} \varphi_i^\alpha(\mathrm{pt}, \omega) \end{aligned}$$

where 'pt' is the unique point of $\Sigma_{\kappa(\alpha)}$. The property comes then from reciprocity for each X_α^i. □

4.2. Pushforward.

Let $f : X \to Y$ be a morphism of finite type and $\mathcal{L}$ a line bundle over Y. For every $x \in X$, f induces $\kappa(f(x)) \to \kappa(x)$ which is an algebraic extension of finite degree if $\dim(f(x)) = \dim(x)$. For a point $y \in Y$, let us define

$$(f_*)^x_y : \operatorname{Cons}(x_r^{\widehat{f^*\mathcal{L}(x)}}, \mathbb{Z}) \longrightarrow \operatorname{Cons}(y_r^{\widehat{\mathcal{L}(y)}}, \mathbb{Z}).$$

Put $(f_*)^x_y = c_{\kappa(x)/\kappa(f(x))}$ if $y = f(x)$ and $\dim(y) = \dim(x)$, $(f_*)^x_y = 0$ else. We obtain for each n,

$$(f_*)_n : C_n(X, f^*\mathcal{L}) \longrightarrow C_n(Y, \mathcal{L}),$$

$$(f_*)_n : AC_n(X, f^*\mathcal{L}) \longrightarrow AC_n(Y, \mathcal{L}).$$

It is functorial: $(g \circ f)_* = g_* \circ f_*$ if $g : Y \to Z$.

PROPOSITION 19. — *If f is proper, f_* commutes with the boundary ∂. Thus f induces morphisms $(f_*)_n : H_n(X, f^*\mathcal{L}) \to H_n(Y, \mathcal{L})$ and $(f_*)_n : H_n^{\mathrm{ac}}(X, f^*\mathcal{L}) \to H_n^{\mathrm{ac}}(Y, \mathcal{L})$.*

Proof. — First assume that $x \in X$ is such that $f(x)$ is of the same dimension, and let z be a specialization of $f(x)$ of codimension 1 and $y_1, \ldots, y_m$ the specializations of x of codimension 1 such that $f(y_i) = z$. We have to compare $\sum_{i=1}^m (f_*)^{y_i}_z \partial^x_{y_i}$ and $\partial_z^{f(x)} (f_*)^x_{f(x)}$.

$$\sum_{i=1}^m (f_*)^{y_i}_z \partial^x_{y_i} = \sum_{i=1}^m c_{\kappa(y_i)/\kappa(z)} \sum_{B \in \mathcal{B}_x(y_i)} c_{k_B/\kappa(y_i)} \sigma_B;$$

$$\partial_z^{f(x)} (f_*)^x_{f(x)} = \sum_{B' \in \mathcal{B}_{f(x)}(z)} c_{k_{B'}/\kappa(z)} \sigma_{B'} c_{\kappa(x)/\kappa(f(x))}.$$

Using Lemma 17, we have to show for each $B' \in \mathcal{B}_{f(x)}(z)$ the equality:

$$c_{k_{B'}/\kappa(z)} \sigma_{B'} c_{\kappa(x)/\kappa(f(x))} = \sum_{i=1}^m c_{\kappa(y_i)/\kappa(z)} \sum_{B \in \mathcal{B}_x(y_i) \cap \mathcal{B}_{B'}} c_{k_B/\kappa(y_i)} \sigma_B.$$

But for $B \in \mathcal{B}_x(y_i) \cap \mathcal{B}_{B'} = \mathcal{B}_i$, we have

$$c_{\kappa(y_i)/\kappa(z)} c_{k_B/\kappa(y_i)} = c_{k_{B'}/\kappa(z)} c_{k_B/k_{B'}}$$

and

$$\sum_{i=1}^m c_{\kappa(y_i)/\kappa(z)} \Big(\sum_{B \in \mathcal{B}_i} c_{k_B/\kappa(y_i)} \sigma_B \Big) = c_{k_{B'}/\kappa(z)} \sum_{i=1}^m \sum_{B \in \mathcal{B}_i} c_{k_B/k_{B'}} \sigma_B.$$

Then it is enough to show

$$\sum_{i=1}^{m} \sum_{B \in \mathcal{B}_i} c_{k_B/k_{B'}} \sigma_B = \sigma_{B'} c_{\kappa(x)/\kappa(f(x))}.$$

This comes from Lemma 8.

Second case: we assume now that $\dim(f(x)) = \dim(x) - 1$; then $(f_*)^x_{f(x)} = 0$. Let $z = f(x)$. We have to show that $\sum_{y \in S} (f_*)^y_z \partial^x_y = 0$ (*) where S is the set of specializations of x of codimension 1 whose image is z. Restricting f to $\overline{\{x\}}$, we obtain a proper morphism still denoted by $f : \overline{\{x\}} \longrightarrow \overline{\{z\}}$ and the computation of (*) is exactly the same. Now each $y \in S$ belongs to the fibre

$$U = \overline{\{x\}}_{\kappa(z)} = \overline{\{x\}} \times_{\overline{\{z\}}} \operatorname{Spec}(\kappa(z)) \xrightarrow{f} \operatorname{Spec}(\kappa(z))$$

and $(\partial_Y)^x_y = (\partial_U)^x_y$ and $y \in S \Leftrightarrow y \in U_{(0)}$. Thus we have to show $\sum_{y \in U_{(0)}} c_{\kappa(y)/\kappa(z)} \circ \partial^x_y = 0$. This is exactly reciprocity for the proper integral curve U on $\kappa(z)$.

In the third case, *i.e.* if $\dim(f(x)) < \dim(x) - 1$, we have $(f_*)^x_{f(x)} = 0$ and for each specialization y of x of codimension 1, $\dim(f(y)) \leq \dim(y) - 1$; then $(f_*)^y_{f(y)} = 0$. □

4.3. Pullback.

Let X, Y be schemes of finite type over R. Let $g : X \to Y$ be a smooth regular morphism of relative dimension p. Let $\mathcal{L}$ be a line bundle over Y. Then (*cf.* [S], (4.2.1)) for each x of dimension $n + p$, we have a canonical isomorphism

$$\Omega^p_{X/Y}(x) \simeq \Omega^p_{\kappa(x)/\kappa(g(x))}.$$

But we have a restriction for such points (*cf.* §2.2)

$$r_{\kappa(x)/\kappa(g(x))} : \operatorname{Cons}\big((g(x)^{\mathcal{L}(g(x))}_r)^\wedge, \mathbb{Z}\big) \longrightarrow \operatorname{Cons}(\widehat{x^{H_x}_r}, \mathbb{Z})$$

where $H_x = g^*\mathcal{L}(x) \otimes_{\kappa(x)} \Omega^p_{\kappa(x)/\kappa(g(x))}$. Thus we can define a morphism

$$g^* : C_n(Y, \mathcal{L}) \longrightarrow C_{n+p}(X, g^*\mathcal{L} \otimes \Omega^p_{X/Y})$$

by the formula $(g^*)^x_{g(x)} = r_{\kappa(x)/\kappa(g(x))}$ and $(g^*)^y_x = 0$ if $g(x) \neq y$.

We have then $(g^*) : AC_n(Y, \mathcal{L}) \longrightarrow AC_{n+p}(X, g^*\mathcal{L} \otimes \Omega^p_{X/Y})$. If $g' : Y \to Z$ is another smooth regular morphism, then $(g' \circ g)^* = g^* \circ g'^*$ (*cf.* [S]).

LEMMA 20. — *Consider a Cartesian square:*

$$\begin{tikzcd} Y \times_X Z \arrow[r, "g'"] \arrow[d, "f'"'] & Z \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

where g is smooth of relative dimension p. Then we have $g^* f_* = f'_* g'^*$.

Proof. — Let $z \in Z_{(n)}$, $y \in Y_{(n+p)}$. If $f(z) \neq g(y)$, we have $(g^*)_y^{f(z)} = 0$, and on the other hand for each $u \in g'^{-1}(z)$, we have $f'(u) \neq y$ thus $(f'_*)_y^u = 0$ and then $(g^* f_*)_y^z = (f'_* g'^*)_y^z = 0$. Now if $f(z) = g(y) = x$, then x is n-dimensional as $\dim(x) \leq \dim(z)$ and $\dim(y) \leq \dim(x) + p$ (g is smooth of relative dimension p); $\kappa(z)$ is then an algebraic extension of finite degree of $\kappa(x)$. As the constructions of pullback and pushforward are local, we can replace respectively X, Z, Y, $Y \times_X Z$ by $\operatorname{Spec} \kappa(x)$, $\operatorname{Spec} \kappa(z)$, $\operatorname{Spec} \kappa(y)$, $\operatorname{Spec} \kappa(y) \otimes_{\kappa(x)} \kappa(z)$ and the property comes directly from Lemma 4. □

PROPOSITION 21. — *If $g: X \to Y$ is étale, g^* commutes with the boundary ∂. Thus g induces pullbacks $g^*: H_n(Y, \mathcal{L}) \to H_n(X, g^*\mathcal{L})$ and $g^*: H_n^{\mathrm{ac}}(Y, \mathcal{L}) \to H_n^{\mathrm{ac}}(X, g^*\mathcal{L})$.*

Proof. — Let $y \in Y_{(n)}$, $x \in X_{(n-1)}$, $z = g(x)$. Let S be the set of points of $g^{-1}(y)$ of codimension 0 specializing to x. Then we have

$$(g^* \circ \partial_Y)_x^y = (g^*)_x^z (\partial_Y)_z^y \quad \text{and} \quad (\partial_X \circ g^*)_x^y = \Big(\sum_{x' \in S} \partial_X\Big)_x^{x'} (g^*)_{x'}^y.$$

If z is not a specialization of y, $(g^*)_x^z = 0$ and S is empty thus everything is zero. Then assume $z \succ y$; this is actually a specialization of codimension 1 as $\dim(x) = \dim(z)$ (g is étale). We still can replace Y with $\overline{\{y\}}$ and X with $X \times_Y \overline{\{y\}}$. We also can assume Y normal; indeed let $\pi : \widetilde{Y} \to Y$ the normalization; in the Cartesian square

$$\begin{tikzcd} Z = X \times_Y \widetilde{Y} \arrow[r, "g'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

we have from Lemma 20, $g^*\pi_* = \pi'_* g'^*$. Thus

$$\pi'_*(\partial_Z \circ g'^*) = \partial_X(\pi'_* \circ g'^*) = (\partial_X g^*)\pi_*,$$
$$\pi'_*(g'^* \circ \partial_{\widetilde{Y}}) = (g^*\pi_*)\partial_{\widetilde{Y}} = (g^*\partial_Y)\pi_*$$

(using the commutation of boundary with pushforward). As for $\tilde{y} \in \pi^{-1}(y)$ and $\tilde{x} \in \pi^{-1}(x)$, $(\pi_*)^{\tilde{y}}_y = c_{\kappa(\tilde{y})/\kappa(y)}$ and $(\pi_*)^{\tilde{x}}_x = c_{\kappa(\tilde{x})/\kappa(x)}$ are isomorphisms, it is enough to show that $(\partial_Z \circ g'^*)^{\tilde{y}}_{\tilde{x}} = (g'^* \circ \partial_{\widetilde{Y}})^{\tilde{y}}_{\tilde{x}}$.

So assume we are in the case where $g : X \to Y = \overline{\{y\}}$ is étale with Y normal. Let $B = \mathcal{O}_{Y,z}$ and $B' = \mathcal{O}_{X,x}$. The scheme $\overline{Y} = \operatorname{Spec} B$ with the structure inherited from Y is integral of dimension 1. It contains the points z and y, and we have clearly $(\partial_Y)^x_y = (\partial_Z)^x_y$, thus we can replace Y with $\overline{Y}$; we can also replace X with $\overline{X}$ =SpecB'. The ring B is a discrete valuation ring, and B' a local ring of dimension ≤ 1. Let $\bar{g} : \overline{X} \longrightarrow \overline{Y}$ be the morphism induced by g; it is flat and non ramified, thus B' is also a discrete valuation ring; $X = \overline{\{x'\}}$ and $(g^* \circ \partial_Y)^y_x = r_{\kappa(x)/\kappa(z)} \circ \sigma_B$ and $(\partial_X \circ g^*)^y_x = \sigma_{B'} \circ r_{\kappa(x')/\kappa(y)}$ and the equality comes from Lemma 9. □

COROLLARY 22 (Long exact sequence). — *Let F a closed Zariski subset in X, $U = X \setminus F$. Assume that $\partial \circ \partial = 0$ for $\partial = \partial_U$ and ∂_F. Then we have exact sequences*:

$$\cdots \to H_{r+1}(U,\mathcal{L}_{|U}) \longrightarrow H_r(F,\mathcal{L}_{|F}) \longrightarrow H_r(X,\mathcal{L}) \longrightarrow H_r(U,\mathcal{L}_{|U}) \to \cdots,$$

$$\cdots \to H^{\mathrm{ac}}_{r+1}(U,\mathcal{L}_{|U}) \longrightarrow H^{\mathrm{ac}}_r(F,\mathcal{L}_{|F}) \longrightarrow H^{\mathrm{ac}}_r(X\mathcal{L}) \longrightarrow H^{\mathrm{ac}}_r(U,\mathcal{L}_{|U}) \to \cdots.$$

Proof. — Let $i : F \hookrightarrow X$ and $j : U \hookrightarrow X$; we have exact sequences:

$$0 \longrightarrow C_r(F, \mathcal{L}_{|F}) \xrightarrow{i_*} C_r(X, \mathcal{L}) \xrightarrow{j^*} C_r(U, \mathcal{L}_{|U}) \longrightarrow 0,$$

$$0 \longrightarrow AC_r(F, \mathcal{L}_{|F}) \xrightarrow{i_*} AC_r(X, \mathcal{L}) \xrightarrow{j^*} AC_r(U, \mathcal{L}_{|U}) \longrightarrow 0,$$

and the commutation of boundary with i and j gives the long exact sequences. □

PROPOSITION 23. — *If $g: X \longrightarrow Y$ is smooth of relative dimension p, g^* commutes with the boundary ∂.*

Thus g induces morphisms

$$g^* : H_n(Y, \mathcal{L}) \longrightarrow H_{n+p}(X, g^*\mathcal{L}), \quad g^* : H^{\mathrm{ac}}_n(Y, \mathcal{L}) \longrightarrow H^{\mathrm{ac}}_{n+p}(X, g^*\mathcal{L}).$$

Proof. — There exists an open covering $\{U_i\}$ of X such that $g_{|U_i}$ factorizes under the form

$$U_i \xrightarrow{g_i'} \mathbb{A}^p_Y = \mathbb{A}^p_{\mathbb{Z}} \times Y \xrightarrow{q} Y$$

where the g_i' are étale and q is the projection. As g_i' commutes with the boundary and the pull-back is functorial, it is enough to show that $q \circ \partial = \partial \circ q$. But q factorizes also in

$$\mathbb{A}^p_Y \longrightarrow \mathbb{A}^{p-1}_Y \longrightarrow \cdots \longrightarrow \mathbb{A}^1_Y \longrightarrow Y$$

and as $\mathbb{A}^k_Y = \mathbb{A}^1_{\mathbb{A}^{k-1}_Y}$ and the pull-back is functorial, we can assume that $p = 1$, that is $q : X = \mathbb{A}^1_Y \to Y$.

Let now $y \in Y_{(n)}$, $x \in X_{(n)}$, $z = q(x)$ such that $y \succ z$ (else everything is zero as in the proof of Proposition 21). If $z = y$, we have also

$$(q^* \circ \partial_Y)^y_x = (q^*)^z_x (\partial_Y)^y_z = 0 \quad \text{and} \quad (\partial_X \circ q^*)^y_x = \Big(\sum_{\substack{x' \in X_{(n+1)} \\ q(x')=y}} \partial_X \Big)^{x'}_x (q^*)^y_{x'} = 0$$

as $\partial^\xi_\eta = 0$ if $\dim(\xi) \le \dim(\eta)$. Assume then that $z \neq y$. Then $z \in \overline{\{y\}}^{(1)}$. As in the proof of Proposition 21 we can consider $Y = \overline{\{y\}}$ normal, thus $B = \mathcal{O}_{Y,z}$ is a discrete valuation ring. If we consider the commutative diagram

$$\begin{tikzcd} \mathbb{A}^1_B \arrow[r, "q'"] \arrow[d, "f'"'] & \operatorname{Spec} B \arrow[d, "f"] \\ \mathbb{A}^1_Y \arrow[r, "q"] & Y \end{tikzcd}$$

we have $f'(\mathbb{A}^1_B) = q^{-1}(f(\operatorname{Spec} B)))$ and for $u \in \mathbb{A}^1_B$ we have a canonical isomorphism $\mathcal{O}_{\mathbb{A}^1_Y, f'(u)} \simeq \mathcal{O}_{\mathbb{A}^1_B, u}$; this enables us to identify

$$\operatorname{Cons}\big(\{u_r^{(qf')^*\mathcal{L}(u)}\}^\wedge, \mathbb{Z}\big) \quad \text{with} \quad \operatorname{Cons}\big(\{(f'(u))_r^{q^*\mathcal{L}(f'(u))}\}^\wedge, \mathbb{Z}\big),$$

and with this identification, we have for each u such that $q(u) = y$, $(\partial_{\mathbb{A}^1_Y})^u_x = (\partial_{\mathbb{A}^1_B})^u_x$ and then $(\partial_{\mathbb{A}^1_Y} \circ q^*)^y_x = (\partial_{\mathbb{A}^1_B} \circ q'^*)^y_x$ and $(q^* \circ \partial_Y)^y_x = (q'^* \circ \partial_{\operatorname{Spec} B})^y_x$; now we just have to consider $q : \mathbb{A}^1_B = \operatorname{Spec} B[t] \to \operatorname{Spec} B$, with y the generic point of $\operatorname{Spec} B$ and z a closed point. If η is the generic point of $q^{-1}(z) = \operatorname{Spec} \kappa(z)[t]$, $\mathcal{O}_{\mathbb{A}^1_B, \eta}$ is a discrete valuation ring and $B \to \mathcal{O}_{\mathbb{A}^1_B, \eta}$ is non ramified, and we finish the proof as the one of Proposition 21. □

Remark. — The proofs of Lemma 20, Proposition 21 and Proposition 23 come directly from Schmidt's [S] analogous proofs.

4.4. Homology of projective and affine spaces.

PROPOSITION 24 (Homotopy property). — *Let $\mathcal{L}$ be a fibre bundle over a scheme X. Let $\pi:\mathbb{A}^n_X \longrightarrow X$. Then*

$$\pi^*:H_p(X,\mathcal{L}) \longrightarrow H_{p+n}(\mathbb{A}^n_X,\Omega^n_{\mathbb{A}^n_X}\otimes\pi^*\mathcal{L}),$$

$$\pi^*:H^{\mathrm{ac}}_p(X,\mathcal{L}) \longrightarrow H^{\mathrm{ac}}_{p+n}(\mathbb{A}^n_X,\Omega^n_{\mathbb{A}^n_X}\otimes\pi^*\mathcal{L})$$

are isomorphisms.

Proof. — As π can be decomposed into

$$\mathbb{A}^n_X \longrightarrow \mathbb{A}^{n-1}_X \longrightarrow \cdots \longrightarrow \mathbb{A}^1_X \longrightarrow X,$$

and as $\mathbb{A}^k_X = \mathbb{A}^1_{\mathbb{A}^{k-1}_X}$, with an induction it is sufficient to show the property for $n=1$.

For each $x\in X$, $\pi^{-1}(x)=\mathbb{A}^1_{\kappa(x)}$ and we have for each p,

$$(\mathbb{A}^1_X)_{(p)} = \bigcup_{x\in X_{(p)}} (\mathbb{A}^1_{\kappa(x)})_{(0)} \cup \bigcup_{x\in X_{(p-1)}} \{\xi_x\}$$

where ξ_x is the generic point of $\mathbb{A}^1_{\kappa(x)}$. As

$$C_p(\mathbb{A}^1_X) = \bigoplus_{x'\in(\mathbb{A}^1_X)_{(p)}} \mathrm{Cons}(\widehat{x'_r},\mathbb{Z}),$$

we have

$$C_p(\mathbb{A}^1_X) = \bigoplus_{x\in X_{(p)}} C_0(\mathbb{A}^1_{\kappa(x)}) \oplus \bigoplus_{x\in X_{(p-1)}} C_1(\mathbb{A}^1_{\kappa(x)}). \tag{1}$$

In fact $C_1(\mathbb{A}^1_{\kappa(x)}) = \mathrm{Cons}(\widehat{(\xi_x)_r},\mathbb{Z})$. As

$$\kappa(\xi_x)=\kappa(x)(u) \quad\text{and}\quad \Omega^1_{\mathbb{A}^1_{\kappa(x)}}(\xi_x)=\Omega^1_{\kappa(x)(u)/\kappa(x)},$$

we can write $C_1(\mathbb{A}^1_{\kappa(x)},\Omega^1_{\mathbb{A}^1_{\kappa(x)}}) = \mathrm{Cons}((\Sigma^{\Omega^1_{\kappa(x)(u)/\kappa(x)}}_{\kappa(x)(u)})^\wedge,\mathbb{Z})$. Write

$$\partial^1 : \bigoplus_{x\in X_{(p)}} C_1(\mathbb{A}^1_{\kappa(x)}) \longrightarrow \bigoplus_{x\in X_{(p-1)}} C_1(\mathbb{A}^1_{\kappa(x)}),$$

$$\partial^0 : \bigoplus_{x\in X_{(p)}} C_1(\mathbb{A}^1_{\kappa(x)}) \longrightarrow \bigoplus_{x\in X_{(p)}} C_0(\mathbb{A}^1_{\kappa(x)})$$

where $\partial_{\mathbb{A}^1_X} = \partial^1+\partial^0$ in restriction to $\bigoplus_{x\in X_{(p)}} C_1(\mathbb{A}^1_{\kappa(x)})$.

We have $\partial^0 = \bigoplus_{x\in X_{(p)}} \partial_{\mathbb{A}^1_{\kappa(x)}}$ from the definition of the boundary.

On the other hand as π is smooth of relative dimension 1, from Proposition 23 we get $\pi^* \circ \partial_X = \partial_{\mathbb{A}^1_X} \circ \pi^*$. We have

$$\pi^* : \bigoplus_{x\in X_{(p)}} \operatorname{Cons}(\widehat{x_r}, \mathbb{Z}) \longrightarrow \bigoplus_{x\in X_{(p)}} \operatorname{Cons}\big((\Sigma_{\kappa(x)(u)}^{\Omega^1_{\kappa(x)(u)/\kappa(x)}})^\wedge, \mathbb{Z}\big)$$

and also $\partial_{\mathbb{A}^1_X} \circ \pi^* = \partial^1 \circ \pi^* + \sum_{x\in X_{(p)}} \partial_{\mathbb{A}^1_{\kappa(x)}} \circ \pi^*$. For each x, we have the following lemma:

Lemma 25. — *The sequences*

$$0 \to \operatorname{Cons}(\widehat{x_r},\mathbb{Z}) \xrightarrow{\pi^*} \operatorname{Cons}\big((\Sigma_{\kappa(x)(u)}^{\Omega^1_{\kappa(x)(u)/\kappa(x)}})^\wedge,\mathbb{Z}\big) \xrightarrow{\partial_{\mathbb{A}^1_{\kappa(x)}}} C_0(\mathbb{A}^1_{\kappa(x)},\Omega^1_{\mathbb{A}^1_{\kappa(x)}}) \to 0,$$

$$0 \to \operatorname{AlgCons}(\widehat{x_r},\mathbb{Z}) \xrightarrow{\pi^*} \operatorname{AlgCons}\big((\Sigma_{\kappa(x)(u)}^{\Omega^1_{\kappa(x)(u)/\kappa(x)}})^\wedge,\mathbb{Z}\big) \xrightarrow{\partial'_{\mathbb{A}^1_{\kappa(x)}}} AC_0(\mathbb{A}^1_{\kappa(x)},\Omega^1_{\mathbb{A}^1_{\kappa(x)}}) \to 0$$

are exact and split.

In particular $\partial_{\mathbb{A}^1_X} \circ \pi^* = \partial^1 \circ \pi^*$ and π^* is injective, thus

$$\operatorname{Ker}(\partial_X) = \operatorname{Ker}(\pi^* \circ \partial_X) = \operatorname{Ker}(\partial^1 \circ \pi^*),$$
$$\pi^*(\operatorname{Im}\partial_X) = \operatorname{Im}(\pi^* \circ \partial_X) = \operatorname{Im}(\partial^1 \circ \pi^*).$$

We have a canonical morphism

$$i : \pi^*\big(\operatorname{Ker}(\partial^1 \circ \pi^*)\big)/\operatorname{Im}(\partial^1 \circ \pi^*) \longrightarrow \operatorname{Ker}(\partial_{\mathbb{A}^1_X})/\operatorname{Im}(\partial_{\mathbb{A}^1_X})$$

where $\operatorname{Im}(\partial^1 \circ \pi^*) \subset \pi^*(\operatorname{Ker}(\partial^1 \circ \pi^*)) \subset \bigoplus_{x\in X_{(p-1)}} C_1(\mathbb{A}^1_{\kappa(x)}, \Omega^1_{\mathbb{A}^1_{\kappa(x)}})$ and $\operatorname{Im}(\partial_{\mathbb{A}^1_X}) \subset \operatorname{Ker}(\partial_{\mathbb{A}^1_X}) \subset C_p(\mathbb{A}^1_X, \Omega^1_{\mathbb{A}^1_X})$. From the preceding, and as $\operatorname{Ker}\partial_X \simeq \pi^* \operatorname{Ker}\partial_X$, the property holds if i is an isomorphism.

It is surjective: take C a cycle in $C_p(\mathbb{A}^1_X)$ decomposed in $C_0 + C_1$ in (1). From Lemma 25, ∂^0 is surjective, thus there is $D \in C_{p+1}(\mathbb{A}^1_X)$ such that $\partial_{\mathbb{A}^1_X}(D) = E_1 + C_0$; put $F_1 = C - \partial_{\mathbb{A}^1_X}(D) = C_1 - E_1$, then $\partial_{\mathbb{A}^1_X}(F_1) = 0$ as C is a cycle, thus from the exact sequence of the lemma, $F_1 \in \pi^*(\operatorname{Ker}(\partial^1 \circ \pi^*))$, and we have $i(\overline{F}_1) = \overline{C}$.

It is injective: take $C = C_1 + 0 \in \pi^*(\operatorname{Ker}(\partial^1 \circ \pi^*))$ such that $C \in \operatorname{Im}(\partial^1 \circ \pi^*)$ (*i.e.* $i(\overline{C}) = 0$); thus $C = \partial_{\mathbb{A}^1_X}(D)$ with $D = D_0 + D_1$, and from surjectivity of ∂^0, $\partial_{\mathbb{A}^1_X}(E) = F_1 + D_0$ with $E \in C_{p+2}(\mathbb{A}^1_X)$. Then $D - \partial_{\mathbb{A}^1_X}(E) = D_1 - F_1$, and $\partial_{\mathbb{A}^1_X}(D - \partial_{\mathbb{A}^1_X}(E)) = C = C_1$, which implies $D - \partial_{\mathbb{A}^1_X}(E) \in \operatorname{Ker}\partial^0$, which also belongs to $\operatorname{Im}\pi^*$ from the exact sequence; this means C belongs to $\operatorname{Im}(\partial^1 \circ \pi^*)$, and $\overline{C} = 0$.

The proof is the same for the algebraically constructible homology. $\square$

Proof of Lemma 25. — Put $F = \kappa(x)$. Fix a transcendence basis $v_1, \ldots, v_m$ of F over R, and put $\omega = \mathrm{d}v_1 \wedge \ldots \wedge \mathrm{d}v_m \in \Omega^m_{F/R}$; let $\tau = \mathrm{d}v_1 \wedge \ldots \wedge \mathrm{d}v_m \wedge \mathrm{d}u \in \Omega^{m+1}_{F(u)/R}$. A point y of $(\mathbb{A}^1_F)_{(0)}$ is the ideal generated by a monic irreducible polynomial P_y of $F[u]$; let t_y be the image of P_y in $B_y = F[u]_{(P)} = \mathcal{O}_{\mathbb{A}^1_F, y}$; this is a uniformizing parameter of B_y; then $\mathrm{d}t_y = e_y\,\mathrm{d}u$ with e_y a unit of B_y. On the other hand for each y, $v_1, \ldots, v_m$ is a transcendence basis of $\kappa(y)$ over R; we still denote by ω the image of ω in $\Omega^m_{\kappa(y)/R}$.

Using shrinking isomorphisms with these coefficients, it is sufficient to show the exact sequences

$$0 \to \operatorname{Cons}(\Sigma_F, \mathbb{Z}) \xrightarrow{r_{F(u)/F}} \operatorname{Cons}(\Sigma_{F(u)}, \mathbb{Z}) \xrightarrow{d} \bigoplus_{y \in (\mathbb{A}^1_F)_{(0)}} \operatorname{Cons}(\Sigma_{\kappa(y)}, \mathbb{Z}) \to 0,$$

$$0 \to \operatorname{AlgCons}(\Sigma_F, \mathbb{Z}) \xrightarrow{r_{F(u)/F}} \operatorname{AlgCons}(\Sigma_{F(u)}, \mathbb{Z}) \xrightarrow{d'} \bigoplus_{y \in (\mathbb{A}^1_F)_{(0)}} \operatorname{AlgCons}(\Sigma_{\kappa(y)}, \mathbb{Z}) \to 0.$$

The application d is defined by $d_y(\varphi) = \sigma_{B_y}(\operatorname{sign} t_y \cdot \varphi)$; d' is $\frac{1}{2}d$.

Clearly $r_{F(u)/F}$ is injective in both cases.

Then $\varphi \in \operatorname{Ker} d$ (or $\operatorname{Ker} d'$ in the second case) if and only if φ is constant on the fibres of $\Sigma_{F(u)} \to \Sigma_F$; indeed let $\alpha \in \Sigma_F$; assume φ is not constant on the fibre of α (which is isomorphic to $\Sigma_{\kappa(\alpha)(u)}$).This implies that there exists $a \in \kappa(\alpha)$ such that φ take two different values in the orderings a_+ and a_- of $\kappa(\alpha)(u)$ (*i.e.* the pullbacks of the ordering of $\kappa(\alpha)$ by the valuation ring $\kappa(\alpha)[u]_{(u-a)}$, positive and negative for $u - a$). Let P be the minimal polynomial of a in the extension $F \longrightarrow \kappa(\alpha)$, and $y = (P)$. Let β be the ordering of $F(a)$ induced by the inclusion $F(a) \hookrightarrow \kappa(\alpha)$. Then a_+ and a_- are the pullbacks in $F(u)$ of β by the valuation ring B_y, positive and negative for t_y. Thus $d_y\varphi(\beta)$ is non zero. In the same way $d'_y\varphi(\beta)$ is non zero.

On the other hand we have retractions ρ and ρ' of $r_{F(u)/F}$: $\rho(\varphi) = d_{(u)}(\mathbf{1}_{\{u<0\}}\varphi)$ if $\varphi \in \mathrm{Cons}(\Sigma_{F(u)}, \mathbb{Z})$ and $\rho'(\varphi) = d'_{(u)}(2 \times \mathbf{1}_{\{u<0\}}\varphi)$ if $\varphi \in \mathrm{AlgCons}(\Sigma_{F(u)}, \mathbb{Z})$. Clearly φ is constant on the fibres if and only if $\varphi = (r_{F(u)/F)} \circ \rho)\varphi$.

Thus $\mathrm{Ker}\, d = \mathrm{Im}\, r_{F(u)/F}$ and $\mathrm{Ker}\, d' = \mathrm{Im}\, r_{F(u)/F}$.

Let us show that d is surjective: by linearity, it is sufficient to show that for a fixed y and $\varphi = \mathbf{1}_{\{f_1>0,\dots,f_k>0\}}$ with $f_i \in \kappa(y)$, there exists $\psi \in \mathrm{Cons}(\Sigma_{F(u)}, \mathbb{Z})$ such that $d_y(\psi) = \varphi$. As $\kappa(y) = F[u]/(P_y)$, we can choose pullbacks of the f_i's in $F[u]$, still denoted by f_i.

Consider the formula

$$\begin{aligned}\exists x_1, \cdots, \exists x_\ell, \quad \big\{ & x_1 < \ldots < x_\ell < u \text{ and } P_y(x_j) = 0 \text{ for } j = 1, \ldots, \ell \\ & \text{and } f_i(x_j) > 0 \text{ for } j = 1, \ldots, \ell \text{ and } i = 1, \ldots, k \\ & \text{and } \forall \xi \ (\xi < u \text{ and } P_y(\xi) = 0 \text{ and } f_i(\xi) > 0 \\ & \text{for } i = 1, \ldots, k) \Longrightarrow (\xi = x_1 \text{ or } \ldots \text{ or } \xi = x_\ell)\big\}.\end{aligned}$$

This formula is equivalent in the theory of real closed fields to a formula with parameter in $F[u]$, thus this defines a constructible set C_ℓ of $\Sigma_{F(u)}$. If $\deg P = n$, then $C_0, \ldots, C_n$ form a constructible partition of $\Sigma_{F(u)}$. Put

$$\psi = \sum_{\imath=1}^{n} \ell\, \mathbf{1}_{C_\ell}.$$

Let us compute $d_y(\psi)$: let $\alpha \in \Sigma_{\kappa(y)}$ such that $f_\imath(\alpha) > 0$ for $i = 1, \ldots, k$; let $x_1, \ldots, x_r$ be the roots of P_y in $\kappa(\alpha)$. The ordering α corresponds to a root x_i of P_y (*cf.* [BCR], Proposition 1.3.7). The pullbacks β_+ and β_- of α in $F(u)$ for B_y are in fact the pullbacks of the ordering of $\kappa(\alpha)$ in $\kappa(\alpha)(u)$ by the valuation ring $\kappa(\alpha)[u]_{(u-x_\imath)}$, positive and negative for $u - x_i$. Thus if β_- is in C_l, then β_+ will be in $C_{\ell+1}$ (the root x_i passes under u). Thus

$$d_y(\psi)(\alpha) = \psi(\beta_+) - \psi(\beta_-) = (\ell+1) - \ell = 1.$$

If α is not in $\{f_1 > 0, \ldots, f_k > 0\}$, then β_+ and β_- are in the same C_ℓ thus $d_y(\psi)(\alpha) = 0$. Thus $d_y(\psi) = \varphi$.

At last, d' is also surjective: take $\varphi = \mathrm{sign}(f) \in \mathrm{AlgCons}(\Sigma_{\kappa(y)}, \mathbb{Z})$, with $f \in F[u]$. There exists a quadratic form on $F(u)$ whose signature counts

$$\sharp\{\xi \text{ real root of } P_y \mid f(\xi)(u-\xi) > 0\} - \sharp\{\xi \text{ real root of } P_y \mid f(\xi)(u-\xi) < 0\}$$

(see for example [KnS]). Let ψ be the signature of this form. Then $d'_y(\psi) = \varphi$. Indeed if α is such that $f(\alpha) > 0$, passing from β_- to β_+, the root x_i passes from the set $\{\xi \mid P_y(\xi) = 0,\ f(\xi)(u-\xi) < 0\}$ to the set $\{\xi \mid P_y(\xi) = 0,\ f(\xi)(u-\xi) > 0\}$, *i.e.*

$$\psi(\beta_+) = (i+1) - (r-i-1) \quad \text{and} \quad \psi(\beta_-) = i - (r-i)$$

thus $d'_y(\psi)(\alpha) = 1$; if $f(\alpha) < 0$, it is the contrary and $d'_y(\psi)(\alpha) = -1$. Finally $d'_y(\psi) = \operatorname{sign}(f)$.

$\square$

PROPOSITION 26 (Homology of projective and affine spaces). — *For $0 \le p \le n$ we have*:

$$H_p(\mathbb{A}^n_R) = \begin{cases} \mathrm{Cons}(\Sigma_R,\mathbb{Z}) & \textit{if } p = n, \\ 0 & \textit{else}; \end{cases}$$

$$H^{\mathrm{ac}}_p(\mathbb{A}^n_R) = \begin{cases} \mathrm{AlgCons}(\Sigma_R,\mathbb{Z}) & \textit{if } p = n, \\ 0 & \textit{else}; \end{cases}$$

$$H_p(\mathbb{P}^n_R) = \begin{cases} \mathrm{Cons}(\Sigma_R,\mathbb{Z}) & \textit{if } p = 0 \textit{ or } p = n \textit{ and } n \textit{ odd}, \\ \mathrm{Cons}(\Sigma_R,\mathbb{Z})/2\,\mathrm{Cons}(\Sigma_R,\mathbb{Z}) & \textit{if } p \textit{ odd and } 0 < p < n, \\ 0 & \textit{else}; \end{cases}$$

$$H^{\mathrm{ac}}_p(\mathbb{P}^n_R) = \begin{cases} \mathrm{AlgCons}(\Sigma_R,\mathbb{Z}) & \textit{if } p = 0 \textit{ or } p = n \textit{ and } n \textit{ odd}, \\ 0 & \textit{else}. \end{cases}$$

When R is real closed,

$$\mathrm{Cons}(\Sigma_R,\mathbb{Z}) = \mathrm{AlgCons}(\Sigma_R,\mathbb{Z}) = \mathbb{Z}$$

and we recover Borel-Moore homology of R^n and $\mathbb{P}^n_R$.

Proof. — For affine spaces, this follows with an induction from homotopy property.

For $n = 0$, $H_0(\mathbb{P}^0_R) = \mathrm{Cons}(\Sigma_R, \mathbb{Z})$ and $H^{\mathrm{ac}}_0(\mathbb{P}^0_R) = \mathrm{AlgCons}(\Sigma_R, \mathbb{Z})$. Next we use an induction on n: consider in $\mathbb{P}^n_R$ an hyperplane at infinity identified with $\mathbb{P}^{n-1}_R$ and apply the long exact sequence and the results for $\mathbb{A}^n_R = \mathbb{P}^n_R \setminus \mathbb{P}^{n-1}_R$; we obtain on one hand $H_p(\mathbb{P}^n_R) = H_p(\mathbb{P}^{n-1}_R)$ and

$H_p^{\mathrm{ac}}(\mathbb{P}_R^n) = H_p^{\mathrm{ac}}(\mathbb{P}_R^{n-1})$ for $p \leq n-2$ and on the other hand the exact sequences

$$0 \to H_n(\mathbb{P}_R^n) \longrightarrow H_n(\mathbb{A}_R^n) \xrightarrow{\alpha} H_{n-1}(\mathbb{P}_R^{n-1}) \longrightarrow H_{n-1}(\mathbb{P}_R^n) \to 0,$$

$$0 \to H_n^{\mathrm{ac}}(\mathbb{P}_R^n) \longrightarrow H_n^{\mathrm{ac}}(\mathbb{A}_R^n) \xrightarrow{\beta} H_{n-1}^{\mathrm{ac}}(\mathbb{P}_R^{n-1}) \longrightarrow H_{n-1}^{\mathrm{ac}}(\mathbb{P}_R^n) \to 0.$$

If $n = 1$, clearly $H_0(\mathbb{P}_R^1) = \mathrm{Cons}(\Sigma_R, \mathbb{Z})$ and $H_0^{\mathrm{ac}}(\mathbb{P}_R^1) = \mathrm{AlgCons}(\Sigma_R, \mathbb{Z})$ and the exact sequences give $H_1(\mathbb{P}_R^1) = \mathrm{Cons}(\Sigma_R, \mathbb{Z})$ and $H_1^{\mathrm{ac}}(\mathbb{P}_R^1) = \mathrm{AlgCons}(\Sigma_R, \mathbb{Z})$.

If n is odd and strictly greater than 1, from the induction hypothesis we get $H_{n-1}(\mathbb{P}_R^{n-1}) = H_{n-1}^{\mathrm{ac}}(\mathbb{P}_R^{n-1}) = 0$ thus $H_n(\mathbb{P}_R^n) = \mathrm{Cons}(\Sigma_R, \mathbb{Z})$ and $H_n^{\mathrm{ac}}(\mathbb{P}_R^n) = \mathrm{AlgCons}(\Sigma_R, \mathbb{Z})$ and $H_{n-1}(\mathbb{P}_R^n) = H_{n-1}^{\mathrm{ac}}(\mathbb{P}_R^n) = 0$.

If the homogeneous coordinates on $\mathbb{P}_R^n$ are $(X_0 : X_1 : \ldots : X_n)$, put $x_i = X_i/X_0$; if $K(\mathbb{P}_R^n)$ is the function field of the variety $\mathbb{P}_R^n$, then the image of the function $\psi : \Sigma_R \to \mathbb{Z}$ by the isomorphisms $\mathrm{Cons}(\Sigma_R, \mathbb{Z}) \xrightarrow{\sim} H_n(\mathbb{P}_R^n)$ and $\mathrm{AlgCons}(\Sigma_R, \mathbb{Z}) \xrightarrow{\sim} H_n^{\mathrm{ac}}(\mathbb{P}_R^n)$ is φ such that $\varphi_{\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n} = r_{K(\mathbb{P}_R^n)/R}\psi$.

If n is even, take $\psi \in \mathrm{Cons}(\Sigma_R, \mathbb{Z})$ and the associated cycle ξ of $\mathbb{A}_R^n$, *i.e.* $\xi_{\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n} = r_{R(x_1, \ldots, x_n)/R}\psi$. To compute its boundary as a chain in $\mathbb{P}_R^n$, put $x'_i = X_i/X_1$ and take the $(n-1)$-form $\mathrm{d}x'_2 \wedge \ldots \wedge \mathrm{d}x'_n$ on $\mathbb{P}_R^{n-1}$ (the homogeneous coordinates of $\mathbb{P}_R^{n-1}$ are $(X_1 : \ldots : X_n)$); this is the Poincaré residue of

$$(1/x'_0)\mathrm{d}x'_2 \wedge \ldots \wedge \mathrm{d}x'_n \wedge \mathrm{d}x'_0 = (-1)^n x_1^{n-2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_n.$$

As n is even, this is always equivalent to $\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$. Then computing the boundary of ξ we obtain $2\xi'$ where $\xi'_{\mathrm{d}x'_2 \wedge \ldots \wedge \mathrm{d}x'_n} = r_{K(\mathbb{P}_R^{n-1})/R}\psi$. Thus $\alpha : \mathrm{Cons}(\Sigma_R, \mathbb{Z}) \to \mathrm{Cons}(\Sigma_R, \mathbb{Z})$ is the multiplication by 2 and $\beta : \mathrm{AlgCons}(\Sigma_R, \mathbb{Z}) \to \mathrm{AlgCons}(\Sigma_R, \mathbb{Z})$ is the identity. □

5. k-algebraically constructible homology.

Once again, R is not necessarily a real closed field.

5.1. The cycle module $\mathcal{AC}(K)$.

If K is an extension of R, denote for each integer $k \geq 0$,

$$\mathcal{AC}_k(K) = \big\{\varphi : \Sigma_K \to \mathbb{Z}/2\mathbb{Z} \mid \exists\, \psi \in \mathrm{AlgCons}(\Sigma_K, \mathbb{Z}),\ \psi \equiv 2^k \varphi \bmod 2^{k+1}\big\}$$

where

$$2^k\varphi : \Sigma_K \longrightarrow \mathbb{Z} : \ \alpha \longmapsto \begin{cases} 2^k & \text{if } \varphi(\alpha) = 1, \\ 0 & \text{else.} \end{cases}$$

We also have

$$\mathcal{A}C_k(K) = \{\varphi : \Sigma_K \to 2^k\mathbb{Z}/2^{k+1}\mathbb{Z} \mid \exists\, \psi \in \operatorname{AlgCons}(\Sigma_K, \mathbb{Z}),\ \varphi = \psi \bmod 2^{k+1}\}.$$

For $k < 0$, $\mathcal{A}C_k(K) = 0$.

Still an other way to see $\mathcal{A}C_k(K)$ is the following: let

$$E_k(K) = \{\varphi \in \operatorname{AlgCons}(\Sigma_K, \mathbb{Z}) \mid 2^k \text{ divides } \varphi\}.$$

Then the $E_k(K)$'s make a filtration of $\operatorname{AlgCons}(\Sigma_K, \mathbb{Z})$, and $\bigoplus_k \mathcal{A}C_k(K)$ is the graded ring associated to it.

In this way we can see $\bigoplus_k \mathcal{A}C_k(K)$ as the "signature" of the graded Witt ring of K. Indeed let $I(K)$ be the fundamental ideal of the Witt ring of K, *i.e.* the ideal of even forms. The graded Witt ring is $\bigoplus_k I^k(K)/I^{k+1}(K)$. A conjecture of Lam, proved by Dickmann and Miraglia from Voevodsky's results (see [DM]), says that the elements of $E_k(K)$ are signatures of elements of $I^k(K)$. Thus we have a "signature" morphism between the graded Witt ring and $\bigoplus_k \mathcal{A}C_k(K)$.

PROPOSITION 27. — *Let $\varphi \in \operatorname{Cons}(\Sigma_K, \mathbb{Z}/2\mathbb{Z})$. Then $\varphi \in \mathcal{A}C_k(K)$ if and only if φ is a sum of characteristic functions of k-basic constructible subsets of Σ_K (a constructible set in Σ_K is k-basic if it can be written $\{\alpha \in \Sigma_K \mid f_1(\alpha) > 0, \dots, f_k(\alpha) > 0\}$ where $f_i \in K$). Consequently:*

1) *$\mathcal{A}C_k(K) = \operatorname{Cons}(\Sigma_K, \mathbb{Z}/2\mathbb{Z})$ for $k \geq s$ where s is the stability index of K (the stability index of K is the least integer s such that every basic set of Σ_K is s-constructible);*

2) *$\mathcal{A}C_0(K) = \{0, \mathbf{1}_{\Sigma_K} \bmod 2\}$;*

3) *$\mathcal{A}C_1(K) = \{\frac{1}{2}(1 + \operatorname{sign} f) \bmod 2 \mid f \in K^\bullet\}$;*

4) *$\mathcal{A}C_k(K) \subset \mathcal{A}C_{k+1}(K)$ for each k.*

Proof. — It comes from the fact that $I^k(K)$ is generated by the k-Pfister forms $\langle\langle a_1, \dots, a_k \rangle\rangle$ whose signature is $2^k \mathbf{1}_{\{a_1<0,\dots,a_n<0\}}$. □

Proposition 28. — *Let $\mathcal{A}C(K) = \bigsqcup_{k\in\mathbb{Z}} \mathcal{A}C_k(K)$. Then $\mathcal{A}C$, together with the following data, is a cycle module (see [R]):*

(D1) *If $i: K \longrightarrow L$, we have*

$$i_* = r_{L/K}: \mathcal{A}C(K) \longrightarrow \mathcal{A}C(L)$$

of degree 0 given in the following way: for $\varphi \in \mathcal{A}C_k(K)$, $i_*\varphi = \varphi \circ i^*$ where $i^*: \Sigma_L \to \Sigma_K$.

(D2) *If $i: K \to L$ is an algebraic extension of finite degree, we have*

$$i^* = c_{L/K}: \mathcal{A}C(L) \longrightarrow \mathcal{A}C(K)$$

of degree 0 given by: for $\varphi \in \mathcal{A}C_k(K)$, $i^*\varphi(\alpha) = \sum_{i^*(\gamma)=\alpha} \varphi(\gamma)$ for $\alpha \in \Sigma_L$.

(D3) *For each field L, we have a structure of K_*L-module (Milnor K-theory) defined for $a_1, \dots, a_n$ in L and $\varphi \in \mathcal{A}C(L)$ by*

$$\{a_1, \dots, a_n\} \cdot \varphi = \mathbf{1}_{\{a_1<0,\dots,a_n<0\}}\varphi.$$

The product respects the graduation, this means that $K_nL \cdot \mathcal{A}C_k(L) \subset \mathcal{A}C_{k+n}(L)$.

(D4) *If v is a discrete valuation on K, with residue field $\kappa(v)$, we have*

$$\partial_v: \mathcal{A}C(K) \longrightarrow \mathcal{A}C(\kappa(v))$$

of degree -1 given by: for $\varphi \in \mathcal{A}C_k(K)$ put

$$\partial_v\varphi = \varphi(\beta_+) + \varphi(\beta_-)$$

where β_+ and β_- are the two pullbacks of $\alpha \in \Sigma_{\kappa(v)}$ through v. In this case we also put, if t is a uniformizing parameter of v, $s_v^t: \mathcal{A}C(K) \longrightarrow \mathcal{A}C(\kappa(v))$: $\varphi \mapsto \partial_v(\{-t\} \cdot \varphi)$.

Proof. — The structure of K_*L-module is well defined as

$$\mathbf{1}_{\{ab<0\}} = \mathbf{1}_{\{a<0\}} + \mathbf{1}_{\{b<0\}} \mod 2$$

(corresponding to $\{ab\} = \{a\} + \{b\}$),

$$\mathbf{1}_{\{a<0\}}\mathbf{1}_{\{1-a<0\}} = 0$$

(corresponding to $\{a\}\{1-a\} = 0$) and

$$2^n\mathbf{1}_{\{a_1<0,\dots,a_n<0\}} = \operatorname{sign}\langle\!\langle a_1, \dots, a_n\rangle\!\rangle$$

which is algebraically constructible .

To prove that this is a cycle premodule, we have to verify the assertions (R1)–(R3) of [R]. Assertions (R1a,b) and (R2a,b,c) are straightforward. The proofs of (R1c) and (R3b) are analogous to the ones of Lemmas 4 and 8. Moreover

(R3a) Let $i : K \to L$, v a discrete valuation on L, w the restriction of v to K, $\bar{\imath} : \kappa(w) \longrightarrow \kappa(v)$ induced by i, e the ramification index; let $\alpha \in \Sigma_{\kappa(v)}$, $\varphi \in \mathcal{A}C_k(K)$. If e is even, the two orderings pullbacks of α in L extend the same ordering on K, thus $(\partial_v \circ i_*\varphi)(\alpha) = 0$. If e is odd, the two orderings restrictions of the pullbacks of α in L, are the same as the two pullbacks of the restriction of α to $\kappa(w)$, thus $(\partial_v \circ i_*\varphi)(\alpha) = \bar{\imath}_* \circ \partial_w\varphi(\alpha)$. In both cases we have $\partial_v \circ i_* = e.\bar{\imath}_* \circ \partial_w$.

(R3c) $i : K \to L$, v valuation on L which is trivial on K. Then for each ordering α on $\kappa(v)$, the restrictions to K of the two pullbacks of α in L, are equal and then $\partial_v \circ i_* = 0$.

(R3d) With the same conditions, if t is a uniformizing parameter of v, then $s_v^t(\varphi)(\alpha) = \varphi(\beta_+)$ where β_+ is the pullback of α in L such that $t(\alpha) > 0$, thus we have $s_v^t \circ i_* = \bar{\imath}_*$

(R3e) If v is a valuation on K, u a unit of $\mathcal{O}_v$, then for $\varphi \in \mathcal{A}C_k(K)$, $\partial_v(\{u\} \cdot \varphi) = \{\bar{u}\} \cdot \partial_v(\varphi)$ as the sign of u for a pullback of an ordering on $\kappa(v)$ is the sign of $\bar{u}$ for this ordering.

To prove that this is a cycle module, we have still to show that $\mathcal{A}C$ verifies (FD) and (C): let X be an excellent scheme, x and y points of X, we define $\partial_y^x : \mathcal{A}C(\kappa(x)) \to \mathcal{A}C(\kappa(y))$: let $Z = \overline{\{x\}}$ and $\pi : \widetilde{Z} \to Z$ the normalization of Z; if y is not a specialization of x of codimension 1, $\partial_y^x = 0$; else $\partial_y^x = \sum_{\tilde{y}\in\pi^{-1}(y)} c_{\kappa(\tilde{y})/\kappa(y)} \circ \partial_{\tilde{y}}$ (if $v_{\tilde{y}}$ is the valuation associated to $\mathcal{O}_{\widetilde{Z},\tilde{y}}$ and $\partial_{\tilde{y}} := \partial_{v_y}$). The proofs of (FD) and (C) are analogous to the ones of Lemma 11 and Proposition 16. □

5.2. Consequences.

The following properties come directly from [R]. We obtain as in [R], complexes $(\mathrm{C}_*^{\mathrm{ac}}(X), \partial)$ and $(\mathrm{C}_*^{k-ac}(X), \partial^k)$ given by

$$C_n^{\mathrm{ac}}(X) = \bigoplus_{x\in X_{(n)}} \mathcal{A}C\big(\kappa(x)\big), \quad C_n^{k-ac}(X) = \bigoplus_{x\in X_{(n)}} \mathcal{A}C_{k+n}\big(\kappa(x)\big).$$

The pushforward for morphism of finite type $f : X \to Y$ is defined by $f_* : C_n^{\mathrm{ac}}(X) \to C_n^{\mathrm{ac}}(Y)$ is such that $(f_*)_y^x = c_{\kappa(x)/\kappa(y)}$ if $y = f(x)$ and $\kappa(x)$ is a finite extension of $\kappa(y)$, 0 else.

For $g : Y \to Y$ a morphism of relative dimension p, we have a pullback $g^* : C_n^{\mathrm{ac}}(X) \to C_{n+p}^{\mathrm{ac}}(Y)$ putting $(g^*)_y^x = a_y^x \cdot r_{\kappa(x)/\kappa(y)}$ if $g(y) = x$, 0 else, where a_y^x is the ramification index of g in x.

Then $(f' \circ f)_* = f'_* \circ f_*$, $(g \circ g')^* = g'^* \circ g^*$, $g^* \circ f_* = f'_* \circ g'^*$ in a Cartesian square

$$\begin{tikzcd} Y \times_X Z \arrow[r, "g'"] \arrow[d, "f'"'] & Z \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

(*cf.* Lemma 20).

If f is proper then f_* commutes to the boundary. If g is flat then g^* commutes to the boundary.

Homotopy property: if $\pi : V \longrightarrow X$ is a n-dimensional affine bundle, then $\pi^* : H_p^{k-\mathrm{ac}}(X) \to H_{n+p}^{(k+p)-\mathrm{ac}}(V)$ is a bijection.

PROPOSITION 29 (Long exact sequence). — *For any closed subscheme F of X, for each integer k we have an exact sequence*

$$\cdots \to H_{r+1}^{k-\mathrm{ac}}(X \backslash F) \longrightarrow H_r^{k-\mathrm{ac}}(F) \longrightarrow H_r^{k-\mathrm{ac}}(X) \longrightarrow H_r^{k-\mathrm{ac}}(X \backslash F) \to \cdots.$$

PROPOSITION 30 (Homology of affine and projective spaces). — *For $0 \le p \le n$, we have:*

$$H_p^{d-\mathrm{ac}}(\mathbb{A}_R^n) = \begin{cases} \mathcal{AC}_{d+n}(R) & \textit{if } p = n \textit{ and } d \ge -n, \\ 0 & \textit{else}; \end{cases}$$

$$H_p^{d-\mathrm{ac}}(\mathbb{P}_R^n) = \begin{cases} 0 & \textit{if } p < -d, \\ \mathcal{AC}_{d+p}(R) & \textit{else}. \end{cases}$$

In particular if R is real closed, we get:

$$H_p^{d-\mathrm{ac}}(\mathbb{A}_R^n) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \textit{if } p = n \textit{ and } d \ge -n, \\ 0 & \textit{else}; \end{cases}$$

$$H_p^{d-\mathrm{ac}}(\mathbb{P}_R^n) = \begin{cases} 0 & \textit{if } p < -d, \\ \mathbb{Z}/2\mathbb{Z} & \textit{else}. \end{cases}$$

Proof. — For affine spaces this is a consequence of homotopy property.

For $n=0$, $H_0^{d-\mathrm{ac}}(\mathbb{P}_R^0)=\mathbb{Z}/2\mathbb{Z}$ if $d\geq 0$, 0 else. Next we use an induction on n: consider in $\mathbb{P}_R^n$ an hyperplane at infinity identified with $\mathbb{P}_R^{n-1}$ and apply the long exact sequence and the results for $\mathbb{A}_R^n=\mathbb{P}_R^n\setminus\mathbb{P}_R^{n-1}$; we obtain on one hand $H_p^{d-\mathrm{ac}}(\mathbb{P}_R^n)=H_p^{d-\mathrm{ac}}(\mathbb{P}_R^{n-1})$ for $p\leq n-2$ and on the other hand the exact sequences

$$0\to H_n^{d-\mathrm{ac}}(\mathbb{P}_R^n)\longrightarrow H_n^{d-\mathrm{ac}}(\mathbb{A}_R^n)\longrightarrow H_{n-1}^{d-\mathrm{ac}}(\mathbb{P}_R^{n-1})\longrightarrow H_{n-1}^{d-\mathrm{ac}}(\mathbb{P}_R^n)\to 0.$$

This gives:

- if $d<-n$, $H_n^{d-\mathrm{ac}}(\mathbb{P}_R^n)=0$ and $H_{n-1}^{d-\mathrm{ac}}(\mathbb{P}_R^n)\simeq H_{n-1}^{d-\mathrm{ac}}(\mathbb{P}_R^{n-1})$;
- if $d=-n$, $H_n^{(-n)-\mathrm{ac}}(\mathbb{P}_R^n)=\mathcal{A}C_0(R)$ and $H_{n-1}^{(-n)-\mathrm{ac}}(\mathbb{P}_R^n)=0$;
- if $n>-d$, the exact sequence is

$$0\to H_n^{d-\mathrm{ac}}(\mathbb{P}_R^n)\longrightarrow \mathcal{A}C_{d+n}(R)\xrightarrow{\alpha}\mathcal{A}C_{d+n-1}(R)\longrightarrow H_{n-1}^{d-\mathrm{ac}}(\mathbb{P}_R^n)\to 0.$$

As $\alpha=0$, we obtain finally $H_n^{d-\mathrm{ac}}(\mathbb{P}_R^n)=\mathcal{A}C_{n+d}(R)$ and $H_{n-1}^{d-\mathrm{ac}}(\mathbb{P}_R^n)=\mathcal{A}C_{n+d-1}(R)$. □

5.3. Case of varieties.

As in Section 3.2 we will consider R real closed and X as an algebraic subset of R^p. Then

$$C_n^{k-\mathrm{ac}}(X)=\bigoplus_{V\in X_{(n)}}\mathcal{A}C_{k+n}\big(\kappa(V)\big).$$

If $d=\dim(X)$, we have $C_n^{k-\mathrm{ac}}(X)=0$ for $k<-d$.

We have an injection $C_n^{k-\mathrm{ac}}(X)\hookrightarrow C_n(X,\mathbb{Z}/2\mathbb{Z})$; Property 1 implies that $C_n^{k-\mathrm{ac}}(X)=C_n(X,\mathbb{Z}/2\mathbb{Z})$ for $k\geq 0$. The boundary

$$\partial^k : C_n^{k-\mathrm{ac}}(X)\longrightarrow C_{n-1}^{k-\mathrm{ac}}(X)$$

is calculated in the following way: for $V\in X_{(n)}$ and $W\in X_{(n-1)}$ such that $W\subset V$, for $\varphi\in\mathcal{A}C_{k+n}(\kappa(V))$ and $\alpha\in V_r$, we have

$$(\partial^k)_W^V=\sum_{B\in\mathcal{B}_V(W)}\sum_{\overline{\beta}^B|_{\kappa(V)}}\varphi(\beta).$$

It is easy to show that the homology of the complex $C_*(X,\mathbb{Z}/2\mathbb{Z})$ (and thus the one of all the complexes $C_*^{k-\mathrm{ac}}(X,\mathbb{Z}/2\mathbb{Z})$ for $k\geq 0$) is Borel-Moore homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. The complex $C_*(X,\mathbb{Z}/2\mathbb{Z})$ is the complex of Corollary 3.2 of [Sch].

On the other hand we have for each n and k, an inclusion $C_n^{k-\mathrm{ac}}(X) \hookrightarrow C_n^{(k+1)-\mathrm{ac}}(X)$ commuting to the boundary ∂^k and if $H_*^{k-\mathrm{ac}}(X)$ denotes the homology, we have for each non negative integer k, a sequence of morphisms:

$$0 \to H_k^{(-k)-\mathrm{ac}}(X) \longrightarrow H_k^{-(k-1)-\mathrm{ac}}(X)$$
$$\to \cdots \to H_k^{0-\mathrm{ac}}(X) = H_k^{1-\mathrm{ac}}(X) = \cdots.$$

Let V be an algebraic subset of dimension k in R^p. The fundamental class of V in $H_k^{\mathrm{BM}}(V, \mathbb{Z}/2\mathbb{Z})$ is the class of the cycle given by the sum of the k-simplexes of a semialgebraic triangulation of the Alexandrov algebraic compactification of V compatible with "the point at infinity". If $V \subset X$, we have a morphism

$$i_* : H_k^{\mathrm{BM}}(V, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H_k^{\mathrm{BM}}(X, \mathbb{Z}/2\mathbb{Z}).$$

The algebraic homology $H_k^{\mathrm{alg}}(X, \mathbb{Z}/2\mathbb{Z})$ of X is the subgroup of $H_*^{\mathrm{BM}}(X, \mathbb{Z}/2\mathbb{Z})$ of the classes of the $i_*[V]$ for $V \in X_{(k)}$ (*cf.* [BCR], §11.3).

Proposition 31. — *Let X be a d-dimensional algebraic subset of R^p. We get a surjective map $H_k^{(-k)-\mathrm{ac}}(X) \to H_k^{\mathrm{alg}}(X, \mathbb{Z}/2\mathbb{Z})$; this is injective if $k = d$ or 0, or if X is defined over $\mathbb{R}$, non singular and compact and $k = d - 1$.*

Proof. — We have

$$C_k^{(-k)-\mathrm{ac}}(X) = \bigoplus_{V \in X_{(k)}} \mathcal{A}C_0(\kappa(V)) = \bigoplus_{V \in X_{(k)}} \{0, \mathbf{1}_{V_r}\}.$$

All the elements of $C_k^{(-k)-\mathrm{ac}}(X)$ are cycles and we have a surjection

$$C_k^{(-k)-\mathrm{ac}}(X) \longrightarrow H_k^{\mathrm{alg}}(X, \mathbb{Z}/2\mathbb{Z}),$$
$$\mathbf{1}_{V_r} \longmapsto i_*[V].$$

If $k = d$, there is no boundary and this map is injective. If $k < d$,

$$C_{k+1}^{(-k)-\mathrm{ac}}(X) = \bigoplus_{x \in X_{(k+1)}} \mathcal{A}C_1\big(\kappa(x)\big)$$
$$= \bigoplus_{x \in X_{(k+1)}} \big\{ \tfrac{1}{2}(1 + \operatorname{sign} f) \bmod 2 \mid f \in \kappa(x)^\bullet \big\}.$$

The map is injective if for each irreducible $(k+1)$-dimensional algebraic subset V of X, and for each k-dimensional irreducible algebraic subset $W_1,\ldots,W_m$ of V verifying $\sum_{i=1}^m [W_i]=0$, there exists $f\in\kappa(V)^*$ such that $\partial(1+\operatorname{sign} f)=\sum_{i=1}^m \mathbf{1}_{W_{ir}}$; if V is defined over $\mathbb{R}$, compact, non singular, this is true from Proposition 12.4.6 of [BCR]. In particular, if X is defined over $\mathbb{R}$ compact non singular and $k=d-1$, the map is injective. $\square$

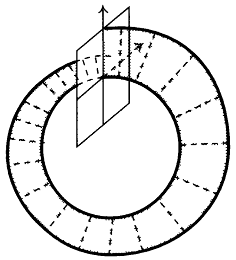
Remark. — If X is singular, the map may be non injective. For example consider the algebraic subset A of $P^1(\mathbb{R})\times\mathbb{R}^2$ given by the equation

$$u_0(x^2u_1-y^2u_1-2yxu_0)=0$$

(where $(u_0:u_1)$ are the coordinates on $P^1(\mathbb{R})$ and (x,y) the coordinates on $\mathbb{R}^2$), then $H_1^{(-1)-\mathrm{ac}}(A)\to H_1^{\mathrm{alg}}(A,\mathbb{Z}/2\mathbb{Z})$ is not injective. Indeed consider the following subsets of A:

$$A^{\varepsilon_1\varepsilon_2}=\big\{((1:u_1)(x,y))\in A\mid \varepsilon_1x>0,\varepsilon_2y>0\big\},$$
$$A_\infty^{\varepsilon_1\varepsilon_2}=\big\{((0:1)(x,y))\in A\mid \varepsilon_1x>0,\varepsilon_2y>0\big\}.$$

The set A can be seen as a fibration over a circle whose fibres are two orthogonal lines, the situation turning of a quarter following the circle; we add the plan containing the two lines over a point of the circle ("∞ point"). We can visualize for example A^{++} on the following drawing:



A^{++}

Then the class of $P^1(\mathbb{R})$ is a boundary for Borel-Moore homology: for example this is the boundary of the 2-chain $[A^{++}]+[A_\infty^{++}]$. But this is not a boundary in (-1)-algebraically constructible homology: assume P is a polynomial of A changing sign passing through $P^1(\mathbb{R})$ (and only there); if P changes sign between A^{++} and A^{--}, then it also changes sign between A^{+-} and A^{-+}. Then $\partial(1+\operatorname{sign}P)=2[P^1(\mathbb{R})]$ is zero.

Example: complement of an hypersurface in R^n.

Let H be an algebraic hypersurface in R^n and $X = R^n \setminus H$. The long exact sequences give for $-n \leq k \leq 0$:

$$\begin{aligned}0 \to \mathbb{Z}/2\mathbb{Z} \longrightarrow H_n^{k-\mathrm{ac}}(X) \longrightarrow H_{n-1}^{k-\mathrm{ac}}(H) \longrightarrow 0 \\ \longrightarrow H_{n-1}^{k-\mathrm{ac}}(X) \longrightarrow H_{n-2}^{k-\mathrm{ac}}(H) \to 0 \longrightarrow \cdots \\ \cdots \longrightarrow 0 \longrightarrow H_1^{k-\mathrm{ac}}(X) \longrightarrow H_0^{k-\mathrm{ac}}(H) \longrightarrow 0 \longrightarrow H_0^{k-\mathrm{ac}}(X) \to 0.\end{aligned}$$

We obtain:

- $H_0^{k-\mathrm{ac}}(X) = 0$;
- $H_p^{k-\mathrm{ac}}(X) \simeq H_{p-1}^{k-\mathrm{ac}}(H)$ for $n-1 \geq p \geq 1$;
- $H_n^{k-\mathrm{ac}}(X) = \mathbb{Z}/2\mathbb{Z} \oplus H_{n-1}^{k-\mathrm{ac}}(H)$.

In particular we recover:

- $H_1^{\mathrm{BM}}(X, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^d$ where d is the number of closed bounded semialgebraically connected components of H;
- $H_n^{\mathrm{BM}}(X, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^c$ where c is the number of semialgebraically connected components of X.

Moreover, we obtain:

- $H_n^{[-(n-1)]-\mathrm{ac}}(X) = (\mathbb{Z}/2\mathbb{Z})^{e+1}$ where e is the number of irreducible components of H;
- $H_p^{(-p)-\mathrm{ac}}(X) = 0$ for $n-1 \geq p \geq 1$;
- $H_n^{(-n)-\mathrm{ac}}(X) = \mathbb{Z}/2\mathbb{Z}$.

This gives entirely the homology of X in the case where X is the complement of a curve in R^2. For example if H is the curve of equation $y^2 = x^3 - x$, we obtain:

- $H_0^{\mathrm{BM}}(X, \mathbb{Z}/2\mathbb{Z}) = H_0^{\mathrm{alg}}(X, \mathbb{Z}/2\mathbb{Z}) = 0$;
- $H_1^{\mathrm{BM}}(X, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \hookleftarrow H_1^{(-1)-\mathrm{ac}}(X) = H_1^{\mathrm{alg}}(X, \mathbb{Z}/2\mathbb{Z}) = 0$;
- $$\begin{aligned}H_2^{\mathrm{BM}}(X, \mathbb{Z}/2\mathbb{Z}) &= (\mathbb{Z}/2\mathbb{Z})^3 \\ &\hookleftarrow H_2^{(-1)-\mathrm{ac}}(X) = (\mathbb{Z}/2\mathbb{Z})^2 \\ &\hookleftarrow H_2^{\mathrm{alg}}(X, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

If we replace H with the union of a line and a circle, we obtain a set semialgebraically homeomorphic to the preceding one, with the same algebraic homology, but $H_2^{(-1)-\mathrm{ac}}(X) = (\mathbb{Z}/2\mathbb{Z})^3$.

6. Lagrangian semialgebraic chains.

Let X be a n-dimensional real algebraic smooth manifold and $\pi : T^*X \to X$ the cotangent bundle. If $(x_1, \dots, x_n)$ is a local coordinate system of X and $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ is the associated coordinate system on T^*X, the canonical 1-form on T^*X is

$$\alpha = \alpha_X = \sum_{i=1}^{n} \xi_i \, \mathrm{d}x_i : T^*X \longrightarrow T^*T^*X.$$

Then T^*X is oriented by the $(2n)$-form

$$(-1)^{n(n-1)/2}(\mathrm{d}\alpha)^n = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}\xi_1 \ldots \wedge \mathrm{d}\xi_n.$$

6.1. Chains with coefficients in $\omega_{T^*X/X}$.

If $f : X \to Y$ is a morphism of smooth manifolds, we denote by $\omega_{X/Y}$ the sheaf $\bigwedge^d \Omega_{X/Y}$ (where $d = \operatorname{codim}(Y, X)$). A *Lagrangian semialgebraic chain* is an element in $\mathrm{C}_n(T^*X, \omega_{T^*X/X})$ whose support is generically a Lagrangian submanifold of T^*X (*cf.* [KS], Chapter 9 and Appendix A2). $\mathcal{L}(X)$ is the set of Lagrangian semialgebraic cycles.

We have canonically $\omega_{T^*X} \simeq \pi_X^*\omega_X \otimes \omega_{T^*X/X}$ and $\omega_{T^*X} \simeq \mathcal{O}_{T^*X}$ so we get a canonical isomorphism $\pi_X^*\omega_X \simeq (\omega_{T^*X/X})^*$. Choosing a form gives an isomorphism $\pi_X^*\omega_X \simeq \omega_{T^*X/X}$.

Remark on the use of coefficients. — If Z is a smooth m-dimensional irreducible algebraic variety, and z is the generic point of Z, we have $\omega_Z(z) = \Omega^m_{\kappa(z)/R}$. If Z is a subvariety of a smooth algebraic variety Y, and if $\mathcal{D}$ is a line bundle over Y, then $\widehat{z_r^{\mathcal{D}(z)}} = z_r^{\omega_Z \otimes \mathcal{D}|_Z(z)}$. Let s be a section of the sheaf $\omega_Z \otimes \mathcal{D}_{|Z}$ on a Zariski open subset U of Z. Then s determines an element of $\omega_Z \otimes \mathcal{D}_{|Z}(z)$. If $\psi : z_r \to \mathbb{Z}$ is a constructible function, then ψ and s determine an element C of $\mathrm{Cons}(\widehat{z_r^{\mathcal{D}(z)}}, \mathbb{Z})$. We shall say that C is given by the function ψ and the coefficient $s \in \omega_Z \otimes \mathcal{D}_{|Z}(U)$ (and even to simplify $s \in \omega_Z \otimes \mathcal{D}$). If the function ψ is associated to the function $\mathbf{1}_S$ for a semialgebraic subset S of Z, we shall say that C is the chain of support S and of coefficient $s \in \omega_Z \otimes \mathcal{D}$.

If Z is not smooth, take Z_0 a Zariski open subset of Z such that Z_0 is smooth, and z_0 the generic point of Z_0. Then $\omega_{Z_0}(z_0) \simeq \Omega^m_{\kappa(z)/R}$ and we use sections of $\omega_{Z_0} \otimes \mathcal{D}_{|Z_0}$.

Geometrically a Lagrangian semialgebraic chain may be seen in the following way: we have a partition of the support in semialgebraic subsets such that the n-dimensional strata Λ and the $\pi_X(\Lambda)$ are orientable; then to a couple $(o_\Lambda, o_{\pi_X(\Lambda)})$ where o_Λ is an orientation of Λ and $o_{\pi_X(\Lambda)}$ is an orientation of $\pi_X(\Lambda)$, we associate an integer; the sign of this integer changes if we change one of the orientations, it does not change if we change both orientations.

If there exists an isomorphism $\omega_X \simeq \mathcal{O}_X$ (this means if X is orientable), $\mathcal{L}(X)$ can be seen as a subgroup of $\mathrm{C}_n(T^*X)$, this means that a Lagrangian semialgebraic chain consists of oriented Nash Lagrangian submanifolds of T^*X with integers.

Example (cf. [KS], *Remark 9.5.8).* — Let Y be a Nash submanifold of X, given by the equations $x_1 = \cdots = x_r = 0$ where $x_1, \ldots, x_n$ are local coordinates on X. Let $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ the associated coordinates on T^*X. Then $[T^*_Y X]$ is the chain whose support is

$$T^*_Y X = \{(x, \xi) \mid x_1 = \cdots = x_r = \xi_{r+1} = \cdots = \xi_n = 0\}$$

and whose coefficient is

$$(-1)^r \mathrm{d}\xi_1 \wedge \ldots \wedge \mathrm{d}\xi_r \wedge \mathrm{d}x_{r+1} \wedge \ldots \wedge \mathrm{d}x_n \otimes \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \in \omega_{T^*_Y X} \otimes \pi^* \omega_X$$

or

$$(-1)^r \mathrm{d}\xi_1 \wedge \ldots \wedge \mathrm{d}\xi_r \wedge \mathrm{d}x_{r+1} \wedge \ldots \wedge \mathrm{d}x_n \otimes \mathrm{d}\xi_1 \wedge \ldots \wedge \mathrm{d}\xi_n \in \omega_{T^*_Y X} \otimes \omega_{T^*X/X}.$$

6.2. Characteristic cycle.

Let $\mathcal{F}(X)$ be the set of constructible functions on X, *i.e.* functions $X \to \mathbb{Z}$ constant on each element of a semialgebraic partition of X. Then we have an isomorphism $C : \mathcal{F}(X) \to \mathcal{L}(X)$ (*cf.* [KS]). If F is a complex of constructible sheaves and $\varphi = \chi(F)$, we have $C(\varphi) = CC(F)$. We would like to describe $C(\varphi)$ without using the constructible sheaves. Assume φ is constructible relatively to a Whitney stratification $\mathcal{S}$ of X; we use the notations of [SV]: write $\bigcup_{S\in\mathcal{S}}(T^*_S X - \bigcup_{R\neq S} \overline{T^*_R X}) = \bigsqcup_{\alpha\in I} \Lambda_\alpha$ (disjoint union of the connected components); then $\Lambda_\alpha \subset T^*_{S_\alpha} X$ for a S_α of $\mathcal{S}$.

$[\Lambda_\alpha]$ is the chain defined by Λ_α with the same coefficients as $[T^*_{S_\alpha} X]$.

$\mathcal{F}(X, \mathcal{S})$ denotes the set of constructible functions relative to the partition $\mathcal{S}$, and $\mathcal{L}(X, \mathcal{S})$ the Lagrangian cycles relative to the partition $\mathcal{S}$, *i.e.* the linear combinations (with integral coefficients) of the $[\Lambda_\alpha]$ which are cycles.

From [SV], we can write

$$C(\varphi) = CC(F) = \sum_{\alpha \in I} m_\alpha [\Lambda_\alpha]$$

where m_α is computed in the following way: take (p, ξ) in Λ_α, and f a Nash function defined in the neighborhood of p in X verifying

- $f(p) = 0$ and $\mathrm{d}f_p = \xi$;
- $\mathrm{d}f$ is transverse to Λ_α in (p, ξ) (*i.e.* p is a non degenerate critical point of $f_{|S_\alpha}$);
- the Hessian of $f_{|S_\alpha}$ is positive definite in p.

Then for a "sufficiently small" ball B centered in p (*cf.* [SV]) and ε small to the radius of B, we have

$$m_\alpha = \varphi(p) - \int_X \varphi \, \mathbf{1}_{\overline{B} \cap f^{-1}(-\varepsilon)}.$$

Indeed from [SV] we get

$$m_\alpha = \chi\big(H^*(B \cap f^{-1}(-\varepsilon, \varepsilon), B \cap f^{-1}(-\varepsilon, 0); F)\big).$$

We have then

$$m_\alpha = \chi\big(R\Gamma(B \cap f^{-1}(-\varepsilon, \varepsilon); F)\big) - \chi\big(R\Gamma(B \cap f^{-1}(-\varepsilon, 0); F)\big).$$

But from Lemma 8.4.7 of [KS], we have

$$\begin{aligned} R\Gamma\big(B \cap f^{-1}(-\varepsilon, \varepsilon); F\big) &\simeq R\Gamma\big(B \cap f^{-1}(0); F\big) \simeq R\Gamma_c\big(\overline{B} \cap f^{-1}(0); F\big) \\ &\simeq R\Gamma\big(X; F \otimes k_{\overline{B} \cap f^{-1}(0)}\big) \end{aligned}$$

and in the same way

$$\begin{aligned} R\Gamma\big(B \cap f^{-1}(-\varepsilon, 0); F\big) &\simeq R\Gamma\big(B \cap f^{-1}(-\varepsilon'); F\big) \simeq R\Gamma_c\big(\overline{B} \cap f^{-1}(-\varepsilon'); F\big) \\ &\simeq R\Gamma\big(X; F \otimes k_{\overline{B} \cap f^{-1}(-\varepsilon')}\big). \end{aligned}$$

Thus $m_\alpha = \int_X \varphi \, \mathbf{1}_{\overline{B} \cap f^{-1}(0)} - \int_X \varphi \, \mathbf{1}_{\overline{B} \cap f^{-1}(-\varepsilon')}$ (with $0 < \varepsilon' < \varepsilon$).

In fact $\int_X \varphi \, \mathbf{1}_{\overline{B} \cap f^{-1}(0)} = \varphi(p)$: from the local conic structure theorem (see [BCR], Theorem 9.3.5), there exists a semialgebraic homeomorphism $h : \overline{B} \to \overline{B}$ such that $h(p * S \cap f^{-1}(0)) = \overline{B} \cap f^{-1}(0)$ ($p * A$ is the cone of vertex p and of base A and S is the boundary of B) and for all $x \in S \cap f^{-1}(0)$ and all t with $0 \leq t < 1$,

$$\varphi\big(h(tp + (1-t)x)\big) = \varphi(x).$$

Using triangulation, we can find contractible compact semialgebraic subsets Y_i of $S \cap f^{-1}(0)$ such that $\varphi_{|S \cap f^{-1}(0)} = \sum m_i \mathbf{1}_{Y_i}$; then

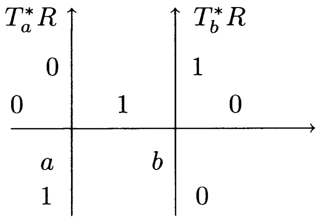
$$\varphi = \sum m_i \mathbf{1}_{p * Y_i} + \big(\varphi(p) - \sum m_i\big) \mathbf{1}_{\{p\}}$$

and $\int_X \varphi \, \mathbf{1}_{\bar{B} \cap f^{-1}(0)} = \sum m_i + \varphi(p) - \sum m_i = \varphi(p)$.

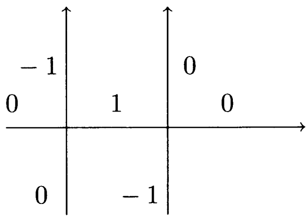
Example. — Let t, τ be the coordinates on T^*R. Orient R by dt. Let $a < b$ be in R. The cycle $[T^*_{\{a\}}R]$ is $T^*_{\{a\}}R$ oriented by $-d\tau$. Then

$$C(\mathbf{1}_{]a,b[}) = [T^*_{]a,b[}R] - [T^*_{\{a\}}R] \cap \{\tau < 0\} - [T^*_{\{b\}}R] \cap \{\tau > 0\}.$$

If T^*R is represented by R^2, we get the following drawings (arrows are both $t > 0$ and $\tau > 0$ and the orientation):



$C(\mathbf{1}_{]a,b[})$



$C(\mathbf{1}_{[a,b]})$

Example. — Take the example of the "cone" of [KS], p. 383: $X = \mathbb{R}^{1+n}$ with coordinates $(t, x_1, \ldots, x_n)$. Put $|x| = \sqrt{\sum x_i^2}$. Let

$$Z_\pm = \big\{\pm t \geq |x|\big\}, \quad Z_0 = \big\{|t| \leq |x|\big\},$$

U_ε the interior of Z_ε for $\varepsilon = 0, +, -$,

$$S_\pm = \big\{\pm t = |x| > 0\big\}.$$

Let $\mathcal{S}$ be the stratification $\{U_0, U_+, U_-, \{0\}, S_+, S_-\}$. Let (t, x, τ, ξ) be the associated coordinates on T^*X. Then $\mathcal{L}(X, \mathcal{S})$ is generated by the following Lagrangian chains:

- $\sigma_\varepsilon = [T^*_{U_\varepsilon} X]$ for $\varepsilon = 0, +, -$;
- $\tau_{\varepsilon_1\varepsilon_2} = [T^*_{S_{\varepsilon_1}} X] \cap \{\varepsilon_2 \tau > 0\}$ for $\varepsilon_i = +, -$;
- $\gamma_\pm = [T^*_{\{0\}} X] \cap \{\pm\tau > |\xi|\}$ and $\gamma_0 = [T^*_{\{0\}} X] \cap \{|\tau| < |\xi|\}$.

If now φ is a constructible function relatively to the stratification $\mathcal{S}$, then $C(\varphi) = \sum \lambda_\varepsilon \sigma_\varepsilon + \sum \mu_{\varepsilon_1\varepsilon_2} \tau_{\varepsilon_1\varepsilon_2} + \sum \nu_\varepsilon \gamma_\varepsilon$.

If

$$\varphi = a\mathbf{1}_{\{0\}} + b_+\mathbf{1}_{S_+} + b_-\mathbf{1}_{S_-} + c_+\mathbf{1}_{U_+} + c_-\mathbf{1}_{U_-} + c_0\mathbf{1}_{U_0},$$

the isomorphism C is given in restriction to these $\mathbb{Z}$-modules by

$$\begin{aligned}
\lambda_\varepsilon &= c_\varepsilon,\\
\nu_+ &= a + ((-1)^n - 1)b_- + (-1)^{n+1}c_-,\\
\nu_- &= a + ((-1)^n - 1)b_+ + (-1)^{n+1}c_+,\\
\nu_0 &= a - b_+ - b_- + c_0,\\
\mu_{++} &= b_+ - c_0, \quad \mu_{--} = b_- - c_0,\\
\mu_{-+} &= b_- - c_-, \quad \mu_{+-} = b_+ - c_+.
\end{aligned}$$

(To compute the coefficients we use for example the function $f(y) = d^2(y,p) + \xi\cdot(y-p)$, considering $\xi = \operatorname{grad}_p f$ as a vector in X; see [KS] for the results.)

A constructible function $\varphi : X \to \mathbb{Z}$ is said to be algebraically constructible if it is the sum of signs of polynomials on X, *cf.* [MP]. They can be characterized by the fans (for every fan F of $\operatorname{Spec}_r \mathcal{P}(X)$, $\sum_{x\in F} \varphi(x) = 0 \bmod |F|$) or by a criterion on the walls, *cf.* [B].

Call $\mathcal{F}^{\mathrm{ac}}(X,\mathcal{S})$ (resp. $\mathcal{L}^{\mathrm{ac}}(X,\mathcal{S})$) the elements of $\mathcal{F}(X,\mathcal{S})$ (resp. $\mathcal{L}(X,\mathcal{S})$) which are algebraically constructible.

In our example, we can find the following characterization of algebraically constructible functions: φ is algebraically constructible if and only if

(i) $b_+ \equiv b_- \bmod 2$;

(ii) $c_+ \equiv c_- \equiv c_0 \bmod 2$;

(iii) $c_+ + c_- + 2c_0 \equiv 0 \bmod 4$.

Indeed if $\pi : Y \to X$ is the blowing up at the origin then φ is algebraically constructible if and only if $\pi^*\varphi$ is algebraically constructible (where $\pi^*\varphi(y) = \varphi(\pi(y))$). The walls of $\pi^*\varphi$ are the strict transform of the cone (a cylinder) and the exceptional divisor; they are smooth and normal crossing, so using the criterion of [B] we obtain: $\pi^*\varphi$ is algebraically constructible if $\pi^*\varphi$ is constant modulo 2 (*i.e.* $c_+ \equiv c_- \equiv c_0 \bmod 2$), if $\pi^*\varphi$ restricted to the cylinder is constant modulo 2 (*i.e.* $b_+ \equiv b_- \bmod 2$) and if the "average" function on the cylinder (whose value is $\frac{1}{2}(c_+ + c_0)$ above and $\frac{1}{2}(c_- + c_0)$ under) is algebraically constructible , *i.e.* $\frac{1}{2}(c_+ + c_0) \equiv \frac{1}{2}(c_- + c_0) \bmod 2$.

In the same way, considering $\mathcal{C} = \sum \lambda_\varepsilon \sigma_\varepsilon + \sum \mu_{\varepsilon_1\varepsilon_2}\tau_{\varepsilon_1\varepsilon_2} + \sum \nu_\varepsilon \gamma_\varepsilon$ as a generically constructible function and applying the generic criterion after blowings up, we obtain that $\mathcal{C}$ is algebraically constructible if and only if

1) $\lambda_0 \equiv \lambda_+ \equiv \lambda_- \bmod 2$;

2) $\nu_0 \equiv \nu_+ \equiv \nu_- \bmod 2$;

3) $\mu_{++} \equiv \mu_{+-} \equiv \mu_{--} \equiv \mu_{-+} \bmod 2$;

4) $\lambda_+ + \lambda_- + 2\lambda_0 \equiv 0 \bmod 4$;

5) $\nu_+ + \nu_- + 2\nu_0 \equiv 0 \bmod 4$;

6) $\mu_{++} + \mu_{+-} + \mu_{--} + \mu_{-+} \equiv 0 \bmod 4$.

Then (i) $\Leftrightarrow$ 1), (ii) $\Leftrightarrow$ 2) and (iii) $\Leftrightarrow$ 4); moreover (i) + (ii) $\Rightarrow$ 3), (i) + (ii) + (iii) $\Rightarrow$ 5) and (i) + (iii) $\Rightarrow$ 6). The conditions are actually equivalent, we have an isomorphism between $\mathcal{F}^{\mathrm{ac}}(X, \mathcal{S})$ and $\mathcal{L}^{\mathrm{ac}}(X, \mathcal{S})$. We shall see in the following that this is always the case.

6.3. Limits of semialgebraic chains.

Let Y be a algebraic smooth manifold, $\rho : Y \times R \to R$ and $\sigma : Y \times R \to Y$ the projections. Let $\mathcal{D}$ a line bundle over Y, and $\mathcal{C}$ a chain in $C_{n+1}(Y \times R, \sigma^*\mathcal{D})$ whose support $\mathcal{S}$ is such that for each $(n+1)$-dimensional irreducible component W of $\mathcal{S}$, $\rho_{|W}$ is dominant.

Let $\varepsilon \in R$. The limit $\mathcal{C}_{\varepsilon^+}$ (resp. $\mathcal{C}_{\varepsilon^-}$) is by definition the chain of $C_n(Y, \mathcal{D})$ given by $\sum_x \sum_{x_\varepsilon} \partial^x_{x_\varepsilon}(\mathbf{1}_{\{t>\varepsilon\}}\psi_x)$ (resp. $-\sum_x \sum_{x_\varepsilon} \partial^x_{x_\varepsilon}(\mathbf{1}_{\{t<\varepsilon\}}\psi_x)$) where x ranges through the set of $(n+1)$-dimensional points of $\mathcal{S}$, $\psi_x \in \mathrm{Cons}(\widehat{x_r}^{\mathcal{D}(x)}, \mathbb{Z})$ is the component of $\mathcal{C}$ on $\overline{\{x\}}$, and x_ε ranges through the set of n-dimensional points of $\mathcal{S}_\varepsilon = \mathcal{S} \cap (Y \times \{\varepsilon\})$. This set is finite as W_ε is n-dimensional from the domination condition.

In particular, if $\mathcal{C}$ is algebraically constructible, then $\partial^x_{x_\varepsilon}(\mathbf{1}_{\{t>\varepsilon\}}\psi_x) = \partial'^x_{x_\varepsilon}(2 \times \mathbf{1}_{\{t>\varepsilon\}}\psi_x)$ is algebraically constructible, thus $\mathcal{C}_{\varepsilon^+}$ is algebraically constructible, and also $\mathcal{C}_{\varepsilon^-}$.

Geometric meaning: let ℓ be a section of $\mathcal{D}$. For each $(n+1)$-dimensional irreducible component of $\mathcal{S}$, fix a $(n+1)$-form $\mathrm{d}y_1\wedge\ldots\wedge\mathrm{d}y_n\wedge\mathrm{d}t$ on $\mathcal{S}$. Let φ be the generically constructible function obtained on $\mathcal{S}$ with $\mathcal{C}$ and the coefficients $\mathrm{d}y_1\wedge\ldots\wedge\mathrm{d}y_n\wedge\mathrm{d}t\otimes\ell$: on a $(n+1)$-dimensional irreducible component $\overline{\{x\}}$ of $\mathcal{S}$, φ is (generically) the function associated to $(\psi_x)_h$ if h is the element of $\Omega^{n+1}_{\kappa(x)/R}\otimes\mathcal{D}(x)$ associated to $\mathrm{d}y_1\wedge\ldots\wedge\mathrm{d}y_n\wedge\mathrm{d}t\otimes\ell$ (*cf.* §3.2). Take a Whitney stratification of $\mathcal{S}$ compatible with the semialgebraic subsets where φ is constant, with the irreducible components of $\mathcal{S}$, with the semialgebraic subsets where the forms $\mathrm{d}y_1\wedge\ldots\wedge\mathrm{d}y_n\wedge\mathrm{d}t$ define an orientation, with $\mathcal{S}_\varepsilon$ and $Y\times\{t\mid t>\varepsilon\}$. Then $\mathcal{C}_{\varepsilon^+}$ is given on a n-dimensional stratum of $\mathcal{S}_\varepsilon$ by the sum of the values of φ on the $(n+1)$-dimensional strata Σ of $\mathcal{S}\cap(Y\times\{t\mid t>\varepsilon\})$ whose boundary is this stratum, and the orientation induced by the one of the Σ (*i.e.*, the coefficients $\mathrm{d}y_1\wedge\ldots\wedge\mathrm{d}y_n\otimes\ell$). On the other side, $\mathcal{C}_{\varepsilon^-}$ is given on this stratum by the opposite of the sum of the values of φ on the $(n+1)$-dimensional strata Σ' of $\mathcal{S}\cap(Y\times\{t\mid t<\varepsilon\})$ whose boundary is this stratum, and the orientation induced by the one of the Σ' (which give here the opposite of the orientation $\mathrm{sign}(\mathrm{d}y_1\wedge\ldots\wedge\mathrm{d}y_n)$); finally, $\mathcal{C}_{\varepsilon^-}$ is also given by the sum of the values of φ on the strata Σ' and the coefficient $\mathrm{d}y_1\wedge\ldots\wedge\mathrm{d}y_n\otimes\ell$.

For generic ε, $\mathcal{C}_{\varepsilon^+}=\mathcal{C}_{\varepsilon^-}=:\mathcal{C}_\varepsilon$. In the general case, we have

$$\mathcal{C}_{\varepsilon^+}=\lim_{\eta\to\varepsilon^+}\mathcal{C}_\eta$$

in the sense of [SV] for example. If $\mathcal{C}$ is a cycle, we always have $\mathcal{C}_{\varepsilon^+}=\mathcal{C}_{\varepsilon^-}$ else the component of the boundary on $\mathcal{S}_\varepsilon$ would be nonzero.

6.4. Pushforward of Lagrangian cycles.

Let $f:Z\longrightarrow X$ be a regular morphism of smooth manifolds. We recall the description of the morphism $f_*:\mathcal{L}(Z)\longrightarrow\mathcal{L}(X)$ given in [SV]. The map f induces the diagram:

$$\begin{tikzcd} T^*Z & Z\times_X T^*X \arrow[l, "\mathrm{d}f^*"'] \arrow[r, "\tau"] \arrow[d, "\pi"] & T^*X \arrow[d, "\pi_X"] \\ & Z \arrow[r, "f"] \arrow[from=ul, "\pi_Z"'] & X. \end{tikzcd}$$

Let d and n the dimensions of Z and X. If C is a Lagrangian cycle on T^*Z, and C_s is a deformation of C such that C is the limit of C_s

when $s \to 0^+$ and that for each s, $|C_s|$ (the support of C_s) is transverse to $\mathrm{d}f^*$, then $\mathrm{d}f^{*-1}(|C_s|)$ is a n-dimensional submanifold of $Z \times_X T^*X$. The coefficients are given by the following isomorphisms: let Λ be a component of $|C_s|$ and $\Lambda_0 \subset \Lambda$ a smooth Zariski open subset; the coefficient of C_s on Λ can be seen as a global section of $\omega_{\Lambda_0} \otimes \pi_Z^*\omega_Z$; if $\mathcal{N}_{\Lambda_0/T^*Z}$ is the normal sheaf of Λ_0 (*cf.* [Ha], p. 182), then we have

$$\omega_{\Lambda_0} \simeq \omega_{T^*Z} \otimes \bigwedge^d \mathcal{N}_{\Lambda_0/T^*Z}$$

(see [Ha], Proposition 8.20); $\bigwedge^d \mathcal{N}_{\Lambda_0/T^*Z}$ corresponds to the coorientation sheaf on Λ_0. As $\omega_{T^*Z} \simeq \mathcal{O}_Z$, we get

$$\omega_{\Lambda_0} \otimes \pi_Z^*\omega_Z \simeq \bigwedge^d \mathcal{N}_{\Lambda_0/T^*Z} \otimes \pi_Z^*\omega_Z.$$

Using the transversality of $\mathrm{d}f^*$ with $|C_s|$, we have $(\mathrm{d}f^*)^*(\bigwedge^d \mathcal{N}_{\Lambda_0/T^*Z}) = \bigwedge^d \mathcal{N}_{df^{*-1}(\Lambda_0)/Z\times_X T^*X}$. Thus

$$(\mathrm{d}f^*)^*(\omega_{\Lambda_0} \otimes \pi_Z^*\omega_Z) \simeq \bigwedge^d \mathcal{N}_{\mathrm{d}f^{*-1}(\Lambda_0)/Z\times_X T^*X} \otimes \pi^*\omega_Z.$$

As $\omega_{Z\times_X T^*X} \otimes (\omega_{Z\times_X T^*X})^* \simeq \mathcal{O}_{Z\times_X T^*X}$, this is also isomorphic to

$$\Big(\bigwedge^d \mathcal{N}_{\mathrm{d}f^{*-1}(\Lambda_0)/Z\times_X T^*X} \otimes \omega_{Z\times_X T^*X}\Big) \otimes \big(\pi^*\omega_Z \otimes (\omega_{Z\times_X T^*X})^*\big).$$

Using once again [Ha], Proposition 8.20, for $\mathrm{d}f^{*-1}(\Lambda_0)$, and as $\omega_{Z\times_X T^*X} \simeq \pi^*\omega_Z \otimes \omega_{Z\times_X T^*X/Z}$, we get

$$\begin{aligned}(\mathrm{d}f^*)^*(\omega_{\Lambda_0} \otimes \pi_Z^*\omega_Z) &\simeq \omega_{\mathrm{d}f^{*-1}(\Lambda_0)} \otimes \omega_{Z\times_X T^*X/Z}\\ &\simeq \omega_{\mathrm{d}f^{*-1}(\Lambda_0)} \otimes \tau^*\omega_{T^*X/X}.\end{aligned}$$

So $\mathrm{d}f^{*-1}(C_s)$ is defined as a cycle with values in $\tau^*\omega_{T^*X/X}$. Then from [SV], f_*C is the limit cycle of $\tau_*\,\mathrm{d}f^{*-1}(C_s)$ when $s \to 0^+$ (τ_* is the usual push-forward).

Example. — Let $f : Z \to \mathbb{R}^2$ be the blowing up at the origin. Let (x, y) be coordinates on $\mathbb{R}^2$ and (x, y, ξ, η) the associated coordinates on $T^*\mathbb{R}^2$. Describe Z with two maps with coordinates (x, t) and (u, y) (with $u = 1/t$); the associated coordinates on T^*Z are (x, t, ζ, τ) and (u, y, ν, θ) (with $u = 1/t$, $\theta = \zeta/t$, $\nu = \zeta xt - t^2\tau$). Finally the coordinates on $Z \times_{\mathbb{R}^2} T^*\mathbb{R}^2$ are (x, t, ξ, η) and (u, y, ξ, η).

The map f is given by $(x,t) \mapsto (x,tx)$ and $(u,y) \mapsto (uy,y)$. The map $\mathrm{d}f^*$ is given by

$$(x,t,\xi,\eta) \mapsto (x,t,\xi+\eta t,\eta x) \quad \text{and} \quad (u,y,\xi,\eta) \mapsto (u,y,\xi y,\xi u+\eta).$$

Let $C = [T^*_E Z]$ where E is the exceptional divisor. Deform $[T^*_E Z]$ into C_ε where $|C_\varepsilon|$ is given by $(0,t,\zeta,\varepsilon t/(1+t^4))$ or $(u,0,-\varepsilon u/(1+u^4),\theta)$ and the coefficient is

$$-\mathrm{d}\zeta \wedge \mathrm{d}t \otimes \mathrm{d}x \wedge \mathrm{d}t \quad \text{or} \quad -\mathrm{d}u \wedge \mathrm{d}\theta \otimes \mathrm{d}u \wedge \mathrm{d}y$$

in $\omega_{|C_\varepsilon|} \otimes \pi_Z^* \omega_Z$. This is a good deformation, and $\mathrm{d}f^{*-1}(C_\varepsilon) = \Lambda_1 \cup \Lambda_2$ where $\Lambda_1 = \{t=0,\ x=0\}$ and $\Lambda_2 = \{u=0,\ y=0\}$. To compute the coefficient, we need to pass through the normal sheaf. For example let us do it in the first map for Λ_1. First, the corresponding coefficient in $\Lambda^2 \mathcal{N}_{|C_\varepsilon|/T^*Z} \otimes \pi_Z^* \omega_Z$ is $\mathrm{d}x \wedge \mathrm{d}\tau \otimes \mathrm{d}x \wedge \mathrm{d}t$ as $(-\mathrm{d}\zeta \wedge \mathrm{d}t) \wedge (\mathrm{d}x \wedge \mathrm{d}\tau)$ gives the canonical orientation of T^*Z. Next step, apply $\mathrm{d}f^*$: as in Λ_1, $\mathrm{d}\tau = -\varepsilon\, \mathrm{d}t$, we obtain $\varepsilon\, \mathrm{d}x \wedge \mathrm{d}t \otimes \mathrm{d}x \wedge \mathrm{d}t$ in $\Lambda^2 \mathcal{N}_{\Lambda_1/Z \times_{\mathbb{R}^2} T^*\mathbb{R}^2} \otimes \pi^* \omega_Z$.

Now fix $\omega = \mathrm{d}x \wedge \mathrm{d}t \wedge \mathrm{d}\xi \wedge \mathrm{d}\eta$ in $\omega_{Z \times_{\mathbb{R}^2} T^*\mathbb{R}^2}$; this fix an isomorphism $\omega_{Z \times_{\mathbb{R}^2} T^*\mathbb{R}^2} \simeq (\omega_{Z \times_{\mathbb{R}^2} T^*\mathbb{R}^2})^*$ and allows us to get on one hand $(-1/\varepsilon)\mathrm{d}\xi \wedge \mathrm{d}\eta$ in ω_{Λ_1} as $(-(1/\varepsilon)\mathrm{d}\xi \wedge \mathrm{d}\eta) \wedge (\varepsilon \mathrm{d}x \wedge \mathrm{d}t) = \omega$ and on the other hand $\mathrm{d}\xi \wedge \mathrm{d}\eta$ in $\omega_{Z \times_{\mathbb{R}^2} T^*\mathbb{R}^2/Z}$ as $(\mathrm{d}x \wedge \mathrm{d}t) \wedge (\mathrm{d}\xi \wedge \mathrm{d}\eta) = \omega$. Thus we have obtained the coefficient $-(1/\varepsilon)\mathrm{d}\xi \wedge \mathrm{d}\eta \otimes (\mathrm{d}\xi \wedge \mathrm{d}\eta)$ in $\omega_{\Lambda_1} \otimes \omega_{Z \times_{\mathbb{R}^2} T^*\mathbb{R}^2/Z}$.

The last step is to apply τ: it is easy to see that the component obtained with Λ_1 is $T^*_{\{0\}}\mathbb{R}^2$ with the coefficient $(-1/\varepsilon)\mathrm{d}\xi \wedge \mathrm{d}\eta \otimes \mathrm{d}\xi \wedge \mathrm{d}\eta$ in $\omega_{T^*_{\{0\}}\mathbb{R}^2} \otimes \omega_{T^*\mathbb{R}^2/\mathbb{R}^2}$. Doing the same with Λ_2 and the coefficients described in the second map, we would have obtained $T^*_{\{0\}}\mathbb{R}^2$ with the opposite coefficient. Thus $\tau_* \mathrm{d}f^{*-1}(C_\varepsilon) = 0$ and the limit is 0. This means that $f_*[T^*_E Z] = 0$.

6.5. Characteristic cycle of algebraically constructible functions.

Let $\varphi : X \to \mathbb{Z}$ be a constructible function. It is algebraically constructible if and only if it can be written $\varphi = \sum f_* \mathbf{1}_Z$ where the sum is finite, the Z's are algebraic varieties and the $f : Z \to X$ are proper. (f_* is defined by $f_* \mathbf{1}_A(x) = \chi(f^{-1}(x) \cap A)$ for $x \in X$). In fact the Z's can be chosen non singular, *cf.* [MP], Lemma 6.4.

If $\psi : Z \to \mathbb{Z}$ is a constructible function and F is a complex of constructible sheaves such that $\psi = \chi(F)$, then we have from [KS], Proposition 9.4.2,

$$C(f_*\psi) = CC(Rf_*F) = f_*CC(F) = f_*C(\psi).$$

Example. — If $f : Z \to \mathbb{R}^2$ is the blowing up at the origin, and $\varphi = \mathbf{1}_E$, then $f_*(\varphi) = 0$, which corresponds to $f_*[T_E^* Z] = 0$.

If $\varphi = \sum f_* \mathbf{1}_Z$ with Z non singular, then

$$C(\varphi) = \sum f_* C(\mathbf{1}_Z) = \sum f_*[T_Z^* Z].$$

Thus to show that $C(\varphi)$ is algebraically constructible it is sufficient to show the following

PROPOSITION 32. — *The cycle $f_*[T_Z^* Z]$ is algebraically constructible.*

First two lemmas:

LEMMA 33. — *Let A, B and X be smooth algebraic varieties, $M \subset X$ an algebraic subvariety and $\mathcal{S}$ a finite Nash stratification of M. Let $f : A \times B \to X$ be a regular map, and assume f is transverse to $\mathcal{S}$. Call $f_a : B \to X$ the map given by $f_a(b) = f(a,b)$. Then there exists a semialgebraic subset Ω of A of positive codimension such that for $a \in A \setminus \Omega$, f_a is transverse to $\mathcal{S}$.*

Proof. — Let S an element of the stratification $\mathcal{S}$. As f is transverse to S, the inverse image $f^{-1}(S)$ is a Nash subvariety of $A \times B \times X$. Consider the projection $\pi : f^{-1}(S) \to A$. If a is a regular value of π, then $\pi^{-1}(a) = \{a\} \times (\operatorname{Im} f_a \cap S)$ is a submanifold of X and f_a is transverse to S. From Sard theorem (*cf.* [BCR], Theorem 9.5.2), the set Ω_S of non regular values of π is a semialgebraic subset of A of positive codimension. Take $\Omega = \bigcup_S \Omega_S$. □

LEMMA 34. — *Let Y be a real algebraic smooth variety, $p : E \to Y$ an algebraic smooth bundle, $s : Y \to E$ a regular section, M an algebraic subvariety of E and $\mathcal{S}$ a finite Nash stratification of M. Then there exist a regular family of regular sections $S = (S_\varepsilon)_{\varepsilon \in R}$ and a Zariski open subset U of R such that $S_0 = s$ and for $\varepsilon \in U$, S_ε is transverse to $\mathcal{S}$.*

Proof (communicated by J. Bochnak). — As E is an algebraic bundle, we can find regular sections $s_1, \ldots, s_k$ of E generating the fibre of E in each point of Y (*cf.* [BCR], Proposition 12.1.7). Put $q_\alpha = s + \sum_{i=1}^k \alpha_i s_i$ for $\alpha = (\alpha_1, \ldots, \alpha_k) \in R^k$, and $q : Y \times R^k \to E : (y, \alpha) \mapsto q_\alpha(y)$. Then q is transverse to $\mathcal{S}$ as it is a submersion: as E is locally isomorphic to $Y \times E_y$, the tangent space $T_{q(y,\alpha)} E$ is locally isomorphic to $T_y Y \times E_y$. The image of

$T_yY \times \{0\}$ by $\mathrm{d}_{(y,\alpha)}q$ is $T_yY \times \{0\}$ as q_α is a section. On the other hand the image of $\{0\} \times R^k$ by $\mathrm{d}_{(y,\alpha)}q$ is generated by the $s_i(y)$'s: this is E_y. Thus $\mathrm{d}_{(y,\alpha)}q(T_yY \times R^k) = T_{q(y,\alpha)}E$.

Apply Lemma 33 to q: there exists a semialgebraic subset Ω of R^k of positive codimension such that $\alpha \notin \Omega$ if and only if q_α is transverse to $\mathcal{S}$, and $q_0 = s$. There is a fibration $\pi : R^k \setminus \{0\} \to \mathbb{P}^{k-1}(R)$ with fibre $R \setminus \{0\}$. As Ω is of dimension strictly less than k, the set V of points of $\mathbb{P}^{k-1}(R)$ for which the fibre of Ω is 1-dimensional, is of dimension strictly less than $k-1$. Choose $v \notin V$ and let D be the line generated by v in R^k. We identify R to D and we take the family $S = q_{|Y\times R}$ and $U = D \setminus \Omega \cap D$. $\square$

Proof of Proposition 32. — Let d and n be the dimensions of Z and X, $(z_1,\dots,z_d)$ and $(x_1,\dots,x_n)$ the coordinates on Z and X, $(z_1,\dots,z_d, \theta_1,\dots,\theta_d)$ and $(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$ the associated coordinates on T^*Z and T^*X, and at last $(z_1,\dots,z_d,\xi_1,\dots,\xi_n)$ the coordinates on $Z \times_X T^*X$. Complete the diagram of §6.4 in the commutative diagram:

$$\begin{tikzcd}
T^*Z \times R \arrow[d, "\sigma_1"] & Z \times_X T^*X \times R \arrow[l, "\widetilde{\mathrm{d}f^*}=\mathrm{d}f^*\times\mathrm{id}"'] \arrow[r, "\tilde\tau=\tau\times\mathrm{id}"] \arrow[d, "\sigma_2"] & T^*X \times R \arrow[d, "\sigma_3"] \\
T^*Z \arrow[dr, "\pi_Z"'] & Z \times_X T^*X \arrow[l, "\mathrm{d}f^*"'] \arrow[r, "\tau"] \arrow[d, "\pi"] & T^*X \arrow[d, "\pi_X"] \\
 & Z \arrow[r, "f"] & X
\end{tikzcd}$$

where the σ_i are the projections. Consider $[T^*_ZZ] \in C_d(T^*Z, \pi_Z^*\omega_Z)$, of support T^*_ZZ (the zero section of T^*Z) and of coefficient $\mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_d \otimes \mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_d$.

First show that there exists a deformation of $[T^*_ZZ]$ which is an algebraically constructible cycle of support transverse to $\mathrm{d}f^*$. As T^*Z is an algebraic bundle (*cf.* [BCR], Proposition 12.1.9), Lemma 34 applied to the zero section of T^*Z gives a regular family of regular section $j_\varepsilon : z \mapsto (z, s^\varepsilon(z))$ and a Zariski open subset U of R such that j_ε is transverse to $\mathrm{d}f^*$ for each $\varepsilon \in U$ (here $M = \mathrm{Im}\ \mathrm{d}f^*$ and $\mathcal{S} = \{M\}$). Then $\mathcal{J}_\varepsilon = j_\varepsilon(Z)$ is an algebraic subvariety of T^*Z and $\mathcal{J} = \{(z,\theta,\varepsilon) \mid (z,\theta) \in j_\varepsilon(Z)\}$ is a smooth algebraic subvariety of $T^*Z \times R$. Fix a coordinate t on R, and let $\mathcal{C}$ the $(d+1)$-chain of $T^*Z \times R$ with coefficients in $\sigma_1^*\pi_Z^*\omega_Z$, of support $\mathcal{J}$ and of coefficient $\mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_d \wedge \mathrm{d}t \otimes \mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_d \in \omega_{\mathcal{J}} \otimes \sigma_1^*\pi_Z^*\omega_Z$. The cycle

$\mathcal{C}_\varepsilon$ is the one of support $\mathcal{J}_\varepsilon$, and of coefficient $\mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_d \otimes \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_d$. Then $[T^*_Z Z]$ is the limit of $\mathcal{C}_\varepsilon$ when $\varepsilon \to 0^+$.

Then $\mathcal{J}' = (\widetilde{\mathrm{d}f^*})^{-1}(\mathcal{J} \cap (T^*Z \times U))$ is a $(n+1)$-dimensional smooth algebraic subvariety of $Z \times_X T^*X \times U$. The chain $(\widetilde{\mathrm{d}f^*})^*(\mathcal{C})$ is the chain of support $\mathcal{J}'$ and of coefficient in $\omega_{\mathcal{J}'} \otimes \sigma_2^* \tau^* \omega_{T^*X/X}$ given by the isomorphism $(\widetilde{\mathrm{d}f^*})^*(\omega_{\mathcal{J}} \otimes \sigma_1^* \pi_Z^* \omega_Z) \simeq \omega_{\mathcal{J}'} \otimes \sigma_2^* \tau^* \omega_{T^*X/X}$ as in the §6.4, which gives if $y_1, \ldots, y_n, t$ are coordinates of $\mathcal{J}'$ (where the y_i are regulars in ξ, z), $A(z, \xi, t)\, \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_n \wedge \mathrm{d}t \otimes \mathrm{d}\xi_1 \wedge \ldots \wedge \mathrm{d}\xi_n$ where A is a regular function. Thus $(\widetilde{\mathrm{d}f^*})^*(\mathcal{C})$ is an algebraically constructible cycle and $((\widetilde{\mathrm{d}f^*})^*(\mathcal{C}))_\varepsilon = (\mathrm{d}f^*)^*(\mathcal{C}_\varepsilon)$.

We also have $(\widetilde{\tau}_*((\widetilde{\mathrm{d}f^*})^*)(\mathcal{C}))_\varepsilon = \tau_*(\mathrm{d}f^*)^*(\mathcal{C}_\varepsilon)$, and $\widetilde{\tau}_*((\widetilde{\mathrm{d}f^*})^*)(\mathcal{C})$ is algebraically constructible as pushforward of algebraically constructible cycle. Thus the limit of $(\tau_*(\mathrm{d}f^*)^*(\mathcal{C}_\varepsilon)$ when $\varepsilon \to 0^+$ is algebraically constructible, this means $f_*[T^*_Z Z]$ is algebraically constructible. □

Let $\mathcal{L}^{\mathrm{ac}}(X) = \mathcal{L}(X) \cap AC_n(T^*X, \omega_{T^*X/X})$ (the set of Lagrangian algebraically constructible cycles) and $\mathcal{F}^{\mathrm{ac}}(X)$ denote the set of algebraically constructible functions on X.

COROLLARY 35. — *The characteristic cycle of an algebraically constructible function is a Lagrangian algebraically constructible cycle (i.e. $C(\mathcal{L}^{\mathrm{ac}}(X)) \subset \mathcal{F}^{\mathrm{ac}}(X)$).*

6.6. Inverse image of a Lagrangian algebraically constructible cycle.

Let $\mathrm{Eu} = C^{-1} : \mathcal{L}(X) \to \mathcal{F}(X)$. Let $\mathcal{C} \in \mathcal{L}(X)$ be an algebraically constructible cycle. Is $\mathrm{Eu}(\mathcal{C})$ an algebraically constructible function? The computation of $\mathrm{Eu}(\mathcal{C})$ is done in the following way (*cf.* [KS], p. 406): for $x \in X$, let $\varphi_x : X \to R$ a Nash function such that $\varphi_x(x) = 0$, $\mathrm{d}\varphi_x(x) = 0$, and the Hessian $H(x)$ of φ_x in x is positive definite. Let $\Lambda_{\varphi_x} = \{(y, \mathrm{d}\varphi_x(y)) \mid y \in X\} \subset T^*X$, and $[\sigma_{\varphi_x}] \in C_n(T^*X, \pi_X^*\omega_X)$ the chain of support Λ_{φ_x} and of coefficient $\omega \otimes \omega \in \omega_{\Lambda_{\varphi_x}} \otimes \pi_X^* \omega_X$ with $\omega_X \simeq \omega_{\Lambda_{\varphi_x}}$ by the projection.

$\mathrm{Eu}(\mathcal{C})(x)$ is the intersection multiplicity of $[\sigma_{\varphi_x}]$ and $\mathcal{C}$ in $(x, 0)$(that will be denoted $\sharp([\sigma_{\varphi_x}] \cap \mathcal{C})_{(x,0)})$. Let $\Sigma \subset T^*X$ be the support of $\mathcal{C}$. We will prove that $\mathrm{Eu}(\mathcal{C})$ is actually algebraically constructible, first generically and then everywhere. We will need the following proposition.

PROPOSITION 36. — *Assume X is of dimension n. Let Y a smooth*

m-dimensional irreducible algebraic variety, $\gamma: Y \to X$ a regular morphism, $\varphi: Y \times X \to R$ a regular function; for $a \in Y$, $\varphi_a: X \to R$ denote the function $x \mapsto \varphi(a,x)$; assume that $\varphi_a(\gamma(a))) = 0$ and $\mathrm{d}\varphi_a(\gamma(a)) = 0$. Let

$$\Lambda_\varphi = \{(a,x,\mathrm{d}\varphi_a(x)) \mid a \in Y, x \in X\} \subset Y \times T^*X.$$

Then there exists a proper algebraic subvariety Z of Y such that $a \mapsto \sharp([\sigma_{\varphi_a}] \cap \mathcal{C})_{(\gamma(a),0)}$ is algebraically constructible outside Z.

Proof. — Let θ be a n-form on X. Let $\Sigma_1, \dots, \Sigma_k$ be the n-dimensional irreducible components of Σ, $\omega_1, \dots, \omega_k$ n-forms on $\Sigma_1, \dots, \Sigma_k$. The cycle $\mathcal{C}$ is algebraically constructible thus there exists regular functions $g_{ij} \in \kappa(\Sigma_i)$ such that $\mathcal{C}$ is given on Σ_i par ψ_i by $(\psi_i)_{\omega_i \otimes \theta} = \sum_j \mathrm{sign}(g_{ij})$.

Let $\mathcal{S}$ be a Whitney stratification of Σ compatible with the singular points of Σ, the points where the g_{ij} vanish, the intersections of irreducible components of Σ, the zeroes and the poles of ω_i. Call $\mathcal{S}_Y$ the Whitney stratification of $Y \times \Sigma$ whose strata are the products of Y with the strata of $\mathcal{S}$.

As $Y \times T^*X \to Y \times X$ is an algebraic bundle, Lemma 34 applied to the section $Y \times X \to Y \times T^*X : (a,x) \mapsto (a,x,\mathrm{d}\varphi_a(x))$ gives a regular family of regular sections $\Phi^\varepsilon : (a,x) \mapsto (a,x,s^\varepsilon(a,x))$ which are transverse to $\mathcal{S}_Y$ for $\varepsilon \in U$, U Zariski open subset of R.

From Lemma 33, for each $\varepsilon \in U$, there exists a semialgebraic subset Ω^ε of Y of positive codimension such that $a \in Y \setminus \Omega^\varepsilon$ if and only if the map $\Psi_a^\varepsilon : x \mapsto (a,x,s^\varepsilon(a,x))$ is transverse to $\mathcal{S}$; if Φ_a^ε denote the section $x \mapsto (x, s^\varepsilon(a,x))$ of T^*X, then Φ_a^ε is transverse to $\mathcal{S}$ for each $a \in Y \setminus \Omega^\varepsilon$ as $\Psi_a^\varepsilon = (\mathrm{id}, \Phi_a^\varepsilon)$. Put $\Omega = \bigsqcup_{\varepsilon \in U} \Omega^\varepsilon$. This is a semialgebraic subset of $U \times Y$. Denote by Z the Zariski closure of $(\{0\} \times Y) \cap \overline{\Omega}^{\mathrm{Zar}}$. Identifying $(\{0\} \times Y)$ to Y we have $Z \subset Y$.

We can choose ω_i as a restriction of a n-form $\overline{\omega}_i$ on T^*X, which gives by pullback by the projection, a n-form $\widetilde{\omega_i}$ on $R \times Y \times T^*X$. On the other hand the n-form θ gives by pullback a n-form $\widetilde{\theta}$ on $R \times Y \times X \simeq \bigsqcup_\varepsilon \mathrm{Im}\, \Phi^\varepsilon \subset R \times Y \times T^*X$. Then there exist regular functions f_i on $R \times Y \times T^*X$ such that $\widetilde{\theta} \wedge \widetilde{\omega}_{i|T^*X} = f_i(-1)^{n(n-1)/2}(\mathrm{d}\alpha_X)^n$. The computation of the intersection number of $[\sigma_{\varphi_a}]$ and $\mathcal{C}$ take count of the sign of f_i (*cf.* [KS], p. 389).

Let $F_i = \sum_j \mathrm{sign}(g_{ij})\, \mathrm{sign}(f_i) \mathbf{1}_{(R \times Y \times \Sigma_i) \cap (\sqcup_\varepsilon \mathrm{Im}\, \Phi^\varepsilon)}$ and $F = \sum_i F_i$. It is an algebraically constructible function on $R \times Y \times T^*X$.

The number $\sharp([\sigma_{\varphi_a}] \cap \mathcal{C})_{(\gamma(a),0)}$ equals, for a sufficiently small ε and $a \notin Z$, the sum of $F(\varepsilon, a, x)$ for the x near $(\gamma(a), 0)$ ($x_1^\varepsilon(a), \dots, x_p^\varepsilon(a)$ denote these x and $m_i = F(\varepsilon, a, x_i^\varepsilon(a))$).

Let $\rho : R \times Y \times T^*X \to R$ be the projection. For ψ a constructible function on $R \times Y \times T^*X$, McCrory and Parusiński [MP] define a constructible function $\Psi_\rho^+\psi$ on $Y \times T^*X$ by

$$\Psi_\rho^+\psi(y,x) = \int_{F_\rho^+(y,x)} \psi$$

where $F_\rho^+(y,x) = \overline{B}((0,y,x),\delta) \cap (\{\varepsilon\} \times Y \times T^*X)$ with $0 < \varepsilon \ll \delta \ll 1$. We have $F_\rho^+(y,x) = \{\varepsilon\} \times \overline{B}((y,x),\delta')$ with $0 < \varepsilon \ll \delta' \ll 1$. Then $\sharp([\sigma_{\varphi_a}] \cap \mathcal{C})_{(\gamma(a),0)}$ equals $\Psi_\rho^+F(a,\gamma(a),0)$ for $a \notin Z$. Indeed in restriction to $\{\varepsilon\} \times \overline{B}((a,\gamma(a),0),\delta')$, $F = m_1\mathbf{1}_{A_1} + \cdots + m_p\mathbf{1}_{A_p}$ where the A_i are described by the x_i^ε; they are homeomorphic to a closed ball of Y, thus

$$\int_{F_\rho^+(a,\gamma(a),0)} F = \sum_i m_i\chi(A_i) = \sum_i m_i.$$

Let $g : Y \to Y \times T^*X : a \mapsto (a,\gamma(a),0)$. The inverse image by g of a constructible function ψ on $Y \times T^*X$ is defined by $g^*\psi(a) = \psi(g(a))$ (*cf.* [MP]). Then

$$\sharp\big([\sigma_{\varphi_a}] \cap \mathcal{C}\big)_{(\gamma(a),0)} = g^*\Psi_\rho^+F(a)$$

outside Z. From [MP], Theorem 2.6, $g^*\Psi_\rho^+F$ is an algebraically constructible function. □

COROLLARY 37. — *The inverse image by C of an algebraically constructible Lagrangian cycle is an algebraically constructible function, i.e.* $\mathrm{Eu}(\mathcal{F}^{\mathrm{ac}}(X)) \subset \mathcal{L}^{\mathrm{ac}}(X)$ *(thus $C_{|\mathcal{L}^{\mathrm{ac}}(X)}$ is an isomorphism from $\mathcal{L}^{\mathrm{ac}}(X)$ onto $\mathcal{F}^{\mathrm{ac}}(X)$).*

Proof. — Embed X in R^m and put $\varphi_x(y) = \|x-y\|^2$ (Euclidean norm). From the preceding proof $\mathrm{Eu}(\mathcal{C})$ is algebraically constructible on $X \setminus Y$ where Y is an algebraic subvariety of X of strictly smaller dimension. For each irreducible component Y_i of Y, let $\delta : \widetilde{Y_i} \to Y_i$ a desingularization of Y_i. Then $a \mapsto \mathrm{Eu}(\mathcal{C})(\delta(a))$ is generically algebraically constructible on $\widetilde{Y_i}$. Let $Z_i \subset \widetilde{Y_i}$ such that it is algebraically constructible on $\widetilde{Y_i} \setminus Z_i$ and that δ in restriction to $\widetilde{Y_i} \setminus Z_i$ is a biregular isomorphism onto its image. Then $\mathrm{Eu}(\mathcal{C})$

is algebraically constructible on $Y_i \setminus \delta(Z_i)$. The dimensions go down and in dimension 0 every integral function is algebraically constructible. $\square$

From this corollary we get easily the following characterization of Lagrangian algebraically constructible cycles:

COROLLARY 38. — *Each chain of $\mathcal{L}^{\mathrm{ac}}(X)$ can be written as $\sum f_*[T_Z^* Z]$, with Z non singular algebraic variety and $f: Z \to X$ proper.*

From this characterization follows the property:

COROLLARY 39. — *If $f: X \to Y$ is a regular proper morphism, then $f_*: \mathcal{L}^{\mathrm{ac}}(X) \to \mathcal{L}^{\mathrm{ac}}(Y)$ (the pushforward of a Lagrangian algebraically constructible cycle is an algebraically constructible cycle).*

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