ANNALES DE L'INSTITUT FOURIER

SHULIM KALIMAN
MIKHAIL ZAIDENBERG

Miyanishi's characterization of the affine 3-space does not hold in higher dimensions

Annales de l'institut Fourier, tome 50, n° 6 (2000), p. 1649-1669
<http://www.numdam.org/item?id=AIF_2000__50_6_1649_0>

Ann. Inst. Fourier, Grenoble
50, 6 (2000), 1649–1669

MIYANISHI'S CHARACTERIZATION OF THE AFFINE 3-SPACE DOES NOT HOLD IN HIGHER DIMENSIONS

by S. KALIMAN and M. ZAIDENBERG

Introduction.

Let X be a smooth, contractible complex affine 3-fold. Recall

MIYANISHI'S THEOREM [Miy] [(1)]. — *$X \simeq \mathbb{C}^3$ if and only if the following two conditions hold:*

(i) *there exists a regular function $f : X \to \mathbb{C}$ and a Zariski open subset $U \subset \mathbb{C}$ such that $f^{-1}(U) \simeq_U U \times \mathbb{C}^2$* (in particular, the general fiber $F_c := f^*(c)$ $(c \in U)$ of f is isomorphic to the affine plane $\mathbb{C}^2$), *and*

(ii) *all the fibers F_c $(c \in \mathbb{C})$ are UFD-s* (that is, for any $c \in \mathbb{C}$ the divisor F_c is reduced and irreducible, and the algebra $A_c := \mathbb{C}[F_c]$ of regular functions on the surface F_c is a UFD).

Research of the first author partially supported by NSA grant MDA904-00-1-0016.
This work was started during the second author's stay at the IHES; he would like to thank IHES for the hospitality.
Keywords: Affine space – Polynomial algebra – Polynomial curve – Locally nilpotent derivation – Exotic structure.
Math. classification: 13A02 – 14J35 – 14R05 – 14R10 – 14R20.

(1) In the original formulation, instead of assuming X to be topologically contractible, it is subjected to the following weaker conditions: $e(X) = 1$, the algebra $\mathbb{C}[X]$ of regular functions on X is a UFD, and all its invertible elements are constants. Likewise, in condition (ii) resp. (ii′) below the fibers themselves are replaced by their irreducible components. But actually, one can show that all the fibers of the function f as in (i) are reduced and irreducible.

By [Ka2], Lemmas I, III, and [KaZa2] *the theorem holds if one only supposes that*

(i′) *the general fibers F_c of f are isomorphic to $\mathbb{C}^2$ and*

(ii′) *each fiber F_c $(c \in \mathbb{C})$ has at most isolated singularities.*

The latter assumption (ii′) is essential, as shows the example of Russell's cubic 3-fold $X \subset \mathbb{C}^4$, $X = p^{-1}(0)$ where $p = x + x^2y + z^2 + t^3$. In this example the fibers $F_c(c \in \mathbb{C})$ of the regular function $f = x|X : X \to \mathbb{C}$ are isomorphic to $\mathbb{C}^2$ except for the fiber F_0 which has non-isolated singularities (and therefore, it is not a UFD). And indeed, the Russell cubic X is not isomorphic to $\mathbb{C}^3$ [ML1], that is, it is an *exotic* $\mathbb{C}^3$ (i.e., a smooth affine variety diffeomorphic to $\mathbb{R}^6$ and non-isomorphic to $\mathbb{C}^3$; see [Za2]).

More generally, the Main Theorem of [Ka2], [KaZa2] provides the following useful supplement to Miyanishi's theorem:

A smooth, contractible affine 3-fold X is an exotic $\mathbb{C}^3$ if there exists a regular function $f : X \to \mathbb{C}$ on X with general fibers isomorphic to $\mathbb{C}^2$, but not all of its fibers being so.(2)

In this paper we prove the following

THEOREM 1. — *The hypersurface X in $\mathbb{C}^5$ given by the equation*

$$p = u^m v + \frac{(xz+1)^k - (yz+1)^l + z}{z} = 0 \tag{1}$$

where $m \geqslant 2$, $k > l \geqslant 3$, $\gcd(k, l) = 1$, is an exotic $\mathbb{C}^4$.

COROLLARY. — *Miyanishi's Theorem does not hold in the dimension four.*

Proof. — The regular function $f = u|X : X \to \mathbb{C}$ on this hypersurface provides a fibration with all the fibers $F_c(c \in \mathbb{C})$ being smooth reduced contractible affine 3-folds, all but the zero one F_0 being isomorphic to $\mathbb{C}^3$. Moreover, the mapping

$$(x, y, z, u) \longmapsto (x, y, z, u, v = u^{-m} q_{k,l}(x, y, z))$$

where

$$q_{k,l} := \frac{(xz+1)^k - (yz+1)^l + z}{z}$$

(2) Notice that such a threefold X admits a birational dominant morphism $\mathbb{C}^3 \to X$.

gives an isomorphism $f^{-1}(U) = X \setminus F_0 \simeq_U U \times \mathbb{C}^3$ where $U = \mathbb{C} \setminus \{0\}$. At the same time, the fiber $F_0 \simeq S_{k,l} \times \mathbb{C}$ is an exotic $\mathbb{C}^3$ (see [Za1], [Za2]). Here $S_{k,l} = q_{k,l}^{-1}(0) \subset \mathbb{C}^3$ is the *tom Dieck-Petrie surface*; this is a smooth contractible affine surface non-homeomorphic to $\mathbb{R}^4$ [tDP]. By Fujita's theorem [Fu], (1.18)–(1.20), [Ka1], (3.2), any smooth, contractible affine variety is a UFD. Hence all the fibers of the function $f = u|X$ are smooth UFD-s. Thus the both conditions (i) and (ii) of the Miyanishi Theorem are fulfilled, whereas due to Theorem 1, $X \not\simeq \mathbb{C}^4$. □

Remark. — Theorem 1 still holds for a triplet (k, l, m) with $l = 2$ if $\gcd(m, 2k) = 1$. The proof of this fact is not difficult but we prefer the argument below since this enables us to demonstrate a nice connection with the Diophantine geometry over function fields (see Section 2). However, we do not know if the statement remains true for (say) the triplet $(k, l, m) = (3, 2, 2)$.

The proof of Theorem 1 is divided in two parts. The first one, concerning the topology of the variety X, is done in [KaZa1]. The second one (which is done in Section 1 below) concerns exoticity of X; it mainly relies on the fact that there are only few regular actions on X of the additive group $\mathbb{C}_+$ of the complex number field and moreover, there are only few polynomial curves in certain affine varieties related to X.

To conclude, recall the following

PROBLEM. — *Let X be a smooth, contractible complex affine n-fold where $n \geqslant 4$, and let $f : X \to \mathbb{C}$ be a regular function on X. Suppose that $f^*(c) \simeq \mathbb{C}^{n-1}$ for every $c \in \mathbb{C}$. Is it true that $X \simeq \mathbb{C}^n$, and that this isomorphism sends f into a variable of the polynomial algebra $\mathbb{C}^{[n]}$?*

The results of [Miy], [Ka2], [KaZa2], [Sa] cited above provide a positive answer for[(3)] $n = 3$.

We are grateful to the referee for useful remarks and suggestions which served us to improve the exposition.

(3) It is worthwile noting that, without the assumption that X is affine, the answer is negative even for $n = 4$. Indeed, consider the smooth non-affine 4-fold $X = \tilde{X} \setminus Z$ where $\tilde{X}$ is the hypersurface $uv = xy + z^2 - 1$ in $\mathbb{C}^5$ and Z is the plane $u = x = z - 1 = 0$ in $\tilde{X}$. Then every fiber of the morphism $(x, u) : X \to \mathbb{C}^2$ is isomorphic to $\mathbb{C}^2$, and every fiber of the regular function $u|X$ is isomorphic to $\mathbb{C}^3$.

1. Proof of Theorem 1.

In [KaZa1], Proposition 4.4, Example 6.2, it is shown that the smooth affine 4-fold X as in Theorem 1 is contractible and moreover, diffeomorphic to $\mathbb{R}^6$. Thus, to prove the theorem it is enough to verify that $X \not\simeq \mathbb{C}^4$. The proof of the latter assertion is based on the computation of the *Makar-Limanov invariant* $\mathrm{ML}(X)$. Recall that $\mathrm{ML}(X)$ denotes the algebra of regular functions on the variety X invariant under any regular $\mathbb{C}_+$-action on X (or in other words, of regular functions on X that are vanished by any locally nilpotent derivation of the algebra $\mathbb{C}[X]$; see e.g., [KaML1], [Za2], or also [De]).

In fact, we prove the following

PROPOSITION 1. — *$\mathrm{ML}(X) \supset \mathbb{C}[\widetilde{u}]$ where $\widetilde{u} = u|X$. Hence $\mathrm{ML}(X) \not\simeq \mathrm{ML}(\mathbb{C}^4) = \mathbb{C}$ and therefore, $X \not\simeq \mathbb{C}^4$.*

Remark. — If $m = 1$ then $\mathrm{ML}(X) = \mathbb{C}$ (and moreover, the group of biregular automorphisms of X generated by the regular $\mathbb{C}_+$-actions on X acts infinitely transitively [KaZa1], Theorem 5.1). The question arises: *is it still true that X is an exotic $\mathbb{C}^4$ when $m = 1$, at least for some values of k and l?*

Notation. — Throughout the proof, we fix a weight degree function d on the polynomial algebra $\mathbb{C}^{[5]} = \mathbb{C}[x, y, z, u, v]$ given by

$$(2) \qquad d_x = l, \quad d_y = k, \quad d_z = 0, \quad d_u = -n\sqrt{2}, \quad d_v = mn\sqrt{2} + kl$$

where $n \in \mathbb{N}$. This degree function d satisfies the following conditions:

$$kd_x + (k-1)d_z = ld_y + (l-1)d_z = md_u + d_v = kl$$
$$> \max_{i=1,\ldots,k-1, j=1,\ldots,l-1} \{0, id_x + (i-1)d_z = il, jd_y + (j-1)d_z = jk\}.$$

It follows that

$$\widehat{p} := u^m v + x^k z^{k-1} - y^l z^{l-1} = u^m v + z^{l-1}(x^k z^{k-l} - y^l)$$

is the principal d-homogeneous part of the polynomial p from (1); indeed,

$$(3) \quad p = u^m v + q_{k,l}(x, y, z) = u^m v + \sum_{i=1}^{k} \binom{k}{i} x^i z^{i-1} - \sum_{j=1}^{l} \binom{l}{j} y^j z^{j-1} + 1.$$

By d_A we denote the induced degree function on the algebra $A := \mathbb{C}[X] = \mathbb{C}^{[5]}/(p)$. Let $\widehat{A}$ be the associate graded algebra, and $d_{\widehat{A}}$ be the induced degree function on $\widehat{A}$. Since the polynomial $\widehat{p}$ is irreducible, by Proposition 4.1 in [KaML3] (see also [Za2], Lemma 7.1), the affine variety $\widehat{X}$:=spec$\widehat{A}$ coincides with the hypersurface in $\mathbb{C}^5$ given by the equation $\widehat{p} = 0$. We denote by $\widehat{x}, \dots, \widehat{v}$ the images in $\widehat{A}$ of the coordinate functions $x, \dots, v$, respectively, whereas their restrictions to X are denoted as $\widetilde{x}, \dots, \widetilde{v}$. Thus in the algebra $\widehat{A}$ the following relation holds:

$$\widehat{u}^m \widehat{v} = \widehat{z}^{l-1}(\widehat{y}^l - \widehat{x}^k \widehat{z}^{k-l}). \tag{4}$$

For an integral domain B of finite type, let LND(B) be the set of all its locally nilpotent derivations. Fix arbitrary $\partial \in \text{LND}(B) \setminus \{0\}$. Recall the following well known facts which we frequently use below (see e.g., [ML1], [KaML1], [Za2]).

LEMMA 0.

(a) *The invariant subalgebra* $\ker \partial \subset B$ *is factorially closed, that is,* $ab \in \ker \partial \setminus \{0\} \Longrightarrow a, b \in \ker \partial$. *Moreover*[(4)],

$$a^k + b^l \in \ker \partial \setminus \{0\} \quad \textit{and} \quad k, l \geqslant 2 \Longrightarrow a, b \in \ker \partial.$$

(b) *Let* $a \in B$ *be an element of* ∂*-degree one, i.e.,* $\partial a \in \ker \partial \setminus \{0\}$. *Then any element* $b \in B$ *can be presented in the form*

$$b = c^{-1} \sum_{i=0}^{N} c_i a^i \tag{5}$$

where $c, c_0, \dots, c_N \in \ker \partial$.

(c) *The invariant subfield* $\operatorname{Frac} \ker \partial \subset \operatorname{Frac} B$ *is algebraically closed in the fraction field* $\operatorname{Frac} B$, *and* $\operatorname{tr} \deg[\operatorname{Frac} B : \operatorname{Frac} \ker \partial] = 1$.

Fix a locally nilpotent derivation $\partial \in \text{LND}(A) \setminus \{0\}$, and let $\widehat{\partial} \in \text{LND}(\widehat{A})$ be the homogeneous locally nilpotent derivation of the graded algebra $\widehat{A}$ associated with ∂ (that is, the principal part of ∂); notice that $\widehat{\partial} \neq 0$ once $\partial \neq 0$ (see [ML1] or also [KaML1], [Za2]).

LEMMA 1. — $\ker \widehat{\partial} \not\subset \mathbb{C}[\widehat{x}, \widehat{y}, \widehat{z}]$.

Proof. — Assume the contrary. Since tr.deg$[\widehat{A} : \ker \widehat{\partial}] = 1$ there exist three algebraically independent elements, say, $a, b, c \in \ker \widehat{\partial}$. Regarding the elements a, b, c as polynomials in x, y, z, consider the morphism $\sigma =$

(4) The latter statement is a lemma due to Makar-Limanov (see e.g., [Za2], Ex.(7.12.d)), which also follows from the Davenport Lemma in Section 2 below.

$(x, a, b, c) : \mathbb{C}^3 \to \mathbb{C}^4$. The Zariski closure of the image of σ being a proper algebraic subvariety of $\mathbb{C}^4$, there is a non-trivial relation $g(x, a, b, c) = 0$ where $g \in \mathbb{C}^{[4]} \setminus \{0\}$. Hence we have a non-trivial relation

$$\sum_{i=0}^{N} g_i(a, b, c)\widehat{x}^i = 0 \tag{6}$$

in the algebra $\widehat{A}$ where $g_i(a, b, c) \in \ker \widehat{\partial}$. Here $N > 0$ (indeed, otherwise the elements a, b, c would be algebraically dependent). It follows from Lemma 0(c) that $\widehat{x} \in \ker \widehat{\partial}$. Similarly, we have $\widehat{y}, \widehat{z} \in \ker \widehat{\partial}$. In virtue of the relation (4) above, also $\widehat{u}^m \widehat{v} \in \ker \widehat{\partial}$. By Lemma 0(a), it follows that $\widehat{u}, \widehat{v} \in \ker \widehat{\partial}$. Therefore, $\ker \widehat{\partial} = \mathbb{C}[\widehat{x}, \widehat{y}, \widehat{z}, \widehat{u}, \widehat{v}] = \widehat{A}$, and so $\widehat{\partial} = 0$, a contradiction. This proves the lemma. □

LEMMA 2. — *The following alternative holds: either $\widehat{u} \in \ker \widehat{\partial}$ or $\widehat{v} \in \ker \widehat{\partial}$.*

Proof. — Due to (4), any element $\widehat{a} \in \widehat{A} = \mathbb{C}[\widehat{X}]$ can be extended to a unique polynomial $\widehat{f} \in \mathbb{C}^{[5]}$ of the form

$$\widehat{f} = \sum_{i \geqslant 0} a_i u^i + \sum_{i=0}^{m-1} \sum_{j>0} b_{ij} u^i v^j \tag{7}$$

where $a_i, b_{ij} \in \mathbb{C}[x, y, z]$. It is known [KaML3] (see also [Za2], Ex.(7.8)) that

$$\begin{aligned} d_{\widehat{A}}(\widehat{a}) &\stackrel{\text{def}}{=} \min\{d(f) | f \in \mathbb{C}^{[5]}, f|\widehat{X} = \widehat{a}\} \\ &= \min\{d(f) | f = \widehat{f} + \widehat{p}g, g \in \mathbb{C}^{[5]}\} \\ &= d(\widehat{f}). \end{aligned}$$

Furthermore, if $\widehat{a} \in \widehat{A}$ is a $d_{\widehat{A}}$-homogeneous element, then the polynomial $\widehat{f}$ is d-homogeneous, too.

Since the derivation $\widehat{\partial}$ is homogeneous (i.e., graded) its kernel $\ker \widehat{\partial}$ is a graded subalgebra of the graded algebra $\widehat{A}$, and so it is generated by homogeneous elements. Let $\widehat{a} \in \ker \widehat{\partial}$ be a non-zero homogeneous element, and let $\widehat{f} \in \mathbb{C}^{[5]}$ be its d-homogeneous extension as in (7) above.

The degree function $d : \mathbb{C}^{[5]} \setminus \{0\} \to \mathbb{Q}[\sqrt{2}]$ can be represented as $d = d' + \sqrt{2}d''$ where $d', d'' : \mathbb{C}^{[5]} \setminus \{0\} \to \mathbb{Q}$. By (2), we have $d|\mathbb{C}[x, y, z] = d'|\mathbb{C}[x, y, z]$ and

$$d''(a_i u^i) = -in, \quad d''(b_{ij} u^i v^j) = (jm - i)n, i = 0, \ldots, m - 1$$

assuming that $a_i, b_{ij} \neq 0$. All these degrees are pairwise distinct. Hence the polynomial $\widehat{f} \neq 0$ being d-homogeneous, the expression (7) for $\widehat{f}$ consists of a single term. Therefore, the following alternative holds:

(i) either $\widehat{a} = a_0 \in \mathbb{C}[\widehat{x}, \widehat{y}, \widehat{z}]$, or

(ii) $\widehat{a} = a_i \widehat{u}^i$ for some $i > 0$ and for some $a_i \in \mathbb{C}[\widehat{x}, \widehat{y}, \widehat{z}] \setminus \{0\}$, or

(iii) $\widehat{a} = b_{ij} \widehat{u}^i \widehat{v}^j$ for some $j > 0$ and for some $b_{ij} \in \mathbb{C}[\widehat{x}, \widehat{y}, \widehat{z}] \setminus \{0\}$.

Since the subalgebra $\ker \widehat{\partial}$ is factorially closed, we have $\widehat{u} \in \ker \widehat{\partial}$ in the case (ii) and $\widehat{v} \in \ker \widehat{\partial}$ in the case (iii). By Lemma 1, (i) cannot happen for all the homogeneous elements $\widehat{a} \in \ker \widehat{\partial}$. Thus, the assertion follows. □

LEMMA 3. — $\widehat{v} \notin \ker \widehat{\partial}$.

Proof. — Assume on the contrary that $\widehat{v} \in \ker \widehat{\partial}$. Then for a general $c \in \mathbb{C}$, the locally nilpotent derivation $\widehat{\partial}$ can be specialized to a locally nilpotent derivation $\widehat{\partial}_c \in \mathrm{LND}(\widehat{X}_c) \setminus \{0\}$ where for $c \in \mathbb{C} \setminus \{0\}$ we denote

$$\widehat{X}_c = \widehat{X} \cap \{v = c\} \simeq \widehat{X}_1 = \{u^m + z^{l-1}(x^k z^{k-l} - y^l) = 0\} \subset \mathbb{C}^4.$$

We keep the same notation $\widehat{\partial}$ for $\widehat{\partial}_1$, and we still denote by $\widehat{\varphi}$ the associated $\mathbb{C}_+$-action $\widehat{\varphi}|\widehat{X}_1$ on the threefold $\widehat{X}_1$.

Note that the threefold $\widehat{X}_1$ has divisorial singularities. Indeed, since by our assumption, $m \geqslant 2$ and $k > l \geqslant 3$, it is singular along the divisor $D_{\widehat{z}}$ of the regular function $\widehat{z} \in \mathbb{C}[\widehat{X}_1]$: $D_{\widehat{z}} \subset \mathrm{sing}\widehat{X}_1$. It follows that the divisor $D_{\widehat{z}}$ is invariant under the $\mathbb{C}_+$-action $\widehat{\varphi}$ on $\widehat{X}_1$. Hence a general $\widehat{\varphi}$-orbit $\mathcal{O}$ does not meet the divisor $D_{\widehat{z}}$, and so the restriction $\widehat{z}|\mathcal{O}$ does not vanish. Therefore, the regular function $\widehat{z}$ is constant along general $\widehat{\varphi}$-orbits, that is, $\widehat{z}$ is a $\widehat{\varphi}$-invariant, or equivalently, $\widehat{z} \in \ker \widehat{\partial}$.

Thus, we are in the position to repeat the specialization descent. Namely, the $\mathbb{C}_+$-action $\widehat{\varphi}$ can be further specialized to the general $\widehat{\varphi}$-invariant surface

$$S_c := \{\widehat{z} = c\} \simeq S_1 = \{u^m + x^k - y^l = 0\} \subset \mathbb{C}^3$$

providing a non-trivial $\mathbb{C}_+$-action on S_c and thereby also on S_1. Now the desired conclusion follows from the next lemma. □

LEMMA 4. — *The Pham-Brieskorn surface*

$$S = S_{k,l,m} = \{x^k + y^l + z^m = 0\} \subset \mathbb{C}^3$$

where $k, l, m \geqslant 2$ admits a non-trivial regular $\mathbb{C}_+$-action if and only if this is a dihedral surface $S_{2,2,m}$. In the latter case $ML(S) = \mathbb{C}$.

Proof. — Assume that $\widehat{\varphi}$ is a non-trivial regular $\mathbb{C}_+$-action on S. Let $\mathcal{O} \subset S$ be a general $\widehat{\varphi}$-orbit. Since $\mathcal{O} \simeq \mathbb{C}$, it can be parameterized by a triple of polynomials $(x(t), y(t), u(t)) \in (\mathbb{C}[t])^3$ satisfying the relation

$$x^k(t) + y^l(t) + z^m(t) = 0.$$

Assume first that $1/k + 1/l + 1/m \leqslant 1$. Then by the Halphen Lemma (a) in the next section these polynomials cannot be relatively prime in pairs, and so a general orbit $\mathcal{O}$ meets one of the axes $x = y = 0$ or $x = z = 0$ or $y = z = 0$, hence it must pass through the origin, a contradiction.

In the remaining cases, S is one of the Platonic surfaces $S_{2,2,m}$, $S_{2,3,3}$, $S_{2,3,4}$ or $S_{2,3,5}$. Anyhow, to exclude the last three cases we will assume in the sequal more generally that $\gcd(m, kl) = 1$, and that on the contrary, $\mathrm{LND}(S) \neq \{0\}$, that is, that the surface S admits a non-trivial regular $\mathbb{C}_+$-action.

Let $\partial_0 \in \mathrm{LND}(S)$, $\partial_0 \neq 0$. Fix a weight degree function d' on the polynomial algebra $\mathbb{C}^{[3]}$ given by $d'_x = 1/k, d'_y = 1/l, d'_z = 1/m$. Since the polynomial $x^k + y^l + z^m$ is d'-homogeneous, the algebra $B = \mathbb{C}[S]$ is graded. The graded locally nilpotent derivation $\widehat{\partial}_0$ of the algebra B associated with ∂_0 is also non-zero. In virtue of the relation $\widehat{z}^m = -(\widehat{x}^k + \widehat{y}^l)$, any element $\widehat{b} \in B$ extends to a unique polynomial $\widehat{f} \in \mathbb{C}^{[3]}$ with $\deg_z \widehat{f} < m$. If the element $\widehat{b}$ is d'_B-homogeneous then also the polynomial $\widehat{f}$ is d'-homogeneous, and the following statement holds.

CLAIM. — *$\widehat{f} = cx^\alpha y^\beta z^\gamma \prod_i (x^{k'} - c_i y^{l'})$ where $k' = k/\gcd(k, l), l' = l/\gcd(k, l), c, c_i \in \mathbb{C}^*$, and $\gamma < m$.*

Proof of the claim. — Letting $d'(x^i y^j z^s) = d'(x^{i'} y^{j'} z^{s'})$ where $0 \leqslant s \leqslant s' \leqslant m - 1$ we will have

$$\frac{s' - s}{m} = \frac{i - i'}{k} + \frac{j - j'}{l}. \tag{8}$$

Since by our assumption $\gcd(m, kl) = 1$, it follows from (8) that $m|(s' - s) \Longrightarrow s = s'$, and hence $\frac{i-i'}{k} = \frac{j'-j}{l} \Longrightarrow \frac{i-i'}{k'} = \frac{j'-j}{l'}$. Now the claim follows. □

The graded subalgebra $\ker \widehat{\partial}_0$ of the algebra B being generated by homogeneous elements, there exists a non-zero homogeneous element $b \in \ker \widehat{\partial}_0$. Since the subalgebra $\ker \widehat{\partial}_0$ is factorially closed, in virtue of the above claim, the following alternative holds:

(i) $\widehat{x} \in \ker \widehat{\partial}_0$ or

(ii) $\widehat{y} \in \ker \widehat{\partial}_0$ or

(iii) $\widehat{z} \in \ker \widehat{\partial}_0$ or

(iv) $\widehat{x}^{k'} - c_i \widehat{y}^{l'} \in \ker \widehat{\partial}_0$ for some $c_i \neq 0$.

Since $\widehat{x}^k + \widehat{y}^l + \widehat{z}^m = 0$, in the case (i) we have $\widehat{y}^l + \widehat{z}^m \in \ker \widehat{\partial}_0$. As $l, m \geqslant 2$, by Lemma 0(a) this implies $\widehat{y}, \widehat{z} \in \ker \widehat{\partial}_0$. Henceforth, $\widehat{\partial}_0 = 0$, a contradiction. Similarly, the cases (ii) and (iii) lead to a contradiction.

If $\min\{k', l'\} \geqslant 2$ then by the same arguments as above, (iv) implies that $\widehat{x}, \widehat{y} \in \ker \widehat{\partial}_0$, and then also $\widehat{z} \in \ker \widehat{\partial}_0$, which again gives a contradiction. Thus it must be $\min\{k', l'\} = 1$; let $l' = 1$. Then $k = lk'$. The regular function $\widehat{x}^{k'} - c_i \widehat{y} \in \mathbb{C}[S]$ being invariant under the associated regular $\mathbb{C}_+$-action $\varphi_{\widehat{\partial}_0}$ on the surface S, its general level sets contain general $\varphi_{\widehat{\partial}_0}$-orbits. Being irreducible, these curves should be isomorphic to $\mathbb{C}$. On the other hand, they are isomorphic to the affine plane curves with the equations

$$x^{lk'} + \left(\frac{x^{k'} - c'}{c_i}\right)^l + z^m = 0$$

where $c' \in \mathbb{C}$ is generic. It is easily seen that such a curve cannot be isomorphic to $\mathbb{C}$ unless $k = l = 2$ and $c_i^2 = -1$, in which case S is a dihedral surface (hint: notice that an irreducible affine curve is isomorphic to $\mathbb{C}$ if and only if it admits a regular $\mathbb{C}_+$-action, and then proceed in the same fashion as above).

To prove the last statement of the lemma, notice that there is an isomorphism $S_{2,2,m} \simeq T_m := \{uv - w^m = 0\}$, and hence $\mathrm{ML}(S_{2,2,m}) \simeq \mathrm{ML}(T_m) = \mathbb{C}$. The latter equality is well known; see e.g. [DanGi], [Be], [ML2], [ML3], [KaZa1]. Indeed, the subgroup $\langle \alpha, \beta \rangle$ of the automorphism group $\mathrm{Aut}T_m$ generated by the following $\mathbb{C}_+$-actions on T_m (restricted from $\mathbb{C}^3$):

$$\alpha : (t, (u, v, w)) \longmapsto \left(u, v + \frac{(w + tu)^m - w^m}{u}, w + tu\right),$$

$$\beta : (t, (u, v, w)) \longmapsto \left(u + \frac{(w + tv)^m - w^m}{v}, v, w + tv\right),$$

has a dense orbit; therefore, $\mathrm{ML}(T_m) = \mathbb{C}$. This concludes the proof. □

From Lemmas 1–3 we obtain such a corollary.

Corollary. — $\widehat{u} \in \ker \widehat{\partial}$.

Lemma 5. — $\widehat{\partial}\widehat{v} \notin \ker \widehat{\partial}$.

Proof. — Assume the contrary. Then by (4) and the above corollary, we have

$$\widehat{\partial}(\widehat{u}^m\widehat{v}) = \widehat{u}^m\widehat{\partial}\widehat{v} \in \ker\widehat{\partial} \Longrightarrow \widehat{\partial}[\widehat{z}^{l-1}(\widehat{y}^l - \widehat{x}^k\widehat{z}^{k-l})] = \widehat{\partial}(g_1g_2) \in \ker\widehat{\partial}$$

where $g_1 := \widehat{z}^{l-1}$ and $g_2 := \widehat{y}^l - \widehat{x}^k\widehat{z}^{k-l} \in \mathbb{C}[\widehat{X}]$. Hence the restriction of the product g_1g_2 onto a general orbit $\mathcal{O}$ serves as a coordinate function of the curve $\mathcal{O} \simeq \mathbb{C}$ (for instance, this follows from Lemma 0(b)). In other words, $\deg_t[(g_1g_2)|\mathcal{O}] = 1$ where t is a coordinate in $\mathcal{O} \simeq \mathbb{C}$ (notice that $\deg_t(f|\mathcal{O}) = \deg_{\widehat{\partial}} f$ where the latter degree is defined below). This provides the following alternative:

– either $\deg_t(g_1|\mathcal{O}) = 0$ and $\deg_t(g_2|\mathcal{O}) = 1$, or

– $\deg_t(g_1|\mathcal{O}) = 1$ and $\deg_t(g_2|\mathcal{O}) = 0$.

Consider each of these two possibilities.

Assuming first that $z|\mathcal{O} = \text{const} \in \mathbb{C}\setminus\{0\}$ (i.e., $\widehat{z} \in \ker\widehat{\partial}$) and $\deg_t[(y^l - x^kz^{k-l})|\mathcal{O}] = 1$, we would have that $\deg_t(y^l(t) - cx^k(t)) = 1$ for two polynomials $x(t), y(t) \in \mathbb{C}[t]$ and for a general constant $c \neq 0$. We may also suppose that $\gcd(x(t), y(t)) = 1$ (i.e., that the orbit $\mathcal{O}$ does not meet the codimension two subvariety $D_{\widehat{x}} \cap D_{\widehat{y}}$ of $\widehat{X}$). Then by the Davenport Lemma in the next section, for a certain $m' \in \mathbb{N}$ the inequalities $1 > m'(kl - k - l) \geqslant 1$ must hold, which is impossible.

In the second case we would have: $\deg_t(g_1|\mathcal{O}) = (l-1)\deg_t(z|\mathcal{O}) = 1$ which is also impossible since by the assumption of Theorem 1, $l - 1 \geqslant 2$. This proves the lemma. □

Recall the notion of the degree function associated with a locally nilpotent derivation $\partial \in \mathrm{LND}(A)$:

$$\deg_\partial a \overset{\text{def}}{=} \max\{n \in \mathbb{N}\cup\{0\} | \partial^n a \neq 0\} \quad \text{if} \quad a \in A\setminus\{0\}; \deg_\partial 0 = -\infty.$$

For the associated locally nilpotent derivation $\widehat{\partial} \in \mathrm{LND}(\widehat{A})$ we have the inequality

$$\deg_\partial a \geqslant \deg_{\widehat{\partial}}\widehat{a}, \quad \forall a \in A,$$

where $\widehat{a} \in \widehat{A}$ denotes the principal d-homogeneous part of a.

Lemma 5 provides the following

COROLLARY. — *$\partial v \notin \ker\partial$; moreover, $\deg_\partial v \geqslant \deg_{\widehat{\partial}}\widehat{v} \geqslant 2$.*

LEMMA 6. — *Let $a \in A$ be an element such that $\deg_\partial a \leqslant 1$. Then a can be extended to a polynomial $f \in \mathbb{C}^{[5]}$ which does not depend on v.*

Proof. — Since $p|X = 0$, in virtue of (3) the restriction $\widetilde{u}^m\widetilde{v}$ of the polynomial $u^m v$ to X can be expressed as a polynomial in $\widetilde{x}, \widetilde{y}$ and $\widetilde{z}$. Hence the element $a \in A$ can be extended (in a unique way) to a polynomial $f \in \mathbb{C}^{[5]}$ written in the form (7). Let us show that this polynomial f does not depend on v. Assume the contrary. Letting the constant n in the definition (2) of the weight degree function d be large enough, we can achieve (by the same arguments as in the proof of Lemma 2) that the principal d-homogeneous part $\widehat{f}$ of f is as in (iii) of this same proof. In particular, v is a factor of the polynomial $\widehat{f}$. Hence $\widehat{v}$ is a factor of $\widehat{a} = \widehat{f}|\widehat{X}$. By Lemma 5, we have the inequalities

$$\deg_\partial a \geqslant \deg_{\widehat{\partial}} \widehat{a} \geqslant \deg_{\widehat{\partial}} \widehat{v} \geqslant 2$$

which contradicts our choice of the element a. The lemma is proven. □

Proof of Proposition 1 (cf. [KaML1], [ML3]). — We have to show that $u \in \ker \partial$ for any $\partial \in \mathrm{LND}(A)$. Fix an element $a \in A$ of ∂-degree 1. Letting in (5) $b = \widetilde{v}$, from (1), (3) and (5) we obtain

$$\text{(9)}\quad \widetilde{v} = c^{-1}\sum_{i=0}^{N} c_i a^i = -\widetilde{u}^{-m} q_{k,l}(\widetilde{x}, \widetilde{y}, \widetilde{z}) \Longrightarrow -\widetilde{u}^m \sum_{i=0}^{N} c_i a^i = c q_{k,l}(\widetilde{x}, \widetilde{y}, \widetilde{z}).$$

By Lemma 6, the element $c \in A$ resp. $\sum_{i=0}^{N} c_i a^i \in A$ can be extended to a polynomial, say, $\eta \in \mathbb{C}[x, y, z, u]$ resp. $\zeta \in \mathbb{C}[x, y, z, u]$. By (9), there exists a polynomial $g \in \mathbb{C}^{[5]}$ such that

$$\text{(10)}\qquad u^m \zeta(x, y, z, u) - q_{k,l}(x, y, z)\eta(x, y, z, u) = pg.$$

The left hand side of (10) does not depend on v but the polynomial p does, hence we must have $g = 0$. Since $\gcd(u, q_{k,l}) = 1$ it follows from (10) that u divides η in the algebra $\mathbb{C}^{[4]}$ and so, $\widetilde{u}$ divides c in the algebra A, that is, $c = \widetilde{u}b$ where $b \in A$. Since $c \in \ker \partial$ and $\ker \partial$ is factorially closed, also $\widetilde{u} \in \ker \partial$, as stated. This completes the proof. □

2. $\mathbb{A}_1$-poor varieties: the lemmas of Mason, Davenport and Halphen.

In course of the proof of Proposition 1 we have used the lemmas of Davenport and Halphen; for the sake of completeness, we provide them below with simple proofs based on the following well known

Mason's abc-Lemma [Mas]. — *Let $a, b, c \in \mathbb{C}[t]$ be three polynomials, not all three constant. For a polynomial $p \in \mathbb{C}[t]$, denote by $d_0(p)$*

the number of its distinct roots (without counting multiplicities). Assume that $a+b+c=0$ *and* $\gcd(a,b)=1$. *Then we have*

$$\max\{\deg a, \deg b, \deg c\} \leqslant d_0(abc)-1. \tag{11}$$

See [La], [Mas], [Si] for an elementary proof. We would like to sketch

An alternative proof. — Let $f : \Gamma_1 \to \Gamma_2$ be a proper, surjective morphism of smooth quasiprojective curves. Then the following inequality for Euler characteristics holds:

$$e(\Gamma_1) \leqslant (\deg f) e(\Gamma_2). \tag{12}$$

This inequality follows from the obvious relations

$$\mathrm{card}(\mathrm{CrPt(f)}) \leqslant (\deg f)\mathrm{card}(\mathrm{CrVa}(f))$$

and

$$e(\Gamma_1 \setminus \mathrm{CrPt}(f)) = (\deg f) e(\Gamma_2 \setminus \mathrm{CrVa}(f))$$

where $\mathrm{CrPt}(f)$ resp. $\mathrm{CrVa}(f)$ denotes the set of critical points resp. critical values of f.

Take $\Gamma_1 = R \setminus S$ where R is a smooth projective curve of genus g and S is a finite subset of R, and let $\Gamma_2 \simeq \mathbb{C} \setminus \{0,1\}$ be realized as $\Gamma_2 = \{u+v=1, u \neq 0, v \neq 0\} \subset \mathbb{C}^2$. Then for a pair $f=(u,v)$ of non-constant rational functions on R with zeros and poles only on S such that $u+v=1$, from (12) we obtain the inequality (see [Mas])

$$\deg u = \deg v \leqslant -e(R \setminus S) = 2g - 2 + \mathrm{card} S. \tag{13}$$

Letting

$$R = \mathbb{P}^1, \quad S = \{\infty\} \cup a^{-1}(0) \cup b^{-1}(0) \cup c^{-1}(0), \quad u = -a/c, \quad v = -b/c$$

(so that the condition $a+b+c=0$ of the lemma becomes $u+v=1$), from (13) we get (11). □

As an immediate corollary, we obtain

Davenport's Lemma ([KlNe], [Dav], Thm. 2)[5]. — *Let three polynomials* $x, y, z \in \mathbb{C}[t]$ *satisfy the relation* $z = x^k - y^l$ *where* k *and* l *are relatively prime*[6], $z \neq 0$, $\gcd(x,y)=1$ *and* $\deg z < \max\{\deg x^k, \deg y^l\}$. *Denote* $n = \deg z$, $lm = \deg x$, $km = \deg y$. *Then we have*

$$n > m(kl - k - l).$$

[5] See also [DvZa] and the literature therein for closely related results.
[6] One can find in [Dav] a general formulation with arbitrary k and l.

Proof (cf. [Pr]). — By Mason's abc-Lemma, we have the inequality

$$\max\{k \deg x, l \deg y\} \leqslant \deg x + \deg y + \deg z - 1.$$

Hence

$$klm \leqslant km + lm + n - 1,$$

and the lemma follows. □

Remark. — It is known [St], [Zn], [Orel] that (whatever k, l and m with $\gcd(k,l) = 1$ are) the bound in Davenport's Lemma is the best possible one. See also [Si] on exactness in Mason's abc-Lemma.

A contemporary exposition of Halphen's results [Ha] is given in [BaDw]. Actually, the original Halphen's Lemma has a broader meaning in the context of our subject. To formulate it in an appropriate way, we introduce the following notions(7).

DEFINITION. — *Let X be an algebraic variety. We say that X is* $\mathbb{A}_1$-poor *if there exists a subvariety Y of X of codimension at least 2 such that every curve (i.e., a non-constant morphism) $f : \mathbb{C} \to X$ meets $Y : f(\mathbb{C}) \cap Y \neq \emptyset$. In contrast, we say that X is* $\mathbb{A}_1$-rich *if, for any two disjoint closed subvarieties $Y, Z \subset X$ with* $\operatorname{codim}_X Y \geqslant 2$ *and* $\dim Z = 0$, *there exists a polynomial curve $\mathbb{C} \to X$ omitting Y and passing through every point of Z.*

Remarks.

1. Evidently, an $\mathbb{A}_1$-poor variety X does not admit non-trivial regular $\mathbb{C}_+$-actions. Or equivalently, $\mathrm{LND}(X) = \{0\} \Longleftrightarrow \mathrm{ML}(X) = \mathbb{C}[X]$. Moreover, the latter equality holds assuming that the algebra $A = \mathbb{C}[X]$ is endowed with a degree function such that for the associated graded algebra $\widehat{A}$, the variety $\widehat{X} = \operatorname{spec}\widehat{A}$ is $\mathbb{A}_1$-poor. This justifies our interest in $\mathbb{A}_1$-poor varieties.

2. The affine space $\mathbb{C}^n (n \geqslant 2)$ is $\mathbb{A}_1$-rich. Indeed, given two disjoint closed subvarieties $Y, Z \subset \mathbb{C}^n$ with $\operatorname{codim}_{\mathbb{C}^n} Y \geqslant 2$ and $\dim Z = 0$, by a theorem due to Gromov and Winkelmann [Grm], [Wi], one can find an automorphism $\alpha \in \operatorname{Aut}\mathbb{C}^n$ such that $\alpha(Y) = Y$ and the image $\alpha(Z)$ is contained in an affine line $L \subset \mathbb{C}^n \setminus Y$. Then the polynomial curve $\mathbb{C} \simeq L \xrightarrow{\alpha^{-1}} \alpha^{-1}(L) \subset \mathbb{C}^n$ omits Y and passes through every point of Z, as required.

(7) *cf.* the notion of abc-variety in [Bu]. Presumably (over the field $\mathbb{C}$) these are the affine varieties X which do not admit non-constant morphisms $\mathbb{C}^* \to X$ [Bu], p. 231.

3. If a variety X admits a finite morphism $X' \to X$ from an $\mathbb{A}_1$-rich affine variety X' (for instance, from $X' = \mathbb{C}^n, n \geqslant 2$), then clearly X is also $\mathbb{A}_1$-rich. Notice also that the family of polynomial curves in an $\mathbb{A}_1$-rich affine variety is unbounded (that is, their degrees are not bounded).

The following two lemmas provide examples of $\mathbb{A}_1$-poor resp. $\mathbb{A}_1$-rich surfaces in $\mathbb{C}^3$.

HALPHEN'S LEMMA [Ha], [Ev], [BaDw]. — *Consider the Pham-Brieskorn surfaces*

$$S_{k,l,m} = \{x^k + y^l + z^m = 0\} \subset \mathbb{C}^3$$

where $k, l, m \geqslant 2$. Then the following statements hold:

(a) *The surface $S_{k,l,m}$ is $\mathbb{A}_1$-poor if and only if $1/k + 1/l + 1/m \leqslant 1$. Actually, under the latter condition any polynomial curve $f : \mathbb{C} \to S_{k,l,m}$ passes through the singular point $\overline{0} \in S_{k,l,m} \subset \mathbb{C}^3$.*

(b) *In contrast, every Platonic surface $S_{k,l,m}$ where $1/k+1/l+1/m > 1$ is $\mathbb{A}_1$-rich.*

Proof.

(a) Suppose first that $1/k+1/l+1/m \leqslant 1$. Let us show that no triple of non-constant relatively prime polynomials $(x(t), y(t), z(t))$ satisfies the relation $x^k + y^l + z^m = 0$. Assuming the contrary, by Mason's abc-Lemma, we have

$$\max\{k \deg x, l \deg y, m \deg z\} \leqslant \deg x + \deg y + \deg z - 1.$$

Thus,

$$\begin{aligned} \deg x &\leqslant 1/k(\deg x + \deg y + \deg z - 1) \\ \deg y &\leqslant 1/l(\deg x + \deg y + \deg z - 1) \qquad (14) \\ \deg z &\leqslant 1/m(\deg x + \deg y + \deg z - 1). \end{aligned}$$

Summing up the three inequalities in (14), in virtue of our assumption, we obtain

$$1/k + 1/l + 1/m \leqslant (1/k + 1/l + 1/m - 1)(\deg x + \deg y + \deg z) \leqslant 0,$$

a contradiction.

(b) Every one of the Platonic surfaces $S = S_{2,2,m}$ $(m \geqslant 2)$, $S_{2,3,3}$, $S_{2,3,4}$ or $S_{2,3,5}$ admits a finite morphism $\mathbb{C}^2 \to S$ (the orbit morphism of the standard linear action on $\mathbb{C}^2$ of the corresponding finite subgroup $\Gamma \subset \mathrm{SU}(2)$; see e.g., [Mil], §4 and Remark 2.1; see also [Schw], [Kl], Ch. I or [Beu], [BaDw], p. 56 for explicit formulas). Since the affine plane $\mathbb{C}^2$ is $\mathbb{A}_1$-rich so is S (see Remark 3 preceding the lemma). This proves the lemma. □

The next lemma is a simple corollary of Theorem 1 in [Sch] (see also [Br], [Ve], [FlZa] for relevant results).

SCHMIDT'S LEMMA [Sch]. — *Let S be a surface in $\mathbb{C}^3$ given by the equation*

$$z^m = f_d(x, y)$$

where $f_d \in \mathbb{C}[x, y]$ is a homogeneous polynomial of degree d without multiple roots. Suppose that $m \geqslant 4$ and $d \geqslant 3$, or $m = 3$ and $d \geqslant 5$, or $m = 2$ and $d \geqslant 17$. Then the surface S is $\mathbb{A}_1$-poor and, moreover, every polynomial curve $f : \mathbb{C} \to S$ passes through the singular point $\bar{0} \in S$.

The purpose of the next lemma is to strengthen the lemmas of Halphen and Schmidt (cf. the examples below). Recall that a regular action of the multiplicative group $\mathbb{C}^*$ on an affine variety X is called *good* if it has a unique fixed point (called *vertex*), and this fixed point is *elliptic*, that is, it belongs to the closure of any orbit. Let S be a normal affine surface with a good $\mathbb{C}^*$-action; denote $S^* = S \setminus V_0$ where V_0 is the vertex, and set $\Gamma = S^*/\mathbb{C}^*$. If the curve Γ is rational then the singularity of the surface S at the origin is called *quasirational* [Ab] (cf. also examples in [Ore2]).

LEMMA 7. — *Let S be a normal affine surface with a good $\mathbb{C}^*$-action. Suppose that the singularity of the surface S at the vertex $V_0 \in S$ is not quasirational. Then any rational curve $r : \mathbb{C} \to S$, as well as any holomorphic entire curve $h : \mathbb{C} \to S$ in the surface S is contained in an orbit closure $\overline{\mathbb{C}^* V}$ for a certain point $V \in S^*$. Consequently, any polynomial curve $f : \mathbb{C} \to S$ passes through the vertex $V_0 \in S$, and so the surface S is $\mathbb{A}_1$-poor.*

Proof. — Let $\widetilde{\Gamma} \to \Gamma$ be a normalization. The rational mapping $g : \mathbb{C} \xrightarrow{r} S \xrightarrow{\pi'} \Gamma$ can be lifted to a morphism $\tilde{g} : \mathbb{C} \to \widetilde{\Gamma}$, which is constant because (by our assumption) the geometric genus $g(\widetilde{\Gamma}) \geqslant 1$. Thus, the image $r(\mathbb{C}) \subset S$ is contained in the closure $\overline{\mathcal{O}}$ of an orbit $\mathcal{O} = \mathbb{C}^* V$ of the $\mathbb{C}^*$-action, as stated. The proof for an entire curve $h : \mathbb{C} \to S$ is similar. □

Remark. — This lemma (with the same proof) remains true also for meromorphic curves $m : \mathbb{C} \to S$ assuming that $e(\Gamma) < 0$ i.e., $g(\widetilde{\Gamma}) \geqslant 2$ (cf. e.g., [Ja], [Grs]).

The following facts will be useful in order to provide examples of quasihomogeneous surfaces in $\mathbb{C}^3$ which satisfy the assumption of Lemma 7. For integers $a_1, \dots, a_n$ denote $[a_1, \dots, a_n] = \mathrm{lcm}(a_1, \dots, a_n)$ and $(a_1, \dots, a_n) = \gcd(a_1, \dots, a_n)$, whereas $\overline{(a_1, \dots, a_n)}$ denotes the vector with the coordinates $a_1, \dots, a_n$.

LEMMA 8. — *Let $f \in \mathbb{C}[x, y, z]$ be a quasihomogeneous polynomial such that*

$$f(\lambda^{q_0} x, \lambda^{q_1} y, \lambda^{q_2} z) = \lambda^d f(x, y, z), \quad \forall \lambda \in \mathbb{C}$$

where $q_0, q_1, q_2, d > 0$ and $(q_0, q_1, q_2) = 1$. Suppose that $d \equiv 0 \bmod q_i$, $i = 0, 1, 2$, and that the surface $S := f^{-1}(0) \subset \mathbb{C}^3$ has an isolated singularity at the origin. Then the singularity $(S, \bar{0})$ is quasirational if and only if one of the following two conditions holds:

(i) *$d = [q_0, q_1, q_2]$, and for some natural numbers p, q, r, s coprime in pairs we have (up to a reordering): $\overline{(q_0, q_1, q_2)} = \overline{(pq, pr, qrs)}$.*

(ii) *$d = 2[q_0, q_1, q_2]$, and for some natural numbers p, q, r coprime in pairs we have: $\overline{(q_0, q_1, q_2)} = \overline{(pq, pr, qr)}$.*

Proof. — By [OrlWa], Prop. 3, $\Gamma = S^*/\mathbb{C}^*$ is a smooth curve of genus

$$g(\Gamma) = \frac{1}{2}\left(\frac{d^2}{q_0 q_1 q_2} - d\left(\frac{1}{[q_0, q_1]} + \frac{1}{[q_0, q_2]} + \frac{1}{[q_1, q_2]}\right) + 2\right).$$

Thus $g(\Gamma) = 0$ if and only if

$$\frac{1}{[q_0, q_1]} + \frac{1}{[q_0, q_2]} + \frac{1}{[q_1, q_2]} = \frac{d}{q_0 q_1 q_2} + \frac{2}{d}. \tag{15}$$

Letting $q_{ij} := (q_i, q_j)$ we can write

$$q_0 = q_{01} q_{02} q_0', \quad q_1 = q_{01} q_{12} q_1', \quad q_2 = q_{02} q_{12} q_2' \tag{16}$$

where the integers $q_{01}, q_{02}, q_{12}, q_0', q_1', q_2'$ are coprime in pairs, since by our assumption $(q_0, q_1, q_2) = 1$. Set $d_0 := q_{01} q_{02} q_{12}$; then $[q_0, q_1, q_2] = d_0 q_0' q_1' q_2'$, and so $d = \rho d_0 q_0' q_1' q_2'$ with $\rho \in \mathbb{N}$ and $q_0 q_1 q_2 = d_0^2 q_0' q_1' q_2'$. Therefore, (15) can be written as

$$\rho^2 - \left(\frac{1}{q_0' q_1'} + \frac{1}{q_0' q_2'} + \frac{1}{q_1' q_2'}\right)\rho + \frac{2}{q_0' q_1' q_2'} = 0.$$

It follows that $\rho = 1$ or $\rho = 2$. In the first case the only solutions of this Diophantine equation are (up to a reordering) $\overline{(q_0', q_1', q_2')} = \overline{(1, 1, s)}$

$(s \in \mathbb{N})$, and in the second case $\overline{(q'_0, q'_1, q'_2)} = \overline{(1,1,1)}$ is the only solution. Letting in (16) $q_{01} =: p, q_{02} =: q, q_{12} =: r$ and taking into account the above observations, we get the desired conclusion. □

The next corollary in the particular case of the Pham-Brieskorn surfaces can be found in [BarKa], p. 117[(8)].

COROLLARY. — *Assume that the polynomial*

$$f = ax^k + by^l + cz^m + \ldots \in \mathbb{C}[x,y,z] \quad (\textit{where} \quad a,b,c \in \mathbb{C}^*)$$

is quasihomogeneous and such that the surface $S := f^{-1}(0) \subset \mathbb{C}^3$ *has an isolated singularity at the origin. Then the singularity* $(S, \bar{0})$ *is quasirational if and only if one of the following two conditions holds:*

(i′) *up to a reordering,* $(k, lm) = 1$, *or*

(ii′) $(k,l) = (k,m) = (l,m) = 2$.

Proof. — Letting $k = \rho k', l = \rho l', m = \rho m'$ where $\rho := (k,l,m)$, set $d'_0 := (k',l')(k',m')(l',m')$ and

$$q_0 := \frac{l'm'}{d'_0}, \quad q_1 := \frac{k'm'}{d'_0}, \quad q_2 := \frac{k'l'}{d'_0}.$$

These are the unique weights with $(q_0,q_1,q_2) = 1$ making f a quasihomogeneous polynomial of degree

$$d := kq_0 = lq_1 = mq_2 = \rho\frac{k'l'm'}{d'_0} = \rho[k',l',m'] = [k,l,m].$$

To apply Lemma 8 assume that

$$\overline{(q_0,q_1,q_2)} = \overline{\left(\frac{l'm'}{d'_0}, \frac{k'm'}{d'_0}, \frac{k'l'}{d'_0}\right)} = \overline{(pq, pr, qrs)}$$

with p,q,r,s coprime in pairs. Then we would have $[q_0,q_1,q_2] = pqrs = [k',l',m']$, whence $d = \rho[q_0,q_1,q_2]$. In view of this observation, it is easily seen that the condition (i) resp. (ii) of Lemma 8 is fulfilled if and only if (i′) resp. (ii′) holds (more precisely, iff $\rho = 1$ and up to a reordering, $\overline{(k,l,m)} = \overline{(rs,qs,p)}$ resp. $\rho = 2$ and up to a reordering, $\overline{(k,l,m)} = 2\overline{(r,q,p)}$). Now the statement follows from Lemma 8. □

(8) cf. [Ev], Thm. 2 where in fact, several possibilities covered by the condition (ii′) below have been omited, because of a gap in the reduction of problem B to problem C.

Examples.

1. By Lemma 7 and the above corollary, the Fermat cubic surface $x^3 + y^3 + z^3 = 0$ in $\mathbb{C}^3$ is $\mathbb{A}_1$-poor, and moreover, any rational curve in it has the diagonal form $t \longmapsto (\varphi(t)x_0, \varphi(t)y_0, \varphi(t)z_0)$ with $\varphi \in \mathbb{C}(t)$. In contrast, the cubic surface $x^3 + y^3 + z^3 = 1$ in $\mathbb{C}^3$ is rich with rational curves, as it is rational [Man]. Furthermore, being non-rational [ClGr], the affine Fermat cubic threefold $x^3 + y^3 + z^3 + u^3 = 0$ in $\mathbb{C}^4$ is unirational, and even is dominated by the affine space $\mathbb{C}^3$; see [PaVa], §3 for explicit formulas(9).

2. For the Pham-Brieskorn surfaces $S_{k,l,m}$, Lemma 7 and the above corollary provide information additional to those given by Halphen's Lemma. Namely, such a surface possesses a non-diagonal polynomial curve (i.e., not of the form

$$t \longmapsto (\varphi_0^{q_0}(t), \varphi_1^{q_1}(t), \varphi_2^{q_2}(t)) \quad \text{with} \quad \varphi_i \in \mathbb{C}[t], i = 0, 1, 2)$$

if and only if one of the conditions (i′) or (ii′) holds. (*cf.* [BoMu], [Beu], [DarGr], [Ev] for similar results, including the more general situation of Pham-Brieskorn type surfaces over function fields. Besides, in [DarGr] one can find a historical account on the subject.)

3. Let a surface $S = \{z^m - f_d(x, y) = 0\}$ in $\mathbb{C}^3$ be as in Schmidt's Lemma. We may choose a coordinate system in $\mathbb{C}^2_{x,y}$ in such a way that neither x nor y divides the polynomial f_d. Then the assumptions of the above corollary are fulfilled with $k = l = d$, and so the singularity $(S, \bar{0})$ is quasirational if and only if either $d = 2$ or $(m, d) = 1$. According to Lemma 7, the conclusion of Schmidt's Lemma remains true for any pair (m, d) with $m \geqslant 2, d \geqslant 2$, except for possibly the pairs $(2, 2), (3, 2), (3, 4)$ and $(2, 2k + 1), k = 1, \ldots, 7$.

Added in proof. — After this paper was written, it was established in [FlZa] that a normal affine surface S with a good $\mathbb{C}^*$-action which possesses a closed rational curve not passing through the vertex $V_0 \in S$, has at most rational singularity (S, V_0) at the vertex. In particular, this nicely fits Halphen's Lemma; indeed, the singularity $(S_{k,l,m}, \bar{0})$ of the Pham-Brieskorn surface $S_{k,l,m}$ is rational precisely for the Platonic surfaces. This also implies that the conclusion of Schmidt's Lemma is true exactly when $d \geqslant 3$ and $(d, m) \neq (3, 2)$.

(9) In a discussion with the authors H. Flenner conjectured that this threefold does not admit a nontrivial $\mathbb{C}_+$-action; however, so far we do not possess a proof of this.

BIBLIOGRAPHY

[Ab] Shreeram S. ABHYANKAR, Quasirational singularities, Amer. J. Math., 101-2 (1979), 267–300.

[BaDw] F. BALDASSARRI, B. DWORK, On second order linear differential equations with algebraic solutions, Amer. J. Math., 101 (1979), 42–76.

[BarKa] G. BARTHEL, L. KAUP, Topologie des surfaces complexes compactes singulières, in: Sur la topologie des surfaces complexes compactes, Sém. Math. Sup., 80, Presses Univ. Montréal, Montréal, Que. (1982), 61–297.

[Be] J. BERTIN, Automorphismes des surfaces non complètes, groupes fuchsiens et singularités quasihomogènes, Sémin. d'algèbre P. Dubreil et M.-P. Malliavin, 36ème Année, Proc., Paris 1983/84, Lect. Notes Math., 1146 (1985), 106–126.

[Beu] F. BEUKERS, The Diophantine equation $Ax^p + By^q = Cz^r$, Duke Math. J., 91-1 (1998), 61–88.

[Bu] A. BUIUM, The abc theorem for abelian varieties, Intern. Math. Res. Notices, 5 (1994), 219–232.

[BoMu] E. BOMBIERI, J. MUELLER, Trinomial equations in function fields, Astérisque, 22 (1995), 19–40.

[Br] B. BRINDZA, On the equation $F(x, y) = z^m$ over function fields, Acta Math. Hung., 49 (1987), 267–275.

[ClGr] C. H. CLEMENS, Ph. A. GRIFFITHS, The intermediate Jacobian of the cubic threefold, Ann. of Math., 95 (1972), 281–356.

[DanGi] V.I. DANILOV, M.H. GIZATULLIN, Automorphisms of affine surfaces. I, II, Math. USSR Izv., 9 (1975), 493–534; ibid., 11 (1977), 51–98.

[DarGr] H. DARMON, A. GRANVILLE, On the equations $z^m = F(x, y)$ and $Ax^p + By^q = Cz^r$, Bull. London Math. Soc., 27 (1995), 513–543.

[Dav] H. DAVENPORT, On $f^3(t) - g^2(t)$, Norske Vid. Selsk. Forh. (Trondheim), 38 (1965), 86–87.

[De] H. DERKSEN, Constructive Invariant Theory and the Linearization Problem, Ph.D. thesis, Basel (1997), 39p.

[tDP] T. tom DIECK, T. PETRIE, Contractible affine surfaces of Kodaira dimension one, Japan J. Math., 16 (1990), 147–169.

[DvZa] R. DVORNICICH, U. ZANNIER, A note on Thue's equation over function fields, Monatsh. für Math., 118 (1994), 219–230.

[Ev] A. EVYATAR (formely A. GUTWIRTH), On polynomial equations, Israel J. Math., 10 (1971), 321–326.

[FlZa] H. FLENNER, M. ZAIDENBERG, Rational curves and rational singularities, in preparation.

[Fu] T. FUJITA, On the topology of non complete algebraic surfaces, J. Fac. Sci. Univ. Tokyo, Sect. IA, 29 (1982), 503–566.

[Grm] M. GROMOV, Oka's principle for holomorphic sections of elliptic bundles, J. Amer. Math. Soc., 2 (1989), 851–897.

[Grs] F. GROSS, On the functional equation $f^n + g^n = h^n$, Amer. Math. Monthly, 73 (1966), 1093–1096.

[Ha] G.H. HALPHEN, Sur la réduction des équations différentielles linéaires aux formes intégrables, Mémoires présentés par divers savants à l'Académie des sciences de l'Institut National de France, T. XXVIII, N 1, Paris, F. Krantz, 1883; Œuvres. Vol. 3, Paris (1921), 1–260.

[Ja] A. V. JATEGAONKAR, Elementary proof of a theorem of P. Montel on entire functions, J. Lond. Math. Soc., 40 (1965), 166–170.

[Ka1] S. KALIMAN, Exotic analytic structures and Eisenman intrinsic measures, Israel Math. J., 88 (1994), 411–423.

[Ka2] S. KALIMAN, Polynomials with general $\mathbb{C}^2$–fibers are variables. I, preprint (1999), 60p.

[KaML1] S. KALIMAN, L. MAKAR-LIMANOV, On Russell–Koras contractible threefolds, J. of Algebraic Geom., 6 (1997), 247–268.

[KaML2] S. KALIMAN, L. MAKAR-LIMANOV, Affine algebraic manifolds without dominant morphisms from Euclidean spaces, Rocky Mount. J. Math., 27-2 (1997), 601–609.

[KaML3] S. KALIMAN, L. MAKAR-LIMANOV, Locally nilpotent derivations of Jacobian type, preprint (1998), 16p.

[KaZa1] S. KALIMAN, M. ZAIDENBERG, Affine modifications and affine varieties with a very transitive automorphism group, Transformation Groups, 4-1 (1999), 53–95.

[KaZa2] S. KALIMAN, M. ZAIDENBERG, Families of affine planes: the existence of a cylinder, MPI, preprint MPI 00-75 (2000), 12p.

[KlNe] M. C. KLAMKIN, D. J. NEWMAN, On the number of distinct zeros of polynomials, Amer. Mathem. Monthly, 66 (1959), 494–496.

[Kl] F. KLEIN, Vorlesungen über das ikosaeder und die auflösung der gleichungen vom fünden grade, Teubner, Leipzig, 1884. English transl.: F. KLEIN, Lectures on the Icosahedron and the solution of equations of fifth degree, Dover, 1956.

[La] S. LANG, Old and new conjectured Diophantine inequalities, Bull. Amer. Math. Soc., 23 (1990), 37–75.

[ML1] L. MAKAR-LIMANOV, On the hypersurface $x+x^2y+z^2+t^3=0$ in $\mathbb{C}^4$ or a $\mathbb{C}^3$-like threefold which is not $\mathbb{C}^3$, Israel J. Math., 96 (1996), 419–429.

[ML2] L. MAKAR-LIMANOV, On groups of automorphisms of a class of surfaces, Israel J. Math., 69 (1990), 250–256.

[ML3] L. MAKAR-LIMANOV, On the group of automorphisms of a surface $x^ny = P(z)$, Preprint (1997), 11p.

[Man] Yu. I. MANIN, Cubic forms. Algebra, geometry, arithmetic, North-Holland Mathematical Library, 4. North-Holland Publishing Co., Amsterdam-New York (1986), 326 p. (The Russian original "Nauka", Moscow, 1972).

[Mas] R. C. MASON, Equations over function fields, Number theory, Noordwijkerhout 1983. Lecture Notes in Math., 1068, Springer, Berlin-New York (1984), 149–157.

[Mil] J. MILNOR, On the 3-dimensional Brieskorn manifolds $M(p,q,r)$, in: Knots, groups, and 3-manifolds, L. P. Neuwirth, ed. Annals of Math. Stud., Princeton Univ. Press, Princeton, NJ (1975), 175–225.

[Miy] M. MIYANISHI, Algebraic characterization of the affine 3-space, Proc. Algebraic Geom. Seminar, Singapore, World Scientific (1987), 53–67.

[Ore1] S. Yu. OREVKOV, Riemann existence theorem and construction of real algebraic curves, preprint (1999), 1–11.

[Ore2] S. Yu. OREVKOV, On singularities that are quasirational in the sense of Abhyankar, Uspekhi Mat. Nauk, 50 (1995), no. 6 (306), 201–202.

[OrlWa] P. ORLIK, P. WAGREICH, Algebraic surfaces with k^*-action, Acta Math., 138 (1977), 43–81.

[PaVa] G. PAYNE, L. VASERSTEIN, Sums of three cubes, The arithmetic of function fields (Columbus, OH, 1991), Ohio State Univ. Math. Res. Inst. Publ., 2, de Gruyter, Berlin (1992), 443–454.

[Pr] V. V. PRASOLOV, Mnogochleny (Polynomials) (in Russian), MCNMO, Moscow, 2000.

[Sa] A. SATHAYE, Polynomial ring in two variables over a D. V. R.: A criterion, Invent. Math., 74 (1983), 159–168.

[Sch] W. M. SCHMIDT, Polynomial solutions of $F(x, y) = z^n$, Proceedings of the Queen's Number Theory Conference, 1979 (Kingston, Ont., 1979), Queen's Papers in Pure and Appl. Math., 54, Queen's Univ., Kingston, Ont. (1980), 33–65.

[Schw] H. A. SCHWARTZ, Ueber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt, J. für die reine und angewandte Mathematik, 75 (1873), 292-335.

[Si] J. H. SILVERMAN, The S-unit equation over function fields, Math. Proc. Cambridge Philos. Soc., 95 (1984), 3–4.

[St] W. W. STOTHERS, Polynomial identities and Hauptmoduln, Quart. J. Math. (2), 32 (1981), 349–370.

[Ve] J. VÉGSÖ, On power values of binary forms over function fields, Publ. Math. Debrecen, 50-1,2 (1997), 145–148.

[Wi] J. WINKELMANN, On automorphisms of complements of analytic subsets in $\mathbf{C}^n$, Math. Zeitschrift, 204 (1990), 117–127.

[Za1] M. ZAIDENBERG, An analytic cancellation theorem and exotic algebraic structures on $\mathbb{C}^n$, $n \geqslant 3$, Astérisque, 217 (1993), 251–282.

[Za2] M. ZAIDENBERG, On exotic algebraic structures on affine spaces, Algebra and Analysis. St. Petersbourg Mathem. J., 10-5 (1999), 60p.

[Zn] U. ZANNIER, On Davenport's bound for the degree of $f^3 - g^2$ and Riemann's existence theorem, Acta Arithm., 71 (1995), 107-137; Addenda, ibid., 74 (1996), 387.

Manuscrit reçu le 7 février 2000,
accepté le 27 avril 2000.

S. KALIMAN,
University of Miami
Department of Mathematics
Coral Gables, FL 33124 (USA).
kaliman@math.miami.edu

M. ZAIDENBERG,
Université de Grenoble I
Institut Fourier
UMR 5582 CNRS-UJF
BP 74
38402 St. Martin d'Hères cedex (France).
zaidenbe@ujf-grenoble.fr