CHRISTOPHER HACON

Remarks on Seshadri constants of vector bundles

REMARKS ON SESHADRI CONSTANTS
OF VECTOR BUNDLES

by Christopher D. HACON

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Notation. — Let X be an irreducible complex projective variety. We will say that a property holds for a very general subvariety Z, if it holds for all Z outside a countable union of proper closed subsets. If $Z \subset X$ is a subvariety, then $\mathcal{I}_Z \subset \mathcal{O}_X$ denotes the ideal sheaf of Z. We will say that a vector bundle $\mathcal{E}$ is ample (respectively very ample, globally generated), if and only if the tautological line bundle $\xi := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is ample (respectively very ample, globally generated) on $\mathbb{P}(\mathcal{E})$.

1. Introduction.

Let X be a smooth complex projective variety and let L be a numerically effective (nef) line bundle on X, let $\pi_x : \tilde{X} \longrightarrow X$ denote the blow up of X at the point x, and E_x the exceptional divisor. For any point $x \in X$ one can define the Seshadri constant of L at x:

$$\epsilon(L, x) := \sup\{\lambda > 0 | \pi_x^* L - \lambda E_x \text{ is nef}\}.$$

Equivalently $\epsilon(L, x)$ is computed by the infimum of $\frac{L.C}{\mathrm{mult}_x(C)}$ over all curves $C \subset X$ containing the point x. This definition is motivated by the following theorem of Seshadri:

THEOREM 1.1 ([13] Theorem 7.1). — *Let X be a complete scheme, and D a divisor on X. Then D is ample if and only if there exists $\epsilon > 0$*

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such that

$$(D.C) \geqslant \epsilon \operatorname{mult}_x(C)$$

for every point x and every integral curve C in X.

In other words L is ample if and only if $\epsilon(L) := \inf\{\epsilon(L,x)|x \in X\}$ is strictly positive. The Seshadri constants of line bundles are very interesting invariants. One of their main applications is the following theorem concerning the generation properties of adjoint bundles.

THEOREM 1.2. — *Let X be a smooth complex projective variety of dimension n, and let L be an ample line bundle on X. Fix a point $x \in X$ and a positive integer $k \geqslant \frac{n}{\epsilon(L,x)}$. If $L^n > (\frac{n}{k})^n$, then $K_X + kL$ has a section which does not vanish at x.*

The Seshadri constants of line bundles have been extensively studied, for example:

THEOREM 1.3 (Ein, Lazarsfeld [7]). — *Let L be an ample line bundle on a smooth projective surface X. Then $\epsilon(L,x) \geqslant 1$ for all except countably many points $x \in X$. More generally, given an integer $e > 1$, suppose that*

$$c_1(L)^2 \geqslant 2e^2 - 2e + 1 \quad \text{and} \quad c_1(L).\Gamma \geqslant e \quad \text{for every curve } \Gamma \subset X.$$

Then $\epsilon(L,x) \geqslant e$ for all but finitely many $x \in X$.

It should also be noted that Miranda has constructed examples of surfaces where $\epsilon(L,x)$ takes arbitrary small values on isolated points. In higher dimensions the picture is similar.

THEOREM 1.4 (Ein, Küechle, Lazarsfeld [9]). — *Let X be a smooth projective variety of dimension n, and let L be an ample line bundle. If, for all very general subvarieties Z^i of codimension i in X,*

$$L^{n-i}Z^i > (n-i)^{n-i}\alpha^{n-i},$$

for all $0 \leqslant i \leqslant n-1$, then $\epsilon(L,x) > \alpha$ for all sufficiently general $x \in X$.

The notion of Seshadri constant may be generalized to ample vector bundles. Let X be a smooth complex projective variety of dimension n, let $\mathcal{E}$ be an ample vector bundle of rank $r + 1$, $p : \mathbb{P}(\mathcal{E}) \longrightarrow X$ the projection from $\mathbb{P}(\mathcal{E})$ to X and $\xi := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ the tautological line bundle on $\mathbb{P}(\mathcal{E})$. Fix

a point $x \in X$ and consider the commutative diagram:

$$\begin{array}{ccc} \mathbb{P}(\pi_x^* \mathcal{E}) & \xrightarrow{\widetilde{\pi}_x} & \mathbb{P}(\mathcal{E}) \\ \widetilde{p} \downarrow & & \downarrow p \\ \widetilde{X} & \xrightarrow{\pi_x} & X \end{array}$$

associated to the blow-up $\pi_x : \widetilde{X} \longrightarrow X$ of X at a point $x \in X$. Let $\widetilde{\xi}_x := \mathcal{O}_{\mathbb{P}(\pi_x^* \mathcal{E})}(1)$ be the tautological line bundle on $\mathbb{P}(\pi_x^* \mathcal{E})$, $F_x = p^{-1}(x)$ the fiber of p over $x \in X$ and $\widetilde{E}_x = \widetilde{\pi}_x^{-1}(F_x)$. The Seshadri constant of $\mathcal{E}$ at $x \in X$ is defined as:

$$\epsilon(\mathcal{E}, x) := \sup\{\lambda > 0 | \widetilde{\xi}_x - \lambda \widetilde{E}_x \text{ is nef}\}.$$

In contrast to the case of line bundles, for vector bundles there are surprisingly few results about these interesting numbers. In [3], Beltrametti, Sommese and Schneider prove (in complete analogy to the line bundle case) that for very ample vector bundles, the Seshadri constant (at any point $x \in X$) is at least 1. They conjecture that this should also hold (for at least one point $x \in X$), under the weaker hypothesis that $\mathcal{E}$ is ample and generated. While this is true for $\dim(X) = 1$, for $\dim(X) \geqslant 2$ it is however not the case, as in [11] we produce examples of ample and spanned vector bundles on a $K3$ surface, for which

$$\epsilon(\mathcal{E}, x) \leqslant \sqrt{\frac{2}{\mathrm{rk}(\mathcal{E})}} \quad \forall x \in X.$$

Furthermore, Beltrametti, Sommese and Schneider in [4], give an example of an ample vector bundle on a smooth curve with Seshadri constant $\epsilon(\mathcal{E}, x) \leqslant \frac{1}{\mathrm{rk}(\mathcal{E})}$ for all $x \in X$, so it is clear that any generalization of the theorem of Ein, Küchle and Lazarsfeld must depend on the rank of the vector bundle $\mathcal{E}$. In this spirit we prove the following:

THEOREM 1.5. — *Let X be a complex projective variety of dimension n, let $\mathcal{E}$ be an ample vector bundle of rank $\mathrm{rk}(\mathcal{E}) = r + 1$, and α and β positive rational numbers.*

a. If for all very general Z^i irreducible subvarieties of X of codimension i,

$$\xi^{n+r-i} \pi^* Z^i > \binom{n+r-i}{r} (\alpha)^{n-i} (n-i)^{n-i}$$

for all $0 \leqslant i \leqslant n-1$. Then

i) If $\xi^\beta \otimes \pi^*(\mathcal{E}^*)$ is ample, then for very general $x \in X$

$$\epsilon(x, \mathcal{E}) \geqslant \frac{\alpha}{\beta}.$$

ii) *If $L^\beta \otimes \mathcal{E}^*$ is ample for some $L \in \mathrm{Pic}(X)$, then for very general $x \in X$*

$$\epsilon(x, \mathcal{E} \otimes L^\beta) \geqslant \alpha.$$

iii) *If $r = 1$, then for very general $x \in X$*

$$\epsilon(x, \mathcal{E} \otimes \det(\mathcal{E})) \geqslant \alpha.$$

b. *Assume that $r = 1$, and that $\mathcal{E}$ is globally generated, and for all very general Z^i irreducible subvarieties of X*

$$\xi^{n+1-i} \pi^* Z^i > (n + 1 - i) 2^{n-i} (n - i)^{n-i},$$

for all $0 \leqslant i \leqslant n - 1$. Then, for very general $x \in X$,

$$\epsilon(x, \mathcal{E}) \geqslant 1.$$

There are two kinds of conditions in the above theorem. The first condition is a positivity condition analogous to the one in the theorem of Ein, Küchle and Lazarsfeld. The main purpose of this condition is to ensure the existence of sections of high multiples of the line bundle ξ on the variety $\mathbb{P}(\mathcal{E})$, and on $\mathbb{P}(\mathcal{E}_{|Z})$ for all very general subvarieties Z. While we expect that these numerical conditions may be weakened considerably, using the current methods, they have the right flavor. The term $\alpha^{n-i}(n - i)^{n-i}$ is completely analogous to the line bundle case. One might expect that an optimal statement would only involve the term α^{n-i}. The coefficient $\binom{n+r-i}{r}$ is new, but in accordance with the conjectural bound:

$$\epsilon(\mathcal{E}, x) \geqslant \frac{1}{n(r + 1)} \quad \text{for all very general } x \in X.$$

In fact, for $r >> n$ we see that

$$\left(\binom{n + r}{r} \alpha^n n^n \right)^{1/n} \cong \frac{nr}{(n!)^{1/n}} \alpha \cong er\alpha,$$

and for $n >> r$ that

$$\left(\binom{n + r}{r} \alpha^n n^n \right)^{1/n} \cong n\alpha.$$

This suggests that for $n >> r$, the Seshadri constants behave as in the line bundle case, and that for $r >> n$, they behave analogously to the curve case.

The second condition (which in the theorem above has three different statements), is one of stability. It is inevitable that we require some

condition of this kind. Consider, for example, the vector bundle $\mathcal{E}_d :=$ $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$. For $d >> 0$ this will satisfy arbitrary positivity conditions (for all $Z \subset \mathbb{P}(\mathcal{E})$), however $\epsilon(\mathcal{E}_d, x) = 1$ for all d. It would be interesting to see if one can replace this second condition with the standard notion of stability.

Of course, as in the line bundle case, results about Seshadri constants, translate into effective results for the global generation (generation of higher jets, etc) of vector bundles of the form
$$K_X^{\otimes a} \otimes L^{\otimes b} \otimes S^c(\mathcal{E}).$$
M. de Cataldo, [5] and [6], has studied this problem for $c = 1$ and $b >> 0$. For example he proves:

THEOREM 1.6 ([6], Theorem 5.2.2.1′). — *If L an ample line bundle, $\mathcal{E}$ is an N-nef vector bundle ($N = \min\{\dim(X), \mathrm{rk}(\mathcal{E})\}$), then for any integer $m \geqslant \frac{1}{2}(n^2 + n + 2)$ the vector bundle $K_X \otimes L^{\otimes m} \otimes \mathcal{E}$ is generated by global sections.*

Our approach however is different in as much as we consider triples (a, b, c), with $a = 1$, $c >> 0$ and $b = O(c)$. For example we obtain the following corollary of Theorem 1.5.a.i).

THEOREM 1.7. — *Let $\mathcal{E}$ be an ample vector bundle and β a positive rational number such that $\xi^\beta \otimes \pi^*(\mathcal{E}^*)$ is ample. Let*
$$M := \mathrm{Min}_{0 \leqslant i \leqslant n-1} \left[\frac{1}{\binom{n+r-i}{r}^{\frac{1}{n-i}}} \cdot \frac{1}{n-i} \right].$$
Then for any integer $\lambda > \frac{n\beta}{M}$, $K_X \otimes S^\lambda(\mathcal{E}) \otimes \det(\mathcal{E})$ is generated by global sections at all very general points $x \in X$.

Proof. — Since $\mathcal{E}$ is ample, for all Z^i irreducible subvarieties of codimension i, $\xi^{n+r-i} \pi^* Z^i \geqslant 1$. It follows that the hypotheses of Theorem 1.5.a are verified for $\alpha < M$. Hence, $\epsilon(X, \mathcal{E}) \geqslant \frac{M}{\beta}$ for very general $x \in X$. For $\lambda > \frac{n\beta}{M}$, the sheaf $K_X \otimes S^\lambda(\mathcal{E}) \otimes \det(\mathcal{E})$ is then generated at x. □

2. Proofs.

Proof of Theorem 1.5.a. — The proof will consist of two parts involving respectively some of the techniques developed in [9] and in [8]. The notation is chosen to be compatible with both papers.

As in [9], we may assume that X is smooth, and we will proceed by induction on the dimension of X. Let

$$U_\lambda := \{x \in X | \epsilon(\mathcal{E}, x) < \lambda\}.$$

We must show that $U_{\frac{\alpha}{\beta}}$ does not contain a Zariski open subset. Suppose in fact that $U_{\frac{\alpha}{\beta}}$ contains a Zariski open subset U of points with Seshadri constant $\epsilon < \frac{\alpha}{\beta}$, i.e. such that for all $x \in U$ there exists an irreducible (Seshadri exceptional) curve $\widetilde{C}_x \subset \mathbb{P}(\pi_x^* \mathcal{E})$ such that $\left(\widetilde{\xi}_x - \frac{\alpha}{\beta} \widetilde{E}_x\right).\widetilde{C}_x < 0$. Then, there exists (see [9] (3.3)) an irreducible quasi-projective variety T, a dominant morphism

$$g : T \longrightarrow X,$$

and an irreducible subvariety

$$C \subset T \times \mathbb{P}(\mathcal{E}),$$

flat over T, such that for all $t \in T$

$$\widetilde{C}_t.(\widetilde{\xi}_{g(t)} - \frac{\alpha}{\beta} \widetilde{E}_{g(t)}) < 0.$$

Here, $\widetilde{C}_t$ denotes the proper transform of C_t under the morphism $\tilde{\pi}_{g(t)} : \mathbb{P}(\pi_{g(t)}^* \mathcal{E}) \longrightarrow \mathbb{P}(\mathcal{E})$. We may also assume that T is smooth and affine and $g : T \longrightarrow X$ is quasi-finite. Let $\Gamma \subset T \times X$ be the graph of g, for any subset $Y \subset T \times \mathbb{P}(\mathcal{E})$ (respectively $Z \subset T \times X$), let $Y_t \subset \mathbb{P}(\mathcal{E})$ (respectively $Z_t \subset X$) denote the fiber of Y (respectively of Z) over $t \in T$, and for any subset $V \subset X$ let F_V denote the fiber of $p : \mathbb{P}(\mathcal{E}) \longrightarrow X$ over V.

LEMMA 2.1 ([9] 3.5.1). — *Let $Z \subset T \times X$ be an irreducible closed variety dominating both X and T. Then one can construct an irreducible closed subvariety*

$$C_{\mathbb{P}(\mathcal{E})} Z \subset T \times \mathbb{P}(\mathcal{E}),$$

with the following properties:

i) $Z \subset (\mathrm{id}, p)(C_{\mathbb{P}(\mathcal{E})} Z)$.

ii) *For general $t \in T$ the fiber $(C_{\mathbb{P}(\mathcal{E})} Z)_t \subset \mathbb{P}(\mathcal{E})$ has the form*

$$(C_{\mathbb{P}(\mathcal{E})} Z)_t = \text{closure} \left(\bigcup_{s \in S_t} C_s \right),$$

where $S_t \subset g^{-1}(Z_t)$ is a closed subset of T, which dominates Z_t.

Property i), implies that $CZ := (\mathrm{id}, p)(C_{\mathbb{P}(E)} Z)$ dominates Z. So we inductively construct a chain of subvarieties

$$\Gamma = Z_0 \subset Z_1 \subset \ ... \ \subset Z_i \subset T \times X,$$

where $Z_0 = \Gamma = \mathrm{graph}(g)$, $Y_{i+1} = C_{\mathbb{P}(\mathcal{E})}Z_i \subset T \times \mathbb{P}(\mathcal{E})$, and $Z_{i+1} = (\mathrm{id}, p)(Y_{i+1})$.

LEMMA 2.2. — Z_i is a proper subvariety of Z_{i+1} for all $0 \leqslant i \leqslant n-1$.

Proof. — Suppose that $Z_i = Z_{i+1}$ for some $i < n$. Then the induction hypothesis applies for general t to $Z_t = (Z_i)_t$, and $\xi_{|Z_t}$, so $\epsilon(\mathcal{E}_{|Z_t}, x) \geqslant \frac{\alpha}{\beta}$ for very general $x \in Z_t$. However, $Z_{i+1} = (\mathrm{id}, p)(C_{\mathbb{P}(\mathcal{E})}Z_i) = Z_i$, so for general $t \in T$ and $s \in S_t$, $C_s \subset F_{Z_t}$ i.e. $C_s \subset p^{-1}(Z_t)$ is a family of Seshadri exceptional curves for the general point $g(s)$ of Z_t. This provides the required contradiction.   □

Therefore, we may assume that $Z_n = T \times X$.

LEMMA 2.3. — For all $k >> 0$ and for all $x \in X$ there exists a section of $k\xi$ vanishing on F_x to order at least $n\alpha k + 1$.

Proof. — Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(k\xi) \otimes \mathcal{I}_{F_x}^{\lambda} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(k\xi) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(k\xi) \otimes (\mathcal{O}_{\mathbb{P}(\mathcal{E})}/\mathcal{I}_{F_x}^{\lambda}) \longrightarrow 0.$$

Then

$$h^0\big(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(k\xi) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}/\mathcal{I}_{F_x}^{\lambda}\big) = \binom{n+\lambda-1}{n}\binom{k+r}{r},$$

$$h^0(\mathbb{P}(\mathcal{E}), k\xi) \cong \frac{k^{n+r}\xi^{n+r}}{(n+r)!}$$

$$> \frac{k^{n+r}(n+r)!n^n\alpha^n}{(n+r)!n!r!}$$

$$= \frac{k^n(n\alpha)^n k^r}{n!r!}$$

$$\cong \binom{n\alpha k}{n}\binom{k+r}{r}$$

$$\cong \binom{n\alpha k + n}{n}\binom{k+r}{r}$$

(here $\cong$ denotes equivalence modulo terms in $o(k^{n+r})$). Moreover, for $\lambda = n\alpha k$, we have $\Gamma(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(k\xi) \otimes \mathcal{I}_{F_x}^{\lambda}) \neq 0$.   □

Let $Pr_1 : T \times \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}(\mathcal{E})$ and $Pr_2 : T \times \mathbb{P}(\mathcal{E}) \longrightarrow T$. For all $k >> 0$ the torsion-free $\mathcal{O}_T$ module

$$\mathcal{F} = Pr_{2*}(Pr_1^*(k\xi) \otimes \mathcal{I}_{\Lambda}^b),$$

has positive rank (where $b = n\alpha k$, and $\mathcal{I}_\Lambda \subset \mathcal{O}_{T \times \mathbb{P}(\mathcal{E})}$ denotes the ideal sheaf of $\Lambda = (\mathrm{id}, p)^{-1}(\Gamma)$). Since T is affine, $\mathcal{F}$ is globally generated. Let $s \in \Gamma(T, \mathcal{F})$ be a non-zero section. Since

$$\Gamma(T, \mathcal{F}) = \Gamma(T \times \mathbb{P}(\mathcal{E}), Pr_1^*(k\xi) \otimes \mathcal{I}_\Lambda^b),$$

s gives rise to a divisor

$$D \in |\mathcal{O}_{T \times \mathbb{P}(\mathcal{E})}(Pr_1^*(k\xi))|$$

with $\mathrm{mult}_\Lambda(D) > b = k\alpha n$. Let $P_i = (\mathrm{id}, p)^{-1}(Z_i)$, in particular $P_0 = \Lambda$, and $P_n = T \times \mathbb{P}(\mathcal{E})$. Since $\mathrm{mult}_{P_0}(D) = \mathrm{mult}_\Lambda(D) > k\alpha n$, and since $\mathrm{mult}_{P_n}(D) = \mathrm{mult}_{T \times \mathbb{P}(\mathcal{E})}(D) = 0$, there must be at least one index $0 \leqslant i \leqslant n - 1$ such that

$$\mathrm{mult}_{P_i}(D) - \mathrm{mult}_{P_{i+1}}(D) > k\alpha.$$

By Proposition 2.3 [9], there exists a divisor

$$D' \in |\mathcal{O}_{T \times \mathbb{P}(\mathcal{E})}(Pr_1^*(k\xi))|$$

such that $\mathrm{mult}_{P_i}(D') > k\alpha$ and $P_{i+1} \not\subset \mathrm{Supp}(D')$.

The next step is to produce a section D'' such that $C_{\mathbb{P}(\mathcal{E})}Z_i \not\subset \mathrm{Supp}(D'')$. To this end, let $\mathcal{N} = T_{\mathbb{P}(\mathcal{E})/X}$ be the relative tangent sheaf of the projection $p : \mathbb{P}(\mathcal{E}) \longrightarrow X$, defined by the exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow T_{\mathbb{P}(\mathcal{E})} \longrightarrow p^*T_X \longrightarrow 0.$$

Let D_B^l be the sheaf of differential operators of order $\leqslant l$ on a line bundle B. Define $D_{\mathcal{N},B}^l$ to be the subsheaf of D_B^l of differential operators of order $\leqslant l$ on the line bundle B with symbols in $T_{\mathbb{P}(\mathcal{E})/X}$. Equivalently $D_{\mathcal{N},B}^l$ corresponds to the subsheaf of D_B^l of $p^*\mathcal{O}_X$-linear differential operators of order $\leqslant l$. These sheaves sit in the exact sequence

$$0 \longrightarrow D_{\mathcal{N},B}^{l-1} \longrightarrow D_{\mathcal{N},B}^l \longrightarrow \mathrm{Sym}^l(\mathcal{N}) \longrightarrow 0.$$

Consider the following exact sequence of vector bundles:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})} \longrightarrow \pi^*\mathcal{E}^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\xi) \longrightarrow \mathcal{N} \longrightarrow 0.$$

Since $\xi^\beta \otimes \pi^*(\mathcal{E}^*)$ is an ample $\mathbb{Q}$-vector bundle, it follows that $\mathcal{N} \otimes \xi^{\beta-1}$ is ample. By an argument analogous to [8] Lemma 2.5, one has:

LEMMA 2.4. — *If l is a sufficiently large and divisible integer such that $\xi^{l\beta}$ is a integral line bundle, then*

$$D_{\mathcal{N},B}^l \otimes \xi^{l(\beta-1)}$$

is globally generated for all line bundles B.

By applying the lemma to the divisor $D' \in |\mathcal{O}_{T \times \mathbb{P}(\mathcal{E})}(Pr_1^*(k\xi))|$, we produce a section

$$D'' \in |\mathcal{O}_{T \times \mathbb{P}(\mathcal{E})}(Pr_1^*(k\beta\xi))|,$$

such that $\text{mult}_{P_i}(D'') > k\alpha$ and $C_{\mathbb{P}(\mathcal{E})}Z_i \not\subset \text{Supp}(D'')$ and this proves (a.i). $\square$

Remark. — We have assumed throughout that X is a smooth variety. Since the statement of the theorem only concerns generic points of X, the result may be recovered more generally by replacing X with an appropriate smooth birational model. There is however one technical point, we may no longer assume that $\xi^\beta \otimes \pi^* \mathcal{E}^*$ is ample. Let L be any ample line bundle on $\mathbb{P}(\mathcal{E})$ and $\delta > 0$ a rational number, then the $\mathbb{Q}$–vector bundle $\xi^\beta \otimes \pi^* \mathcal{E}^* \otimes L^\delta$ is ample. Eventually we take the limit as $\delta \longrightarrow 0$, and hence this does not affect the preceding computations.

The proofs of (a.ii) and (a.iii) are analogous, for the latter observe that if the rank of $\mathcal{E}$ is 2, then $\mathcal{E}^* \otimes \det(\mathcal{E}) \cong \mathcal{E}$.

Proof of b. — We maintain the notation of the proof of a., in particular let $C \subset T \times \mathbb{P}(\mathcal{E})$ be a family of Seshadri exceptional curves (ie such that $(\xi_x - \widetilde{E}_x).\widetilde{C}_x < 0$). Since $\mathcal{E}$ is globally generated, there exists a map $\varphi : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^N$ such that $\xi = \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)$. The vector bundle $\mathcal{E}$ is ample, and hence the map φ is finite onto its image. In particular φ does not contract curves. If $\varphi(C_x) \not\subset \varphi(F_x)$, then there exists a hyperplane section H containing the linear space $\varphi(F_x)$ and not containing the curve $\varphi(C_x)$. The divisor $\varphi^* H$ will then contain the fiber F_x but not the curve C_x. This contradicts the fact that C_x is a Seshadri exceptional curve. We may therefore assume that $C_x \subset \varphi^{-1}\varphi(F_x)$. Since φ is finite, $\varphi(C_x) = \varphi(F_x)$ is a line in $\mathbb{P}^N$. Consequently, the family of curves C dominates the image of $\mathbb{P}(\mathcal{E})$ in $\mathbb{P}^N$, but since φ is a finite map, the family of curves C must dominate $\mathbb{P}(\mathcal{E})$.

As in the proof of a., there exists a divisor

$$D' \in |\mathcal{O}_{T \times \mathbb{P}(\mathcal{E})}(Pr_1^*(k\xi))|$$

such that $\text{mult}_{P_i}(D') \geqslant 2k$ and $P_{i+1} \not\subset \text{Supp}(D')$ so in particular $\text{mult}_{C_{\mathbb{P}(\mathcal{E})}Z_i}(D') \leqslant k$.

Since the family of curves C dominates $\mathbb{P}(\mathcal{E})$, for a fixed $t_0 \in T$ and a general $t \in T$, the divisor $D_t' \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(k\xi)|$ does not vanish along $\widetilde{C}_{t_0}$. So by taking the k–th derivative (again it suffices to apply [9] Proposition 2.3

to $V = Y_{i+1} \subset T \times \mathbb{P}(\mathcal{E}))$, we produce a section

$$D'' \in |\mathcal{O}_{T \times \mathbb{P}(\mathcal{E})}(Pr_1^*(k\xi))|,$$

such that $\text{mult}_{P_i}(D') \geqslant k$ and $C_{\mathbb{P}(\mathcal{E})} Z_i \nsubseteq \text{Supp}(D'')$. Hence $(k\widetilde{\xi}_x - k\widetilde{E}_x).\widetilde{C}_x > 0$, which proves that $\epsilon(x, \mathcal{E}) \geqslant 1$. $\qquad\square$

3. Ample vector bundles on curves.

In this section we treat the case of ample vector bundles over a curve C. We first show that for all ample vector bundles over a curve C, (and for all points $x \in C$)

$$\epsilon(\mathcal{E}, x) \geqslant \frac{1}{\text{rk}(\mathcal{E})}.$$

We begin by proving this using the techniques of [12] for vector bundles of rank 2. Then using the techniques of [14], we treat the general case. Next, we illustrate a construction (due to Beltrametti, Sommese and Schneider [4]) of vector bundles where this bound is actually achieved. Finally we show that if $\mathcal{E}$ is an ample and globally generated vector bundle on a smooth complex projective curve C, then $\epsilon(\mathcal{E}, x) \geqslant 1$ for all points $x \in X$. This directly confirms the conjectures of [3] and the conjecture of [1] when X is a curve.

PROPOSITION 3.1. — Let $\mathcal{E}$ be an ample vector bundle of rank 2 on a smooth complex projective curve C. Then $\epsilon(\mathcal{E}, x) \geqslant \frac{1}{2}$ for all points $x \in C$.

Proof. — $\text{Pic}(\mathbb{P}(\mathcal{E})) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by ξ, the class of the tautological line bundle, and by f the class of a fiber. As usual $f^2 = 0$, $f.\xi = 1$ and $\xi^2 = \deg(c_1(\mathcal{E})) = e > 0$. Let $D = a\xi + bf$ be an irreducible, reduced curve on $\mathbb{P}(\mathcal{E})$, we claim that one of the following statements holds:

- $a = 0$ and D is a fiber

- $a = 1$ and D is a section

- $a \geqslant 2$ and $b \geqslant -ae/2$.

To see this, assume that $a \geqslant 2$ and view D as a finite cover of C of

degree a. By Riemann-Hurwitz we have

$$a(2g(C) - 2) \leqslant 2P_D - 2 = D.(D + K_{\mathbb{P}(\mathcal{E})})$$
$$= (a\xi + bf).((a - 2)\xi + (b + 2g(C) - 2 + e)f)$$
$$= a(a - 2)e + b(a - 2) + a(b + 2g(C) - 2 + e)$$
$$= (a - 1)(2b + ae) + a(2g(C) - 2).$$

Hence $0 \leqslant (a - 1)(ae + 2b)$, which proves the claim.   □

If $D = f$, then we have $D.\xi = 1$ and $D.f = 0$. If $a = 1$, then $D.f = 1$ and $D.\xi \geqslant 1$. In both cases $(\xi - f).D \geqslant 0$. Finally if $a \geqslant 2$:

$$D.(\xi - \frac{1}{2}f) = ae + b - a/2$$
$$\geqslant \frac{1}{2}(ae - a)$$
$$= \frac{1}{2}a(e - 1) \geqslant 0.$$

Therefore, $\xi - \frac{1}{2}f$ is numerically effective for all fibers f, and hence $\epsilon(\mathcal{E}, x) \geqslant \frac{1}{2}$.

Let $\mu(\mathcal{E}) := \deg(\mathcal{E})/\operatorname{rk}(\mathcal{E})$ be the slope of the vector bundle $\mathcal{E}$, and let $\mu_{\max}$ and $\mu_{\min}$ be the largest (respectively the smallest) slopes of a subbundle (respectively a quotient bundle) of $\mathcal{E}$. These coincide with the slopes of the first and last associated gradeds of the Harder-Narasimhan filtration. Let us recall the following well known facts [14]:

$$\operatorname{rk}(S^k\mathcal{E}) = \binom{k + \operatorname{rk}(\mathcal{E}) - 1}{\operatorname{rk}(\mathcal{E}) - 1},$$
$$\deg(S^k\mathcal{E}) = k\frac{\deg(\mathcal{E})}{\operatorname{rk}(\mathcal{E})}\binom{k + \operatorname{rk}(\mathcal{E}) - 1}{\operatorname{rk}(\mathcal{E}) - 1},$$
$$\mu(S^k\mathcal{E}) = k\mu(\mathcal{E}).$$

THEOREM 3.1. — *Let $\mathcal{E}$ be an ample vector bundle on a smooth projective curve, then*

$$\epsilon(\mathcal{E}, x) = \mu_{\min}.$$

Proof. — Suppose that $\mathcal{E} \otimes \mathcal{O}_C(-\epsilon x)$ is an ample $\mathbb{Q}$-vector bundle, then for any quotient Q of $\mathcal{E}$, we have that $Q \otimes \mathcal{O}_C(-\epsilon x)$ is also an ample $\mathbb{Q}$-vector bundle. In fact for any sufficiently big and divisible integer $k > 0$, we have a surjection of (integral) vector bundles

$$S^k\mathcal{E} \otimes \mathcal{O}_C(-k\epsilon x) \longrightarrow S^kQ \otimes \mathcal{O}_C(-k\epsilon x).$$

Since $S^k\mathcal{E} \otimes \mathcal{O}_C(-k\epsilon x)$ is ample, it follows that $S^kQ \otimes \mathcal{O}_C(-k\epsilon x)$ is also ample and hence has positive degree, i.e.

$$k\left(\frac{\deg Q}{\mathrm{rk}\,Q} - \epsilon\right)\binom{k + \mathrm{rk}\,Q - 1}{\mathrm{rk}\,Q - 1} \geqslant 0.$$

It follows that $\mu(Q) - \epsilon > 0$. This implies that $\epsilon(\mathcal{E}, x) \leqslant \mu_{\min}(\mathcal{E})$. We will now show that for all rational numbers $0 < \epsilon < \mu_{\min}$, we have $\epsilon(\mathcal{E}, x) \geqslant \epsilon$. By Proposition 2.4 of [12], a vector bundle $\mathcal{E}$ is ample if and only if its symmetric powers $S^n(\mathcal{E})$ are ample for all large enough n. By Theorem 2.4 [14], it suffices to show for all $0 < \epsilon < \mu_{\min}$, and some $k >> 0$ sufficiently divisible (ie such that $k\epsilon$ is an integer), that all quotients of the bundle $S^k\mathcal{E} \otimes \mathcal{O}_C(-k\epsilon)$ have positive degree. This is clear since if Q is a quotient of $S^k\mathcal{E}$, then $Q \otimes \mathcal{O}_C(-k\epsilon)$ is a quotient of $S^k\mathcal{E} \otimes \mathcal{O}_C(-k\epsilon)$. By direct computation

$$\mu(Q \otimes \mathcal{O}_C(-k\epsilon)) = \frac{\deg(Q) - \mathrm{rk}(Q) \cdot k\epsilon}{\mathrm{rk}(Q)} = \mu(Q) - k\epsilon,$$

and this is positive for all $0 < \epsilon < \frac{\mu_{\min}(S^k\mathcal{E})}{k}$. The theorem now follows as on a curve $\mu_{\min}(S^k\mathcal{E}) = k\mu_{\min}(\mathcal{E})$.   □

Of course it follows that if $\mathcal{E}$ is an ample and stable vector bundle, then $\epsilon(\mathcal{E}) = \mu(\mathcal{E})$.

COROLLARY 3.1. — *Let $\mathcal{E}$ be an ample vector bundle on a smooth complex projective curve. Then*

$$\epsilon(\mathcal{E}, x) \geqslant \frac{1}{\mathrm{rk}(\mathcal{E})}.$$

Proof. — Since $\mathcal{E}$ is ample, every quotient bundle of $\mathcal{E}$ has positive degree, and in particular $\mu_{\min} \geqslant \frac{1}{\mathrm{rk}(\mathcal{E})}$.   □

Next, following the construction of Beltrametti, Sommese and Schneider, we exhibit certain stable and ample vector bundles over a curve, whose Chern class has degree 1. These are examples of vector bundles with Seshadri constants $\epsilon(\mathcal{E}, x) = \frac{1}{\mathrm{rk}(\mathcal{E})}$.

LEMMA 3.1 (Beltrametti, Sommese, Schneider [4]). — *Given any smooth projective curve, C, of genus $g \geqslant 2$, any $r \geqslant 0$, and any point $x \in C$, there exists an ample, stable, rank r vector bundle $\mathcal{E}_r$, with $\det(\mathcal{E}_r) \cong \mathcal{O}_C(x)$.*

These vector bundles may be constructed as follows; for $r = 1$ let $\mathcal{E}_1 = \mathcal{O}_C(x)$. For $r \geqslant 2$, let F be a stable rank $r - 1$ vector bundle with $\det(F) = \mathcal{O}_C$. Pick any nontrivial extension

$$0 \longrightarrow F \longrightarrow \mathcal{E}_r \longrightarrow \mathcal{O}_C(x) \longrightarrow 0$$

corresponding to an appropriate section in $H^1(C, F(-x)) \neq 0$.

Since $c_1(\mathcal{E}_r) = \mathcal{O}_C(x)$, it follows that $\epsilon(\mathcal{E}_r, x) \leqslant \frac{1}{\mathrm{rk}(\mathcal{E}_r)}$. By Corollary 3.1, we deduce that

$$\epsilon(\mathcal{E}_r, x) = \frac{1}{\mathrm{rk}(\mathcal{E}_r)}.$$

PROPOSITION 3.2. — *Let $\mathcal{E}$ be an ample and generated vector bundle on a smooth complex projective curve C. Then $\epsilon(\mathcal{E}, x) \geqslant 1$ for all points $x \in C$.*

Proof. — Since all fibers are numerically equivalent, it suffices to prove this for one (any) fiber. The vector bundle $\mathcal{E}$ is globally generated, so there exists a map $f : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^N$, such that $\xi = f^* \mathcal{O}_{\mathbb{P}^N}(1)$. Since $\mathcal{E}$ is ample, this map is finite onto its image, and in particular it does not contract curves. Let $D \subset \mathbb{P}(\mathcal{E})$ be a Seshadri exceptional curve, i.e. a curve such that

$$(\xi - F_p).D < 0.$$

If $f(D) \not\subset f(F_p)$, then there exists a hyperplane H containing the linear space $f(F_p)$ and not containing the curve $f(D)$. The divisor $f^*H - F_p$ is then effective and does not contain D, contradicting the fact that D is Seshadri exceptional. We may therefore assume that $f(D)$ is contained in the image of all of the fibers of $\mathbb{P}(\mathcal{E})$. It follows that $\dim f^{-1}(f(D)) \geqslant 2$ and this contradicts the finiteness of the map f.   □

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Christopher D. HACON,
University of Utah
Department of Mathematics
Salt Lake City, UT 84112-0090 (USA).
hacon@math.utah.edu