ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 50, nº 2 (2000), p. 677-687
<http://www.numdam.org/item?id=AIF_2000__50_2_677_0>

Ann. Inst. Fourier, Grenoble
50, 2 (spécial Cinquantenaire) (2000), 677–687

AN APPROXIMATION THEOREM RELATED TO GOOD COMPACT SETS IN THE SENSE OF MARTINEAU

by Jean-Pierre ROSAY[1] & Edgar Lee STOUT

This paper is about a theorem (Theorem 1) stated, but in our opinion not fully proved, by J.E. Björk and a theorem (Theorem 2) proved but not stated by E. Bishop.

In [2] Björk states the following theorem (in a footnote):

THEOREM 1. — *Let U be an open set in $\mathbb{C}^n$, and let K be a compact subset of U. Then there exists a neighborhood W of K such that if $f \in \mathcal{O}(U)$ and if, for every $I \in \mathbb{N}^n$, $\frac{\partial^{|I|} f}{\partial Z^I}$ is the uniform limit on K of a sequence of polynomials, then f is the uniform limit on W of a sequence of polynomials.*

In this paper $\mathcal{O}(U)$ denotes the algebra of holomorphic functions on U and *polynomial* will always mean holomorphic polynomial.

Theorem 1 has an important corollary, called the Main Theorem in [2], which says that every compact set in $\mathbb{C}^n$ is a good compact set in the sense of Martineau [5]. It implies, by an argument given in [5], that every compact subset of every Stein manifold is a good compact set.

COROLLARY. — *With K and U as above, if $f \in \mathcal{O}(U)$, and if f is uniformly approximable by polynomials on some neighborhood of K, then f is uniformly approximable by polynomials on the fixed neighborhood W.*

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(1) Research supported in part by a grant from the NSF.
Keywords: Polynomial hulls – Polynomial approximation – Analytic functionals – Good compact sets.
Math. classification: 32E20 – 32E30 – 46F15 – 46A22.

This approximation property is a necessary and sufficient condition in order to extend every analytic functional carried by K to a "local analytic functional", which is useful for decomposing analytic functionals into sums of analytic functionals carried by smaller sets. See [5], Theorem 1.2 in Chapter 1 and 2.A in [6].

Björk's arguments do not differ substantially from Martineau's argument, establishing the theorem under an additional hypothesis [5], Theorem 1.1′ in Chapter 1. In our view, Björk's proof has a gap in that it does not establish a compactness property of the topology that he introduces on a maximal ideal space. (Alternatively, it has to be shown that, on the polynomial hull, this topology coincides with the $\mathbb{C}^n$ topology.) This is however the heart of the matter. (In similar situations see [1] page 48, lines 19–25, or [5].)

It is true that Björk himself gives an indication leading to a complete proof. On page 495 in [2], he mentions Bishop's work, claiming that "we shall not need this result". Our point is that, apparently, one does need it.

The next, and not at all easy, question is: Which result of Bishop do we need? The paper [1] is extremely deep but hard to read. The following theorem is not to be found stated in [1], nor in [3] or [4], where a very helpful introduction to part of Bishop's work is given. Nonetheless, we undoubtedly can attribute the result to Bishop.

THEOREM 2. — *Let L be a compact set, and let U be an open set in $\mathbb{C}^n$ with $L \subset U$. Let $f \in \mathcal{O}(U)$. Let L_1 be the graph in $\mathbb{C}^{n+1}$ defined by*

$$L_1 = \{(z, f(z)) \in \mathbb{C}^{n+1} : z \in L\}.$$

Let $\hat{L}_1$ denote the polynomially convex hull of L_1. For every $\epsilon > 0$, there exist $k \in \mathbb{N}$ and an exceptional set $E_\epsilon \subset \mathbb{C}^n$ of measure not more than ϵ such that, for every $z \in \mathbb{C}^n - E_\epsilon$, the fiber of $\hat{L}_1$ over z has at most k elements. Moreover k depends only on L, U and ϵ, not on f.

The fiber of $\hat{L}_1$ over z is the set of $(z, w) \in \mathbb{C}^{n+1}$, $w \in \mathbb{C}$, such that $(z, w) \in \hat{L}_1$.

We propose this statement (which could easily be generalized to Riemann domains) as an extremely readable one, one that may well suffice for many applications of Bishop's theory, while avoiding extensions of norms, etc.

In Section 2 we establish Theorem 2, but assuming Theorem 2 we first prove Theorem 1 in Section 1. Although the proof has been sketched in [6] (Appendix on good compact sets) it seems desirable to have a clear complete proof written down for such an important result.

We should point out the paper [7], by W.R. Zame. In that paper the author has an interesting systematic discussion of algebras invariant under differentiation. He uses considerably more of Bishop's theory than we do. A main result in [7] is Theorem 4.1. It is claimed that a slight modification of its proof, which, the reader should be warned, is marred by misprints, leads to Theorem 4.2 in [7], a special case of which is Theorem 1 of this paper.

1. Proof of Theorem 1.

It is enough to prove that there exists a neighborhood W_1 of $\hat{K}$, the polynomial hull of K in $\mathbb{C}^n$, such that for any f as in the statement of the theorem, there exists $\hat{f} \in \mathcal{O}(W_1)$ such that $f = \hat{f}$ near K. The theorem follows then from the Oka-Weil Theorem.

The proof will be presented in three steps.

In Step 1, we recall some basic facts on algebras of holomorphic functions closed under differentiation. (See [1] Section 4.) In Step 2, consequences are drawn from Theorem 2. In Step 3, we apply the above to the situation of Theorem 1. Some details in the proof are left to the reader who can also look at [2].

Step 1). For an open subset U of $\mathbb{C}^n$, we consider a subalgebra A of $\mathcal{O}(U)$, that contains the polynomials and such that for every $f \in A$ and for every $j \in \{1, \dots, n\}$, $\frac{\partial f}{\partial z_j} \in A$. If H is a compact subset of U, we denote by $\operatorname{Spec}_H A$ the set of algebra homomorphisms from A onto $\mathbb{C}$ that are continuous with respect to sup norm on H. The coordinate functions in $\mathbb{C}^n$ are denoted $Z_1, \dots, Z_n$. If $h \in \operatorname{Spec}_H A$, then $\pi(h)$ is the point in $\mathbb{C}^n$ defined by $\pi(h) = (h(Z_1), \dots, h(Z_n))$. We also write $\pi(h) = z = (z_1, \dots, z_n) = h(Z)$.

Fix compact sets $K \subset L \subset U$, with K included in the interior of L. Let $r = \operatorname{dist}(K, U \setminus L)$, a positive number. We shall establish the following two simple facts i) and ii), which are familiar and basic to these kinds of considerations.

i) If $h \in \operatorname{Spec}_K A$ and $h(Z) = z$, there exists a holomorphic map φ, a section of π, of the ball $B(z, r)$, centered at z with radius r, into

$\operatorname{Spec}_L A$,

$$\varphi : B(z,r) \to \operatorname{Spec}_L A,$$

such that $\pi \circ \varphi = \mathrm{id}$ and $\varphi(z) = h$.

That φ is a holomorphic map means that for every $f \in A$, the map $z \mapsto \varphi(z)(f)$ is a holomorphic function.

This is an immediate consequence of the Cauchy estimates on derivatives.

For $\zeta \in B(z,r)$, simply set

$$\varphi(\zeta)(f) = \sum_I \frac{1}{I!} h\left(\frac{\partial^{|I|} f}{\partial z_I}\right) (\zeta - z)^I.$$

ii) If φ' is another such map, associated to a homomorphism h' (possibly h itself) with $h'(Z) = z'$, then: either $\varphi \equiv \varphi'$ on $B(z,r) \cap B(z',r)$ or $\varphi(B(z,r))$ and $\varphi'(B(z',r))$ are disjoint.

Indeed if $\varphi(\zeta) = \varphi'(\zeta)$, then not only $\varphi(\zeta)(f) = \varphi'(\zeta)(f)$ for all $f \in A$, but also $\frac{\partial^{|I|}}{\partial \zeta^I}(\varphi(\zeta)(f)) = \frac{\partial^{|I|}}{\partial \zeta^I}(\varphi'(\zeta)(f))$ for all multi-indices I, so $\varphi(\zeta)(f) = \varphi'(\zeta)(f)$ on $B(z,r) \cap B(z',r)$.

Step 2). In the setting of 1), Theorem 2 has the following consequence:

COROLLARY. — *There exists $k \in \mathbb{N}$ such that for every $z \in \mathbb{C}^n$, the fiber of* $\operatorname{Spec}_L A$ *over z has at most k elements.*

Proof of the corollary. — Although we stated the corollary as it is to be used, for $\operatorname{Spec}_L A$, we will rather prove it for $\operatorname{Spec}_K A$, just in order to keep the same notations. This is of course inconsequential.

Let $r > 0$ be as in 1), let $|B(0,r)|$ denote the volume of the ball $B(0,r)$, let $\epsilon < |B(0,r)|$, and fix k as in Theorem 2. (The choice of k depends, in part, on that of ϵ.) If the fiber of Spec_K over z had $(k+1)$ distinct points $h_1, \ldots, h_{k+1}$, there would exist $f \in A$ such that

$$h_j(f) \neq h_\ell(f) \quad \text{for} \quad 1 \leqslant j < \ell \leqslant k+1.$$

For each $j \in \{1, \ldots, k+1\}$ let then φ_j be a holomorphic map defined on $B(z,r)$ as in 1) with $\varphi_j(z) = h_j$.

Let

$$L_1 = \{(\zeta, f(\zeta)) \in \mathbb{C}^{n+1} : \zeta \in L\}.$$

For every $\zeta \in B(z,r)$, the restriction of $\varphi_j(\zeta)$ to the algebra generated by $Z_1, \ldots, Z_n$ and f defines an element $\varphi_j^*(\zeta) \in \hat{L}_1$, which is in the fiber of $\hat{L}_1$

over ζ. (Indeed the algebra of functions on L generated by $Z_1, \dots, Z_n$ and f can be identified with the algebra of functions on L_1 generated by the polynomials, whose maximal ideal space is $\hat{L}_1$.) Except on a set of measure not more than ϵ, this fiber over ζ has at most k elements.

Since $\epsilon < |B(z,r)|$, there exist $1 \leqslant j < \ell \leqslant k+1$, and a set G of positive measure in $B(z,r)$ such that for every $\zeta \in G$:

$$\varphi_j(\zeta)(f) = \varphi_\ell(\zeta)(f).$$

It follows that $\varphi_j(\zeta)(f) = \varphi_\ell(\zeta)(f)$ for all $\zeta \in B(z,r)$, in particular for $\zeta = z$, a contradiction.

Step 3). Following [2], we apply 1) and 2) to the algebra A of holomorphic functions on U that can be uniformly approximated on K by polynomials and whose derivatives can also be so approximated.

If $z \in \hat{K}$ (the polynomial hull of K in $\mathbb{C}^n$) the map $P \mapsto P(z)$ defined on the algebra of polynomials, extends naturally to a homomorphism of A onto $\mathbb{C}$, that we denote by h_z. This is the natural inclusion $\hat{K} \subset \operatorname{Spec}_K A$.

Fix L as in 1), with $K \subset \operatorname{int} L \Subset U$. For each $z \in \hat{K}$, denote by φ_z the map from $B(z,r)$ into $\operatorname{Spec}_L A$ that satisfies $\varphi_z(z) = h_z$ as defined in 1). If $z \in K$, then for $\zeta \in B(z,r)$ we have $\varphi_z(\zeta)(f) = f(\zeta)$ for all f in our algebra.

The following lemma corresponds to what we consider to be the gap in [2]:

LEMMA 1. — *There exists $\delta > 0$ such that if z and $z' \in \hat{K}$ and $|z - z'| < \delta$ then $h_{z'} = \varphi_z(z')$.*

Proof of the lemma. — Using the uniqueness result in ii (Step 1) and the Bolzano-Weierstrass Theorem, it is enough to show that if (z_j) is a sequence of points $z_j \in \hat{K}$ with limit $z \in \hat{K}$, then for j large enough $h_z = \varphi_{z_j}(z)$.

Indeed by Schwarz's Lemma, $\varphi_{z_j}(z) \underset{j\to\infty}{\longrightarrow} h_z$. However $\varphi_{z_j}(z)$ belongs to the fiber of $\operatorname{Spec}_L A$ over z, which is a finite set, so the sequence must be stationary.

This argument is taken from [1] page 488, lines 19–25.

End of the proof of Theorem 1. — The proof ends exactly as in [1], [2], [5]. For every $z \in \hat{K}$, and every $f \in A$ the function $\zeta \mapsto \varphi_z(\zeta)(f)$ is a holomorphic function defined for $|\zeta - z| < r$, such that $\varphi_z(\zeta)(f) = f(\zeta)$ if $z \in K$. With δ as above, set $\delta_1 = \min(r, \frac{\delta}{2})$. Denote by ψ_z the restriction of φ_z to the ball $B(z, \delta_1)$. It follows from the lemma above and from the uniqueness result 1 ii) that the functions ψ_z agree: On the intersection of their domains $\psi_z \equiv \psi_{z'}$. So the functions $\zeta \mapsto \varphi_z(\zeta)(f)$ define a holomorphic function on the δ_1 neighborhood of $\hat{K}$ in $\mathbb{C}^n$, which agrees with f on some neighborhood of K. Since δ_1 does not depend on f, Theorem 1 follows from the Oka-Weil Theorem.

Remarks. — 1) Theorem 2 has allowed us to overcome a problem of multivaluedness, which is the main difficulty in the proof. The proof is otherwise rather immediate. The problem of multivaluedness was the difficulty met by Martineau.

2) For $0 < \epsilon < 1$ consider:

$$K = \{(e^{i\theta}, 0) \in \mathbb{C}^2\} \cup \{(4e^{i\theta}, \epsilon) \in \mathbb{C}^2\}$$

and

$$\begin{aligned} U = &\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 2,\ |z_2| < 1\} \cup \\ &\{(z_1, z_2) \in \mathbb{C}^2 : 3 < |z_1| < 5,\ |z_2| < 1\}. \end{aligned}$$

By letting ϵ tend to 0, one sees in this simple example that the size of the neighborhood W of K in Theorem 1 does not depend on simple data such as the diameter of K, the distance from K to the boundary of U, etc.

2. Proof of Theorem 2.

We could simply say that the proof strictly follows [4] pages 52–55, or [3] pages 85–91, sources which are much easier to read than [1]. However it seems better to go through the proof in some detail. In addition to having the conclusion stated differently, our exposition presents some minor differences from [3] or [4]: We avoid Lemma 9 in [3], page 85, and appropriate normalizations lead to some simplification of notations.

LEMMA 2. — *Let $A_1, \dots, A_N \in U \subset \mathbb{C}^n$. For α and $\beta \in \mathbb{N}$, there exists a non-zero polynomial $P_{\alpha\beta}$ in $(n+1)$ variables $(z_1, \dots, z_n, w)$ of degree not more than α in each variable $z_1, \dots, z_n$ and of degree not more than β in w such that the function of n variables*

$$F_{\alpha\beta}(z) = P_{\alpha\beta}(z, f(z)),$$

which is defined on U, satisfies

$$\frac{\partial^{|I|} F_{\alpha\beta}}{\partial z^{|I|}}(A_j) = 0$$

for all multi-indices I with $|I| < (N^{-1/n}\alpha\beta^{1/n}) - 1$.

Proof. — A simple count of dimension. There are fewer than $(N^{-1/n}\alpha\beta^{1/n})^n$ such multiindices, and since there are N points A_j we have fewer than $\alpha^n\beta$ linear constraints on a vector space of polynomials of dimension $(\alpha+1)^n(\beta+1)$.

We use the notations of Theorem 2. Independently of f we now *select the points* $A_1, \dots, A_N$ to be used in applying Lemma 2. Let $\rho < \operatorname{dist}(L, bU)$. Choose $A_1, \dots, A_N \in L$ such that

$$L \subset \bigcup_{j=1}^{N} B(A_j, \rho/2),$$

where $B(A_j, \rho/2)$ denotes the open ball centered at A_j and of radius $\rho/2$.

Normalizations. — Without loss of generality we will assume that $U \subset B(0,1)$ and that $|f| < 1$. (By shrinking U one can first restrict to bounded holomorphic functions f.) A polynomial $Q = \Sigma a_{Iq} z^I w^q$ is said to be *normalized* if $\max |a_{Iq}| = 1$. By multiplying by appropriate constant, we can take the polynomial $P_{\alpha\beta}$ in Lemma 2 to be normalized.

LEMMA 3. — *With N and $A_1, \dots, A_N$ as above, fix $C > 2^{-(N^{-1/n})}$. For each (α, β) let $P_{\alpha\beta}$ be a normalized polynomial as provided by Lemma 2. There exists $\beta_0 \in \mathbb{N}$ such that for every $\alpha \geqslant 1$ and every $\beta \geqslant \beta_0$*

$$\sup_{z \in L} |P_{\alpha\beta}(z, f(z))| \leqslant C^{\alpha\beta^{1/n}}.$$

The lemma will be used by taking an arbitrary value of C with $2^{-(N^{-1/n})} < C < 1$. Notice that

$$\sup_{z \in L} |P_{\alpha\beta}(z, f(z))| = \sup_{L_1} |P_{\alpha\beta}|.$$

Proof. — By counting the number of monomials one has the estimate:

$$|P_{\alpha\beta}(z, w)| \leqslant (\alpha+1)^n(\beta+1) \text{ if each } |z_j| \leqslant 1 \text{ and } |w| \leqslant 1.$$

Applying Schwarz's Lemma to $P_{\alpha\beta}(z, f(z))$ on each ball $B(A_j, \rho)$ yields the estimate on the half balls that

$$\sup_{B(a_j, \rho/2)} |P_{\alpha,\beta}(z, f(z))| \leqslant (\alpha+1)^n(\beta+1) \;\; 2^{-p_n}$$

where p_n is the largest integer less than $N^{-1/n}\alpha\beta^{1/n} - 1$. For $\alpha \geqslant 1$ and $\beta \geqslant \beta_0$, β_0 large enough,

$$(\alpha+1)^n(\beta+1)2^{-p_n} \leqslant C^{\alpha\beta^{1/n}},$$

whence the lemma.

LEMMA 4. — *There exists $\gamma > 0$ such that for every $\alpha \in \mathbb{N}$ and for every normalized polynomial Q in $\mathbb{C}^n$ of degree at most α in each variable and for every $t \in (0,1)$*

$$|\{z \in B(0,1) : |Q(z)| < t^\alpha\}| \leqslant \frac{\gamma}{\log \frac{1}{t}}.$$

(The above notation is for the measure of the set defined by $|Q(z)| < t^\alpha$.)

The important features are: $\frac{\gamma}{\log \frac{1}{t}}$ does not depend on the degree and can be made arbitrarily small by appropriate choice of t.

This lemma is given in [3], p. 88; the proof we give, which is included for the convenience of the reader, is a minor adaptation of the proof given there.

Proof. — Set

$$F_t = \{z \in B(0,1) : |Q(z)| \leqslant t^\alpha\}.$$

By simply counting the number of monomials and using the rough estimate that $|z^J| \leqslant 2^{|J|}$, one gets that for $z \in B(0,2)$

$$|Q(z)| \leqslant (\alpha+1)^n 2^{n\alpha}.$$

If $Q(z) = \sum a_J z^J$, let $K = (k_1, \ldots, k_n)$ be such that $|a_K| = 1$. We have

$$a_K = \sqrt{n}^{|K|} \int Q\left(\frac{e^{i\theta_1}}{\sqrt{n}}, \cdots, \frac{e^{i\theta_n}}{\sqrt{n}}\right) e^{-i(k_1\theta_1 + \cdots + k_n\theta_n)} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi}$$

so

$$\max_{z\in B(0,1)} |Q(z)| \geqslant \frac{1}{(\sqrt{n})^{n\alpha}}.$$

Fix $z^o \in B(0,1)$ such that $|Q(z^o)| \geqslant \frac{1}{n^{\frac{n\alpha}{2}}}$. As $\log|Q|$ is subharmonic, the mean value property on the ball $B(z^o, 1)$ gives

$$\log \frac{1}{n^{\frac{n\alpha}{2}}} \leqslant \frac{1}{|B(0,1)|} \int_{B(z^o,1)} \log|Q|.$$

The estimates given above then yield

$$-\frac{n\alpha}{2}\log n \leqslant \log\bigl((\alpha+1)^n 2^{n\alpha}\bigr) + \frac{1}{|B(0,1)|}\int_{F_t}\log|Q|$$
$$\leqslant \log\bigl[(\alpha+1)^n 2^{n\alpha}\bigr] - \frac{|F_t|}{|B(0,1)|}\,\alpha\log\frac{1}{t}.$$

Thus

$$|F_t| \leqslant \frac{|B(0,1)|\bigl(n\frac{\log(\alpha+1)}{\alpha} + n\log 2 + \frac{n}{2}\log n\bigr)}{\log\frac{1}{t}}.$$

As $\frac{\log(\alpha+1)}{\alpha} \leqslant 1$, we can take

$$\gamma = n|B(0,1)|\left(1 + \log 2 + \frac{\log n}{2}\right).$$

Proof of Theorem 2. — We use the notations of Lemmas 2, 3 and 4.

Fix $\epsilon > 0$ and let $t > 0$ be small enough that $\frac{\gamma}{\log\frac{1}{t}} \leqslant \epsilon$. In Lemma 2, fix $C < 1$. Fix β, $\beta \geqslant \beta_0$, such that $C^{\beta^{1/n}} < t$. For each $\alpha \geqslant 1$, let $P_{\alpha\beta}$ as in Lemma 3. Write

$$P_{\alpha\beta} = \sum_{j=0}^{\beta} Q_j(z)w^j.$$

At least one of the polynomials Q_j must be a normalized polynomial in the n variables $z_1,\dots,z_n$ of degree not more than α in each variable.

Thus the set

$$S_\alpha = \{z \in B(0,1) \subset \mathbb{C}^n : \max_j|Q_j(z)| \leqslant t^\alpha\}$$

has measure at most ϵ.

If $z \in B(0,1)\setminus S_\alpha$ we set

$$P^z_{\alpha\beta}(\cdot) = \frac{P_{\alpha\beta}(\cdot)}{\max_j|Q_j(z)|}.$$

Notice that if $z = (z_1,\dots,z_n)$, then $w \mapsto P^z_{\alpha\beta}(z_1,\dots,z_n,w)$ is a normalized polynomial of degree not more than β in the variable w.

If $z \in B(0,1) \subset \mathbb{C}^n$,

$$\begin{cases} \bullet \quad \text{either } z \in S_\alpha, \quad \text{an exceptional set of measure not more than } \epsilon \\ \bullet \quad \text{or } \sup_{L_1}|P^z_{\alpha\beta}| \leqslant \left(\frac{C^{\beta^{1/n}}}{t}\right)^\alpha. \end{cases}$$

Remember that $\frac{C^{\beta^{1/n}}}{t} < 1$.

The set of z which belong to all but finitely many S_α has measure not more than ϵ; this is the exceptional set E_ϵ in the theorem.

If there are infinitely many α such that $z \notin S_\alpha$, by passing to a subsequence we get a sequence of polynomials R_j in $n+1$ variables such that

$$\begin{cases} \bullet \quad \sup_{L_1} |R_j| \underset{j\to\infty}{\longrightarrow} 0 \\ \bullet \quad G_j(w) = R_j(z,w) \text{ is a normalized polynomial of degree} \\ \quad\ \text{not more than } \alpha, \text{ in one variable, and the sequence } (G_j) \\ \quad\ \text{converges to a normalized polynomial } G. \end{cases}$$

Of course the sequence R_j (thus G_j) depends on z.

By the very definition of polynomial hull we immediately get that if $(z,w) \in \hat{L}_1$, then $G(w) = 0$. So there are at most β points in the fiber of $\hat{L}_1$ above z. (Here β is the k in the statement of the Theorem).

Remarks. — 1) In Theorem 2 it is not true that $\hat{L}_1$ is necessarily finitely sheeted over $\mathbb{C}^n$, so that the exceptional set E_ϵ is, in general, nonempty. This is illustrated by the following simple example. Let $L = \{(e^{i\theta}, \lambda e^{i\theta}) \in \mathbb{C}^2, |\lambda| \leqslant 1\}$. The function $(z_1, z_2) = \frac{z_2}{z_1}$ is holomorphic in a neighborhood of L. Here $L_1 = \{(e^{i\theta}, \lambda e^{i\theta}, \lambda) \in \mathbb{C}^3, |\lambda| \leqslant 1\}$. For every $\lambda \in \mathbb{C}$, such that $|\lambda| \leqslant 1$, $(0,0,\lambda) \in \hat{L}_1$. So the fiber of $\hat{L}_1$ above $(0,0)$ is the unit disk.

2) In an arbitrary complex manifold, good compact sets are defined simply by replacing polynomials by global holomorphic functions. In Stein manifolds all compact sets are good compact sets, but there are examples of complex manifolds where even a compact set consisting of a single point fails to be a good compact set. (See [6].) [6] contains also a related example of indecomposable analytic functional, an analytic functional carried by a compact set not reduced to a point, that cannot be decomposed into the sum of analytic functionals carried by smaller compact sets.

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Manuscrit reçu le 23 juin 1999.

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