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WHEN IS A PSEUDO-DIFFERENTIAL EQUATION SOLVABLE?

by Nicolas LERNER

1. Introduction.

In this introduction, we survey informally the state of the art in matters of solvability for pseudo-differential equations. The second section of the paper contains a technical result which is used in the third section to prove a new solvability result. We start with

The Hans Lewy operator and condition (ψ). — In 1957, Hans Lewy[1] published a paper entitled *An example of a smooth linear partial differential equation without solution* [Lw]. The Hans Lewy operator L_0 is

$$L_0 = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + i(x_1 + ix_2)\frac{\partial}{\partial x_3}. \tag{1.1}$$

One proves that there exists $f \in C^\infty$ such that the equation $L_0 u = f$ has no distribution solution, even locally [2]. Several things were quite shocking about this example. First of all, L_0 has a very simple expression and is not a cooked-up example since this is the Cauchy-Riemann operator on the boundary of the pseudo-convex set

$$\{|z_1|^2 + 2\operatorname{Im} z_2 < 0\}.$$

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(1) Hans Lewy (1904–1988).

(2) In fact H. Lewy proves in his paper that there is no $C^{1,\alpha}$ solution and this version was obtained in 1960 by L. Hörmander [H1].

Morever, the symplectic geometry of the characteristics is amazingly simple: the symbol of $-iL_0$ is

$$\xi_1 + i\xi_2 + i(x_1 + ix_2)\xi_3 = \xi_1 - x_2\xi_3 + i(\xi_2 + x_1\xi_3) \tag{1.2}$$

and the Poisson bracket of the real and imaginary part is

$$\{\xi_1 - x_2\xi_3, \xi_2 + x_1\xi_3\} = 2\xi_3. \tag{1.3}$$

In particular one can note that at $(x;\xi) = (x_1, x_2, x_3; x_2, -x_1, 1)$, the symbol vanishes and the Poisson bracket (1.3) is positive. This implies that the bicharacteristic curve of the real part $\xi_1 - x_2\xi_3$ goes from a region where the imaginary part $\xi_2 + x_1\xi_3$ is negative to a region where it is positive (with a simple zero in this case). Also L_0 is a non vanishing vector field, so no multiple characteristics pathology is to be expected for this operator. It is certainly interesting to notice that this example triggered the efforts of many mathematicians to get an understanding of the geometrical conditions on the symbol of an operator which could be equivalent to the property of local solvability for the quantization of this symbol. We refer the reader to the survey paper [H3] for a historical perspective on this topic.

Let us restrict our attention for a while to smooth non-vanishing complex-valued vector fields, like L_0 is. If the vector field is real-valued and non vanishing, there is no difficulty finding local solutions, using for instance the flow of the vector field. When the vector field is complex-valued, Nirenberg and Treves [NT1] gave a complete geometric explanation for the solvability properties of two-dimensional models studied also by Mizohata [Mi]

$$M_k = D_t + it^k D_x, \qquad N_k = D_t + it^k|D_x|, \qquad k \in \mathbb{N}. \tag{1.4}$$

Note that N_k is a pseudo-differential operator. It turns out that for $k \in \mathbb{N}$, the operators $M_{2k}, N_{2k}, N^*_{2k+1}$ are solvable whereas M_{2k+1}, N_{2k+1} are non-solvable. The geometric explanation was made transparent on these models in [NT2]. The integral curves of the real part are straight lines with direction $\partial/\partial t$ and the imaginary part of the symbol of M_k (resp. N_k) is $t^k\xi$ (resp. $t^k|\xi|$). For fixed ξ, the function $t \mapsto t^k|\xi|$ changes sign from $-$ to $+$ when k is odd which implies that M_{2k+1}, N_{2k+1} are non-solvable. On the other hand, for $M_{2k}, N_{2k}, N^*_{2k+1}$, we may have a change of sign, but not from $-$ to $+$. This led Nirenberg and Treves to define condition (ψ) for p as *the imaginary part of p does not change sign from $-$ to $+$ along the oriented bicharacteristics of the real part of p.* These authors conjectured (this is the Nirenberg-Treves conjecture) that local solvability of a principal-type operator with principal symbol p should be equivalent to condition (ψ) for

p. An elementary way to get a crude understanding of this condition is to look at the ODE–like[3]

$$i(D_t + it|D_x|)^* = \partial_t + t|D_x|.$$

The non-solvability of $D_t + it|D_x|$ is linked to a non-injectivity property of the adjoint operator $\partial_t + t|D_x|$ which indeed has a non-trivial kernel, given by the inverse Fourier transform of

$$e^{-t^2|\xi|/2}.$$

Note that for $\partial_t - t|D_x|$ it would not be possible to construct such a solution since one would have to take the inverse Fourier transform of $e^{+t^2|\xi|/2}$. A more refined intuitive approach is linked to propagation of singularities for the adjoint operator P^* with P satisfying condition (ψ). Whenever this condition is satisfied, following the integral curves of $\operatorname{Re} p = \operatorname{Re} \bar{p}$, we can get from negative values of $-\operatorname{Im} p = \operatorname{Im} \bar{p}$ to positive values of $\operatorname{Im} \bar{p}$. Since the open set $\{\operatorname{Im} \bar{p} < 0\}$ is a *backward* region for propagation and $\{\operatorname{Im} \bar{p} > 0\}$ is a *forward* region, the singularities should go away. On the other hand, if following the integral curve of $\operatorname{Re} p$, one can get from the forward $\{\operatorname{Im} \bar{p} > 0\}$ to the backward $\{\operatorname{Im} \bar{p} < 0\}$, we obtain a trapped singularity, which will provide an (approximate) solution of $P^*u = 0$, triggering non-solvability for P.

Main results on the Nirenberg-Treves conjecture.

• The necessity of condition (ψ) for solvability was proved by Moyer in two dimensions and by Hörmander in the general case (see [H2], section 26.4).

• Condition (ψ) is sufficient for solvability of principal type *differential* operators. This result was proved by Nirenberg and Treves [NT2] under an analyticity assumption which was removed by Beals and Fefferman [BF]. In fact for differential operators of order m with principal symbol p, since

$$p(x, -\xi) = (-1)^m p(x, \xi), \tag{1.5}$$

condition (ψ) for p is equivalent to condition (P): *the imaginary part of p does not change sign along the bicharacteristics of the real part of p.*

[3] One should take a look at the models (1.13-14) below to get an actual PDE behaviour. In fact, the operators (1.4) provide useful but oversimplified examples. The same remark holds for subellipticity: the operators N_k are easily shown subelliptic with loss of $k/(k+1)$ derivatives whereas it is more difficult to tackle the models

$$D_t + it^{2p}(D_{x_1} + t^{2q+1}x_1^{2r}|D_x|),$$

where p, q, r are non-negative integers, precisely because they actually correspond to a PDE, here the vector field with large parameter Λ, $D_t + it^{2p}(D_{x_1} + t^{2q+1}x_1^{2r}\Lambda)$.

Condition (ψ) was proved sufficient for solvability in two dimensions in [L1] (see also [S]) and for the classical oblique-derivative problem ([L2]).

• The Nirenberg-Treves conjecture is proven true for differential operators and in two dimensions, but the sufficiency of condition (ψ) for solvability of pseudo-differential equations in three or more dimensions is an open problem. We describe below the reasons for which condition (ψ) does not imply the "natural" estimates (1.6-7).

Counting the loss of derivatives and optimal solvability. — We shall say that the equation $Lu = f$, where L is a pseudo-differential operator of order m is locally solvable with a loss of μ derivatives if for $f \in H^s_{\text{loc}}$, one can find a solution $u \in H^{s+m-\mu}_{\text{loc}}$. The loss is 0 iff the operator is elliptic. The "optimal" loss is 1 since this is the loss for a real-valued non-vanishing vector field: the equation $\partial u/\partial x_1 = f$ has a (local) solution $u \in L^2 = H^{0+1-1}$ for $f \in H^0 = L^2$. In fact, when L satisfies condition (P) and is of order m, the adjoint operator[(4)] satisfies the estimate with optimal loss 1,

$$\|L^*u\|_{H^s} \geq C\,\|u\|_{H^{s+m-1}}\,, \tag{1.6}$$

where u is smooth compactly supported, with a small support, C is a "large" constant. The existence of smooth solutions for smooth right-hand-sides, as well as semi-global existence results for operators satisfying condition (P) were obtained by Hörmander (see [H2], chapter 26). Moreover, the known semi-global existence theorems only give existence with loss of $1+\epsilon$ derivatives for all $\epsilon > 0$.

Going back to (1.6), if we consider a first-order pseudo-differential operator satisfying condition (P) the following estimate is true, for u smooth compactly supported, with a small support:

$$\|L^*u\|_{L^2} \geq C\,\|u\|_{L^2}\,. \tag{1.7}$$

Since the constant C could be made arbitrarily large by shrinking the support of u, the estimate (1.7) is rather natural in the sense that one can absorb without difficulty L^2 bounded perturbation of L in the right-hand side of (1.7). In particular, one should not expect lower order terms to play any role in the solvability of a principal-type operator. The hopes of getting estimate (1.7) assuming condition (ψ) for L were ruined by the paper [L3], in which it is proved that one can find L satisfying condition (ψ) such that

(4) Since then L^* also satisfies condition (P), (1.6) is true for L as well.

(1.7) is not true (see also Section 6 in [H3]). This means for this example that one can find $f \in L^2$ such that

$$Lu = f$$

has no local solution u in L^2. However, Dencker [D1] proved that for this example, one can find a solution in H^{-1}, that is loosing two derivatives. Finally solvability holds for this example but not optimal solvability with loss of one derivative as it is true under condition (P) and in many other cases ([L4], [L5], [H3]).

Miscellaneous results. — Let us consider operators L such that(5)

$$L^* = D_t + iq(t, x, D_x) \tag{1.8}$$

with L satisfying condition (ψ), i.e. q is a first-order real-valued symbol such that

$$q(t, x, \xi) > 0 \text{ and } s > t \quad \text{imply} \quad q(s, x, \xi) \geq 0. \tag{1.9}$$

The estimate (1.7) is actually true ([L4], [L5]) under the following additional assumption (assuming q is first-order homogeneous): there exists a constant C such that

$$|\delta q|^2 = |\xi|^{-1}|\nabla_x q|^2 + |\xi||\nabla_\xi q|^2 \leq C q'_t \quad \text{at } q = 0. \tag{1.10}$$

It means that the regular points of $\{q = 0\}$, that is the points at which $\delta q \neq 0$ are transversal for $\partial/\partial t$. In fact one can check directly that (1.10) is violated for the counterexamples in [L3] as well as generically for operators satisfying condition (P) such as degenerate Cauchy-Riemann operators e.g.

$$D_t + ia_0(t, x, D_x)D_{x_1}, \tag{1.11}$$

where a_0 is non-negative of order 0. On the other hand, if (1.7) is not satisfied, the set on which q changes sign should be symplectically non trivial: whenever in the above setting, for all (t, x, y, ξ), one has

$$q(t, x, \xi)q(t, y, \xi) \geq 0, \tag{1.12}$$

it was proved in [H3] (Theorem 8.4) that the estimate (1.7) holds. This result generalizes the theorem in [L2] for the oblique derivative problem. Condition (1.12) is satisfied for operators such as (1.11) and thus appears as some kind of extension for condition (P). However as soon as changes of sign occur on a symplectic plane as it is the case for

$$D_t + ia_0(t, x, D_x)(D_{x_1} + tx_1^2|D_x|) \tag{1.13}$$

(5) It is in fact the general case, using the Malgrange preparation theorem and the Egorov theorem on quantization of canonical transformations.

where a_0 is non-negative of order 0, condition (1.12) is violated. As a matter of fact, the example in [L3] (for which (1.7) is violated) has a structure quite close to the previous operator, and can be written as

(1.14)

$$D_t + ia_0(t,x,D_x)(D_{x_1} + V(t,x)|D_x|), \text{ with } \partial V/\partial t \geq 0,\ a_0(t,x,\xi) \geq 0.$$

Summing up, one can say that transversal changes of sign related to condition (1.10) are too far from condition (P) and that condition (1.12) stays symplectically too close to condition (P). So it is a significant difficulty to find a condition weaker than condition (P) and accepting operators such as (1.14). Of course, it is then natural to expect a weaker estimate than (1.7), since the latter does not hold for (1.14). A precise result in this direction will be given in Section 2 (see also [D3]). We shall see in particular that operators (1.14) are locally solvable with loss of two derivatives. Even for the simpler models (1.13), no better result seems available, if no additional assumption is made on the non-negative symbol a_0.

A perturbative result is given in [L5]: if L is a first-order pseudo-differential operator which satisfies condition (ψ), there exists an L^2 bounded perturbation R such that $L + R$ is solvable with loss of two derivatives. This result was proved using a factorization theorem for operators satisfying condition (ψ). We could use this result of factorization to get sufficiency of condition (ψ) for solvability of the general "two-step" case. The operator L is solvable (with loss of 3/2 derivatives) whenever

$$L^* = D_t + iQ(t),\quad Q(t) = q_1(x,D_x) \text{ for } t < 0,\quad Q(t) = q_2(x,D_x) \text{ for } t > 0,$$

where q_j are first-order real-valued symbols such that L satisfies condition (ψ) which should mean here that

$$q_1(x,\xi) > 0 \ \text{ implies } \ q_2(x,\xi) \geq 0.$$

The rather technical details on this matter will be given elsewhere.

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2. Factorization and the energy method.

Let us give some notations before proceeding with our statements.

DEFINITION 2.1. — *Let n be an integer and $m \in \mathbb{R}$. A function*

$$a : E = \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi \times [1,\infty) \longrightarrow \mathbb{C}$$

belongs to S^m *if it is measurable with respect to* t*, smooth with respect to the variables* x, ξ *and such that for each integer* k

$$\gamma_k(a) = \sup_{E, |\alpha|+|\beta| \le k} |D_x^\alpha D_\xi^\beta a(t, x, \xi, \Lambda)| \Lambda^{-m+\frac{|\alpha|+|\beta|}{2}} < +\infty. \tag{2.1}$$

Since the dimension n is fixed throughout the section, we omit it everywhere. The elements of S^m will be called symbols of order m. Moreover, we shall also omit most of the time the dependence of the symbols on the large parameter Λ, and refer to the constants $\gamma_k(a)$ as the semi-norms of a. We recall also that the Weyl quantization associates to a symbol $a(t, x, \xi, \Lambda) \in S^m$ the $\mathcal{L}(L^2(\mathbb{R}^n))$ operator given by $v \in L^2(\mathbb{R}^n) \mapsto a(t)^w v$, with

$$\big(a(t)^w v\big)(x) = \iint e^{2i\pi(x-y)\cdot\xi} a\Big(t, \frac{x+y}{2}, \xi, \Lambda\Big) v(y) dy d\xi. \tag{2.2}$$

Of course there the $\mathcal{L}(L^2(\mathbb{R}^n))$ norm is bounded by $\gamma_{k_n}(a) \times \Lambda^m$, $k_n = [\frac{n}{2}] + 1$. Our main result for this section is the following

THEOREM 2.2. — *Let* a_0 *be a symbol of order 0 taking non-negative values,* b_1 *in* S^1*. We assume that* $t \mapsto b_1(t, x, \xi)$ *is real-valued non-decreasing for each* $(x, \xi) \in \mathbb{R}^{2n}$*. Let* r_0 *be a (complex-valued) symbol of order 0. Then, there exists* $C_0, T_0 > 0$ *depending only on the semi-norms of* a_0, b_1, r_0 *such that for any* $u \in C_0^{0,1}(\mathbb{R}_t, L^2(\mathbb{R}^n))$, $\operatorname{supp} u \subset \{|t| \le T_0\}$,

$$C_0 \|D_t u + i(a_0 b_1)(t)^w u(t) + i r_0(t)^w u(t)\|_{L^1(\mathbb{R}_t, L^2(\mathbb{R}^n_x))} \ge \Lambda^{-1} \|u\|_{L^\infty(\mathbb{R}_t, L^2(\mathbb{R}^n_x))}, \tag{2.3}$$

where $D_t = -i\partial/\partial t$.

It should be emphasized that we do not require any smoothness in the t-variable for the symbols involved in this theorem. It gives a proof of solvability with a loss of two derivatives for a class of operators satisfying condition (ψ). In fact, (1.9) is obviously satisfied by $q = a_0 b_1$ since $a_0 \ge 0$ and $\partial b_1/\partial t \ge 0$. In particular, it contains the counterexamples of [L3] and is a generalization of Lemma 5.2 in [D1]. It is interesting to notice that it contains somewhat naturally operators satisfying condition (P) (requiring $q(t, x, \xi) q(s, x, \xi) \ge 0$ instead of (1.9) above): it was shown by Beals and Fefferman in [BF] that a non homogeneous microlocalization led to such a factorization (with b_1 independent of t), where the smoothness in the t-variable was lost along the way (in fact, after this microlocalization no new control of the t-derivative of the symbol q was available). We prove next a Hilbertian lemma.

LEMMA 2.3. — *Let $\mathbb{H}$ be a Hilbert space, $A_0(t), B_1(t), N_0(t)$ bounded operators in $\mathbb{H}$, depending measurably on a real parameter t and on $\Lambda \geq 1$. We assume that there exist non negative numbers $(\gamma_j)_{1\leq j\leq 6}$ such that for any $\Lambda \geq 1$ and any real t,*

$$\text{(2.4)}\quad \operatorname{Re} A_0(t)+\gamma_1\Lambda^{-2} \geq 0,\ B_1(t) = B_1^*(t),\ \dot{B}_1(t)+\gamma_2 \geq 0,\ N_0(t) = N_0^*(t),$$

$$\text{(2.5)}\qquad \|\,[\operatorname{Im} A_0(t), B_1(t)]\,\| \leq \gamma_3,\ \|A_0(t)\| \leq \gamma_4,\ \|B_1(t)\| \leq \gamma_5\Lambda,\quad \|\,[N_0(t), B_1(t)]\,\| \leq \gamma_6,$$

where $\| \ \|$ stands for the $\mathcal{L}(\mathbb{H})$ operator-norm, $\operatorname{Re} Q = (Q+Q^*)/2$, $\operatorname{Im} Q = (Q-Q^*)/(2i)$, $[,]$ is the commutator, $\dot{B}_1(t)$ is the weak derivative. Then there exists a constant C_0 and a positive constant T_0 (depending only on the $(\gamma_j)_{1\leq j\leq 6}$) such that for any $u \in C_0^{0,1}(\mathbb{R}, \mathbb{H})$ with $\operatorname{supp} u \subset \{|t| \leq T_0\}$,

$$\text{(2.6)}\qquad C_0 \|D_t u + N_0(t)u + iA_0(t)B_1(t)u\|_{L^1(\mathbb{R},\mathbb{H})} \geq \Lambda^{-1} \|u\|_{L^\infty(\mathbb{R},\mathbb{H})}.$$

Proof. — We compute the $L^2(\mathbb{R}, \mathbb{H})$ dot products, Y standing for the Heaviside function (characteristic function of $\mathbb{R}_+$),

$$\begin{aligned}
&2\operatorname{Re}\langle D_t u + N_0(t)u + iA_0(t)B_1(t)u, iB_1(t)u + iY(t-T)u\rangle \\
&= \langle \dot{B}_1 u, u\rangle + |u(T)|^2_{\mathbb{H}} + \langle i[N_0(t), B_1(t)]u, u\rangle + 2\langle B_1(t)\operatorname{Re} A_0(t)B_1(t)u, u\rangle \\
&\quad + 2\langle Y(t-T)\operatorname{Re}\big(A_0(t)B_1(t)\big)u, u\rangle \\
&= \langle (\dot{B}_1 + \gamma_2)u, u\rangle + |u(T)|^2_{\mathbb{H}} + \langle (i[N_0(t), B_1(t)] + \gamma_6)u, u\rangle \\
&\quad + 2\langle B_1(t)(\operatorname{Re} A_0(t) + \gamma_1\Lambda^{-2})B_1(t)u, u\rangle \\
&\quad + \Big\langle Y(t-T)\Big((\operatorname{Re} A_0(t))B_1(t) + B_1(t)\operatorname{Re} A_0(t)\Big)u, u\Big\rangle \\
&\quad - \big((\gamma_2+\gamma_6)\|u\|^2_{L^2(\mathbb{R},\mathbb{H})} + 2\gamma_1\Lambda^{-2}\|B_1 u\|^2_{L^2(\mathbb{R},\mathbb{H})}\big) \\
&\quad + \langle Y(t-T)\big[i\operatorname{Im} A_0(t), B_1(t)\big]u, u\rangle.
\end{aligned}$$

Let us check, with $A = \operatorname{Re} A_0 + \gamma_1\Lambda^{-2} \geq 0$,

$$\begin{aligned}
&2B_1AB_1 + Y(\operatorname{Re} A_0)B_1 + YB_1\operatorname{Re} A_0 \\
&= 2B_1AB_1 + YAB_1 + YB_1A - 2Y\gamma_1\Lambda^{-2}B_1 \\
&= (B_1A^{1/2}2^{1/2} + 2^{-1/2}YA^{1/2})(2^{1/2}A^{1/2}B_1 + 2^{-1/2}YA^{1/2}) \\
&\quad - \frac{1}{2}YA - 2Y\gamma_1\Lambda^{-2}B_1 \geq -\frac{1}{2}YA - 2Y\gamma_1\Lambda^{-2}B_1.
\end{aligned}$$

From the assumptions of the lemma and the previous identities, one gets

$$\begin{aligned}
&2\,\|D_t u + N_0(t)u + iA_0(t)B_1(t)u\|_{L^1(\mathbb{R},\mathbb{H})}\,(\gamma_5\Lambda+1)\,\|u\|_{L^\infty(\mathbb{R},\mathbb{H})}\\
&\geq 2\mathrm{Re}\langle D_t u + N_0(t)u + iA_0(t)B_1(t)u, iB_1(t)u + iY(t-T)u\rangle\\
&\geq \|u\|^2_{L^\infty(\mathbb{R},\mathbb{H})} - \|u\|^2_{L^2(\mathbb{R},\mathbb{H})}\,(\gamma_2+\gamma_6+2\gamma_1\gamma_5^2+\gamma_3+\frac{\gamma_4+\gamma_1\Lambda^{-2}}{2}\\
&\quad + 2\gamma_1\gamma_5\Lambda^{-1})\\
&\geq \|u\|^2_{L^\infty(\mathbb{R},\mathbb{H})}\Big(1-2T_0(\gamma_2+\gamma_6+2\gamma_1\gamma_5^2+\gamma_3+\frac{\gamma_4+\gamma_1}{2}+2\gamma_1\gamma_5)\Big).
\end{aligned}$$

We can then choose T_0 positive small enough to get

$$1-2T_0(\gamma_2+\gamma_6+2\gamma_1\gamma_5^2+\gamma_3+\frac{\gamma_4+\gamma_1}{2}+2\gamma_1\gamma_5)\geq 1/2,$$

yielding $C_0 = 2(\gamma_5+1)$ in (2.6). The proof of the lemma is complete. □

We need now to recall some facts on the so-called Wick quantization, as used in [L4], [L5].

DEFINITION 2.4. — *Let $Y=(y,\eta)$ be a point in $\mathbb{R}^{2n}$. The operator Σ_Y is defined as $\big[2^n e^{-2\pi|\cdot-Y|^2}\big]^w$. This is a rank-one orthogonal projection: $\Sigma_Y u = (Wu)(Y)\tau_Y\varphi$ with $(Wu)(Y) = \langle u,\tau_Y\varphi\rangle_{L^2(\mathbb{R}^n)}$, where $\varphi(x) = 2^{n/4}e^{-\pi|x|^2}$ and*

$$(\tau_{y,\eta}\varphi)(x) = \varphi(x-y)e^{2i\pi\langle x-\frac{y}{2},\eta\rangle}.$$

Let a be in $L^\infty(\mathbb{R}^{2n})$. The Wick quantization of a is defined as

$$a^{\mathrm{Wick}} = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_Y dY. \tag{2.7}$$

The following proposition is classical and easy (see e.g. [L4], [L5]).

PROPOSITION 2.5. — *Let a be in $L^\infty(\mathbb{R}^{2n})$. Then $a^{\mathrm{Wick}} = W^* a^\mu W$ and $1^{\mathrm{Wick}} = Id_{L^2(\mathbb{R}^n)}$ where W is the isometric mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ given above, and a^μ the operator of multiplication by a in $L^2(\mathbb{R}^{2n})$. The operator $\pi_H = WW^*$ is the orthogonal projection on a closed proper subspace H of $L^2(\mathbb{R}^{2n})$. Moreover, we have*

$$\|a^{\mathrm{Wick}}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \|a\|_{L^\infty(\mathbb{R}^{2n})} \text{ and } a(X)\geq 0 \text{ for all } X \text{ implies } a^{\mathrm{Wick}}\geq 0. \tag{2.8}$$

PROPOSITION 2.6. — *Let m be a real number and p be a symbol in S^m (see Definition 2.1). Then $p^{\mathrm{Wick}} = p^w + r(p)^w$, with $r(p)\in S^{m-1}$ so that the mapping $p\mapsto r(p)$ is continuous. Moreover, $r(p)=0$ if p is a linear form or a constant.*

Proof. — We omit the dependence on t and Λ of the symbols. From the definition above, one has $p^{\text{Wick}} = \tilde{p}^w$, with

$$(2.9)\quad \tilde{p}(X) = \int_{\mathbb{R}^{2n}} p(X+Y)\, e^{-2\pi|Y|^2} 2^n dY$$
$$= p(X) + \underbrace{\int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta) p''(X+\theta Y) Y^2 e^{-2\pi|Y|^2} 2^n dY d\theta}_{r(p)(X)}.$$

Thus we get from the estimates on p that,

$$|r(p)^{(k)}(X)| \le \gamma_{k+2}(p) \Lambda^{m-\frac{k+2}{2}} \int_{\mathbb{R}^{2n}} |Y|^2 e^{-2\pi|Y|^2} 2^{n-1} dY,$$

which implies $r \in S^{m-1}$. The last point in the proposition follows from the formula (2.9) showing that $r(p)$ depends linearly on p''. □

We are now able to prove Theorem 2.2. Using the notations of this theorem and of Proposition 2.6, we look at

$$(2.10)\quad (a_0 b_1)^w + r_0^w = a_0^w b_1^w + s_0^w + r_0^w$$
$$= \left(a_0^{\text{Wick}} - r(a_0)^w\right)\left(b_1^{\text{Wick}} - r(b_1)^w\right) + s_0^w + r_0^w$$
$$= a_0^{\text{Wick}} b_1^{\text{Wick}} + r^w,$$

where $s_0, r_0, r(b_1) \in S^0$ and $r(a_0) \in S^{-1}$ so that $r \in S^0$, with semi-norms depending only on those of a_0, b_1, r_0. To apply Lemma 2.3 we need only to get rid of the real part of r. We shall conjugate D_t by a suitable operator to obtain this, up to operators of order -1. We set, with r defined by (2.10),

$$(2.11)\qquad u_0(t,X) = \exp\Big(-\int_0^t \operatorname{Re} r(s,X) ds\Big).$$

The symbols $u_0, 1/u_0$ belong to S^0 locally in t (i.e. (2.1) is satisfied for t in a bounded set) with semi-norms depending only on those of r, that is in fact on semi-norms of a_0, b_1, r_0. We calculate

$$(2.12)\qquad (u_0)^w D_t\ (1/u_0)^w = (u_0)^w (1/u_0)^w D_t - i(u_0)^w \Big(\frac{\operatorname{Re} r}{u_0}\Big)^w$$
$$= (u_0)^w (1/u_0)^w D_t - i(\operatorname{Re} r)^w + \omega_{-1}^w,$$

where $\omega_{-1} \in S^{-1}$, with semi-norms depending only on those of a_0, b_1, r_0. We obtain from (2.10), (2.12), with $\equiv$ standing for equality modulo operators with symbols in S^{-1} with semi-norms depending only on those of a_0, b_1, r_0,

$$(2.13)\quad (u_0)^w \Big(D_t + i(a_0 b_1)^w + i r_0^w\Big)\ (1/u_0)^w$$

$$
\begin{aligned}
&= (u_0)^w\Big(D_t + ia_0^{\text{Wick}}b_1^{\text{Wick}} + ir^w\Big)\ (1/u_0)^w\\
&= (u_0)^w(1/u_0)^w D_t - i(\operatorname{Re} r)^w + \omega_{-1}^w + (u_0)^w\\
&\quad \Big(ia_0^{\text{Wick}}b_1^{\text{Wick}} + i(\operatorname{Re} r)^w - (\operatorname{Im} r)^w\Big)(1/u_0)^w\\
&\equiv (u_0)^w(1/u_0)^w D_t - (\operatorname{Im} r)^w + \big[u_0^w, ia_0^{\text{Wick}}\big]\ b_1^{\text{Wick}}\ (1/u_0)^w\\
&\quad + a_0^{\text{Wick}}\ \big[u_0^w, ib_1^{\text{Wick}}\big]\ (1/u_0)^w + ia_0^{\text{Wick}}b_1^{\text{Wick}}u_0^w(1/u_0)^w.
\end{aligned}
$$

Let us now recall the following simple formula from the Weyl symbolic calculus: for $p_1, p_2 \in S^{m_1}, S^{m_2}$, one has

$$
(2.14)\quad p_1^w p_2^w = (p_1 \sharp p_2)^w = \Big(p_1 p_2 + \frac{1}{4i\pi}\{p_1, p_2\} + \rho\Big)^w, \quad \rho \in S^{m_1+m_2-2}.
$$

Since the Poisson bracket $\{u_0, u_0^{-1}\} = 0$, we get

$$
(2.15)\qquad u_0^w(1/u_0)^w = \text{Id} + \Lambda^{-2}\omega_0^w, \quad \text{with } \omega_0 \in S^0.
$$

In fact, with $U_0(t) = \big(u_0(t)\big)^w, V_0(t) = \big(1/u_0(t)\big)^w$ we have

$$
U_0(t)V_0(t) = \text{Id} + \Lambda^{-2}\Omega_0(t), \quad V_0(t)U_0(t) = \text{Id} + \Lambda^{-2}\Omega_0(t)^*,
$$

where the $\mathcal{L}(L^2(\mathbb{R}^n))$ norm of $\Omega_0(t)$ is estimated by semi-norms of a_0, b_1, r_0. Consequently, there exists Λ_0 depending only on the semi-norms of a_0, b_1, r_0, such that for $\Lambda \geq \Lambda_0$, $U_0(t), V_0(t)$ are (bounded) invertible operators with norms of the inverse operators controlled by the semi-norms of a_0, b_1, r_0. Since (2.3) is obvious for bounded Λ, we can assume from now on that $\Lambda \geq \Lambda_0$. Moreover from Proposition 2.6 we get

$$
\big[u_0^w, ia_0^{\text{Wick}}\big]\ b_1^{\text{Wick}}\ (1/u_0)^w \equiv \big(\underbrace{\{u_0, a_0\}b_1/2\pi u_0}_{\text{real-valued}}\big)^w,
$$

$$
a_0^{\text{Wick}}\ \big[u_0^w, ib_1^{\text{Wick}}\big]\ (1/u_0)^w \equiv \big(\underbrace{\{u_0, b_1\}a_0/2\pi u_0}_{\text{real-valued}}\big)^w.
$$

This implies from (2.13), (2.15)

$$
\begin{aligned}
(2.16)\qquad &(u_0)^w\Big(D_t + i(a_0b_1)^w + ir_0^w\Big)\ (1/u_0)^w\\
&\equiv (u_0)^w(1/u_0)^w D_t + \rho_0^w + ia_0^{\text{Wick}}b_1^{\text{Wick}},\\
&\equiv (\text{Id} + \Lambda^{-2}\omega_0^w)\Big(D_t + \rho_0^w + ia_0^{\text{Wick}}b_1^{\text{Wick}}\Big)
\end{aligned}
$$

with $\rho_0 \in S^0$ and real-valued. We can now apply Lemma 2.3 with $\mathbb{H} = \mathbb{L}^{\not{\kappa}}(\mathbb{R}^{\kappa})$, and with

$$
A_0(t) = a_0(t)^{\text{Wick}} = A_0(t)^* \geq 0
$$

from (2.8) since $a_0(t, X) \geq 0$: one can take $\gamma_1{=}0$. We take

$$
B_1(t) = b_1(t)^{\text{Wick}} = B_1^*(t),
$$

since b_1 is real-valued. Moreover, from (2.8) and the fact that $b_1(t,X)$ is non-decreasing as a function of t for fixed X, we get that

$$(B_1(t)-B_1(s))(t-s)\geq 0,$$

so we can choose $\gamma_2=0$. We set $N_0(t)=\rho_0(t)^w$ which is self-adjoint since ρ_0 is real-valued. Going on to the line (2.5), we can take $\gamma_3=0$ since A_0 is self-adjoint. Now γ_4,γ_5 can be chosen respectively as semi-norms of the symbols a_0,b_1. Eventually, the commutator $\left[\rho_0^w, b_1^{\mathrm{Wick}}\right]$ is an operator with a symbol of order 0, whose semi-norms are controlled by those of a_0,b_1,r_0. From Lemma 2.3 and (2.16) we get the estimate

$$\text{(2.17)}\quad C_1\Lambda^{-1}\|u\|_{L^1(\mathbb{R},\mathbb{H})}+C_0\left\|(\mathrm{Id}+\Lambda^{-2}\omega_0^w)^{-1}(u_0)^w\Big(D_t+i(a_0b_1)^w+ir_0^w\Big)(1/u_0)^w u\right\|_{L^1(\mathbb{R},\mathbb{H})}\geq\Lambda^{-1}\|u\|_{L^\infty(\mathbb{R},\mathbb{H})},$$

where C_1 is a constant depending only on the semi-norms of a_0,b_1,r_0. Shrinking the T_0 given by Lemma 2.3, one can absorb the term $C_1\Lambda^{-1}\|u\|_{L^1(\mathbb{R},\mathbb{H})}$ in the right-hand-side. To reach the conclusion of Theorem 2.2, one just needs to look back at (2.15), which ensures that u_0^w, $(1/u_0)^w$ and $(\mathrm{Id}+\Lambda^{-2}\omega_0^w)$ are invertible operators, provided Λ is large enough with respect to a finite number of semi-norms of a_0,b_1,r_0. Then using (2.17) with $u(t)$ replaced by $V_0(t)^{-1}u(t)$, we get the result. The proof of Theorem 2.2 is complete.

3. Patching together weak estimates.

We consider the phase space $\mathbb{R}^{2n}=\mathbb{R}^n_x\times\mathbb{R}^n_\xi$ equipped with its canonical symplectic structure $\sigma=\sum_{1\leq j\leq n}d\xi_j\wedge dx_j$. For a positive definite quadratic form g on $\mathbb{R}^{2n}$ we define the symplectic inverse g^σ of g as

$$g^\sigma(T)=\sup_{g(U)=1}\sigma(T,U)^2.$$

DEFINITION 3.1. — *Let $X\in\mathbb{R}^{2n}\mapsto g_X$ be a mapping from $\mathbb{R}^{2n}$ to the set of positive definite quadratic forms on $\mathbb{R}^{2n}$. The metric g is said to be admissible if for each $X\in\mathbb{R}^{2n}$, $g_X\leq g_X^\sigma$, and if g is slowly varying and temperate: there exists $C>0$ and $N\geq 0$, such that for all $X,Y,T\in\mathbb{R}^{2n}$,*

$$g_X(Y-X)\leq C^{-1}\Longrightarrow C^{-1}\leq g_X(T)/g_Y(T)\leq C,$$

$$g_X(T)/g_Y(T)\leq C\big(1+g_X^\sigma(X-Y)\big)^N.$$

A positive function m defined on $\mathbb{R}^{2n}$ will be called a weight if there exists N' such that

$$\sup_{X,Y} m(X)m(Y)^{-1}\big(1+g_X^\sigma(X-Y)\big)^{-N'} < \infty$$

and if there exists a positive C such that

$$g_X(Y-X) \le C^{-1} \Longrightarrow C^{-1} \le m(X)/m(Y) \le C.$$

The class of symbols $S(m,g)$ is defined as the functions $a \in C^\infty(\mathbb{R}^{2n})$ such that for all integers k, $\sup_X \|a^{(k)}(X)\|_{g_X} m(X)^{-1} < \infty$. These quantities are called the semi-norms of the symbol a.

We refer the reader to chapter 18 in [H2] for basic properties of such metrics as well as for the following lemma.

LEMMA 3.2. — *Let g be an admissible metric on $\mathbb{R}^{2n}$. There exists a sequence $(X_\nu)_{\nu\in\mathbb{N}}$ of points in the phase space $\mathbb{R}^{2n}$ and positive numbers r_0, N_0, such that the following properties are satisfied. We define $U_\nu, U_\nu^*, U_\nu^{**}$ as the $g_\nu = g_{X_\nu}$ balls with center X_ν and radius $r_0, 2r_0, 4r_0$. There exist two families of nonnegative smooth functions on $\mathbb{R}^{2n}$, $(\chi_\nu)_{\nu\in\mathbb{N}}$, $(\psi_\nu)_{\nu\in\mathbb{N}}$ such that*

$$\sum_\nu \chi_\nu(X) = 1,\ \operatorname{supp}\chi_\nu \subset U_\nu, \quad \psi_\nu \equiv 1 \ \text{ on } U_\nu^*,\ \operatorname{supp}\psi_\nu \subset U_\nu^{**}.$$

Moreover, $\chi_\nu, \psi_\nu \in S(1, g_\nu)$ with semi-norms bounded independently of ν. The overlap of the balls U_ν^{**} is bounded, i.e.

$$\bigcap_{\nu\in\mathcal{N}} U_\nu^{**} \neq \emptyset \implies \#\mathcal{N} \le N_0.$$

Moreover, $g_X \sim g_\nu$ all over U_ν^{**} (i.e. the ratios $g_X(T)/g_\nu(T)$ are bounded above and below by a fixed constant, provided that $X \in U_\nu^{**}$).

The next lemma in proved in [BC] (see also [L6]).

LEMMA 3.3. — *Let g be an admissible metric on $\mathbb{R}^{2n}$ and $\sum_\nu \chi_\nu(x,\xi) = 1$ be a partition of unity related to g as in the previous lemma. There exists a positive constant C such that for all $u \in L^2(\mathbb{R}^n)$*

$$C^{-1}\|u\|^2_{L^2(\mathbb{R}^n)} \le \sum_\nu \|\chi_\nu^w u\|^2_{L^2(\mathbb{R}^n)} \le C\|u\|^2_{L^2(\mathbb{R}^n)},$$

where a^w stands for the Weyl quantization of the symbol a.

Let us consider now $q(t,x,\xi)$ a real-valued continuous function defined on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ satisfying (1.9) and such that $q(t,\cdot,\cdot) \in S(\lambda, g)$ where g is an admissible metric and

$$\lambda(X) = \inf_T \big(g_X^\sigma(T)/g_X(T)\big)^{1/2}. \tag{3.1}$$

We set, using the notations of the previous lemmas,

$$Q(t) = q(t)^w, \qquad Q_\nu(t) = (\psi_\nu q(t))^w.$$

We assume that we have a weak energy inequality and more precisely that for all ν, there exists an operator $M_\nu(t)$ with a real-valued Weyl symbol uniformly[6] in $S(\lambda_\nu = \lambda(X_\nu), g_\nu)$ such that [7] with a uniform C,

$$\dot{M}_\nu(t) + Q_\nu(t)M_\nu(t) + M_\nu(t)Q_\nu(t) \geq \delta(t-T) - C. \tag{3.2}$$

We consider now

$$M(t) = \sum_\nu \chi_\nu^w M_\nu(t)\chi_\nu^w. \tag{3.3}$$

From the symbolic calculus, we get that $M(t)$ is an operator with Weyl symbol in $S(\lambda, g)$. We obtain with $Q_\nu(t) = (\psi_\nu q(t))^w$ and R with symbol in $S(\lambda^{-\infty}, g) = \cap_N S(\lambda^{-N}, g)$, the identities

$$\begin{aligned}
&\dot{M} + Q(t)M(t) + M(t)Q(t) \\
&= \sum_\nu \chi_\nu^w \dot{M}_\nu(t)\chi_\nu^w + Q(t)\chi_\nu^w M_\nu(t)\chi_\nu^w + \chi_\nu^w M_\nu(t)\chi_\nu^w Q(t) \\
&= \sum_\nu \chi_\nu^w \dot{M}_\nu(t)\chi_\nu^w + (\psi_\nu q(t))^w \chi_\nu^w M_\nu(t)\chi_\nu^w \\
&\quad + \chi_\nu^w M_\nu(t)\chi_\nu^w(\psi_\nu q(t))^w + R \\
&= \sum_\nu \chi_\nu^w \Big\{\dot{M}_\nu(t) + Q_\nu(t)M_\nu(t) + M_\nu(t)Q_\nu(t)\Big\}\chi_\nu^w \\
&\quad + \sum_\nu [Q_\nu(t), \chi_\nu^w]M_\nu(t)\chi_\nu^w - \chi_\nu^w M_\nu(t)[Q_\nu(t), \chi_\nu^w] + R.
\end{aligned} \tag{3.4}$$

At this point we notice that the principal symbol of the first-order operator

$$[Q_\nu(t), \chi_\nu^w]M_\nu(t)\chi_\nu^w - \chi_\nu^w M_\nu(t)[Q_\nu(t), \chi_\nu^w] \tag{3.5}$$

actually vanishes, which makes it an operator with symbol in $S(1, g)$. Eventually, we get from (3.4) that

$$\dot{M} + Q(t)M(t) + M(t)Q(t) \geq r_0^w + \sum_\nu \chi_\nu^w \delta(t-T)\chi_\nu^w,$$

with $r_0 \in S(1, g)$. Consequently we have with a fixed constant C_0

$$\begin{aligned}
&2\operatorname{Re}\langle D_t u + iQ(t)u(t), iM(t)u(t)\rangle \\
&\geq -C_0 \int \|u(t)\|^2_{L^2(\mathbb{R}^n)} dt + \sum_\nu \|\chi_\nu^w u(T)\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{3.6}$$

---

[6] It means that the semi-norms of the symbol of M_ν are bounded above independently of ν.

[7] $A(t) \geq \delta(t-T)$ means $\int \langle A(t)u(t), u(t)\rangle_{L^2(\mathbb{R}^n)} dt \geq |u(T)|^2_{L^2(\mathbb{R}^n)}$ for all $u \in C_0^0(\mathbb{R}, L^2(\mathbb{R}^n))$.

Moreover, using Lemma 3.3 and the Bony-Chemin definition of the Sobolev spaces $H(m, g)$ in [BC], we obtain with positive C_1, C_2, $P = D_t + iQ(t)$,

$$2C_1 \int \|Pu(t)\|_{H(\lambda,g)}\, dt \sup_t \|u(t)\|_{L^2(\mathbb{R}^n)} + C_0 \int \|u(t)\|^2_{L^2(\mathbb{R}^n)}\, dt \geq C_2^{-1} \sup_t \|u(t)\|^2_{L^2(\mathbb{R}^n)}.$$

Shrinking the support of u in the t-variable, we get

$$C_3 \int \|Pu(t)\|_{H(\lambda,g)}\, dt \geq \sup_t \|u(t)\|_{L^2(\mathbb{R}^n)}, \tag{3.7}$$

yielding solvability for P^*. Now, if we assume that $\psi_\nu q(t) = a_{0\nu}(t) b_{1\nu}(t)$ with $a_{0\nu} \in S(1, g_\nu), b_{1\nu} \in S(\lambda_\nu, g_\nu)$ uniformly, and with $a_0 \geq 0, \partial b_1/\partial t \geq 0$ (as in theorem 2.2), we have, with $\iota = 2i\pi$,

$$a_{0\nu} \sharp b_{1\nu} = a_{0\nu} b_{1\nu} + \frac{1}{2\iota}\{a_{0\nu}, b_{1\nu}\} + \rho_{-1}, \quad \text{with } \rho_{-1} \in S(\lambda_\nu^{-1}, g_\nu). \tag{3.8}$$

Thus we get

$$2Q_\nu = 2(\psi_\nu q)^w = a_{0\nu}^w b_{1\nu}^w + b_{1\nu}^w a_{0\nu}^w + R'_{-1\nu}, \quad \text{with } R'_{-1\nu} \in \operatorname{Op} S(\lambda_\nu^{-1}, g_\nu).$$

Using the Fefferman-Phong inequality, we have with a uniform constant C_0

$$A_{0\nu} = a_{0\nu}^w + C_0 \lambda_\nu^{-2} \geq 0, \tag{3.9}$$

so that

$$\begin{aligned} 2Q_\nu = 2(\psi_\nu q)^w &= (A_{0\nu} - C_0\lambda_\nu^{-2}) b_{1\nu}^w + b_{1\nu}^w (A_{0\nu} - C_0 \lambda_\nu^{-2}) + R'_{-1\nu} \\ &= A_{0\nu} b_{1\nu}^w + b_{1\nu}^w A_{0\nu} + R_{-1\nu}, \text{ with } R_{-1\nu} \in \operatorname{Op} S(\lambda_\nu^{-1}, g_\nu). \end{aligned}$$

We define now Γ_ν to be the unique positive definite quadratic form such that

$$g_\nu \leq \Gamma_\nu = \Gamma_\nu^\sigma \leq g_\nu^\sigma \tag{3.10}$$

and we use the Wick quantization related(8) to Γ_ν. We set then

$$B_{1\nu} = b_{1\nu}^{\text{Wick}} = b_{1\nu}^w + r_{0\nu}^w, \quad \text{with } r_{0\nu} \in S(1, g_\nu).$$

We obtain

$$\begin{aligned} 2Q_\nu &= A_{0\nu} B_{1\nu} + B_{1\nu} A_{0\nu} + A_{0\nu} R_{0\nu} + R_{0\nu} A_{0\nu} + R_{-1\nu}, \\ &\quad R_{0\nu} = R_{0\nu}^* \in \operatorname{Op} S(1, g_\nu), R_{-1\nu} = R_{-1\nu}^* \in \operatorname{Op} S(\lambda_\nu^{-1}, g_\nu), \\ &\quad A_{0\nu}^* = A_{0\nu} \geq 0, B_{1\nu} = B_{1\nu}^*, \frac{d}{dt} B_{1\nu} \geq 0, \\ &\quad A_{0\nu} \in \operatorname{Op} S(1, g_\nu), B_{1\nu} \in \operatorname{Op} S(\lambda_\nu, g_\nu). \end{aligned}$$

(8) In Definition 2.4, one should take as Σ_Y the Weyl quantization of $2^n \exp -2\pi\Gamma_\nu(X - Y)$.

As in the proof of Lemma 2.3, we calculate with

$$M_\nu = B_{1\nu} + Y(t-T), \tag{3.11}$$

the expression in which $R'_{0\nu}, R''_{0\nu} \in \operatorname{Op} S(1, g_\nu)$,

$$\begin{aligned}
&\dot{M}_\nu + 2\operatorname{Re}(M_\nu Q_\nu)\\
&= \dot{B_{1\nu}} + \delta(t-T) + \operatorname{Re}\big(B_{1\nu}(A_{0\nu}B_{1\nu} + B_{1\nu}A_{0\nu} + A_{0\nu}R_{0\nu} + R_{0\nu}A_{0\nu}\\
&\qquad\qquad + R_{-1\nu})\big)\\
&\quad + Y(t-T)(A_{0\nu}B_{1\nu} + B_{1\nu}A_{0\nu} + A_{0\nu}R_{0\nu} + R_{0\nu}A_{0\nu} + R_{-1\nu})\\
&= \dot{B_{1\nu}} + \delta(t-T) + B_{1\nu}A_{0\nu}B_{1\nu} + \operatorname{Re}(B_{1\nu}B_{1\nu}A_{0\nu})\\
&\quad + \operatorname{Re}(B_{1\nu}A_{0\nu}R_{0\nu} + B_{1\nu}R_{0\nu}A_{0\nu}) + R'_{0\nu} + Y(A_{0\nu}B_{1\nu} + B_{1\nu}A_{0\nu})\\
&= \dot{B_{1\nu}} + \delta(t-T) + 2B_{1\nu}A_{0\nu}B_{1\nu} + \operatorname{Re}(B_{1\nu}A_{0\nu}R_{0\nu} + B_{1\nu}R_{0\nu}A_{0\nu})\\
&\quad + R''_{0\nu} + Y(A_{0\nu}B_{1\nu} + B_{1\nu}A_{0\nu})\\
&= \dot{B_{1\nu}} + \delta(t-T) + 2B_{1\nu}A_{0\nu}B_{1\nu}\\
&\quad + \operatorname{Re}(B_{1\nu}A_{0\nu}R_{0\nu} + [B_{1\nu}, R_{0\nu}]A_{0\nu} + R_{0\nu}[B_{1\nu}, A_{0\nu}] + R_{0\nu}A_{0\nu}B_{1\nu})\\
&\quad + R''_{0\nu} + Y(A_{0\nu}B_{1\nu} + B_{1\nu}A_{0\nu}).
\end{aligned}$$

Using the identities (omitting the subscripts)

$$BAB + SAB + BAS = (BA^{1/2} + SA^{1/2})(A^{1/2}B + A^{1/2}S) - SAS \geq -SAS,$$

we obtain with a uniform constant C_1

$$\dot{M}_\nu + 2\operatorname{Re}(M_\nu Q_\nu) \geq \delta(t-T) - C_1. \tag{3.12}$$

This implies the solvability of the adjoint of

$$D_t + iQ(t) + S_0(t) + S_{-1}(t), \tag{3.13}$$

where $S_j \in \operatorname{Op} S(\lambda^j, g)$ and S_0 has a real-valued symbol; in fact the left-hand-side of (3.6) is modified by an operator with symbol in $S(1,g)$, when iQ is replaced by $iQ + S_0 + S_{-1}$ since the term S_0 will generate a bracket $[S_0, M] \in \operatorname{Op} S(1, g)$. Finally, we have proved the following

THEOREM 3.4. — *Let G be an admissible metric on $\mathbb{R}^{2n}_X$ and $q(t,X)$ a continuous real-valued function defined on $\mathbb{R}_t \times \mathbb{R}^{2n}_X$ such that $q(t,\cdot) \in S(\Lambda, G)$ uniformly in t (Λ is defined by (3.1) for G) and such that (1.9) is satisfied. Let us assume that there exists an admissible metric $g \geq G$ on $\mathbb{R}^{2n}$ such that $q(t,\cdot) \in S(\lambda, g)$ uniformly in t and for each $Y \in \mathbb{R}^{2n}$, there* *exists* $a_{0Y}(t,\cdot), b_{1Y}(t,\cdot)$, *respectively uniformly in* $S(1,g_Y), S(\lambda(Y),g_Y)$, *with* $a_{0Y} \geq 0$, $t \mapsto b_{1Y}(t,\cdot)$ *non-decreasing*

$$q(t,X) = a_{0Y}(t,X)b_{1Y}(t,X) \quad \text{for } g_Y(X-Y) \leq r_0, |t| \leq 1.$$

Then the operator $P = D_t + iq(t,\cdot)^w$ *satisfies the estimate (3.7) and thus* P^* *is locally solvable.*

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