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UPPER ENVELOPES OF INNER PREMEASURES

by Heinz KÖNIG

In the recent book [18] on measure and integration (cited as MI) and in subsequent papers [19] - [24] the present author attempted to restructure the domain of their basic extension and representation procedures, and to develop the implications on various issues in measure and integration and beyond. There were essential connections with the work [7] [8] of Gustave Choquet. Thus MI Section 10 obtained an extended version of his capacitability theorem. The representation theories in MI chapter V and [22] [24] were based on the Choquet integral introduced in [7] Section 48, in form of the so-called horizontal integral of MI Section 11, while [24] Section 1 obtained a comprehensive version of his fundamental theorem [7] 54.1. The present paper wants to resume another theme initiated in [7] Section 53.7, that is the representation of certain non-additive set functions and functionals as upper envelopes of appropriate measures.

We quote the definitive result due to Topsøe [28] Section 8 Theorem 2, subsequent to papers of Strassen [26], Dellacherie [9], Anger [3], Fuglede [12], and Huber-Strassen [15]. See also Anger [4] [5], Dellacherie [10], and Topsøe [29].

THEOREM. — *Let X be a Hausdorff topological space with the obvious set systems* $\mathrm{Comp}(X)$ *and* $\mathrm{Op}(X)$. *Assume that the set function* $\beta : \mathrm{Comp}(X) \to [0,\infty[$ *is isotone with* $\beta(\varnothing) = 0$ *and submodular, and* continuous from above *in the sense that*

for $A \in \mathrm{Comp}(X)$ *and* $\varepsilon > 0$ *there exists* $U \in \mathrm{Op}(X)$ *with* $A \subset U$

such that $\beta(K) < \beta(A) + \varepsilon$ *for all* $K \in \mathrm{Comp}(X)$ *with* $K \subset U$.

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Then for each $A \in \mathrm{Comp}(X)$ *there exists a Radon premeasure* $\varphi : \mathrm{Comp}(X) \to [0,\infty[$ *such that* $\varphi \leqq \beta$ *and* $\varphi(A) = \beta(A)$. *Likewise if* $\sup \beta < \infty$ *then there exists a Radon premeasure* $\varphi : \mathrm{Comp}(X) \to [0,\infty[$ *such that* $\varphi \leqq \beta$ *and* $\sup \varphi = \sup \beta$.

Then Adamski [1] transferred the theorem to the frame of abstract measures. He assumed certain pairs of lattices $\mathfrak{S}$ and $\mathfrak{T}$ of subsets in an abstract set X to take the place of $\mathrm{Comp}(X)$ and $\mathrm{Op}(X)$. With new ideas he was able to obtain fortified results. Now the present paper wants to show that the frame of MI leads to even more fortified and simpler forms of the results. One of the simplifications is that Adamski [1] assumed $\mathfrak{S}$ to be stable under countable intersections and the initial set function $\beta : \mathfrak{S} \to [0,\infty[$ to be σ continuous at $\varnothing$, which at once leads to the level of measures, whereas we shall see that the adequate level is the so-called finitely additive one, that is the level of contents. Also we shall compare our basic result with the main theorem of MI Section 18, at that place called the extended Henry-Lembcke-Bachman-Sultan-Lipecki-Adamski theorem, this time after Adamski [2]. There will be some remarkable consequences and examples.

Besides measure and integration the basic methodical device for the present area is an appropriate Hahn-Banach type theorem (or an equivalent assertion), as it became clear in particular from Topsøe [28] Section 8. Now the relevant Hahn-Banach theorems in the literature are of delicate nature and proof; see the work of Anger-Lembcke [6] referred to in Topsøe [28], or Fuchssteiner-König [11], and in particular the theorem of Rodé [25] [17]. Thus it is perhaps not superfluous to present a certain special case of the Rodé theorem which suffices for the present purpose and has a short and simple proof.

Basic notions and notations. — We adopt the terms of MI but shall recall the less familiar ones. Let X be a nonvoid set. For $S \subset X$ the complement is denoted S', and for a set system $\mathfrak{S}$ in X we write $\mathfrak{S}\perp := \{S' : S \in \mathfrak{S}\}$. For set systems $\mathfrak{S}$ and $\mathfrak{T}$ in X we form the *transporter* $\mathfrak{S}\top\mathfrak{T} := \{A \subset X : A \cap S \in \mathfrak{T}\ \forall S \in \mathfrak{S}\}$. For a set function $\Theta : \mathfrak{P}(X) \to [0,\infty]$ with $\Theta(\varnothing) = 0$ we recall the *Carathéodory class*

$$\mathfrak{C}(\Theta) := \{A \subset X : \Theta(M) = \Theta(M \cap A) + \Theta(M \cap A') \text{ for all } M \subset X\}.$$

$\mathfrak{C}(\Theta)$ turns out to be an algebra, and $\Theta|\mathfrak{C}(\Theta)$ to be modular.

The extension and representation theories in MI and in the subsequent [22] [23] [24] come in three simultaneous versions. They are marked $\bullet = \star\sigma\tau$, where $\star$ is to be read as *finite*, σ as *sequential* or countable, and τ as *nonsequential* or arbitrary (or as the respective adverbs). Moreover the theories come in parallel *outer* and *inner* versions, that means in versions which are based on outer and inner regularity. The extension theories for *set functions* are summarized in [22] Section 1 and [23] Section 1. The present paper will concentrate on the inner $\star$ version, but it is of course vital to note the consequences for the inner $\sigma\tau$ versions which result from routine combinations with the means of MI.

We recall the relevant envelope formations: Let $\mathfrak{S}$ be a lattice of subsets of X with $\varnothing \in \mathfrak{S}$, and $\varphi : \mathfrak{S} \to [0,\infty]$ be an isotone set function with $\varphi(\varnothing) = 0$. One defines the *outer* and *inner* $\star$ *envelopes* $\varphi^\star, \varphi_\star : \mathfrak{P}(X) \to [0,\infty]$ of φ to be

$$\varphi^\star(A) = \inf\{\varphi(S) : S \in \mathfrak{S} \text{ with } S \supset A\},$$
$$\varphi_\star(A) = \sup\{\varphi(S) : S \in \mathfrak{S} \text{ with } S \subset A\}.$$

The inner versions require that $\varphi : \mathfrak{S} \to [0,\infty[$ be *finite*. We consider the *inner* $\star$ *version*: One defines an *inner* $\star$ *extension* of φ to be a content $\alpha : \mathfrak{A} \to [0,\infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}$ which extends φ and is inner regular $\mathfrak{S}$. One defines φ to be an *inner* $\star$ *premeasure* iff it admits inner $\star$ extensions. The subsequent *inner* $\star$ *theorem* characterizes those φ which are inner $\star$ premeasures, and then describes all inner $\star$ extensions of φ. The theorem is in terms of $\varphi_\star$; its essence can be found earlier in Topsøe [27] Section 4.

INNER $\star$ THEOREM.— *Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ is isotone with $\varphi(\varnothing) = 0$. Then the following are equivalent:*

1) *φ is an inner $\star$ premeasure.*

2) *$\varphi_\star|\mathfrak{C}(\varphi_\star)$ is an inner $\star$ extension of φ.*

3) *$\varphi_\star|\mathfrak{C}(\varphi_\star)$ is an extension of φ.*

4) *φ is supermodular, and* inner $\star$ tight *in the sense that*

$$\varphi(B) \leqq \varphi(A) + \varphi_\star(B \setminus A) \quad \textit{for all } A \subset B \textit{ in } \mathfrak{S}.$$

In this case all inner $\star$ extensions of φ are restrictions of $\varphi_\star|\mathfrak{C}(\varphi_\star)$. Moreover $\mathfrak{S}\top\mathfrak{C}(\varphi_\star) \subset \mathfrak{C}(\varphi_\star)$.

1. The Hahn-Banach type theorems.

The first theorem is an obvious special case of the Hahn-Banach theorem due to Rodé [25] (with Zusatz p. 480). The point here is its short and simple proof.

THEOREM 1.1.— *Assume that the nonvoid set E carries an associative and commutative addition $+$ and an order relation $\leqq$, which are compatible in the sense that $u \leqq v \Rightarrow u+x \leqq v+x$ for all $u,v,x \in E$. Let*

$$\begin{aligned} Q : E \to]-\infty,\infty] &\quad \text{be subadditive and isotone,} \\ P : E \to]-\infty,\infty] &\quad \text{be superadditive and isotone,} \end{aligned}$$

and $P \leqq Q$. Then there exists an additive and isotone function $f : E \to]-\infty,\infty]$ such that $P \leqq f \leqq Q$.

Proof. — Let $Q : E \to]-\infty,\infty]$ be subadditive and isotone, and define $\mathrm{M}(Q)$ to consist of all superadditive and isotone functions $f : E \to]-\infty,\infty]$ with $f \leqq Q$.

0) $\mathrm{M}(Q)$ is upward inductive in the pointwise order. In fact, if $H \subset \mathrm{M}(Q)$ is nonvoid and upward directed then the pointwise supremum $f := \sup_{h\in H} h$ is in $\mathrm{M}(Q)$. Thus each $P \in \mathrm{M}(Q)$ has maximal members $f \in \mathrm{M}(Q)$ with $P \leqq f$. It remains to show that each maximal $f \in \mathrm{M}(Q)$ must be subadditive and hence additive.

1) We claim that $f(nx) = nf(x)$ for $x \in E$ and $n \in \mathbb{N}$, where of course $nx := x+\cdots+x$ (n terms). First of all $f(nx) \geqq nf(x)$ and $Q(nx) \leqq nQ(x)$. For fixed $n \in \mathbb{N}$ now consider $F : F(x) = \frac{1}{n}f(nx)$ for $x \in E$. Then $F \in \mathrm{M}(Q)$ and $F \geqq f$, and hence $F = f$ as claimed.

2) We claim that $f(x+a) \leqq f(x) + Q(a)$ for $x, a \in E$. To see this fix $a \in E$ with $Q(a) < \infty$, and define $F : E \to]-\infty,\infty]$ to be

$$F(x) = \sup\{f(x+na) - nQ(a) : n \geqq 0\} \quad \text{for } x \in E,$$

with the obvious rôle of $n = 0$. Then F is isotone and $F \geqq f$. We have $F \leqq Q$ because

$$f(x+na) - nQ(a) \leqq Q(x+na) - nQ(a) \leqq Q(x) + Q(na) - nQ(a) \leqq Q(x),$$

and F is superadditive because for $u, v \in E$ and $m, n \geqq 0$ we have

$$\begin{aligned} F(u+v) &\geqq f\big((u+v)+(m+n)a\big)-(m+n)Q(a) \\ &= f\big((u+ma)+(v+na)\big)-(m+n)Q(a) \\ &\geqq \big(f(u+ma)-mQ(a)\big)+\big(f(v+na)-nQ(a)\big). \end{aligned}$$

Thus $F \in \mathrm{M}(Q)$ and $F \geqq f$, and hence $F = f$. For $n = 1$ the assertion follows.

3) We claim that $f(x+a) \leqq f(x) + f(a)$ for $x, a \in E$, which will complete the proof. To see this fix $a \in E$ with $f(a) < \infty$, and define $F : E \to]-\infty, \infty]$ to be

$$\begin{aligned} F(x) &= \sup\{f(x+na)-nf(a) : n \geqq 1\} \\ &= \lim_{n\to\infty} \Big(f(x+na)-nf(a)\Big) \text{ for } x \in E; \end{aligned}$$

note that the expression in the last brackets increases with $n \geqq 1$. Then F is isotone and $F \geqq f$, and $F \leqq Q$ from 1)2). As in 2) one proves that F is superadditive. Thus $F \in \mathrm{M}(Q)$ and $F \geqq f$, and hence $F = f$. For $n = 1$ the assertion follows. □

We combine the above result with the sub/superadditivity theorem for the Choquet integral in the version MI 11.11 to obtain our basic device. This is a known theorem too; see Kindler [16] Section 5 Example 1 and MI 11.24. The present proof has been sketched in MI 11.14.

THEOREM 1.2.— *Let $\mathfrak{S}$ be a lattice of subsets in X with $\varnothing \in \mathfrak{S}$. Assume that*

$$\beta : \mathfrak{S} \to [0, \infty] \textit{ is isotone with } \beta(\varnothing) = 0 \textit{ and submodular,}$$

$$\alpha : \mathfrak{S} \to [0, \infty] \textit{ is isotone with } \alpha(\varnothing) = 0 \textit{ and supermodular,}$$

and $\alpha \leqq \beta$. Then there exists an isotone and modular set function $\varphi : \mathfrak{S} \to [0, \infty]$ such that $\alpha \leqq \varphi \leqq \beta$.

Proof.— Let E consist of the finite linear combinations of the characteristic functions χ_S of the $S \in \mathfrak{S}$ with coefficients $\geqq 0$, that is $E = \mathrm{S}(\mathfrak{S})$ in the sense of MI Section 11, equipped with pointwise addition $+$ and order relation $\leqq$. Define $Q, P : E \to [0, \infty]$ to be $Q(u) = ⨍ u d\beta$ and $P(u) = ⨍ u d\alpha$ for $u \in E$, where $⨍$ denotes the *horizontal integral* of MI Section 11. After MI 11.11 then Q and P are as required in 1.1. Hence there exists an additive and isotone functional $f : E \to [0, \infty]$ such that

$P \leqq f \leqq Q$. Define $\varphi : \mathfrak{S} \to [0,\infty]$ to be $\varphi(S) = f(\chi_S)$ for $S \in \mathfrak{S}$. Then φ is isotone and modular and fulfils $\alpha \leqq \varphi \leqq \beta$. □

We conclude the section with a short supplement which will not be needed in the sequel. The next assertion is the central result of Horn-Tarski [14] specialized to lattices (but with the value ∞ admitted).

THEOREM 1.3.— *Let $\mathfrak{S}$ be a lattice in X with $\varnothing \in \mathfrak{S}$. Then each isotone and modular set function $\varphi : \mathfrak{S} \to [0,\infty]$ with $\varphi(\varnothing) = 0$ can be extended to a content $\phi : \mathfrak{P}(X) \to [0,\infty]$ with $\phi(X) = \sup \varphi$.*

Proof.— The extension $\vartheta : \mathfrak{S} \cup \{X\} \to [0,\infty]$ of φ, in case $X \notin \mathfrak{S}$ with $\vartheta(X) = \sup \varphi$, retains the assumptions. The assertion then follows from 1.2 applied to the pair $\vartheta_\star \leqq \vartheta^\star$. □

Example 1.4.— Let $X = [0,1]$. Define $\mathfrak{S}$ to consist of $\varnothing$ and of the $[0,t[$ with $0 < t \leqq 1$, and $\varphi : \mathfrak{S} \to [0,\infty[$ to be $\varphi(\varnothing) = \varphi([0,t[) = 0$ for $0 < t < 1$ and $\varphi([0,1[) = 1$. Thus $\mathfrak{S}$ is totally ordered under the inclusion $\subset$ and hence a lattice, and φ is isotone and modular. Let $\phi : \mathfrak{P}(X) \to [0,\infty]$ be as in 1.3, and $\vartheta := \phi|\mathrm{Comp}(X)$. Then $\vartheta : \mathrm{Comp}(X) \to [0,\infty[$ is isotone and modular with $\vartheta(\varnothing) = 0$, but not inner $\star$ tight, because $\vartheta([0,1]) = 1$ and $\vartheta(\{1\}) = \vartheta_\star([0,1[) = 0$. Thus ϑ is not an inner $\star$ premeasure, that is not a Radon premeasure. We note that we have not seen such examples in the literature so far. Their existence has been overlooked for example in [27] p.4 l.14-24.

2. Some preliminaries.

The present section assumes a pair of lattices $\mathfrak{S}$ and $\mathfrak{T}$ in X with $\varnothing \in \mathfrak{S}, \mathfrak{T}$. An illustrative example is $\mathfrak{S} = \mathrm{Comp}(X)$ and $\mathfrak{T} = \mathrm{Op}(X)$ in a Hausdorff topological space X, as discussed in the introduction and resumed in 2.7 below. After some simple remarks we consider certain properties of separation.

Remark 2.1.— For isotone set functions $\varphi : \mathfrak{S} \to [0,\infty]$ and $\psi : \mathfrak{T} \to [0,\infty]$ with $\varphi(\varnothing) = \psi(\varnothing) = 0$ we have $\varphi_\star|\mathfrak{T} \leqq \psi \Leftrightarrow \varphi \leqq \psi^\star|\mathfrak{S}$.

Proof.— Both relations mean that all pairs $S \in \mathfrak{S}$ and $T \in \mathfrak{T}$ with $S \subset T$ fulfil $\varphi(S) \leqq \psi(T)$. □

Remark 2.2. — For an isotone $\varphi : \mathfrak{S} \to [0,\infty]$ with $\varphi(\varnothing) = 0$ we have $\varphi \leqq (\varphi_\star|\mathfrak{T})^\star|\mathfrak{S}$. Moreover $\varphi = (\varphi_\star|\mathfrak{T})^\star|\mathfrak{S}$ iff $\varphi = \psi^\star|\mathfrak{S}$ for some isotone $\psi : \mathfrak{T} \to [0,\infty]$ with $\psi(\varnothing) = 0$.

Proof. — The first assertion is clear from the definitions. We prove the second one. $\Rightarrow$) $\psi := \varphi_\star|\mathfrak{T}$ is as required. $\Leftarrow$) We have $\varphi_\star|\mathfrak{T} = (\psi^\star|\mathfrak{S})_\star|\mathfrak{T} \leqq \psi$ as in the first assertion, and hence $(\varphi_\star|\mathfrak{T})^\star|\mathfrak{S} \leqq \psi^\star|\mathfrak{S} = \varphi$.

□

Remark 2.3. — Assume that the isotone and submodular $\psi : \mathfrak{T} \to [0,\infty]$ with $\psi(\varnothing) = 0$ fulfils $\vartheta := \psi^\star|\mathfrak{S} < \infty$, so that $\vartheta : \mathfrak{S} \to [0,\infty[$ is isotone with $\vartheta(\varnothing) = 0$. If $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$ then ϑ is inner $\star$ tight.

Proof. — To be shown is $\vartheta(B) \leqq \vartheta(A) + \vartheta_\star(B \setminus A)$ for $A \subset B$ in $\mathfrak{S}$. For fixed $c > \vartheta(A)$ there exists $T \in \mathfrak{T}$ with $T \supset A$ and $\psi(T) < c$. Then $T' \cap B \in \mathfrak{S}$ with $T' \cap B \subset A' \cap B = B \setminus A$. Therefore

$$\begin{aligned}\vartheta(B) = \psi^\star(B) \leqq \psi^\star(B \cup T) &\leqq \psi^\star(T) + \psi^\star(T' \cap B)\\ &= \psi(T) + \vartheta(T' \cap B) < c + \vartheta_\star(B \setminus A).\end{aligned}$$

The assertion follows. □

We turn to the announced properties of separation. Let as before $\mathfrak{S}$ and $\mathfrak{T}$ be lattices in X with $\varnothing \in \mathfrak{S}, \mathfrak{T}$. We say as usual that $\mathfrak{T}$ *separates* $\mathfrak{S}$ iff for each disjoint pair $A, B \in \mathfrak{S}$ there exists a disjoint pair $U, V \in \mathfrak{T}$ with $A \subset U$ and $B \subset V$. We need two further properties. On the one hand we define a pair $A, B \subset X$ to be *separated* $\mathfrak{T}$ iff for each $M \in \mathfrak{T}$ with $A \cap B \subset M$ there exists a pair $U, V \in \mathfrak{T}$ with $A \subset U$ and $B \subset V$ such that $U \cap V \subset M$. On the other hand we define a pair $A, B \subset X$ to be *coseparated* $\mathfrak{S}$ iff for each $M \in \mathfrak{S}$ with $M \subset A \cup B$ there exists a pair $P, Q \in \mathfrak{S}$ with $P \subset A$ and $Q \subset B$ such that $M \subset P \cup Q$. These two properties came up in MI 4.2 and MI 6.4. We recall the consequences obtained at these places.

Remark 2.4. — 1) Let $\varphi : \mathfrak{S} \to [0,\infty]$ be isotone and submodular with $\varphi(\varnothing) = 0$. If the pair $A, B \subset X$ is coseparated $\mathfrak{S}$ then

$$\varphi_\star(A \cup B) + \varphi_\star(A \cap B) \leqq \varphi_\star(A) + \varphi_\star(B).$$

2) Let $\psi : \mathfrak{T} \to [0,\infty]$ be isotone and supermodular with $\psi(\varnothing) = 0$. If the pair $A, B \subset X$ is separated $\mathfrak{T}$ then

$$\psi^\star(A \cup B) + \psi^\star(A \cap B) \geqq \psi^\star(A) + \psi^\star(B).$$

Thus each time the semimodular behaviour carries over to the less natural and unexpected $\star$ envelope. We come to the basic relations which connect the notions defined above.

PROPOSITION 2.5. — *Assume that $\mathfrak{T}$ separates $\mathfrak{S}$, and that $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$. 1) Each pair $A, B \in \mathfrak{T}$ is coseparated $\mathfrak{S}$. 2) Each pair $A, B \in \mathfrak{S}$ is separated $\mathfrak{T}$.*

Proof. — 1) Fix $A, B \in \mathfrak{T}$ and $M \in \mathfrak{S}$ with $M \subset A \cup B$. Then $A', B' \in \mathfrak{S}\top\mathfrak{S}$ and hence $M \cap A', M \cap B' \in \mathfrak{S}$, and these two sets are disjoint. Hence there exist disjoint $U, V \in \mathfrak{T}$ with $M \cap A' \subset U$ and $M \cap B' \subset V$, that is with $M \subset A \cup U$ and $M \subset B \cup V$. Thus $P := M \cap U' \in \mathfrak{S}$ and $Q := M \cap V' \in \mathfrak{S}$ fulfil $P \subset A$ and $Q \subset B$. Moreover $U' \cup V' = X$ implies that $P \cup Q = M$.

2) Fix $A, B \in \mathfrak{S}$ and $M \in \mathfrak{T}$ with $A \cap B \subset M$. Then $M' \in \mathfrak{S}\top\mathfrak{S}$ and hence $A \cap M', B \cap M' \in \mathfrak{S}$, and these two sets are disjoint. Hence there exist disjoint $P, Q \in \mathfrak{T}$ with $A \cap M' \subset P$ and $B \cap M' \subset Q$, that is with $A \subset M \cup P$ and $B \subset M \cup Q$. Thus $U := M \cup P \in \mathfrak{T}$ and $V := M \cup Q \in \mathfrak{T}$ are as required. □

CONSEQUENCE 2.6. — *Assume that $\mathfrak{T}$ separates $\mathfrak{S}$, and that $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$. 1) Let $\varphi : \mathfrak{S} \to [0, \infty]$ be isotone and submodular with $\varphi(\varnothing) = 0$. Then $\varphi_\star|\mathfrak{T}$ is submodular. 2) Let $\psi : \mathfrak{T} \to [0, \infty]$ be isotone and supermodular with $\psi(\varnothing) = 0$. Then $\psi^\star|\mathfrak{S}$ is supermodular.*

Example 2.7. — Let X be a Hausdorff topological space. Then $\mathfrak{S} := \mathrm{Comp}(X)$ and $\mathfrak{T} := \mathrm{Op}(X)$ fulfil both $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$ and $\mathfrak{S} \subset (\mathfrak{T}\top\mathfrak{T})\perp$, and $\mathfrak{T}$ separates $\mathfrak{S}$.

We conclude the section with the overall remark that part of its results can be extended to the common frame $\bullet = \star\sigma\tau$, while others lead to serious problems. Thus the situation is different from [23] part I, but similar to MI chapter VI.

3. The basic results.

As before we assume a pair of lattices $\mathfrak{S}$ and $\mathfrak{T}$ in X with $\varnothing \in \mathfrak{S}, \mathfrak{T}$. For an isotone set function $\varphi : \mathfrak{S} \to [0, \infty]$ with $\varphi(\varnothing) = 0$ we write $\hat{\varphi} := (\varphi_\star|\mathfrak{T})^\star|\mathfrak{S}$. Thus $\hat{\varphi} : \mathfrak{S} \to [0, \infty]$ is isotone with $\hat{\varphi}(\varnothing) = 0$. The above 2.2 says that $\varphi \leqq \hat{\varphi}$ and has a simple equivalence for the case $\varphi = \hat{\varphi}$, in which case φ is sometimes called *semi-regular.*

Example 3.1. — In a Hausdorff topological space X let $\mathfrak{S} = \mathrm{Comp}(X)$ and $\mathfrak{T} = \mathrm{Op}(X)$. Then for an isotone $\varphi : \mathfrak{S} \to [0,\infty]$ with $\varphi(\varnothing) = 0$ the relation $\varphi = \hat{\varphi}$ means that φ is *continuous from above* as defined in the theorem quoted in the introduction.

Example 3.2. — . Assume that $\mathfrak{S}$ is *upward enclosable* $\mathfrak{T}$, defined to mean that each member of $\mathfrak{S}$ is contained in some member of $\mathfrak{T}$. Define $\varphi : \mathfrak{S} \to [0,\infty[$ to be $\varphi(\varnothing) = 0$ and $\varphi(S) = 1$ for nonvoid $S \in \mathfrak{S}$. Thus φ is isotone and submodular. It is immediate that $\varphi = \hat{\varphi}$.

The next result extends [27] Lemma 2.4 and [1] Lemma 3.1(b)(c). In its present form it can be considered as an abstract version of classical results in Halmos [13] Sections 53-54.

PROPOSITION 3.3. — *Assume that $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$, and that $\mathfrak{T}$ separates $\mathfrak{S}$. For an isotone $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi(\varnothing) = 0$ then*

if φ is modular and $\hat{\varphi} < \infty$: $\hat{\varphi}$ is an inner $\star$ premeasure.

Moreover

if φ is submodular: $\varphi = \hat{\varphi} \Longrightarrow \varphi$ is inner $\star$ tight and $\hat{\varphi} < \infty$,

if φ is supermodular: $\varphi = \hat{\varphi} \Longleftarrow \varphi$ is inner $\star$ tight and $\hat{\varphi} < \infty$;

both times the converse need not be true. In particular

if φ is modular: $\varphi = \hat{\varphi} \Longleftrightarrow \varphi$ is inner $\star$ tight and $\hat{\varphi} < \infty$.

Proof. — We have $\hat{\varphi} = \psi^\star|\mathfrak{S}$ for $\psi := \varphi_\star|\mathfrak{T}$, where $\psi : \mathfrak{T} \to [0,\infty]$ is isotone with $\psi(\varnothing) = 0$.

1) Let φ be supermodular. Then ψ is supermodular, and hence $\hat{\varphi}$ is supermodular by 2.6.2).

2) Let φ be submodular. Then ψ is submodular by 2.6.1), and hence $\hat{\varphi}$ is submodular. If moreover $\hat{\varphi} < \infty$, then $\hat{\varphi}$ is inner $\star$ tight by 2.3.

3) The first assertion follows from 1)2), and the second one from 2). We turn to the third assertion. The converses will be dealt with in 3.4 below.

4) Let φ be supermodular and inner $\star$ tight, that is an inner $\star$ premeasure, with $\hat{\varphi} < \infty$. To be shown is $\hat{\varphi} \leqq \varphi$. We fix $S \in \mathfrak{S}$, and then $T \in \mathfrak{T}$ with $T \supset S$ and $\varphi_\star(T) < \infty$. For fixed $\varepsilon > 0$ we take $K \in \mathfrak{S}$ with $K \subset T \setminus S$ and $\varphi(K) > \varphi_\star(T \setminus S) - \varepsilon$. In view of $\mathfrak{S} \subset \mathfrak{C}(\varphi_\star)$ and

$\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp \subset \mathfrak{C}(\varphi_\star)$ the last relation can be written $\varphi(S) > \varphi_\star(T\backslash K)-\varepsilon$. Now for the two disjoint $S, K \in \mathfrak{S}$ there exist disjoint $U, V \in \mathfrak{T}$ with $S \subset U$ and $K \subset V$; we can of course assume that $U, V \subset T$. Then $U \subset T \cap V' \subset T \cap K' = T \setminus K$ and hence

$$\hat{\varphi}(S) \leqq \varphi_\star(U) \leqq \varphi_\star(T \setminus K) < \varphi(S) + \varepsilon.$$

The assertion follows. □

Example 3.4.— There are simple counterexamples which disprove the two converse assertions.

1) Assume that $\mathfrak{S}$ contains the one-point subsets of X, and fix a subset $K \in \mathfrak{S}$ which is not in $\mathfrak{T}$. Define $\varphi : \mathfrak{S} \to [0,\infty[$ to be $\varphi(S) = 0$ for $S \subset K$ and $\varphi(S) = 1$ for $S \not\subset K$. Then φ is isotone and submodular, and one verifies that φ is inner $\star$ tight and $\hat{\varphi} \leqq 1 < \infty$. But $\varphi(K) = 0$ and $\hat{\varphi}(K) = 1$, so that $\varphi \neq \hat{\varphi}$.

2) In a set X of more than one element let $\mathfrak{S}$ consist of the finite subsets and $\mathfrak{T} \supset \mathfrak{S}$; then $\mathfrak{S}$ and $\mathfrak{T}$ are as required. Define $\varphi : \mathfrak{S} \to [0,\infty[$ to be $\varphi(S) = (\#(S))^2$ for $S \in \mathfrak{S}$. Then φ is isotone with $\varphi(\varnothing) = 0$, and $\varphi = \hat{\varphi} < \infty$ in view of $\mathfrak{T} \supset \mathfrak{S}$. For $A, B \in \mathfrak{S}$ with $\#(A) = m, \#(B) = n$ and $\#(A \cap B) = p$ we have

$$\begin{aligned}\varphi(A \cup B) + \varphi(A \cap B) - \varphi(A) - \varphi(B) &= (m+n-p)^2 + p^2 - m^2 - n^2 \\ &= 2(m-p)(n-p).\end{aligned}$$

Thus φ is supermodular. But since $\mathfrak{S}$ is a ring the case $p = 0$ shows that φ is not inner $\star$ tight.

For the remainder of the section we fix an isotone set function $\beta : \mathfrak{S} \to [0,\infty[$ with $\beta(\varnothing) = 0$. We define $\mathrm{M}(\beta)$ to consist of the isotone and supermodular set functions $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi \leqq \beta$.

Remark 3.5.— 1) *$\mathrm{M}(\beta)$ is upward inductive in the setwise order.*

2) *If β is submodular then each maximal member of $\mathrm{M}(\beta)$ is modular.*

Proof.— 1) If $H \subset \mathrm{M}(\beta)$ is nonvoid and upward directed then the setwise supremum $\vartheta := \sup_{\varphi \in H} \varphi$ is in $\mathrm{M}(\beta)$.

2) Follows from the Hahn-Banach Theorem 1.2. □

The next theorem and the subsequent consequence are the basic results in the present context.

THEOREM 3.6.— *Assume that* $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$, *and that* $\mathfrak{T}$ *separates* $\mathfrak{S}$. *Let* β *be submodular and* $\beta = \hat{\beta}$. *Then each maximal member* $\varphi \in \mathrm{M}(\beta)$ *is an inner* $\star$ *premeasure and fulfils* $\varphi = \hat{\varphi}$.

Proof.— i) We see from 3.5.2) that φ is submodular.

ii) $\psi := \varphi_\star|\mathfrak{T}$ is supermodular, and hence $\hat{\varphi} = \psi^\star|\mathfrak{S}$ is supermodular by 2.6.2). Moreover $\varphi \leqq \hat{\varphi} \leqq \hat{\beta} = \beta$. Therefore $\varphi = \hat{\varphi}$.

iii) From i)ii) combined with 3.3 we see that φ is inner $\star$ tight and hence an inner $\star$ premeasure. □

CONSEQUENCE 3.7.— *Assume that* $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$, *and that* $\mathfrak{T}$ *separates* $\mathfrak{S}$. *Let* β *be submodular and* $\beta = \hat{\beta}$. *If* $\mathfrak{M} \subset \mathfrak{S}$ *is a lattice such that* $\beta|\mathfrak{M}$ *is modular then there exists an inner* $\star$ *premeasure* $\varphi : \mathfrak{S} \to [0, \infty[$ *with* $\varphi \leqq \beta$ *and* $\varphi|\mathfrak{M} = \beta|\mathfrak{M}$.

Proof.— We can assume that $\varnothing \in \mathfrak{M}$. Then $\alpha := (\beta|\mathfrak{M})_\star|\mathfrak{S}$ is in $\mathrm{M}(\beta)$. Thus there exists a maximal member $\varphi \in \mathrm{M}(\beta)$ with $\alpha \leqq \varphi \leqq \beta$, that is with $\varphi \leqq \beta$ and $\varphi|\mathfrak{M} = \beta|\mathfrak{M}$. □

SPECIALIZATION 3.8.— *Assume that* $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$, *and that* $\mathfrak{T}$ *separates* $\mathfrak{S}$. *Let* β *be submodular and* $\beta = \hat{\beta}$. *If* $\mathfrak{M} \subset \mathfrak{S}$ *is nonvoid and totally ordered under inclusion then there exists an inner* $\star$ *premeasure* $\varphi : \mathfrak{S} \to [0, \infty[$ *with* $\varphi \leqq \beta$ *and* $\varphi|\mathfrak{M} = \beta|\mathfrak{M}$.

The case $\mathfrak{M} = \{A\} \subset \mathfrak{S}$ in the special situation of Example 3.1 is the first assertion in the theorem of Topsøe [28] quoted in the introduction. Its second assertion is contained in 3.9 below. The case that $\mathfrak{M}$ consists of the members of a sequence $S_1 \subset \cdots \subset S_n \subset \cdots$ in $\mathfrak{S}$ is in Adamski [1] Corollary 3.7, in essence under fortified assumptions. This special case has the consequence which follows.

CONSEQUENCE 3.9.— *Assume that* $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$, *and that* $\mathfrak{T}$ *separates* $\mathfrak{S}$. *Let* β *be submodular and* $\beta = \hat{\beta}$. *For each* $M \subset X$ *there exists an inner* $\star$ *premeasure* $\varphi : \mathfrak{S} \to [0, \infty[$ *with* $\varphi \leqq \beta$ *and* $\varphi_\star(M) = \beta_\star(M)$.

Proof.— Fix a sequence $S_1 \subset \cdots \subset S_n \subset \cdots \subset M$ in $\mathfrak{S}$ such that $\beta(S_n) \uparrow \beta_\star(M)$. The result follows from 3.8 applied to $\mathfrak{M} := \{S_n : n \in \mathbb{N}\}$.

□

So far the basic results in the present context. It is of interest to compare 3.6 with the main theorem MI 18.10 of MI Section 18. We present this result (in the so-called *conventional situation* of MI) in the reformulation which follows. Let as above $\mathfrak{S}$ and $\mathfrak{T}$ be lattices in X with $\varnothing \in \mathfrak{S}, \mathfrak{T}$ and $\beta : \mathfrak{S} \to [0, \infty[$ be isotone with $\beta(\varnothing) = 0$.

THEOREM 3.10.— *Assume that $\beta = \hat{\beta}$. Then each maximal member $\varphi \in \mathrm{M}(\beta)$ such that $\varphi_\star|\mathfrak{T}$ is an inner $\star$ premeasure is itself an inner $\star$ premeasure.*

It is not obvious that the above 3.10 is equivalent to MI 18.10. Therefore we shall include the proof of this equivalence. For an isotone and supermodular set function $\psi : \mathfrak{T} \to [0, \infty[$ with $\psi(\varnothing) = 0$ let $\star(\psi)$ as in MI Section 18 consist of the isotone and supermodular $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi(\varnothing) = 0$ such that $\varphi_\star|\mathfrak{T} = \psi$. Then MI 18.4 says that $\star(\psi)$ is upward inductive in the setwise order, and MI 18.10 can be formulated as follows: *Assume that $\mathfrak{S}$ is upward enclosable $\mathfrak{T}$. If $\psi : \mathfrak{T} \to [0, \infty[$ is an inner $\star$ premeasure, then each maximal member $\varphi : \mathfrak{S} \to [0, \infty[$ of $\star(\psi)$ is an inner $\star$ premeasure.*

Proof of MI 18.10 $\Rightarrow$ 3.10.— Let β and φ be as assumed in 3.10. Then first of all $\beta < \infty$ implies that $\mathfrak{S}$ is upward enclosable $\mathfrak{T}$. By assumption $\psi := \varphi_\star|\mathfrak{T}$ is an inner $\star$ premeasure, and by definition $\varphi \in \star(\psi)$. It suffices to show that φ is a maximal member of $\star(\psi)$. Thus let $\vartheta \in \star(\psi)$ with $\varphi \leqq \vartheta$. To be shown is $\varphi = \vartheta$. From $\vartheta_\star|\mathfrak{T} = \psi$ and 2.1 we have $\vartheta \leqq \psi^\star|\mathfrak{S} = \hat{\varphi} \leqq \hat{\beta} = \beta$ and hence $\vartheta \in \mathrm{M}(\beta)$. By assumption φ is a maximal member of $\mathrm{M}(\beta)$, and hence $\varphi = \vartheta$. □

Proof of 3.10 $\Rightarrow$ MI 18.10.— Assume that $\mathfrak{S}$ is upward enclosable $\mathfrak{T}$, and let ψ and φ be as assumed in MI 18.10. Then $\beta := \psi^\star|\mathfrak{S}$ is $< \infty$ and isotone with $\beta(\varnothing) = 0$, and $\beta = \hat{\beta}$ by 2.2. From $\varphi_\star|\mathfrak{T} = \psi$ and 2.1 we have $\varphi \leqq \psi^\star|\mathfrak{S} = \beta$ and hence $\varphi \in \mathrm{M}(\beta)$. It remains to show that φ is a maximal member of $\mathrm{M}(\beta)$. Thus let $\vartheta \in \mathrm{M}(\beta)$ with $\varphi \leqq \vartheta$. To be shown is $\varphi = \vartheta$. We have $\psi = \varphi_\star|\mathfrak{T} \leqq \vartheta_\star|\mathfrak{T} \leqq \beta_\star|\mathfrak{T}$; and from $\beta = \psi^\star|\mathfrak{S}$ and 2.1 also $\beta_\star|\mathfrak{T} \leqq \psi$. Thus $\vartheta_\star|\mathfrak{T} = \psi$ and hence $\vartheta \in \star(\psi)$. By assumption φ is a maximal member of $\star(\psi)$, and hence $\varphi = \vartheta$. □

In MI sections 18 and 19 there were numerous applications of MI 18.10. In the present context of 3.10 we shall restrict ourselves to the particular case $\mathfrak{T} := \{\varnothing, X\}$, because then each set function $\psi : \mathfrak{T} \to [0, \infty[$ with $\psi(\varnothing) = 0$ is an inner $\star$ premeasure. In this case $\beta = \hat{\beta}$ means that $\beta(S) = \sup \beta < \infty$ for all nonvoid $S \in \mathfrak{S}$. Thus we fix $\beta : \mathfrak{S} \to [0, \infty[$ to be $\beta(\varnothing) = 0$ and $\beta(S) = 1$ for $S \neq \varnothing$. Then 3.10 says that each maximal member of $\mathrm{M}(\beta)$ is an inner $\star$ premeasure. We emphasize one particular consequence.

Consequence 3.11. — *Let $\mathfrak{S}$ be a lattice in X with $\varnothing \in \mathfrak{S}$. Assume that the nonvoid $\mathfrak{M} \subset \mathfrak{P}(X)$ is downward directed and upward enclosable $\mathfrak{S}$ with $\varnothing \notin \mathfrak{M}$. Then there exists an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi \leqq 1$ and $\varphi^\star | \mathfrak{M} = 1$.*

Proof. — Define $\alpha : \mathfrak{S} \to [0, \infty[$ to be $\alpha(S) = 0$ when there is no $M \in \mathfrak{M}$ with $M \subset S$, and $\alpha(S) = 1$ otherwise. It is clear that $\alpha \in \mathrm{M}(\beta)$ for the above $\beta : \mathfrak{S} \to [0, \infty[$. Thus each maximal member $\varphi \in \mathrm{M}(\beta)$ with $\alpha \leqq \varphi$ is as required. □

Remark 3.12. — In 3.10 it cannot be achieved as in 3.6 that $\varphi = \hat{\varphi}$. In fact, in the particular case 3.11 this would mean that $\varphi(S) = 1$ for all nonvoid $S \in \mathfrak{S}$, so that φ could not be modular whenever there are nonvoid $A, B \in \mathfrak{S}$ with $A \cap B = \varnothing$.

It is remarkable that the above 3.11 can also be obtained from the basic Theorem 3.6, this time via $\mathfrak{T} = (\mathfrak{S} \top \mathfrak{S})\perp$ and with the same proof. However, one needs the additional assumption that $(\mathfrak{S} \top \mathfrak{S})\perp$ separates $\mathfrak{S}$, but in return one can achieve that $\varphi = \hat{\varphi}$.

We conclude with two further consequences which underline the wealth of inner $\star$ premeasures. The results are in the terms of MI Sections 6 and 11.

Proposition 3.13. — *Let $\mathfrak{S}$ be a lattice in X with $\varnothing \in \mathfrak{S}$ which for some $\bullet = \sigma\tau$ is not $\bullet$ compact. Then there exists an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ which is not $\bullet$ continuous at $\varnothing$.*

Proof. — The assumption means that there exists a nonvoid and downward directed set system $\mathfrak{M} \subset \mathfrak{S}$ of type $\bullet$ such that $\varnothing \notin \mathfrak{M}$ but $\mathfrak{M} \downarrow \varnothing$. Thus the assertion follows from 3.11. □

Proposition 3.14. — *Let $\mathfrak{S}$ be a lattice in X with $\varnothing \in \mathfrak{S}$. Assume that $f : X \to [0, \infty[$ with $\sup f = 1$ has $\mathfrak{M} := \{[f \geqq t] : 0 < t < 1\}$ upward enclosable $\mathfrak{S}$. Then there exists an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi \leqq 1$ such that $\fint f d\varphi^\star = 1$.*

Proof. — Follows from 3.11 applied to $\mathfrak{M}$. □

Example 3.15. — The last assertion becomes false when instead of $\fint f d\varphi^\star = 1$ one requires that $\fint f d\varphi_\star = 1$. As an example take $X = [0, 1]$ and $\mathfrak{S} = \mathrm{Comp}(X)$, and $f : f(x) = x$ for $0 \leqq x < 1$ and $f(1) = 0$.

4. Extension of the basic consequence.

The present section combines the basic consequence 3.7 with the ideas of Adamski [1] Lemma 3.2(b)(c) in order to obtain an extended version. We use the ideas in question as expressed in the next two lemmata. The initial part of the section assumes a lattice $\mathfrak{S}$ in X with $\varnothing \in \mathfrak{S}$.

LEMMA 4.1.— *Let $\varphi : \mathfrak{S} \to [0,\infty[$ be an inner $\star$ premeasure. For fixed $\mathfrak{M} \subset \mathfrak{S}$ nonvoid and upward directed define $\alpha : \mathfrak{S} \to [0,\infty[$ to be*

$$\alpha(S) = \sup\{\varphi(S \cap M) : M \in \mathfrak{M}\} \quad \textit{for } S \in \mathfrak{S}.$$

Then α is an inner $\star$ premeasure. Note that $\alpha \leqq \varphi$, and $\alpha(S) = \varphi(S)$ for $S \in \mathfrak{S}$ upward enclosable $\mathfrak{M}$; therefore

$$\alpha(S) = \sup\{\alpha(S \cap M) : M \in \mathfrak{M}\} \quad \text{for } S \in \mathfrak{S}.$$

Proof.— It is obvious that α is isotone and as claimed in the last sentence, and a routine verification that α is supermodular. Now let $A \subset B$ in $\mathfrak{S}$. For $M \in \mathfrak{M}$ then

$$\begin{aligned}\varphi(B \cap M) &\leqq \varphi(A \cap M) + \sup\{\varphi(S) : S \in \mathfrak{S} \text{ with } S \subset (B \cap M) \setminus (A \cap M)\}\\ &= \varphi(A \cap M) + \sup\{\varphi(S \cap M) : S \in \mathfrak{S} \text{ with } S \subset B \setminus A\}\\ &\leqq \alpha(A) + \sup\{\alpha(S) : S \in \mathfrak{S} \text{ with } S \subset B \setminus A\},\end{aligned}$$

so that $\alpha(B) \leqq \alpha(A) + \alpha_\star(B \setminus A)$. □

LEMMA 4.2.— *Let $\varphi : \mathfrak{S} \to [0,\infty[$ be isotone and submodular with $\varphi(\varnothing) = 0$. For fixed $\mathfrak{M} \subset \mathfrak{S}$ nonvoid and upward directed define $\alpha : \mathfrak{S} \to [0,\infty[$ to be*

$$\alpha(S) = \inf\{\varphi(S \cup M) - \varphi(M) : M \in \mathfrak{M}\} \quad \textit{for } S \in \mathfrak{S}.$$

0) *For fixed $S \in \mathfrak{S}$ the set function $M \mapsto \varphi(S \cup M) - \varphi(M)$ on $\mathfrak{S}$ is antitone.*

1) *α is isotone and submodular with $\alpha \leqq \varphi$.*

2) *In case $c := \sup\{\varphi(M) : M \in \mathfrak{M}\} < \infty$ we have*

$$\alpha(S) = \sup\{\varphi(S \cup M) : M \in \mathfrak{M}\} - c \quad \textit{for } S \in \mathfrak{S}.$$

3) *We have $\alpha_\star(T) - \alpha(S) \leqq \varphi_\star(T) - \varphi(S)$ for all $S \in \mathfrak{S}$ and $T \supset S$. Thus if $\mathfrak{T}$ is a lattice in X with $\varnothing \in \mathfrak{T}$ such that $\varphi = \hat{\varphi}$ then $\alpha = \hat{\alpha}$ as well.*

Proof. — 0) For $P \subset Q$ in $\mathfrak{S}$ we have

$$\begin{aligned}\varphi(S\cup Q)-\varphi(Q)&=\varphi((S\cup P)\cup Q)-\varphi(Q)\\&\leqq\varphi(S\cup P)-\varphi((S\cup P)\cap Q)\leqq\varphi(S\cup P)-\varphi(P).\end{aligned}$$

1) It is obvious that α is isotone with $\alpha \leqq \varphi$. To see that α is submodular let $A, B \in \mathfrak{S}$. For $P, Q \in \mathfrak{M}$ and $M \in \mathfrak{M}$ with $P, Q \subset M$ then 0) implies that

$$\begin{aligned}\alpha(A\cup B)+\alpha(A\cap B)&\leqq\big(\varphi((A\cup B)\cup M)-\varphi(M)\big)+\big(\varphi((A\cap B)\cup M)-\varphi(M)\big)\\&\leqq\big(\varphi(A\cup M)-\varphi(M)\big)+\big(\varphi(B\cup M)-\varphi(M)\big)\\&\leqq\big(\varphi(A\cup P)-\varphi(P)\big)+\big(\varphi(B\cup Q)-\varphi(Q)\big).\end{aligned}$$

The assertion follows.

2) To prove $\geqq$ let $P, Q \in \mathfrak{M}$ and $M \in \mathfrak{M}$ with $P, Q \subset M$. From 0) then

$$\begin{aligned}\varphi(S\cup P)&\leqq\varphi(S\cup M)=\varphi(M)+\big(\varphi(S\cup M)-\varphi(M)\big)\\&\leqq c+\big(\varphi(S\cup M)-\varphi(M)\big)\leqq c+\big(\varphi(S\cup Q)-\varphi(M)\big).\end{aligned}$$

The assertion follows. The direction $\leqq$ is an immediate consequence of $\varphi(M) + \alpha(S) \leqq \varphi(S \cup M)$ for $M \in \mathfrak{M}$.

3) Fix $S \in \mathfrak{S}$ and $T \supset S$. For $K \in \mathfrak{S}$ with $K \subset T$ and $M \in \mathfrak{M}$ we have

$$\begin{aligned}\varphi(K\cup M)-\varphi(S\cup M)&\leqq\varphi((S\cup K)\cup(S\cup M))-\varphi(S\cup M)\\&\leqq\varphi(S\cup K)-\varphi((S\cup K)\cap(S\cup M))\\&\leqq\varphi(S\cup K)-\varphi(S),\\\varphi(K\cup M)-\varphi(M)&\leqq\big(\varphi(S\cup M)-\varphi(M)\big)+\varphi(S\cup K)-\varphi(S),\\\alpha(K)&\leqq\alpha(S)+\varphi(S\cup K)-\varphi(S)\leqq\alpha(S)+\varphi_\star(T)-\varphi(S).\end{aligned}$$

It follows that $\alpha_\star(T) \leqq \alpha(S) + \varphi_\star(T) - \varphi(S)$. □

Next we need some further terms. For $\mathfrak{M} \subset \mathfrak{P}(X)$ nonvoid and upward directed we define $\mathfrak{S}(\mathfrak{M}) \subset \mathfrak{S}$ to consist of the $S \in \mathfrak{S}$ which are upward enclosable $\mathfrak{M}$. Thus $\mathfrak{S}(\mathfrak{M})$ is a lattice with $\varnothing \in \mathfrak{S}(\mathfrak{M})$. For isotone set functions $\beta : \mathfrak{S} \to [0, \infty[$ with $\beta(\varnothing) = 0$ and $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi(\varnothing) = 0$ we say that φ *supports* β *at* $\mathfrak{M}$ iff $\varphi \leqq \beta$ and $\varphi_\star|\mathfrak{M} = \beta_\star|\mathfrak{M}$. We say that φ *minisupports* β *at* $\mathfrak{M}$ iff in addition

$$\varphi(S) = \sup\{\varphi(S \cap H) : H \in \mathfrak{S}(\mathfrak{M})\} \quad \text{for } S \in \mathfrak{S};$$

this means that φ achieves to support β at $\mathfrak{M}$ with minimum expenditure. We note an immediate consequence of 4.1, and then deduce from 4.2 the main device for the sequel.

Remark 4.3. — Let $\varphi : \mathfrak{S} \to [0, \infty[$ be an inner $\star$ premeasure which supports β at $\mathfrak{M}$. Define $\alpha : \mathfrak{S} \to [0, \infty[$ to be

$$\alpha(S) = \sup\{\varphi(S \cap H) : H \in \mathfrak{S}(\mathfrak{M})\} \quad \text{for } S \in \mathfrak{S}.$$

Then α is an inner $\star$ premeasure which minisupports β at $\mathfrak{M}$.

LEMMA 4.4. — *Assume that $\mathfrak{M}, \mathfrak{N} \subset \mathfrak{P}(X)$ are nonvoid and upward directed with $M \subset N$ for all $M \in \mathfrak{M}$ and $N \in \mathfrak{N}$. Let $\beta : \mathfrak{S} \to [0, \infty[$ be isotone and submodular with $\beta(\varnothing) = 0$, and define $\alpha : \mathfrak{S} \to [0, \infty[$ as in 4.2 to be*

$$\alpha(S) = \inf\{\beta(S \cup H) - \beta(H) : H \in \mathfrak{S}(\mathfrak{M})\} \quad \textit{for } S \in \mathfrak{S}.$$

Assume that

$\vartheta : \mathfrak{S} \to [0, \infty[$ *is an inner $\star$ premeasure which minisupports β at $\mathfrak{M}$,*

$\varphi : \mathfrak{S} \to [0, \infty[$ *is an inner $\star$ premeasure which minisupports α at $\mathfrak{N}$.*

Then $\theta := \vartheta + \varphi$ is an inner $\star$ premeasure which minisupports β at $\mathfrak{M} \cup \mathfrak{N}$.

Proof. — 0) It is clear that $\theta_\star = \vartheta_\star + \varphi_\star$ on $\mathfrak{P}(X)$ and that θ is an inner $\star$ premeasure.

1) We claim that $\theta \leqq \beta$. For $S \in \mathfrak{S}$ and $H \in \mathfrak{S}(\mathfrak{M})$ we have

$$\begin{aligned} \vartheta(S \cap H) + \varphi(S) &\leqq \vartheta(S \cap H) + \alpha(S) \\ &\leqq \beta(S \cap H) + \beta(S \cup H) - \beta(H) \leqq \beta(S), \end{aligned}$$

and hence in fact $\theta(S) = \vartheta(S) + \varphi(S) \leqq \beta(S)$.

2) From the definition we have $\vartheta \leqq \theta$; on $\mathfrak{S}(\mathfrak{M})$ moreover $\alpha = 0$ and hence $\varphi = 0$, so that $\vartheta = \theta$.

3) For $S \in \mathfrak{S}$ we have

$$\begin{aligned} \sup\{\theta(S \cap R) &= \vartheta(S \cap R) + \varphi(S \cap R) : R \in \mathfrak{S}(\mathfrak{M} \cup \mathfrak{N})\} \\ &= \sup\{\vartheta(S \cap H) + \varphi(S \cap R) : H \in \mathfrak{S}(\mathfrak{M}) \text{ and } R \in \mathfrak{S}(\mathfrak{N})\} \\ &= \vartheta(S) + \varphi(S) = \theta(S). \end{aligned}$$

4) We claim that $\theta_\star(M) = \beta_\star(M)$ for $M \in \mathfrak{M}$. In fact, from 2) we have $\theta_\star(M) = \vartheta_\star(M) = \beta_\star(M)$.

5) We claim that $\theta_\star(N) = \beta_\star(N)$ for $N \in \mathfrak{N}$. First of all note that

$$\begin{aligned} c &:= \sup\{\beta(H) : H \in \mathfrak{S}(\mathfrak{M})\} = \sup\{\beta_\star(M) : M \in \mathfrak{M}\} \\ &= \sup\{\vartheta_\star(M) : M \in \mathfrak{M}\} \leqq \vartheta_\star(N), \end{aligned}$$

so that the assertion is clear when $c = \infty$. Thus we can assume that $c < \infty$. Then 4.2.2) furnishes

$$\begin{aligned} \theta_\star(N) &= \vartheta_\star(N) + \varphi_\star(N) = \vartheta_\star(N) + \alpha_\star(N) \\ &= \vartheta_\star(N) + \sup\{\alpha(S) : S \in \mathfrak{S} \text{ with } S \subset N\} \\ &= \vartheta_\star(N) + \sup\{\beta(S \cup H) - c : S \in \mathfrak{S} \text{ with } S \subset N \text{ and } H \in \mathfrak{S}(\mathfrak{M})\} \\ &= \big(\vartheta_\star(N) - c\big) + \sup\{\beta(S) : S \in \mathfrak{S} \text{ with } S \subset N\} \geqq \beta_\star(N), \end{aligned}$$

because $\vartheta_\star(N) \geqq c$ as shown above. The assertion follows. □

For the remainder of the section we fix a pair of lattices $\mathfrak{S}$ and $\mathfrak{T}$ in X with $\varnothing \in \mathfrak{S}, \mathfrak{T}$ and an isotone set function $\beta : \mathfrak{S} \to [0, \infty[$ with $\beta(\varnothing) = 0$.

THEOREM 4.5. — *Assume that $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$, and that $\mathfrak{T}$ separates $\mathfrak{S}$. Let β be submodular and $\beta = \hat{\beta}$. If $\mathfrak{M} \subset \mathfrak{P}(X)$ is nonvoid and well-ordered under inclusion then there exists an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ which minisupports β at $\mathfrak{M}$.*

Proof. — We can assume that $\varnothing \in \mathfrak{M}$.

1) A nonvoid subsystem $\mathfrak{P} \subset \mathfrak{M}$ is called *initial* iff $A \subset B$ for $A \in \mathfrak{M}$ and $B \in \mathfrak{P}$ implies that $A \in \mathfrak{P}$. Thus 1.i) $\{\varnothing\} \subset \mathfrak{M}$ is initial. 1.ii) If $\mathfrak{P}, \mathfrak{Q} \subset \mathfrak{M}$ are initial then either $\mathfrak{P} \subset \mathfrak{Q}$ or $\mathfrak{Q} \subset \mathfrak{P}$. In fact, if $\mathfrak{Q} \not\subset \mathfrak{P}$ and $B \in \mathfrak{Q}$ but $B \notin \mathfrak{P}$, then an $A \in \mathfrak{P}$ cannot fulfil $B \subset A$ and hence must fulfil $A \subset B$, so that $A \in \mathfrak{Q}$. Thus $\mathfrak{P} \subset \mathfrak{Q}$.

2) Define Σ to consist of all pairs $(\mathfrak{P}, \xi)$, where $\mathfrak{P} \subset \mathfrak{M}$ is an initial subsystem and $\xi : \mathfrak{S} \to [0, \infty[$ is an inner $\star$ premeasure which minisupports β at $\mathfrak{P}$. Σ is nonvoid, because the pair $(\mathfrak{P}, \xi)$ with $\mathfrak{P} = \{\varnothing\}$ and $\xi = 0$ is in Σ. For $(\mathfrak{P}, \xi), (\mathfrak{Q}, \eta) \in \Sigma$ we define $(\mathfrak{P}, \xi) \sqsubseteq (\mathfrak{Q}, \eta)$ iff $\mathfrak{P} \subset \mathfrak{Q}$, and $\xi \leqq \eta$ on $\mathfrak{S}$ and $\xi = \eta$ on $\mathfrak{S}(\mathfrak{P})$. This is of course an order relation on Σ.

3) We claim that Σ is upward inductive in $\sqsubseteq$. To see this let $\Lambda \subset \Sigma$ be nonvoid and totally ordered under $\sqsubseteq$.

3.i) First of all $\mathfrak{V} := \bigcup_{(\mathfrak{P},\xi)\in\Lambda} \mathfrak{P} \subset \mathfrak{M}$ is an initial subsystem. Define $\vartheta := \sup_{(\mathfrak{P},\xi)\in\Lambda} \xi$, so that $\vartheta : \mathfrak{S} \to [0, \infty[$ with $\vartheta \leqq \beta$. It is a routine verification that $\vartheta_\star = \sup_{(\mathfrak{P},\xi)\in\Lambda} \xi_\star$ on $\mathfrak{P}(X)$ and that ϑ is an inner $\star$ premeasure.

3.ii) For $V \in \mathfrak{V}$ there exists $(\mathfrak{P}, \xi) \in \Lambda$ such that $V \in \mathfrak{P}$ and hence $\vartheta_\star(V) = \beta_\star(V)$.

3.iii) For $S \in \mathfrak{S}$ we have

$$\begin{aligned} &\sup\{\vartheta(S \cap H) : H \in \mathfrak{S}(\mathfrak{V})\} \\ &\geqq \sup\{\xi(S \cap H) : H \in \mathfrak{S}(\mathfrak{P})\} = \xi(S) \quad \text{for all } (\mathfrak{P}, \xi) \in \Lambda, \end{aligned}$$

and hence $\sup\{\vartheta(S \cap H) : H \in \mathfrak{S}(\mathfrak{V})\} = \vartheta(S)$.

3.iv) All this proves that $(\mathfrak{V}, \vartheta) \in \Sigma$.

3.v) It remains to show that $(\mathfrak{P}, \xi) \sqsubseteq (\mathfrak{V}, \vartheta)$ for each $(\mathfrak{P}, \xi) \in \Lambda$. It is clear that $\mathfrak{P} \subset \mathfrak{V}$ and $\xi \leqq \vartheta$ on $\mathfrak{S}$. On $\mathfrak{S}(\mathfrak{P})$ we have

$$\begin{aligned} &\text{for } (\mathfrak{Q}, \eta) \in \Lambda \text{ with } (\mathfrak{Q}, \eta) \sqsubseteq (\mathfrak{P}, \xi)\text{: } \eta \leqq \xi, \\ &\text{for } (\mathfrak{Q}, \eta) \in \Lambda \text{ with } (\mathfrak{P}, \xi) \sqsubseteq (\mathfrak{Q}, \eta)\text{: } \eta = \xi; \end{aligned}$$

therefore $\eta \leqq \xi$ for all $(\mathfrak{Q}, \eta) \in \Lambda$, so that $\vartheta \leqq \xi$ and hence $\vartheta = \xi$. Thus in fact $(\mathfrak{P}, \xi) \sqsubseteq (\mathfrak{V}, \vartheta)$.

4) Now let $(\mathfrak{V}, \vartheta)$ be a maximal member of Σ. We have to prove that $\mathfrak{V} = \mathfrak{M}$. If not, then there is a smallest $M \in \mathfrak{M}$ which is not in $\mathfrak{V}$. Then

4.i) $V \subset M$ for all $V \in \mathfrak{V}$, because $\mathfrak{V} \subset \mathfrak{M}$ is an initial subsystem.

4.ii) By the choice of M the subsystem $\mathfrak{V} \cup \{M\} \subset \mathfrak{M}$ is initial as well.

4.iii) We shall invoke 4.4 for the set systems $\mathfrak{V}$ and $\{M\}$ and for β; the respective $\alpha : \mathfrak{S} \to [0, \infty[$ is submodular and fulfils $\alpha = \hat{\alpha}$ from 4.2.1)3). Thus from 3.9 and 4.3 we obtain an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ which minisupports α at $\{M\}$. Now 4.4 asserts that $\theta := \vartheta + \varphi$ is an inner $\star$ premeasure which minisupports β at $\mathfrak{V} \cup \{M\}$. Combined with 4.ii) this says that $(\mathfrak{V} \cup \{M\}, \theta) \in \Sigma$. Moreover $(\mathfrak{V}, \vartheta) \sqsubseteq (\mathfrak{V} \cup \{M\}, \theta)$, because on $\mathfrak{S}(\mathfrak{V})$ one has $\alpha = 0$ and hence $\theta = \vartheta$. This contradicts the assumption that $(\mathfrak{V}, \vartheta)$ be a maximal member of Σ. The proof is complete. □

The final result then reads as follows.

THEOREM 4.6. — *Assume that* $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$, *and that* $\mathfrak{T}$ *separates* $\mathfrak{S}$. *Let* β *be submodular and* $\beta = \hat{\beta}$. *Assume that*

$\mathfrak{M} \subset \mathfrak{S}$ *is a lattice such that* $\beta|\mathfrak{M}$ *is modular, and*

$\mathfrak{N} \subset \mathfrak{P}(X)$ *is nonvoid and well-ordered under inclusion,*

such that $M \subset N$ *for all* $M \in \mathfrak{M}$ *and* $N \in \mathfrak{N}$. *Then there exists an inner* $\star$ *premeasure* $\varphi : \mathfrak{S} \to [0, \infty[$ *which minisupports* β *at* $\mathfrak{M} \cup \mathfrak{N}$.

Proof. — After the model of part 4.iii) in the last proof we shall invoke 4.4 for $\mathfrak{M}$ and $\mathfrak{N}$ and for β; the respective $\alpha : \mathfrak{S} \to [0, \infty[$ is submodular and fulfils $\alpha = \hat{\alpha}$ from 4.2.1)3). From 3.7 and 4.3 we obtain an inner $\star$ premeasure $\vartheta : \mathfrak{S} \to [0, \infty[$ which minisupports β at $\mathfrak{M}$, and from 4.5 we obtain an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ which minisupports α at $\mathfrak{N}$. Then 4.4 asserts that $\theta := \vartheta + \varphi$ is an inner $\star$ premeasure which minisupports β at $\mathfrak{M} \cup \mathfrak{N}$. □

The result obtained in Adamski [1] Proposition 3.8 is for the case that $\mathfrak{M}$ consists of the members of a sequence $S_1 \subset \cdots \subset S_n \subset \cdots$ in $\mathfrak{S}$ and that $\mathfrak{N} = \{X\}$, in essence under fortified assumptions.

In this connection we note that the lattice $\mathfrak{T}$ with $\varnothing \in \mathfrak{T}$ which in 3.7 and 4.6 occurs in the assumptions does no more occur in the conclusions. When the assumptions hold true for some such $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$ then for the particular choice $\mathfrak{T} = (\mathfrak{S}\top\mathfrak{S})\perp$ as well. Thus one has the full theorems when one restricts oneself to $\mathfrak{T} = (\mathfrak{S}\top\mathfrak{S})\perp$. In this case there is also the opposite relation $\mathfrak{S} \subset (\mathfrak{T}\top\mathfrak{T})\perp$, which in [1] occurs in the overall assumptions. It can thus be said that the added assumption $\mathfrak{S} \subset (\mathfrak{T}\top\mathfrak{T})\perp$ does not narrow the results of [1].

However, it is an essential point that we removed the overall assumptions in [1] that $\mathfrak{S}$ be stable under countable intersections and that β be σ continuous at $\varnothing$. On the one hand this produces no loss, because after the fundamentals in MI 6.31 an inner $\star$ premeasure which for some $\bullet = \sigma\tau$ is $\bullet$ continuous at $\varnothing$ proves to be an inner $\bullet$ premeasure. On the other hand this extension permits to obtain specific results on inner $\star$ premeasures as in the final part of the last section.

There are some further comments which are related to nontrivial counterexamples. They will be presented in the final section below.

5. Some further comments and counterexamples.

We want to ask two questions. The first question comes from the direct comparison of 3.8 and 4.5. Under identical conditions we have proved

in 3.8: If $\mathfrak{M} \subset \mathfrak{S}$ is nonvoid and *totally ordered* under inclusion,

in 4.5: If $\mathfrak{M} \subset \mathfrak{P}(X)$ is nonvoid and *well-ordered* under inclusion,

then there exists an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ which (mini)supports β at $\mathfrak{M}$. Of course the question is whether in 4.5 (and hence in 4.6 too) the assumption *well-ordered* can be weakened to *totally ordered.* We shall see that the answer is no.

Example 5.1. — In a Hausdorff topological space X let $\mathfrak{S} = \mathrm{Comp}(X)$ and $\mathfrak{T} = \mathrm{Op}(X)$. Define $\beta : \mathfrak{S} \to [0,\infty[$ to be $\beta(\varnothing) = 0$ and $\beta(S) = 1$ for $S \neq \varnothing$. After 2.7 and 3.2 we are in the situation of 3.8 and 4.5. Assume that X is infinite. We assert that there exists a nonvoid and totally ordered system $\mathfrak{M} \subset \mathfrak{P}(X)$ of countable subsets of X such that there is no inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ which supports β at $\mathfrak{M}$.

Proof. — 1) Let $(a_l)_l$ be a sequence of pairwise different points in X, and define $f : X \to [0,\infty[$ to be $f(a_l) = 1 - \frac{1}{l}$ for $l \in \mathbb{N}$ and $f(x) = 0$ for the other $x \in X$. Then $\mathfrak{M} := \{[f \geqq t] : 0 < t < 1\} \subset \mathfrak{P}(X)$ consists of nonvoid countable subsets of X, and is nonvoid and totally ordered.

2) Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ is an inner $\star$ premeasure which supports β at $\mathfrak{M}$, that is $\varphi \leqq 1$ and $\varphi_\star([f \geqq t]) = \beta_\star([f \geqq t]) = 1$ for $0 < t < 1$. Now φ is a Radon premeasure and hence $\lambda := \varphi_\star|\mathrm{Bor}(X)$ a finite Borel-Radon measure. It fulfils $\lambda([f \geqq t]) = 1$ for $0 < t < 1$, whereas $[f \geqq t] \downarrow \varnothing$ for $t \uparrow 1$. This is a contradiction. □

In order to formulate the other question let as usual $\mathfrak{S}$ and $\mathfrak{T}$ be lattices in X with $\varnothing \in \mathfrak{S}, \mathfrak{T}$ and $\beta : \mathfrak{S} \to [0,\infty[$ be isotone with $\beta(\varnothing) = 0$. The results 3.7-3.8 and 4.5-4.6 were under the common conditions that on the one hand $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$ and $\mathfrak{T}$ separates $\mathfrak{S}$, and on the other hand that β is submodular and $\beta = \hat{\beta}$. Our question is whether these conditions are *necessary* for the respective conclusions, that is for the existence of inner $\star$ premeasures with the respective supportive properties. We restrict ourselves to the conditions on β, because there were some relevant considerations on $\mathfrak{S}$ and $\mathfrak{T}$ in the final part of Section 3. We shall see that the condition that β be submodular is indeed a necessary one, whereas the condition $\beta = \hat{\beta}$ is not, even when one assumes that $\hat{\beta} < \infty$. For the latter result we shall exhibit a Hausdorff topological space X and a submodular $\beta : \mathfrak{S} \to [0,\infty[$ on $\mathfrak{S} = \mathrm{Comp}(X)$ such that in case $\mathfrak{T} = \mathrm{Op}(X)$ one has $\beta \neq \hat{\beta}$ and $\hat{\beta} < \infty$, while in case $\mathfrak{T} = (\mathfrak{S}\top\mathfrak{S})\perp \supset \mathrm{Op}(X)$ one has $\beta = \hat{\beta}$, which implies that the above conclusions are all valid. We know of no counterexample where $\mathfrak{T} = (\mathfrak{S}\top\mathfrak{S})\perp$.

We refer to the related but different considerations in Anger [4] [5], where in the mainstream X is a locally compact Hausdorff topological space. We remark that the condition $\beta = \hat{\beta}$ becomes a necessary one, and is in fact an immediate consequence of the classical Dini theorem, whenever one can, in an appropriate topology on the space of all inner $\star$ premeasures $\varphi : \mathfrak{S} \to [0, \infty[$, invoke appropriate compact subsets. For these ideas we also refer to the work of Topsøe exemplified in [27] [28] [29].

Remark 5.2. — Assume that for each pair of subsets $A \subset B$ in $\mathfrak{S}$ there exists an inner $\star$ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ which supports β at $\{A, B\}$. Then β is submodular.

Proof. — For fixed $A, B \in \mathfrak{S}$ let $\varphi : \mathfrak{S} \to [0, \infty[$ be an inner $\star$ premeasure which supports β at $\{A \cap B, A \cup B\}$. Then $\beta(A \cup B) + \beta(A \cap B) = \varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B) \leqq \beta(A) + \beta(B)$. □

Example 5.3. — We fix a Hausdorff topological space X which is not discrete and hence not finite, but in which all compact subsets are finite; thus $\mathfrak{S} = \mathrm{Comp}(X)$ consists of the finite subsets of X. There are well-known countable X of this kind; see for example MI 9.8. Also fix $a \in X$ such that $\{a\}$ is not open. Define $\beta : \mathfrak{S} \to [0, \infty[$ to be $\beta(S) = 0$ for $S \subset \{a\}$ and $\beta(S) = 1$ for $S \not\subset \{a\}$. Then β is isotone and submodular, and $\beta_\star(T) = 0$ for $T \subset \{a\}$ and $\beta_\star(T) = 1$ for $T \not\subset \{a\}$. In case $\mathfrak{T} = \mathrm{Op}(X)$ therefore $\hat{\beta}(\{a\}) = 1$ while $\beta(\{a\}) = 0$, so that $\beta \neq \hat{\beta}$ and $\hat{\beta} \leqq 1 < \infty$. But in case $\mathfrak{T} = (\mathfrak{S} \top \mathfrak{S})\perp = \mathfrak{P}(X)$ one has of course $\beta = \hat{\beta}$. Therefore β is as required above.

BIBLIOGRAPHY

[1] W. ADAMSKI, Capacitylike set functions and upper envelopes of measures, Math. Ann., 229 (1977), 237–244.

[2] W. ADAMSKI, On regular extensions of contents and measures, J. Math. Analysis Appl., 127 (1987), 211–225.

[3] B. ANGER, Approximation of capacities by measures, in: Lecture Notes Math., 226, pp. 152–170, Springer, 1971.

[4] B. ANGER, Kapazitäten und obere Einhüllende von Maßen. Math. Ann., 199 (1972), 115–130.

[5] B. ANGER, Representation of capacities. Math. Ann., 229 (1977), 245–258.

[6] B. ANGER and J. LEMBCKE, Hahn-Banach type theorems for hypolinear functionals, Math. Ann., 209 (1974), 127–152.

[7] G. CHOQUET, Theory of capacities, Annales Inst. Fourier, Grenoble, 5 (1953-54), 131-295.

[8] G. CHOQUET, Forme abstraite du théorème de capacitabilité, Annales Inst. Fourier, Grenoble, 9 (1959), 83–89.

[9] C. DELLACHERIE, Quelques commentaires sur les prolongements de capacités, in: Lecture Notes Math., 191, 77–81, Springer, 1971.

[10] C. DELLACHERIE, Ensembles Analytiques, Capacités, Mesures de Hausdorff, Lecture Notes Math., 295, Springer, 1972.

[11] B. FUCHSSTEINER and H. KÖNIG, New versions of the Hahn-Banach theorem, in: Intern. Ser.Num. Math., 47, 255-266, Birkhäuser 1980.

[12] B. FUGLEDE, Capacity as a sublinear functional generalizing an integral, Danske Vid. Selsk. Mat. -Fys. Medd., 38 (1971), 1–44.

[13] P.R. HALMOS, Measure Theory. Van Nostrand 1950.

[14] A. HORN and A. TARSKI, Measures on Boolean algebras, Trans. Amer. Math. Soc., 64 (1948), 467–497.

[15] P.J. HUBER and V. STRASSEN, Minimax tests and the Neyman-Pearson lemma for capacities, Ann. Statist., 1 (1973), 251–263.

[16] J. KINDLER, A MAZUR-ORLICZ type theorem for submodular set functions, J. Math. Analysis Appl., 120 (1986), 533–546.

[17] H. KÖNIG, On the abstract Hahn-Banach theorem due to Rodé., Aequat. Math., 34 (1987), 89–95.

[18] H. KÖNIG, Measure and Integration: An Advanced Course in Basic Procedures and Applications, Springer, 1997.

[19] H. KÖNIG, Image measures and the so-called image measure catastrophe, Positivity, 1 (1997), 255–270.

[20] H. KÖNIG, What are signed contents and measures ? Math. Nachr., 204 (1999), 101–124.

[21] H. KÖNIG, The product theory for inner premeasures, Note di Mat., to appear.

[22] H. KÖNIG, Measure and integration: Comparison of old and new procedures, Arch. Math., 72 (1999), 192–205.

[23] H. KÖNIG, Measure and integration: Mutual generation of outer and inner premeasures, Annales Univ. Saraviensis, Ser. Math., 9 (1998), 99–122.

[24] H. KÖNIG, Measure and integration: Integral representations of isotone functionals, Annales Univ. Saraviensis, Ser. Math., 9 (1998), 123–153.

[25] G. RODÉ, Eine abstrakte Version des Satzes von Hahn-Banach, Arch. Math., 31 (1978), 474–481.

[26] V. STRASSEN, Meßfehler und Information, Z. Wahrsch. Verw. Geb., 2 (1964), 273–305.

[27] F. TOPSØE, Topology and Measure, Lecture Notes Math., 133, Springer, 1970.

[28] F. TOPSØE, On construction of measures, in: Topology and Measure I (Zinnowitz 1974) Part 2, 343-381, Univ. Greifswald 1978.

[29] F. TOPSØE, Radon measures, some basic constructions, in: Lecture Notes Math., 1033, 303–311, Springer, 1983.

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