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# DIRECT IMAGES IN NON-ARCHIMEDEAN ARAKELOV THEORY

**by Henri GILLET and Christophe SOULÉ**

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In this paper we develop a formalism of direct images for metrized vector bundles in the context of the non-archimedean Arakelov theory introduced in our joint work [BGS] with S. Bloch, and we prove a Riemann-Roch-Grothendieck theorem for this direct image. The new ingredient in the construction of the direct image is a non archimedean "analytic torsion current".

Let $K$ be the fraction field of a discrete valuation ring $\Lambda$, and $X$ a smooth projective variety over $K$. In [BGS] we defined the codimension $p$ arithmetic Chow group of $X$ as the inductive limit

$$\widehat{\mathrm{CH}}^p(X) = \varinjlim \mathrm{CH}^p(\mathcal{X})$$

of the Chow groups of the models $\mathcal{X}$ of $X$ over $\Lambda$. Assuming resolution of singularities (cf. 1.1 below) we proved that these groups can also be defined as rational equivalence classes of pairs $(Z, g)$, where $Z$ is a codimension $p$ cycle on $X$, and $g$ is a "Green current" for $Z$. Here a "current" is a projective system of cycle classes on the special fibers of all possible models of $X$. We have shown in [BGS] that many concepts and results in complex geometry and arithmetic intersection theory [GS1] have analogs in this context: differential forms, $\partial\bar{\partial}$-lemma, Poincaré-Lelong formula, intersection product, inverse and direct image maps in arithmetic Chow groups etc.

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On the other hand, we defined a metrized vector bundle on $X$ to be a bundle $E$ on $X$, together with a bundle $E_{\mathcal{X}}$ on some model $\mathcal{X}$ of $X$ which restricts to $E$ on $X$. The theory of characteristic classes (resp. Bott-Chern secondary characteristic classes) for hermitian vector bundles on arithmetic varieties [GS2] is replaced here by characteristic classes with values in the Chow groups of $\mathcal{X}$ (resp. the Chow groups of $\mathcal{X}$ with supports in its special fiber) ([BGS], (1.9), and §2 below). These classes are contravariant for maps of varieties over $K$.

However, we were not able in [BGS] to define direct images of metrized vector bundles. Recall that in Arakelov geometry, if $f : X \to Y$ is a map of varieties over $\mathbb{Z}$ which is smooth on the set of complex points of $X$, and if $\overline{E}$ is an hermitian vector bundle on $X$, once we choose a metric on $Tf$, the $L^2$-metric on the determinant line bundle $\det(Rf_* E)$ needs *not* be smooth in general. For this reason, following an idea of Quillen [Q], one is led to modify the $L^2$-metric on the determinant line bundle by multiplying it by the Ray-Singer analytic torsion of the Dolbeault complex, which results in a smooth metric. One of the key features of the Quillen metric, is that it gives a Riemann-Roch formula for the first Chern class of the determinant line bundle which is an equality of forms. More generally, if one chooses a complex of vector bundles $F.$, a quasi-isomorphism $F. \to Rf_*E$ and hermitian metrics on all the $F_i$'s, one can define a form $\theta$ on the complex points of $Y$, called the higher analytic torsion ([GS3], [BK]), which is well defined up to boundaries, and such that $-dd^c(\theta)$ is equal to the difference between the Chern character form of $\overline{F.}$ and the direct image of the product of the Chern character form of $\overline{E}$ with the Todd form of $Tf$. This form $\theta$ is the key ingredient when defining direct images for the "arithmetic Grothendieck groups" [GS2] [GS3].

In the non-archimedean case we face a similar difficulty. Assume $f : X \to Y$ is a morphism of projective varieties over $K$, induced by a map of models $\underline{f} : \mathcal{X} \to \mathcal{Y}$. Let $E_{\mathcal{X}}$ be a vector bundle on $\mathcal{X}$, with restriction $E$ to $X$. A natural candidate for a (non-archimeadean) metric on $Rf_* E$ is then the complex of vector bundles $R\underline{f}_* E_{\mathcal{X}}$ on $\mathcal{Y}$. But this choice will *not*, in general, be compatible with changes of models for both $X$ and $Y$. Indeed, consider a commutative diagram of models

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\pi} & \mathcal{X} \\ {\scriptstyle \underline{f}'}\downarrow & & \downarrow{\scriptstyle \underline{f}} \\ \mathcal{Y}' & \xrightarrow[p]{} & \mathcal{Y} \end{array}$$

where both $\underline{f}$ and $\underline{f}'$ induce $f : X \to Y$, when $\pi$ (resp. $p$) induces the identity on $X$ (resp. $Y$). The canonical map

$$Lp^* R\underline{f}_* E_{\mathcal{X}} \to R\underline{f}'_* L\pi^* E_{\mathcal{X}}$$

need not be an isomorphism. We are led to use the Chern character with supports of a cone of this map to define the $\mathcal{Y}'$-component $\theta_{\mathcal{Y}'}$ of the "higher analytic torsion" $\theta$, which is a current on $Y$ (see (50) and Prop. 4 for a precise definition).

We then define a Grothendieck group $\overset{\vee}{K}_0(X)$, generated by triples $(E, h, \eta)$ where $E$ is a bundle on $X$, $h$ is a metric on $E$ and $\eta$ is a sum of currents of all degrees on $X$. The relations in $\overset{\vee}{K}_0(X)$ come from exact sequences of vector bundles on $X$ (§2.6, (37)). By imposing that $\eta$ be smooth (i.e. $\eta$ consists of an inductive system of cycle classes and not only a projective one, see [BGS] and 1.2. below), we also define a subgroup $\widehat{K}_0(X) \subset \overset{\vee}{K}_0(X)$. Now let $f : X \to Y$ be any morphism, and choose a (virtual) metric on the relative tangent complex $Tf$. We attach to these data a direct image morphism

$$f_* : \overset{\vee}{K}_0(X) \to \overset{\vee}{K}_0(Y).$$

If $\underline{f} : \mathcal{X} \to \mathcal{Y}$ is a map of models inducing $f$, if $E$ is a bundle on $X$, if the metric $h$ on $E$ (resp. the virtual metric on $Tf$) is defined by a bundle $E_{\mathcal{X}}$ on $\mathcal{X}$ (resp. by $T\underline{f}$), and if $R^q \underline{f}_* E_{\mathcal{X}} = 0$, $q > 0$, the direct image $f_*(E, h, 0)$ is the class of $(f_* E, \underline{f}_* E_{\mathcal{X}}, \theta)$, where $\theta$ is the higher analytic torsion of $E_{\mathcal{X}}$ (Prop. 4, Th. 1).

When $f$ is flat, $f_*$ maps $\widehat{K}_0(X)$ into $\widehat{K}_0(Y)$ and a Riemann-Roch-Grothendieck theorem holds for Chern characters with values in $\widehat{\mathrm{CH}}^* \otimes \mathbb{Q}$ (Th. 1, i) and Th. 2, ii)). This is not so surprising, as it follows from the definition of $\theta$ and the Riemann-Roch-Grothendieck theorem with values in Chow groups of projective schemes over $\Lambda$. What is more involved is, first, to show that the family $\theta = (\theta_{\mathcal{Y}'})$ does define a current on $Y$ (a form when $f$ is flat) (Prop. 4) and, second, to check that the expected anomaly formulae for the change of metrics on either $E$ or $Tf$ are true in our case (Th. 1, (64) and (66)). These facts rely upon the vanishing of the direct image of the relative Todd class with support of birational maps (Prop. 3 ii)). This key lemma is itself a consequence of the proof by Franke of a refined Riemann-Roch formula conjectured by Saito (this proof of Franke [Fr] remains unfortunately unpublished). When $\Lambda$ is the localization of an algebra of finite type over a field of charactristic zero, we also give an alternative proof using the recent proof of the weak factorization conjecture by Wlodarczyk et al. ([W], [AKMW]). Finally we prove, in Thm. 2 i) and iii), that our direct image is functorial and satisfies the projection formula.

The paper is organized as follows. In §1 we review the arithmetic intersection theory of [BGS], and we also introduce a group $\check{\mathrm{CH}}^p(X)$ containing $\widehat{\mathrm{CH}}^p(X)$ which is always covariant (this definition was inspired by similar definitions by Burgos [B] and Zha [Z]). In §2 we develop the theory of secondary characteristic classes for metrized complexes of vector bundles, we introduce a notion of virtual metric for objects in the derived category of vector bundles $X$, and we define arithmetic Grothendieck groups $\widehat{K}_0(X) \subset \check{K}_0(X)$ together with Chern characters from $\check{K}_0(X)$ (resp. $\widehat{K}_0(X)$) to $\oplus_{p\geq 0}\check{\mathrm{CH}}^p(X) \otimes \mathbb{Q}$ (resp. $\oplus_{p\geq 0}\widehat{\mathrm{CH}}^p(X) \otimes \mathbb{Q}$). In §3 we prove several facts about the secondary Todd classes of birational maps of models. In §4, after defining the higher analytic torsion currents (Prop. 4), we define the direct images $f_*$ on $\check{K}_0$, and we give properties of this map, including a Riemann-Roch-Grothendieck theorem with values in $\check{\mathrm{CH}}$ (Th. 1 and Th. 2).

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## 1. Cycles.

**1.1.** We first recall some definitions and results in "non-archimedean Arakelov theory", from our joint work with S. Bloch [BGS], to which we refer the reader for more details.

Let $\Lambda$ be an excellent discrete valuation ring with quotient field $K$ and residue field $k$. Let $X$ be a smooth projective scheme over Spec$(K)$. Let $\mathcal{X}$ be a regular scheme, projective and flat over Spec$(\Lambda)$, together with an isomorphism of its generic fiber $\mathcal{X}_K$ with $X$. We denote by $\mathcal{X}_0 = \mathcal{X} \underset{\mathrm{Spec}(\Lambda)}{\times} \mathrm{Spec}(k)$ the special fiber of $\mathcal{X}$ and by $i : \mathcal{X}_0 \to \mathcal{X}$ its closed immersion into $\mathcal{X}$. We shall say that $\mathcal{X}$ is a *model* of $X$ when $\mathcal{X}_0^{\mathrm{red}}$ is a divisor with normal crossings (such a scheme was called a "DNC model" in [BGS]).

A *map of varieties* $X \to Y$ will mean a morphism over $K$ between smooth projective schemes over $K$. When $\mathcal{X}$ and $\mathcal{Y}$ are models of $X$ and $Y$ respectively, a *map of models* $\mathcal{X} \to \mathcal{Y}$ is any morphism defined over $\Lambda$. When $X = Y$, a map of models $\mathcal{X} \to \mathcal{Y}$ which induces the identity on $X$ will be called a *morphism of models*. A morphism between models is *good* when it is the composite of blow ups with integral regular centers contained in the special fiber.

We shall assume that axioms (M1) and (M2) of [BGS] (1.1) hold. Axiom (M1) says that, given any scheme $\mathcal{X}$, projective and flat over $\Lambda$, with smooth generic fiber $X$, there exists a model $\mathcal{X}'$ of $X$ and a morphism $\mathcal{X}' \to \mathcal{X}$ over $\Lambda$. Axiom (M2) says that, given two models $\mathcal{X}$ and $\mathcal{X}'$ of $X$, there exists a third model $\mathcal{X}''$, a morphism of models $\mathcal{X}'' \to \mathcal{X}'$, and a good morphism of models $\mathcal{X}'' \to \mathcal{X}$.

By Hironaka [H], these axioms are satisfied when $\Lambda$ is a localization of an algebra of finite type over a field of characteristic zero.

We write $\mathcal{M}(X)$ for the category of models of $X$.

**1.2**. Under the assumption of 1.1, let $\mathcal{X}$ be a model of $X$ and let $p \geq 0$ be an integer. Denote by $\mathrm{CH}_p(\mathcal{X}_0)$ the Chow homology group of dimension $p$ algebraic cycles on $\mathcal{X}_0$ modulo rational equivalence, and by $\mathrm{CH}^p(\mathcal{X}_0)$ the Chow cohomology group of codimension $p$ of $\mathcal{X}_0$, i.e. the bivariant group $\mathrm{CH}^p(\mathcal{X}_0 \xrightarrow{\mathrm{id}} \mathcal{X}_0)$ of [F], 17.1.

When $X$ is fixed, any morphism $\pi : \mathcal{X}' \to \mathcal{X}$ between models of $X$ induces both direct images

$$\pi_* : \mathrm{CH}_p(\mathcal{X}'_0) \to \mathrm{CH}_p(\mathcal{X}_0)$$
$$\pi_* : \mathrm{CH}^p(\mathcal{X}'_0) \to \mathrm{CH}^p(\mathcal{X}_0)$$

and inverse images

$$\pi^* : \mathrm{CH}_p(\mathcal{X}_0) \to \mathrm{CH}_p(\mathcal{X}'_0)$$
$$\pi^* : \mathrm{CH}^p(\mathcal{X}_0) \to \mathrm{CH}^p(\mathcal{X}'_0)$$

([BGS] (1.4). Notice the existence of $\pi_* : \mathrm{CH}^p(\mathcal{X}'_0) \to \mathrm{CH}^p(\mathcal{X}_0)$ and $\pi^* : \mathrm{CH}_p(\mathcal{X}_0) \to \mathrm{CH}_p(\mathcal{X}'_0)$ is due to the fact that the map $\pi$ is a local complete intersection morphism These were denoted $\pi_!$ and $\pi^!$ respectively in [BGS]). The projection formula implies

$$\pi_* \pi^* = \mathrm{id}. \tag{1}$$

If $d$ is the dimension of $\mathcal{X}_0$ over $k$, we may consider the inductive limits with respect to $\pi^*$:

$$A^{pp}_{\text{closed}}(X) := \varinjlim_{\mathcal{M}(X)} \mathrm{CH}^p(\mathcal{X}_0)$$

$$\widetilde{A}^{pp}(X) := \varinjlim_{\mathcal{M}(X)} \mathrm{CH}_{d-p}(\mathcal{X}_0),$$

as well as the projective limits under $\pi_*$:

$$D^{pp}_{\text{closed}}(X) := \varprojlim_{\mathcal{M}(X)} \mathrm{CH}^p(\mathcal{X}_0)$$

$$\widetilde{D}^{pp}(X) := \varprojlim_{\mathcal{M}(X)} \mathrm{CH}_{d-p}(\mathcal{X}_0).$$

By analogy with Arakelov theory [GS1] these groups are called, respectively, closed $(p,p)$-forms, $(p,p)$-forms modulo boundaries, closed $(p,p)$-currents, and $(p,p)$-currents modulo boundaries.

From (1) it follows that there are canonical inclusions

$$A^{pp}_{\text{closed}}(X) \subset D^{pp}_{\text{closed}}(X)$$

and

$$\widetilde{A}^{pp}(X) \subset \widetilde{D}^{pp}(X)$$

of forms into currents.

We shall also denote by $A^{pp}_{\text{closed}}(X)_{\mathbb{Q}}$, $\widetilde{A}^{pp}(X)_{\mathbb{Q}}, \ldots$ the tensor products $A^{pp}_{\text{closed}}(X) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, $\widetilde{A}^{pp}(X) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \ldots$ Furthermore we let

$$A_{\text{closed}}(X)_{\mathbb{Q}} = \bigoplus_{p \geq 0} A^{pp}_{\text{closed}}(X)_{\mathbb{Q}},$$

and we define similarly $\widetilde{A}(X)_{\mathbb{Q}}$ etc.

**1.3.** Given a model $\mathcal{X}$, there is a morphism

$$i^*\, i_* : \mathrm{CH}_{d-p}(\mathcal{X}_0) \to \mathrm{CH}^{p+1}(\mathcal{X}_0),$$

obtained by composing the direct image in Chow homology

$$i_* : \mathrm{CH}_{d-p}(\mathcal{X}_0) \to \mathrm{CH}_{d-p}(\mathcal{X}),$$

the Poincaré duality

$$\mathrm{CH}_{d-p}(\mathcal{X}) \simeq \mathrm{CH}^{p+1}(\mathcal{X}),$$

and the restriction map in Chow cohomology

$$i^* : \mathrm{CH}^{p+1}(\mathcal{X}) \to \mathrm{CH}^{p+1}(\mathcal{X}_0).$$

When $\mathcal{X}$ varies, these maps are compatible with $\pi_*$ and they induce a morphism

$$dd^c : \widetilde{D}^{pp}(X) \to D^{p+1,p+1}(X)$$

on projective limits. Using resolution of singularities, one gets the following result ([BGS], Th. 2.3.1):

PROPOSITION 1.

i) *A current $g \in \widetilde{D}^{pp}(X)$ lies in $\widetilde{A}^{pp}(X)$ if and only if $dd^c(g)$ lies in the subgroup $A^{p+1,p+1}_{\text{closed}}(X)$ of $D^{p+1,p+1}_{\text{closed}}(X)$.*

ii) *The kernel (resp. cokernel) of $dd^c$ coincides with the kernel (resp. cokernel) of $i^*\, i_*$ on any model of $X$.*

**1.4.** Let $Y \subset X$ be a closed integral subvariety of codimension $p$. For any model $\mathcal{X}$ we may consider the Zariski closure $\overline{Y}$ of $Y$ in $\mathcal{X}$, and the restriction $i^*[\overline{Y}] \in \mathrm{CH}^p(\mathcal{X}_0)$ of its cycle class on $\mathcal{X}$. These are compatible with $\pi_*$ and we get this way a closed current

$$\delta_Y = (i^*[\overline{Y}]) \in D^{pp}_{\text{closed}}(X).$$

When $Z = \sum_\alpha n_\alpha\, Y_\alpha \in Z^p(X)$ is any algebraic cycle of codimension $p$ on $X$, we let

$$\delta_Z = \sum_\alpha n_\alpha\, \delta_{Y_\alpha}.$$

A *Green current* for $Z$ is any current $g \in \widetilde{D}^{p-1,p-1}(X)$ such that $dd^c(g) + \delta_Z$ lies in $A^{pp}_{\text{closed}}(X)$.

For example, let $W \subset X$ be a closed integral subvariety of codimension $p-1$ and $f \in k(W)^*$ a non trivial rational function on $W$. Let $\mathrm{div}(f)$ be the divisor of $f$ on $W$, viewed as a codimension $p$ cycle on $X$, let $\overline{\mathrm{div}(f)}$ be its Zariski closure on a model $\mathcal{X}$, and let $\mathrm{div}_{\mathcal{X}}(f)$ be the divisor of $f$ on the Zariski closure of $W$ in $\mathcal{X}$. Consider the difference

$$\mathrm{div}_v(f)_{\mathcal{X}} = \overline{\mathrm{div}(f)} - \mathrm{div}_{\mathcal{X}}(f).$$

The family $-\mathrm{div}_v(f) = (-\mathrm{div}_v(f)_{\mathcal{X}})$ is then a Green current for the cycle $\mathrm{div}(f)$ ([BGS], (3.1)).

**1.5.** For any $p \geq 0$, the *arithmetic Chow group* $\widehat{\mathrm{CH}}^p(X)$ is defined as the quotient of the abelian group of pairs $(Z, g)$, where $Z \in Z^p(X)$ is a codimension $p$ algebraic cycle on $X$ and $g$ is a Green current for $X$, by the subgroup generated by the set of all pairs $(\mathrm{div}(f), -\mathrm{div}_v(f))$, where

$f$ is a non zero rational function on a codimension $(p-1)$ closed integral subvariety in $X$.

Let us also introduce the group $\check{\mathrm{CH}}^p(X)$, equal to the quotient of $Z^p(X) \oplus \widetilde{D}^{p-1,p-1}(X)$ by the group generated by pairs $(\mathrm{div}(f), -\mathrm{div}_\nu(f))$ as above (compare [B] and [Z]). Clearly there is an inclusion

$$\widehat{\mathrm{CH}}^p(X) \subset \check{\mathrm{CH}}^p(X).$$

Let

$$\omega : \check{\mathrm{CH}}^p(X) \to D^{pp}_{\mathrm{closed}}(X)$$

be the morphism sending the class of $(Z, g)$ to $dd^c(g) + \delta_Z$ (this kills the relations in $\check{\mathrm{CH}}^p(X)$, cf. [BGS] Prop. (3.1.1)). The subgroup $\widehat{\mathrm{CH}}^p(X)$ consists of those $x$ in $\check{\mathrm{CH}}^p(X)$ such that $\omega(x)$ lies in the subgroup $A^{pp}_{\mathrm{closed}}(X)$ of $D^{pp}_{\mathrm{closed}}(X)$.

We also denote by

$$a : \widetilde{D}^{p-1,p-1}(X) \to \check{\mathrm{CH}}^p(X)$$

the morphism sending $\eta$ to the class of $(0, \eta)$. By Proposition 1 i), $\eta$ lies in $\widetilde{A}^{p-1,p-1}(X)$ if and only if its image $a(\eta)$ lies in $\widehat{\mathrm{CH}}^p(X)$.

As shown in [BGS] Th. 3.3.3, there is a canonical isomorphism

$$\varinjlim_{\mathcal{M}(X)} \mathrm{CH}^p(\mathcal{X}) \xrightarrow{\sim} \widehat{\mathrm{CH}}^p(X) \tag{2}$$

and (taking inverse limit in the diagram of op.cit., p. 461) it extends to an isomorphism

$$\varprojlim_{\mathcal{M}(X)} \mathrm{CH}^p(\mathcal{X}) \xrightarrow{\sim} \check{\mathrm{CH}}^p(X). \tag{3}$$

Given $\eta = (\eta_{\mathcal{X}}) \in \widetilde{D}^{p-1,p-1}(X)$ this isomorphism sends $(i_* \eta_{\mathcal{X}})$ to $a(\eta)$, and if $Z \in Z^p(X)$ it sends the projective system of Zariski closures of $Z$ in $\mathcal{X}$ to the class of $(Z, 0)$.

**1.6.** Let $f : X \to Y$ be a map of varieties. We know from [BGS] 1.6 that forms are contravariant and currents are covariant. Furthermore, it follows from (2) that $f$ induces pull-back morphisms

$$f^* : \widehat{CH}^p(Y) \to \widehat{CH}^p(X)$$

and from (3) we get direct image morphisms

$$f_* : \check{\mathrm{CH}}^p(X) \to \check{\mathrm{CH}}^{p-\delta}(X),$$

where $\delta$ is the relative dimension of $X$ over $Y$. We may also describe $f_*$ as mapping the class of $(Z,g)$ to the class of $(f_*(Z), f_*(g))$, where $f_*(Z)$ is the usual direct image of the cycle $Z$ [F]. Given two maps of varieties $f : X \to Y$ and $h : Y \to Z$, we have $(hf)_* = h_* f_*$ and $(hf)^* = f^* h^*$.

Assume furthermore that $f$ is flat. Then, as was shown in [BGS], Thms. (4.1.1) and (4.2.1), the morphism $f_*$ maps forms to forms and respects $\widehat{\mathrm{CH}}$:

$$f_*\big(\widetilde{A}^{pp}(X)\big) \subset \widetilde{A}^{p-\delta,p-\delta}(X),$$
$$f_*\big(A^{pp}_{\mathrm{closed}}(X)\big) \subset A^{p-\delta,p-\delta}_{\mathrm{closed}}(X),$$
$$f_*\big(\widehat{\mathrm{CH}}^p(X)\big) \subset \widehat{\mathrm{CH}}^{p-\delta}(X).$$

**1.7**. From formula (2) we also deduce a graded ring structure

$$\widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q(X) \to \widehat{\mathrm{CH}}^{p+q}(X) \tag{4}$$

on arithmetic Chow groups. From (3) and the projection formula

$$\pi_*(x\,\pi^*(y)) = \pi_*(x)y$$

for any morphism $\pi : \mathcal{X}' \to \mathcal{X}$ between models of $X$, $x \in \mathrm{CH}^p(\mathcal{X}')$, $y \in \mathrm{CH}^q(\mathcal{X})$, we deduce a pairing

$$\check{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q(X) \to \check{\mathrm{CH}}^{p+q}(X) \tag{5}$$

extending (4), and turning $\check{\mathrm{CH}}(X)$ into a graded module on $\widehat{\mathrm{CH}}(X)$.

When $f : X \to Y$ is a map of varieties, the formulae

$$f^*(xy) = f^*(x)f^*(y)$$

and

$$f_*(xf^*(y)) = f_*(x)y$$

hold when $x$ lies in $\widehat{\mathrm{CH}}^p(Y)$ (resp. $\check{\mathrm{CH}}^p(X)$) and $y$ lies in $\widehat{\mathrm{CH}}^q(Y)$.

Similar facts are true for pairings between forms and currents ([BGS] 1.5 and Proposition (1.6.2)).

## 2. Vector bundles and characteristic classes.

**2.1.** We keep the notation of Section 1.1. Let $E$ be a vector bundle on $X$. A *metric* on $E$ is determined by the choice of a vector bundle $h = E_{\mathcal{X}}$ on some model $\mathcal{X}$ of $X$, together with an isomorphism $E \simeq E_{\mathcal{X}|X}$ of $E$ with the restriction of $E_{\mathcal{X}}$ to $X$. By convention, given any morphism $\pi : \mathcal{X}' \to \mathcal{X}$ between models of $X$, $\pi^* E_{\mathcal{X}}$ defines the same metric as $E_{\mathcal{X}}$ (see [BGS] (1.9.1)). Notice that (by [RG] Part I, Th. 5.2.2 together with [M] Th. 7.10) any bundle $E$ on $X$ has a metric.

Let $\phi$ denote either the Chern character ch or the Todd genus. Given any metrized vector bundle $\overline{E} = (E, h)$ on $X$, we can attach to $\overline{E}$ a closed form

$$\phi(\overline{E}) \in A_{\text{closed}}(X)_{\mathbb{Q}}.$$

If $E_{\mathcal{X}}$ is an extension of $E$ defining $h$, this form $\phi(\overline{E})$ is defined as the image in the direct limit of Chow cohomology groups of

$$\phi(i^* E_{\mathcal{X}}) \in \mathrm{CH}^\bullet(\mathcal{X}_0)_{\mathbb{Q}} = \bigoplus_{p \geq 0} \mathrm{CH}^p(\mathcal{X}_0)_{\mathbb{Q}}.$$

The Chern character is such that

$$\mathrm{ch}(\overline{E} \oplus \overline{F}) = \mathrm{ch}(\overline{E}) + \mathrm{ch}(\overline{F})$$

and

$$\mathrm{ch}(\overline{E} \otimes \overline{F}) = \mathrm{ch}(\overline{E})\, \mathrm{ch}(\overline{F}),$$

while the Todd class is multiplicative

$$\mathrm{Td}(\overline{E} \oplus \overline{F}) = \mathrm{Td}(\overline{E})\, \mathrm{Td}(\overline{F}).$$

Here, given $\overline{E} = (E, E_{\mathcal{X}})$ and $\overline{F} = (F, F_{\mathcal{X}'})$, their sum $\overline{E} \oplus \overline{F}$ is defined as $E \oplus F$ extended to $\pi^* E_{\mathcal{X}} \oplus (\pi')^* F_{\mathcal{X}'}$ on any model $\mathcal{X}''$ with morphisms of models $\pi : \mathcal{X}'' \to \mathcal{X}$ and $\pi' : \mathcal{X}'' \to \mathcal{X}'$.

From (2) we may also define a class

$$\widehat{\phi}(\overline{E}) \in \widehat{\mathrm{CH}}^\bullet(X)_{\mathbb{Q}}$$

such that $\omega(\widehat{\phi}(\overline{E})) = \phi(\overline{E})$. This is just the image of $\phi(E_{\mathcal{X}}) \in \mathrm{CH}^\bullet(\mathcal{X})_{\mathbb{Q}}$ in the inductive limit.

**2.2.** Let $X$ be as above and let $\mathcal{X}$ be a model of $X$. Consider a bounded acyclic complex

$$(E^\bullet, d) = (E^0 \xrightarrow{d} E^1 \xrightarrow{d} \cdots \xrightarrow{d} E^k)$$

of bundles over $X$. For every $n \geq 0$, let $E^n_{\mathcal{X}}$ be a vector bundle on $\mathcal{X}$ restricting to $E^n$ on $X$. Let $\phi$ be either ch or $\mathrm{Td}-1$. We denote by $\phi(E^\bullet_{\mathcal{X}})$ the elements

$$\mathrm{ch}(E^\bullet_{\mathcal{X}}) = \sum_{n=0}^{k} (-1)^n \, \mathrm{ch}(E^n_{\mathcal{X}}) \tag{6}$$

and

$$(\mathrm{Td}-1)(E^\bullet_{\mathcal{X}}) = \Big(\prod_{n=0}^{k} \mathrm{Td}(E^n_{\mathcal{X}})^{(-1)^n}\Big) - 1 \tag{7}$$

in $\mathrm{CH}(\mathcal{X})_{\mathbb{Q}}$.

PROPOSITION 2. — *There exists a unique class*

$$\phi_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}}) \in \mathrm{CH}_\bullet(\mathcal{X}_0)_{\mathbb{Q}}$$

*with the following three properties:*

i) *One has*

$$i_* \, \phi_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}}) = \phi(E^\bullet_{\mathcal{X}}).$$

ii) *Let $f : \mathcal{Y} \to \mathcal{X}$ be any map of models as in §1.1. Then*

$$f^* \, \phi_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}}) = \phi_{\mathcal{X}_0}(f^*(E^\bullet_{\mathcal{X}})).$$

iii) *Assume that, for all $n \geq 0$, the differential $d : E^n \to E^{n+1}$ extends to $d_{\mathcal{X}} : E^n_{\mathcal{X}} \to E^{n+1}_{\mathcal{X}}$ and that $(E^\bullet_{\mathcal{X}}, d_{\mathcal{X}})$ is acyclic on $\mathcal{X}$. Then*

$$\phi_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}}) = 0.$$

*Furthermore*

iv) *Let*

$$0 \to S^\bullet_{\mathcal{X}} \to E^\bullet_{\mathcal{X}} \to Q^\bullet_{\mathcal{X}} \to 0$$

*be an exact sequence of bounded complexes of vector bundles over $\mathcal{X}$. Assume that the restriction of $S^\bullet_{\mathcal{X}}$, $E^\bullet_{\mathcal{X}}$ and $Q^\bullet_{\mathcal{X}}$ to $X$ are acyclic complexes. Then the following identities hold:*

$$\mathrm{ch}_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}}) = \mathrm{ch}_{\mathcal{X}_0}(S^\bullet_{\mathcal{X}}) + \mathrm{ch}_{\mathcal{X}_0}(Q^\bullet_{\mathcal{X}}), \tag{8}$$

$$\begin{aligned}(\mathrm{Td}-1)_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}}) &= (\mathrm{Td}-1)_{\mathcal{X}_0}(S^\bullet_{\mathcal{X}}) \cdot (i^* \, \mathrm{Td}(Q^\bullet_{\mathcal{X}})) + (\mathrm{Td}-1)_{\mathcal{X}_0}(Q^\bullet_{\mathcal{X}}) \\ &= (\mathrm{Td}-1)_{\mathcal{X}_0}(S_{\mathcal{X}}) + i^*(\mathrm{Td}(S^\bullet_{\mathcal{X}})) \cdot (\mathrm{Td}-1)_{\mathcal{X}_0}(Q^\bullet_{\mathcal{X}}).\end{aligned} \tag{9}$$

v) *If $F_{\mathcal{X}}$ is any bundle on $\mathcal{X}$,*

$$\mathrm{ch}_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}} \otimes F_{\mathcal{X}}) = \mathrm{ch}_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}}) \, i^* \, \mathrm{ch}(F_{\mathcal{X}}). \tag{10}$$

vi) *If* $E^\bullet_{\mathcal{X}}[1]$ *denotes the shift by one of* $E^\bullet_{\mathcal{X}}$, *the following two equalities hold:*

$$\mathrm{ch}_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}}[1]) = -\mathrm{ch}_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}}) \tag{11}$$

*and*

$$(\mathrm{Td}-1)_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}}[1]) = -(\mathrm{Td}-1)_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}})\,\mathrm{Td}(E^\bullet_{\mathcal{X}})^{-1}. \tag{12}$$

*Proof.* — We use the Grassmann-graph construction [BFM] (see also [F] §18.1, and [GS4] §1 for a variant of this construction). Let $e_n$ be the rank of $E^n_{\mathcal{X}}$, consider the Grassmann variety $G_n = \mathrm{Grass}_{e_n}(E^n_{\mathcal{X}} \oplus E^{n+1}_{\mathcal{X}})$, $n \geq 0$, and let

$$G = G_0 \underset{\mathcal{X}}{\times} G_1 \underset{\mathcal{X}}{\times} \cdot \underset{\mathcal{X}}{\times} G_{k-1}.$$

The acyclic complex $(E^\bullet, d)$ defines a section

$$\varphi : X \underset{K}{\times} \mathbb{P}^1 \to G \underset{\Lambda}{\times} \mathbb{P}^1$$

of the projection

$$G \underset{\Lambda}{\times} \mathbb{P}^1 \to \mathcal{X} \underset{\Lambda}{\times} \mathbb{P}^1$$

on the generic fiber $X \underset{K}{\times} \mathbb{P}^1$. This map $\varphi$ is given by the graphs of the maps $\lambda d$ at the point $(x, \lambda) \in X \underset{K}{\times} \mathbb{A}^1$; the fact that it extends to $X \underset{K}{\times} \mathbb{P}^1$ is shown in [F], proof of Lemma 18.1, p. 342. We let $\mathcal{W}^1$ be the Zariski closure of $\varphi(X \underset{K}{\times} \mathbb{P}^1)$ in $G \underset{\Lambda}{\times} \mathbb{P}^1$, and $\mathcal{W} \to \mathcal{W}^1$ a resolution of $\mathcal{W}^1$. The scheme $\mathcal{W}$ is a model of $X \underset{K}{\times} \mathbb{P}^1$. For each $n \geq 0$, the tautological bundle of rank $e_n$ on $G$ defines a vector bundle $E^n_{\mathcal{W}}$ on $\mathcal{W}$, and there exists an acyclic complex of vector bundles $(\widetilde{E}^\bullet, \widetilde{d})$ on $X \underset{K}{\times} \mathbb{P}^1$ where, for each $n \geq 0$, $\widetilde{E}^n$ is the restriction of $E^n_{\mathcal{W}}$ to $X \underset{K}{\times} \mathbb{P}^1$.

Let $\mathcal{W}^0$ (resp. $\mathcal{W}^\infty$) be the Zariski closure of $X \times \{0\}$ (resp. $X \times \{\infty\}$) in $\mathcal{W}$. Denote by

$$j^0 : \mathcal{X}^0 \to \mathcal{W}$$

(resp. $j^\infty : \mathcal{X}^\infty \to \mathcal{W}$) the composite of a resolution of singularities $\mathcal{X}^0 \to \mathcal{W}^0$ (resp. $\mathcal{X}^\infty \to \mathcal{W}^\infty$) with the inclusion $\mathcal{W}^0 \to \mathcal{W}$ (resp. $\mathcal{W}^\infty \to \mathcal{W}$). Let $p : \mathcal{W} \to \mathcal{X}$, $\pi^0 : \mathcal{X}^0 \to \mathcal{X}$ and $\pi^\infty : \mathcal{X}^\infty \to \mathcal{X}$ be the projection maps. Note that both $\pi^0$ and $\pi^\infty$ are morphisms of models.

One has $j^{0*}(E^\bullet_{\mathcal{W}}) = \pi^{0*}(E^\bullet_{\mathcal{X}})$ and there exists a split acyclic complex $(j^{\infty *}(E^\bullet_{\mathcal{W}}), d^\infty)$ on $\mathcal{X}^\infty$ which restricts to $j^{\infty *}(\widetilde{E}^\bullet, \widetilde{d})$ on $X \times \{\infty\}$ ([F], proof of Lemma 18.1).

The standard parameter $z$ on $\mathbb{P}^1$ defines a rational function on $\mathcal{W}$, hence a class

$$\ell(z) = \mathrm{div}_\nu\,(z)_{\mathcal{W}}$$

such that

$$i_*\,\ell(z) = [\mathcal{W}^0] - [\mathcal{W}^\infty] = j_*^0[\mathcal{X}^0] - j_*^\infty[\mathcal{X}^\infty]. \tag{14}$$

Assume a class $\widetilde{\phi}_{\mathcal{X}_0}(E_{\mathcal{X}}^\bullet)$ satisfying properties i), ii), iii) has been defined. We get

$$\begin{aligned}
p_*(i^*\,\phi(E_{\mathcal{W}}^\bullet)\cdot\ell(z)) &= p_*(i^*\,i_*\,\phi_{\mathcal{W}_0}(E_{\mathcal{W}}^\bullet)\cdot\ell(z))\\
&= p_*(\phi_{\mathcal{W}_0}(E_{\mathcal{W}}^\bullet)\cdot i^*\,i_*\,\ell(z))\\
&= p_*(\phi_{\mathcal{W}_0}(E_{\mathcal{W}}^\bullet)(j_*^0[\mathcal{X}^0] - j_*^\infty[\mathcal{X}^\infty]))\\
&= \pi_*^0\phi_{\mathcal{X}_0^0}(j^{0*}\,E_{\mathcal{W}}^\bullet) - \pi_*^\infty\phi_{\mathcal{X}_0^\infty}(j^{\infty*}\,E_{\mathcal{W}}^\bullet)\\
&= \phi_{\mathcal{X}_0}(E_{\mathcal{X}}^\bullet).
\end{aligned} \tag{15}$$

This proves the uniqueness of $\phi_{\mathcal{X}_0}(E_{\mathcal{X}}^\bullet)$. Note that this proof of uniqueness is the same as the one in [GS2] 1.3.2 for the archimedean analog.

Conversely, if we define $\phi_{\mathcal{X}_0}(E_{\mathcal{X}}^\bullet)$ by formula (15) we can check properties i) to v) as in loc.cit.. Indeed, i) follows from the equalities

$$\begin{aligned}
i_*p_*(i^*\,\phi(E_{\mathcal{W}}^\bullet)\cdot\ell(z)) &= p_*(\phi(E_{\mathcal{W}}^\bullet)\cdot i_*\ell(z))\\
&= p_*(\phi(E_{\mathcal{W}}^\bullet)(j_*^0[\mathcal{X}^0] - j_*^\infty[\mathcal{X}^\infty]))\\
&= \pi_*^0\phi(j^{0*}\,E_{\mathcal{W}}^\bullet) = \phi(E_{\mathcal{X}}^\bullet).
\end{aligned} \tag{16}$$

Property ii) is clear. Under the assumption of iii), there exists a split acyclic complex $(E_{\mathcal{W}}^\bullet, d_{\mathcal{W}})$ extending $(E_{\mathcal{X}}^\bullet, d_{\mathcal{X}})$ to $\mathcal{W}$, hence $\phi(E_{\mathcal{W}}^\bullet) = 0$ and, by (15), $\phi_{\mathcal{X}_0}(E_{\mathcal{X}}^\bullet)$ vanishes. To prove (10), we note that

$$p_*(i^*\,ch(E_{\mathcal{W}}^\bullet\otimes p^*(F_{\mathcal{X}}))\cdot\ell(z)) = \mathrm{ch}_{\mathcal{X}_0}(E_{\mathcal{X}}^\bullet\otimes F_{\mathcal{X}}).$$

It remains to prove the identities (8) and (9) for the behaviour of $\phi_{\mathcal{X}_0}$ in exact sequences. By deformation as in [F], proof of Proposition 18.1 b), or by iii) and the analog of [GS2] Prop. 1.3.4, we are reduced to the case where $(E_{\mathcal{X}}^\bullet, d_{\mathcal{X}}) = (S_{\mathcal{X}}^\bullet, d_{\mathcal{X}})\oplus(Q_{\mathcal{X}}^\bullet, d_{\mathcal{X}})$. But then we can copy the proof of [GS2] Prop. 1.3.2. Namely let $\phi(S_{\mathcal{X}}^\bullet, Q_{\mathcal{X}}^\bullet)$ be the class

$$\phi(S_{\mathcal{X}}^\bullet, Q_{\mathcal{X}}^\bullet) = \mathrm{ch}_{\mathcal{X}_0}(S_{\mathcal{X}}^\bullet) + \mathrm{ch}_{\mathcal{X}_0}(Q_{\mathcal{X}}^\bullet)$$

when $\phi = \mathrm{ch}$, and

$$\phi(S_{\mathcal{X}}^\bullet, Q_{\mathcal{X}}^\bullet) = (\mathrm{Td}-1)_{\mathcal{X}_0}(S_{\mathcal{X}}^\bullet)(i^*\,\mathrm{Td}(Q_{\mathcal{X}}^\bullet)) + (\mathrm{Td}-1)_{\mathcal{X}_0}(Q_{\mathcal{X}}^\bullet)$$

when $\phi = \mathrm{Td} - 1$. This class is functorial and such that

$$i_* \phi(S^\bullet_{\mathcal{X}}, Q^\bullet_{\mathcal{X}}) = \phi(S^\bullet_{\mathcal{X}} \oplus Q^\bullet_{\mathcal{X}}).$$

Furthermore it vanishes when there exist acyclic complexes $(S^\bullet_{\mathcal{X}}, d_{\mathcal{X}})$ and $(Q^\bullet_{\mathcal{X}}, d_{\mathcal{X}})$. Let $\mathcal{W}$ be a model of $X \underset{K}{\times} \mathbb{P}^1$ together with maps $j^0 : \mathcal{X}^0 \to \mathcal{W}$, $j^\infty : \mathcal{X}^\infty \to \mathcal{W}$, $\pi^0 : \mathcal{X}^0 \to \mathcal{X}$ and $\pi^\infty : \mathcal{X}^\infty \to \mathcal{X}$ as above, where $\mathcal{W}$ maps to the closures of both $\varphi_S(X \underset{K}{\times} \mathbb{P}^1)$ and $\varphi_Q(X \underset{K}{\times} \mathbb{P}^1)$, where

$$\varphi_S : X \underset{K}{\times} \mathbb{P}^1 \to G \underset{\Lambda}{\times} \mathbb{P}^1$$

and

$$\varphi_Q : X \underset{K}{\times} \mathbb{P}^1 \to G' \underset{\Lambda}{\times} \mathbb{P}^1$$

are the maps defined as in [F] 18.1 from the complexes $(S^\bullet, d)$ and $(Q^\bullet, d)$ respectively. Let $S^\bullet_{\mathcal{W}}$ and $Q^\bullet_{\mathcal{W}}$ be the bundles on $\mathcal{W}$ defined as was $E^\bullet_{\mathcal{W}}$ at the beginning of this proof, and let $p : \mathcal{W} \to \mathcal{X}$ be the projection map. We get successively

$$\begin{aligned}
\phi_{\mathcal{X}_0}(S^\bullet_{\mathcal{X}} \oplus Q^\bullet_{\mathcal{X}}) &= p_*(i^* \phi(S^\bullet_{\mathcal{W}} \oplus Q^\bullet_{\mathcal{W}}) \cdot \ell(z)) \\
&= p_*(i^* i_* \phi(S^\bullet_{\mathcal{W}}, Q^\bullet_{\mathcal{W}}) \cdot \ell(z)) \\
&= \pi^0_* j^{0*} \phi(S^\bullet_{\mathcal{W}}, Q^\bullet_{\mathcal{W}}) - \pi^\infty_* j^{\infty*} \phi(S^\bullet_{\mathcal{W}}, Q^\bullet_{\mathcal{W}}) \\
&= \phi(S^\bullet_{\mathcal{X}}, Q^\bullet_{\mathcal{X}}).
\end{aligned}$$

q.e.d.

**2.3**. Given any regular noetherian scheme $S$ of finite dimension, we let $D^b(S)$ be the derived category of bounded complexes of vector bundles on $S$. It is equivalent to the derived category of bounded complexes of coherent sheaves on $S$ ([SGA6], II). When $\mathcal{X}$ is a model as in 1.1, we denote by $D^b_{\mathcal{X}_0}(\mathcal{X})$ the full subcategory of $D^b(\mathcal{X})$ consisting of complexes which are acyclic outside $\mathcal{X}_0$.

Let $\phi = \mathrm{ch}$ or $\mathrm{Td}-1$. It follows from Proposition 2 iv) that, if $(E^\bullet_{\mathcal{X}}, d_{\mathcal{X}})$ and $(F^\bullet_{\mathcal{X}}, d_{\mathcal{X}})$ are quasi-isomorphic complexes which are both acyclic on $X$,

$$\phi_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}}) = \phi_{\mathcal{X}_0}(F^\bullet_{\mathcal{X}}).$$

In other words, given an object $K_{\mathcal{X}}$ in $D^b_{\mathcal{X}_0}(\mathcal{X})$, the element $\phi_{\mathcal{X}_0}(K_{\mathcal{X}}) \in \mathrm{CH}_\bullet(\mathcal{X}_0)_{\mathbb{Q}}$ depends only on the isomorphism class of $K_{\mathcal{X}}$. Also, given a distinguished triangle

$$K'_{\mathcal{X}} \to K_{\mathcal{X}} \to K''_{\mathcal{X}} \to K'_{\mathcal{X}}[1]$$

in $D^b_{\mathcal{X}_0}(\mathcal{X})$, the formulae

$$(17) \qquad \mathrm{ch}_{\mathcal{X}_0}(K_{\mathcal{X}}) = \mathrm{ch}_{\mathcal{X}_0}(K'_{\mathcal{X}}) + \mathrm{ch}_{\mathcal{X}_0}(K''_{\mathcal{X}})$$

and

$$(18)\qquad (\mathrm{Td}-1)_{\mathcal{X}_0}(K_{\mathcal{X}}) = (\mathrm{Td}-1)_{\mathcal{X}_0}(K'_{\mathcal{X}})\,\mathrm{Td}(K''_{\mathcal{X}}) + (\mathrm{Td}-1)_{\mathcal{X}_0}(K''_{\mathcal{X}})$$

hold, as well as

$$(19)\qquad i_*\phi_{\mathcal{X}_0}(K^\bullet_{\mathcal{X}}) = \phi(K^\bullet_{\mathcal{X}}).$$

Now let

$$(20)\qquad \mathcal{E} : 0 \to E' \to E \to E'' \to 0$$

be an exact sequence of bundles on $X$ and assume that $E'$, $E$ and $E''$ are equipped with arbitrary metrics. We can attach to these data classes $\widetilde{\mathrm{ch}}(\overline{\mathcal{E}})$, $\mathrm{Td}_0(\overline{\mathcal{E}})$ and $\widetilde{\mathrm{Td}}(\overline{\mathcal{E}})$ in $\widetilde{A}(X)_{\mathbb{Q}}$ which are defined as follows. Choose a model $\mathcal{X}$ and bundles $E^n_{\mathcal{X}}$ on $\mathcal{X}$, $n = 0,1,2$, which restrict to $E^0 = E'$, $E^1 = E$ and $E^2 = E''$ respectively on the generic fiber $X$. We then define, using Proposition 2,

$$(21)\qquad \widetilde{\mathrm{ch}}(\overline{\mathcal{E}}) := \mathrm{ch}_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}})$$

$$(22)\qquad \mathrm{Td}_0(\overline{\mathcal{E}}) := (\mathrm{Td}-1)_{\mathcal{X}_0}(E^\bullet_{\mathcal{X}})$$

$$(23)\qquad \widetilde{\mathrm{Td}}(\overline{\mathcal{E}}) := \mathrm{Td}_0(\overline{\mathcal{E}})\,\mathrm{Td}(\overline{E})$$

in $\widetilde{A}(X)_{\mathbb{Q}}$, where $\mathrm{Td}(\overline{E}) \in A_{\mathrm{closed}}(X)_{\mathbb{Q}}$ is the Todd form of $E$ with its chosen metric (cf. §2.1). When $\phi$ denotes ch or Td, Proposition 2 i) implies:

$$(24)\qquad a(\widetilde{\phi}(\overline{\mathcal{E}})) = \widehat{\phi}(\overline{E}' \oplus \overline{E}'') - \widehat{\phi}(\overline{E}).$$

Let $h_0$ and $h_1$ be two metrics on a given vector bundle $E$ on $X$. Consider the exact sequence $\mathcal{E}$ as in (20) where $E'' = 0$, $E' = E$, and $E' \to E$ is the identity. Let $E'$ be metrized by $h_0$ and $E$ by $h_1$. When $\phi = \mathrm{ch}$ or Td we define

$$(25)\qquad \widetilde{\phi}(h_0,h_1) := \widetilde{\phi}(\overline{\mathcal{E}}),$$

hence

$$(26)\qquad a(\widetilde{\phi}(h_0,h_1)) = \widehat{\phi}(E,h_0) - \widehat{\phi}(E,h_1).$$

It follows from (17) and from (11) that, given three metrics $h_0$, $h_1$, $h_2$, we have

$$(27)\qquad \widetilde{\phi}(h_0,h_1) = \widetilde{\phi}(h_0,h_1) + \widetilde{\phi}(h_1,h_2)$$

and that

$$(28)\qquad \widetilde{\phi}(h_0,h_1) = -\widetilde{\phi}(h_1,h_0).$$

The classes $\widetilde{ch}(\mathcal{E})$ and $\widetilde{ch}(h_0, h_1)$ satisfy all properties of the Bott-Chern classes enumerated by Deligne in [D], (5.2.3)-(5.2.8). For example, let us check property (5.2.5) in loc.cit., i.e. let $E_3 = 0 \subset E_2 \subset E_1 \subset E_0 = E$ be a three-step filtration of a vector bundle on $X$ and choose arbitrary metrics on $E_i/E_j, 0 \leq i \leq j \leq 3$. Consider the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_1/E_2 & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_2 & \longrightarrow & E & \longrightarrow & E/E_2 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E/E_1 & = & E/E_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Let $\overline{C_i}$ and $\overline{L_i}$, $i = 1, 2, 3$ be its columns and rows with the chosen metrics. By the argument of [GS2] Proposition 1.3.4, we know that

$$\widetilde{ch}(\overline{L_2}) - \widetilde{ch}(\overline{L_1} \oplus \overline{L_3}) = \widetilde{ch}(\overline{C_2}) - \widetilde{ch}(\overline{C_1} \oplus \overline{C_3}).$$

Using Proposition 2 iii) and iv) we get

$$\widetilde{ch}(\overline{L_1} \oplus \overline{L_3}) = \widetilde{ch}(\overline{L_1})$$

and

$$\widetilde{ch}(\overline{C_1} \oplus \overline{C_3}) = \widetilde{ch}(\overline{C_3}).$$

The equality

$$\widetilde{ch}(\overline{L_2}) - \widetilde{ch}(\overline{L_1}) = \widetilde{ch}(\overline{C_2}) - \widetilde{ch}(\overline{C_3})$$

is the same as (5.2.5) in loc.cit. (with different notation).

**2.4.** If $S$ is any regular noetherian finite dimensional scheme, we let $\underline{K}(S)$ be the Picard category of virtual vector bundles over $S$, as defined in [D] §4.12. By the universal property defining $\underline{K}(S)$ in op. cit., 4.6, to any vector bundle $E$ over $S$ is associated an object $[E] \in \underline{K}(S)$. As explained in op.cit., 4.10, this map extends to a functor

$$[\ \ ] : (D^b(S), \mathrm{is}) \to \underline{K}(S);$$

this follows from the arguments in [KM] Thm. 2 concerning determinant line bundles. In addition, using our hypotheses on $S$ and the same proof as in [KM] Corollary 2, given a distinguished triangle

$$\mathcal{T} : K' \to K \to K'' \to K'[1]$$

in $D^b(S)$, one defines an isomorphism

$$[K] = [K'] + [K'']$$

in $\underline{K}(S)$, which depends functorially on $\mathcal{T}$.

Since $\widetilde{ch}$ satisfies all the properties of Bott-Chern classes (5.2.3)-(5.2.8) in [D], by §5 in op.cit. we get a Picard category $\underline{KM}(X)$ of *virtual metrized bundles* on $X$ and a faithful functor

$$\underline{KM}(X) \to \underline{K}(X)$$

the fiber of which is $\widetilde{A}(X)_{\mathbb{Q}}$. As explained in loc.cit., to give $\underline{KM}(X)$ is the same as giving a functor $\mu$ from $\underline{K}(X)$ to the commutative Picard category of $\widetilde{A}(X)_{\mathbb{Q}}$-torsors. By definition, given any exact sequence $\overline{\mathcal{E}}$ as in (20), the identity

$$\overline{E} = \overline{E'} + \overline{E''} + \widetilde{\mathrm{ch}}(\overline{\mathcal{E}}) \tag{29}$$

holds in $\mu([E])$ ([D] (5.2.2)).

Given any model $\mathcal{X}$ of $X$, there is a canonical morphism of commutative Picard categories

$$m : \underline{K}(\mathcal{X}) \to \underline{KM}(X) \tag{30}$$

which sends the class $[E_{\mathcal{X}}]$ of a vector bundle on $\mathcal{X}$ to the metric $[h]$ that it defines on its restriction to $X$. Given any exact sequence

$$0 \to E'_{\mathcal{X}} \to E_{\mathcal{X}} \to E''_{\mathcal{X}} \to 0$$

of vector bundles on $\mathcal{X}$, the corresponding metrics satisfy

$$[h] = [h'] + [h'']$$

as follows from (29) and Proposition 2 iii). This equality has the properties b), c), d) in [D], §4.3, therefore, by the universal property of $\underline{K}(\mathcal{X})$, the morphism $m$ is well defined.

Given an object $K$ in $D^b(X)$, we call *virtual metric* on $K$ any element $h$ in $\mu([K])$. In particular, any $K_{\mathcal{X}}$ in $D^b(\mathcal{X})$ defines a virtual metric $m([K_{\mathcal{X}}])$ on its restriction $K$ to the generic fiber. When $K_{\mathcal{X}}$ lies in $D^b_{\mathcal{X}_0}(\mathcal{X})$, $m([K_{\mathcal{X}}])$ lies in the fiber of $\underline{KM}(X) \to \underline{K}(X)$, and (29) implies that its class in $\widetilde{A}(X)_{\mathbb{Q}}$ coincides with the image of $\mathrm{ch}_{\mathcal{X}_0}(K_{\mathcal{X}})$.

**2.5**. Since the Chern character $\widehat{\mathrm{ch}}$ is additive and satisfies formula (26), it extends to virtual metrics. Given a distinguished triangle

$$\mathcal{T} : K' \to K \to K'' \to K'[1] \tag{31}$$

in $D^b(X)$, equipped with virtual metrics $h', h, h''$, the formula

$$[h] = [h'] + [h''] + \widetilde{\mathrm{ch}}(\overline{\mathcal{T}})$$

in $\mu([K]) = \mu([K']) + \mu([K''])$ defines a class $\widetilde{\mathrm{ch}}(\overline{\mathcal{T}}) \in \widetilde{A}(X)_{\mathbb{Q}}$ which generalizes the class $\widetilde{\mathrm{ch}}(\overline{\mathcal{E}})$ defined in 2.3. In particular, given two virtual metrics $h_0$ and $h_1$ on $K$, we get a Bott-Chern class $\widetilde{\mathrm{ch}}(h_0, h_1)$. Similar constructions can also be made with the Todd class and the following lemma holds:

LEMMA 1. — *Let $\phi = \mathrm{Td}$ or $\mathrm{ch}$. To any virtual metric $h$ one can attach functorial classes $\widehat{\phi}(h) \in \widehat{\mathrm{CH}}^{\bullet}(X)_{\mathbb{Q}}$, and to a metrized distinguished triangle $\mathcal{T}$ as in (31) one can attach functorial classes $\widetilde{\phi}(\overline{\mathcal{T}}) \in \widetilde{A}(X)_{\mathbb{Q}}$ which generalizes the classes $\widehat{\phi}(\overline{E})$ of §2.1 and the classes $\widetilde{\phi}(\overline{\mathcal{E}})$ defined in (21) and (22). They have the following properties:*

i) $a(\widetilde{\phi}(\overline{\mathcal{T}})) = \widehat{\phi}(\overline{K}' \oplus \overline{K}'') - \widehat{\phi}(\overline{K})$.

ii) *Let $\mathcal{T}$ be a distinguished triangle as in (31). Let $h_0'$, $h_0$, $h_0''$ and $h_1'$, $h_1$, $h_1''$ be two sets of virtual metrics on $K'$, $K$, $K''$ respectively. Let $\widetilde{\phi}(\overline{\mathcal{T}}_0)$ and $\widetilde{\phi}(\overline{\mathcal{T}}_1)$ be the classes above for each choice of virtual metrics. If $\phi = \mathrm{ch}$ we have*

$$\widetilde{\mathrm{ch}}(\overline{\mathcal{T}}_0) - \widetilde{\mathrm{ch}}(\overline{\mathcal{T}}_1) = \widetilde{\mathrm{ch}}(h_0', h_1') - \widetilde{\mathrm{ch}}(h_0, h_1) + \widetilde{\mathrm{ch}}(h_0'', h_1''). \tag{32}$$

iii) *Assume furthermore that $\phi = \mathrm{Td}$, $h_0 = h_1$ and $h_0'' = h_1''$. Then*

$$\widetilde{\mathrm{Td}}(\overline{\mathcal{T}}_0) - \widetilde{\mathrm{Td}}(\overline{\mathcal{T}}_1) = \widetilde{\mathrm{Td}}(h_0', h_1')\,\mathrm{Td}(\overline{K}''). \tag{33}$$

*Proof.* — When $\phi = \mathrm{ch}$ we gave the definitions before stating the lemma. These are clearly compatible with pull back morphisms. To check ii), we use the identities

$$[h_0] = [h_1] + \widetilde{\mathrm{ch}}(h_0, h_1) = [h_1'] + [h_1''] + \widetilde{\mathrm{ch}}(\overline{\mathcal{T}}_1) + \widetilde{\mathrm{ch}}(h_0, h_1)$$

and

$$[h_0] = [h_0'] + [h_0''] + \widetilde{\mathrm{ch}}(\overline{\mathcal{T}}_0) = [h_1'] + [h_1''] + \widetilde{\mathrm{ch}}(\overline{\mathcal{T}}_0) + \widetilde{\mathrm{ch}}(h_0', h_1') + \widetilde{\mathrm{ch}}(h_0'', h_1'')$$

in $\mu([K])$.

We deduce the case where $\phi = \mathrm{Td}$ from the case $\phi = \mathrm{ch}$, since the Todd class can be computed from the Chern character. More precisely, there exists a unique universal formal power series

$$e(X_1, X_2, \cdots) = 1 + X_1/2 + X_1^2/8 - X_2/12 + \cdots$$

with rational coefficients such that, for any bundle $V$ (on a smooth variety say),

$$\mathrm{Td}(V) = e(ch_1(V), ch_2(V), \cdots)$$

(see for example [F] 3.2.4). The Chern character (resp. the Todd class) being additive (resp. multiplicative), one can show that this series turns addition into multiplication: given any graded commutative $\mathbb{Q}$-algebra

$$\Gamma = \bigoplus_{p \geq 0} \Gamma^p,$$

it induces a map

$$e : \bigoplus_{p \geq 1} \Gamma^p \to \Gamma$$

such that $e(\alpha + \beta) = e(\alpha)e(\beta)$.

Given any virtual metric $h$ in $\underline{KM}(X)$, we let

$$\widehat{\mathrm{Td}}(h) = e(\widehat{\mathrm{ch}}(h)) \in \widehat{\mathrm{CH}}^{\bullet}(X)_{\mathbb{Q}}. \tag{34}$$

On the other hand, on the graded $\mathbb{Q}$-vector space

$$\Gamma = \mathbb{Q} \oplus \widetilde{A}(X)_{\mathbb{Q}},$$

where $\mathbb{Q}$ has degree zero and $\widetilde{A}(X)_{\mathbb{Q}}^{(p-1,p-1)}$ has degree $p$, we define a product by the formula

$$(\lambda, \alpha) * (\mu, \beta) = (\lambda\mu, \lambda\beta + \mu\alpha + \alpha dd^c\beta). \tag{35}$$

Given a metrized distinguished triangle $\overline{\mathcal{T}}$ as in (31), we let

$$1 + \mathrm{Td}_0(\overline{\mathcal{T}}) = e(\widetilde{\mathrm{ch}}(\overline{\mathcal{T}})) \in \Gamma$$

and

$$\widetilde{\mathrm{Td}}(\overline{\mathcal{T}}) = \mathrm{Td}_0(\overline{\mathcal{T}})\mathrm{Td}(\overline{K}) \in \widetilde{A}(X)_{\mathbb{Q}}.$$

In particular, given two virtual metrics $h_0$ and $h_1$ on $K$, we get a secondary class $\widetilde{\mathrm{Td}}(h_0, h_1)$. This construction is clearly functorial and to check that it is an extension (22) and (23), one can use the axiomatic description of the Todd class with supports in Proposition 2. To check i), note that

$$a(\widetilde{\mathrm{Td}}(\overline{\mathcal{T}})) = a(\mathrm{Td}_0(\overline{\mathcal{T}}))\widehat{\mathrm{Td}}(\overline{K})$$

since $\omega(\widehat{\mathrm{Td}}(\overline{K}) = \mathrm{Td}(\overline{K})$ (see [BGS] (1.5.2)), and

$$\begin{aligned} 1 + a(\mathrm{Td}_0(\overline{\mathcal{T}})) &= e \circ a(\widetilde{\mathrm{ch}}(\overline{\mathcal{T}})) = e(\widehat{ch}(\overline{K}' \oplus \overline{K}'') - \widehat{ch}(\overline{K})) \\ &= \widehat{\mathrm{Td}}(\overline{K}' \oplus \overline{K}'')/\widehat{\mathrm{Td}}(\overline{K}). \end{aligned}$$

Therefore

$$a(\widetilde{\mathrm{Td}}(\overline{\mathcal{T}})) = \widehat{\mathrm{Td}}(\overline{K}' \oplus \overline{K}'') - \widehat{\mathrm{Td}}(\overline{K}). \tag{36}$$

Similarly, to check iii), notice that, under these hypotheses, ii) reads

$$\widetilde{\mathrm{ch}}(\overline{\mathcal{T}}_0) = \widetilde{\mathrm{ch}}(\overline{\mathcal{T}}_1) + \widetilde{\mathrm{ch}}(h_0', h_1').$$

By applying $e$ to this equality we get

$$1 + \mathrm{Td}_0(\overline{\mathcal{T}}_0) = (1 + \mathrm{Td}_0(\overline{\mathcal{T}}_1)) * (1 + \widetilde{\mathrm{Td}}(h_0', h_1')/\mathrm{Td}(h_1')).$$

Therefore, by (35),

$$\begin{aligned}\mathrm{Td}_0(\overline{\mathcal{T}}_0) = {} & \mathrm{Td}_0(\overline{\mathcal{T}}_1) + \widetilde{\mathrm{Td}}(h_0', h_1')/\mathrm{Td}(h_1') \\ & + [\mathrm{Td}(h_1')\mathrm{Td}(\overline{K''})/\mathrm{Td}(\overline{K}) - 1]\widetilde{\mathrm{Td}}(h_0', h_1')/\mathrm{Td}(h_1').\end{aligned}$$

Multiplying this equality by $\mathrm{Td}(\overline{K})$ gives

$$\widetilde{\mathrm{Td}}(\overline{\mathcal{T}}_0) - \widetilde{\mathrm{Td}}(\overline{\mathcal{T}}_1) = \widetilde{\mathrm{Td}}(h_0', h_1')\,\mathrm{Td}(\overline{K}''),$$

i.e. iii) holds.

q.e.d.

**2.6**. We now define the (non-archimedean) *arithmetic $K$-groups*. The group $\widehat{K}_0(X)$ is the group of isomorphism classes in $\underline{KM}(X)$. It is generated by triples $(E, h, \eta)$, where $E$ is a vector bundle on $X$, $h$ is a metric on $E$, and $\eta \in \widetilde{A}(X)_{\mathbb{Q}}$. These generators are required to satisfy the following relations. Let

$$\mathcal{E} : 0 \to S \to E \to Q \to 0$$

be any exact sequence of bundles on $X$ and suppose that $S$, $E$ and $Q$ are equipped with arbitrary metrics. Let $\widetilde{\mathrm{ch}}(\overline{\mathcal{E}}) \in \widetilde{A}(X)_{\mathbb{Q}}$ be the corresponding secondary class, defined as in (21). Then, for any $\eta' \in \widetilde{A}(X)_{\mathbb{Q}}$ and $\eta'' \in \widetilde{A}(X)_{\mathbb{Q}}$, we have

$$(\overline{S}, \eta') + (\overline{Q}, \eta'') = (\overline{E}, \eta' + \eta'' + \widetilde{\mathrm{ch}}(\overline{\mathcal{E}})) \tag{37}$$

in $\widehat{K}_0(X)$.

If, in this definition, we allow $\eta$, $\eta'$, $\eta''$ to be any currents in $\widetilde{D}(X)_{\mathbb{Q}}$, we get another group, denoted $\overset{\vee}{K}_0(X)$, which clearly contains $\widehat{K}_0(X)$ as a subgroup.

There are maps

$$\alpha : \widetilde{D}(X)_{\mathbb{Q}} \to \overset{\vee}{K}_0(X)$$

and

$$\mathrm{ch} : \overset{\vee}{K}_0(X) \to D_{\mathrm{closed}}(X)_{\mathbb{Q}}$$

defined by $\alpha(\eta) = (0, \eta)$ and

$$\mathrm{ch}(\overline{E}, \eta) = \mathrm{ch}(\overline{E}) + dd^c\, \eta. \tag{38}$$

Note the equalities

$$\mathrm{ch} \circ \alpha = \omega \circ a = dd^c.$$

By Proposition 1 i), this implies that $x \in \overset{\vee}{K}_0(X)$ lies in $\widehat{K}_0(X)$ if and only if $\mathrm{ch}(x)$ lies in $A_{\mathrm{closed}}(X)_{\mathbb{Q}}$. Furthermore $\alpha(\eta)$ lies in $\widehat{K}_0(X)$ if and only if $\eta$ lies in $\widetilde{A}(X)_{\mathbb{Q}}$.

Since forms and metrized vector bundles are contravariant, any map of varieties $f : Y \to X$ induces a pull-back morphism

$$f^* : \widehat{K}_0(X) \to \widehat{K}_0(Y).$$

The formula

$$(\overline{E}, \eta) \cdot (\overline{F}, \xi) = (\overline{E} \otimes \overline{F}, \mathrm{ch}(\overline{E})\, \xi + \eta\, \mathrm{ch}(\overline{F}) + \eta\, dd^c\, \xi)$$

defines both a ring structure on $\widehat{K}_0(X)$ and a module structure of $\overset{\vee}{K}_0(X)$ over $\widehat{K}_0(X)$ (that this map is compatible with (37) is a consequence of Proposition 2 v)). Note that $f^*(xy) = f^*(x)f^*(y)$ when $x$ and $y$ lie in $\widehat{K}_0(X)$, and that

$$x\alpha(\eta) = \alpha(\mathrm{ch}(x)\eta) \tag{39}$$

when $\eta \in \widetilde{D}(X)_{\mathbb{Q}}$ and $x \in \widehat{K}_0(X)$.

There is a Chern character map

$$\overset{\vee}{\mathrm{ch}} : \overset{\vee}{K}_0(X) \to \overset{\vee}{\mathrm{CH}}{}^{\bullet}(X)_{\mathbb{Q}},$$

defined by mapping $(\overline{E}, \eta)$ to the class of $\widehat{\mathrm{ch}}(\overline{E}) + a(\eta)$. It induces a ring homomorphism

$$\widehat{\mathrm{ch}} : \widehat{K}_0(X) \to \widehat{\mathrm{CH}}{}^{\bullet}(X)_{\mathbb{Q}}$$

which commutes with pull-backs. Note that

$$\omega \circ \overset{\vee}{\mathrm{ch}} = \mathrm{ch} \qquad \text{and} \qquad \overset{\vee}{\mathrm{ch}} \circ \alpha = a. \tag{40}$$

Our main goal will be to define push-forward morphisms $f_*$ on arithmetic $K$-groups, satisfying a Riemann-Roch formula. For that purpose we need preliminaries on tangent complexes.

## 3. Secondary Todd classes of tangent complexes.

**3.1.** Given any map of varieties $\varphi : X \to Y$ (resp. any map of models $f : \mathcal{X} \to \mathcal{Y}$) we can attach to it a tangent complex $T\varphi \in D^b(X)$ (resp. $Tf \in D^b(\mathcal{X})$). Since $X$ and $Y$ are smooth, $T\varphi$ is canonically isomorphic to the complex of vector bundles $TX \to \varphi^*(TY)$ with $TX$ in degree zero. Similarly, when both models are smooth over some base, $Tf$ is canonically isomorphic to $T\mathcal{X} \to f^*(T\mathcal{Y})$; in general it is defined by the construction dual to the one of the cotangent complex in [SGA6] Exp. VIII, §2.

Given any map of models $f : \mathcal{X} \to \mathcal{Y}$, we shall denote by

$$\mathrm{Td}(f) \in \mathrm{CH}^\bullet(\mathcal{X}_0)_\mathbb{Q} \subset A^\bullet_{\mathrm{closed}}(X)_\mathbb{Q}$$

the Todd class $i^*\,\mathrm{Td}(Tf)$. When $\pi : \mathcal{X}' \to \mathcal{X}$ is a morphism between models of a given variety $X$, the tangent complex $T\pi$ is acyclic on $X$. Therefore, as in §2.3, it defines a class

$$\mathrm{Td}_0(\pi) := (\mathrm{Td} - 1)_{\mathcal{X}'_0}(T\pi) \in \mathrm{CH}_\bullet(\mathcal{X}'_0)_\mathbb{Q} \subset \widetilde{A}(X)_\mathbb{Q}.$$

Finally, when $f : \mathcal{X} \to \mathcal{Y}$ and $f' : \mathcal{X}' \to \mathcal{Y}'$ are two maps of models which induce the same map $\varphi : X \to Y$ of varieties, we let $h_0$ (resp. $h_1$) be the virtual metric induced by $Tf$ (resp. $Tf'$) on $T\varphi$ and we define

$$\widetilde{\mathrm{Td}}(f, f') := \widetilde{\mathrm{Td}}(h_0, h_1) \in \widetilde{A}(X)_\mathbb{Q}, \tag{41}$$

where the right hand side is defined as in (25), using Lemma 1 to extend this definition to the case of virtual metrics.

When $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are two maps of models, there is a distinguished triangle in $D^b(\mathcal{X})$:

$$Tf \to T(gf) \to f^*\,Tg \to Tf[1]. \tag{42}$$

It follows that

$$\mathrm{Td}(gf) = \mathrm{Td}(f)\, f^*\,\mathrm{Td}(g). \tag{43}$$

This also implies that, if $\pi : \mathcal{X}' \to \mathcal{X}$ is a morphism between two models of the variety $X$ and $f : \mathcal{X} \to \mathcal{Y}$ is any map of models, by (23) and (41), the following identity holds true in $\widetilde{A}(X)_\mathbb{Q}$:

$$\mathrm{Td}_0(\pi)\, \pi^*(\mathrm{Td}(f)) = \widetilde{\mathrm{Td}}(f\pi, f). \tag{44}$$

Similarly, if $g : \mathcal{Y} \to \mathcal{X}'$ is any map of models, the identity

$$g^*(\mathrm{Td}_0(\pi)) \cdot \mathrm{Td}(g) = \widetilde{Td}(\pi g, g) \tag{45}$$

holds in $\widetilde{A}(Y)_\mathbb{Q}$.

Finally, when $\pi : \mathcal{X}' \to \mathcal{X}$ and $\rho : \mathcal{X}'' \to \mathcal{X}'$ are two morphisms between models of a given variety $X$, we deduce from (42) and (18) that

$$\mathrm{Td}_0(\pi\rho) = \mathrm{Td}_0(\rho)\, \rho^* \,\mathrm{Td}(\pi) + \rho^* \,\mathrm{Td}_0(\pi). \tag{46}$$

**3.2.** PROPOSITION 3. — *Let $\pi : \mathcal{X}' \to \mathcal{X}$ be a morphism between two models of $X$.*

i) *For any $K \in D^b(\mathcal{X})$ the adjunction map $K \to R\pi_* \, L\pi^* \, K$ is an isomorphism.*

ii) *Let $\mathrm{Td}_0(\pi) = (\mathrm{Td} - 1)_{\mathcal{X}_0'}(T\pi) \in \mathrm{CH}_\bullet(\mathcal{X}_0')_{\mathbb{Q}}$. Then, in $\mathrm{CH}_\bullet(\mathcal{X}_0)_{\mathbb{Q}}$, we have*

$$\pi_* \,\mathrm{Td}_0(\pi) = 0. \tag{47}$$

*Proof.* — To prove i), note that $R\pi_* \, L\pi^* \, K$ is the derived tensor product of $K$ with $R\pi_* \, \mathcal{O}_{\mathcal{X}'}$, hence it is enough to consider the case where $K = \mathcal{O}_{\mathcal{X}}$. If we assume furthermore that $\pi$ is the blow up of a closed regular subscheme in $\mathcal{X}$, the assertion follows from [SGA6] Exp. VII, Lemme 3.5, p. 441. Therefore i) is true when $\pi$ is a good morphism of models in the sense of §1.1. Using the axiom (M2) of loc. cit., we may find two models $\mathcal{X}_1, \mathcal{X}_2$, and morphisms of models $\mathcal{X}_2 \to \mathcal{X}_1$ and $\mathcal{X}_1 \to \mathcal{X}'$ so that the composite morphisms $\mathcal{X}_2 \to \mathcal{X}'$ and $\mathcal{X}_1 \to \mathcal{X}$ are good. Let $\pi_1 : \mathcal{X}_1 \to \mathcal{X}$ and $\pi_2 : \mathcal{X}_2 \to \mathcal{X}$ be the two obvious morphisms of models. We get a sequence of morphisms in $D^b(\mathcal{X})$:

$$\mathcal{O}_{\mathcal{X}} \to R\pi_* \, \mathcal{O}_{\mathcal{X}'} \to R\pi_{1*} \, \mathcal{O}_{\mathcal{X}_1} \to R\pi_{2*} \, \mathcal{O}_{\mathcal{X}_2}.$$

The composite of the first two maps is the isomorphism $\mathcal{O}_{\mathcal{X}} \to R\pi_{1*}\mathcal{O}_{\mathcal{X}_1}$ (since $\pi_1$ is good) and the composite of the last two maps is also an isomorphism $R\pi_*\mathcal{O}_{\mathcal{X}'} \to R\pi_{2*}\mathcal{O}_{\mathcal{X}_2}$ since $\mathcal{X}_2 \to \mathcal{X}'$ is good. It follows that all morphisms in the sequence above are isomorphisms. This proves i).

To prove ii), we apply the refined Riemann-Roch formula conjectured by T. Saito [S] p. 163, and proved by J. Franke [Fr] §3.3 (at least when $Y = T$). Consider the statement in [S] loc. cit. when $Y = T = \mathcal{X}$, $R = \mathcal{X}_0$, $Z = \mathcal{X}'$, $h = id_{\mathcal{X}}$, $\pi = g$, and $F = \mathcal{O}_{\mathcal{X}}$. By i) above, the canonical map

$$Rh_*F \to Rg_*L\pi^*F$$

is an isomorphism in that case, so the left hand side of [S], loc. cit., vanishes. On the other hand, the right hand side is precisely $\pi_*\mathrm{Td}_0(\pi)$.

q.e.d.

**3.3**. When $\pi$ is a good morphism of models, one can also prove Proposition 3 ii) directly. Indeed, by definition, $\pi$ is then the composite of a sequence of blow ups with integral regular centers contained in the special fiber. By (46) and the projection formula, it is enough to check (47) when $\pi$ is one of these good blow ups. Let $\mathcal{Y}$ be the center of this blow up. We let $j_0 : \mathcal{Y} \to \mathcal{X}_0$ and $j : \mathcal{Y} \to \mathcal{X}$ be the obvious inclusions and denote by $\mathcal{Y}' = \pi^{-1}(\mathcal{Y})$ the exceptional divisor of $\pi$. If $N$ is the normal bundle of $\mathcal{Y}$ in $\mathcal{X}$, we know that $\mathcal{Y}' = \mathbb{P}(N)$. Let $j' : \mathcal{Y}' \to \mathcal{X}'$ be the inclusion. According to [F], Lemma 15.4 (iv), the tangent complex $T\pi$, when shifted by one, is canonically isomorphic the direct image $j'_*(F)$ of the universal quotient bundle $F$ on $\mathbb{P}(N)$. Therefore, if we apply the Grothendieck-Riemann-Roch theorem with supports to $j'_*$, we see that $\mathrm{Td}_0(\pi)$ is $j'_*(\tau)$, where $\tau$ is a universal polynomial in the Chern classes of $p^*N$ and the Chern class of the canonical line bundle $\mathcal{O}(1)$ on $\mathbb{P}(N)$. It follows that $\pi_* \mathrm{Td}_0(\pi) = j_{0*}(\sigma)$, where $\sigma$ is a universal polynomial $R(c_1(N), \dots, c_r(N))$ in the Chern classes of $N$, where $R$ depends only on the rank $r$ of $N$.

On the other hand, if we apply the Grothendieck-Riemann-Roch theorem to $\pi$ and $\mathcal{O}_{\mathcal{X}'}$, since $R\pi_*\mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}}$, we obtain

$$1 = \mathrm{ch}(\mathcal{O}_{\mathcal{X}}) = \pi_* \mathrm{Td}(\pi),$$

therefore, by (19),

$$i_* \pi_* \mathrm{Td}_0(\pi) = \pi_*(\mathrm{Td}(\pi) - 1) = 0.$$

In particular Proposition 3 ii) holds as soon as $i_*$ is injective, and $\sigma = 0$ when $j_*$ is injective. The polynomial $R$ is the same for any regular closed immersion $j$ ($\mathcal{X}$ need not defined over $\Lambda$), for instance the standard section $j : \mathcal{Y} \to \mathbb{P}(N \oplus 1)$ of the completed projective bundle of $N$, for which $j_*$ is injective. Therefore this universal polynomial $R$ must vanish. Hence we always have $\sigma = 0$, and Proposition 3 holds.

**3.4**. When $\Lambda$ is a localization of an algebra of finite type over a field $k$ of characteristic zero, the general case of Proposition 3 ii) follows from §3.3 and the weak factorization conjecture for birational maps proved in [W] and [AKMW], Th. (0.1.1). Indeed, by a standard inductive limit argument, one is reduced to proving (47) when $\pi : \mathcal{X}' \to \mathcal{X}$ is a birational map between two smooth projective varieties over $k$ which is the identity on the open complement $U$ of the closed subset $\mathcal{X}_0 \subset \mathcal{X}$. But then, according to op. cit., there exist a a sequence of smooth projective varieties $\mathcal{X}_i$, $1 \leq i \leq n$, where $\mathcal{X}_n = \mathcal{X}'$ and $\mathcal{X}_1 = \mathcal{X}$, and, for each $i$, a map $\mathcal{X}_i \to \mathcal{X}$ which is the identity on $U$, with the following property: for any index $i < n$ there exists either a morphism $\mathcal{X}_i \to \mathcal{X}_{i+1}$ over $\mathcal{X}$ or a morphism $\mathcal{X}_{i+1} \to \mathcal{X}_i$ over $\mathcal{X}$ which is a blow up of a smooth center disjoint from $U$. By §3.3, the identity (47) is true for such blow ups. Using (46) and the projection formula, it follows that (47) holds for the map $\mathcal{X}_{i+1} \to \mathcal{X}$ if and only if it holds for $\mathcal{X}_i \to \mathcal{X}$. So, by induction on $i$, we conclude that each morphism $\mathcal{X}_i \to \mathcal{X}$ satisfies (47).

## 4. Direct images.

**4.1.** Let $\varphi : X \to Y$ be a map of varieties and consider a commutative diagram of maps of models

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\pi} & \mathcal{X} \\ f' \downarrow & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{p} & \mathcal{Y} \end{array} \tag{48}$$

where $\mathcal{X}$ and $\mathcal{X}'$ (resp. $\mathcal{Y}$ and $\mathcal{Y}'$) are models of $X$ (resp. $Y$) and both $f$ and $f'$ induce the same map $\varphi$ from $X$ to $Y$, when $\pi$ and $p$ are morphisms of models. Let $E_{\mathcal{X}}$ be a vector bundle on $\mathcal{X}$. The elements $Lp^* \, Rf_* \, E_{\mathcal{X}}$ and $Rf'_* \, L\pi^* \, E_{\mathcal{X}}$ in $D^b(\mathcal{Y}')$ both restrict to $R\varphi_* \, E$ on $Y$, when $E$ is the restriction of $E_{\mathcal{X}}$ to $X$. Furthermore there is a canonical morphism

$$Lp^* \, Rf_* \, E_{\mathcal{X}} \to Rf'_* \, L\pi^* \, E_{\mathcal{X}}. \tag{49}$$

Indeed, let $\mathcal{X}_1$ be the fiber product of $\mathcal{X}$ and $\mathcal{Y}'$ over $\mathcal{Y}$, and $\varepsilon : \mathcal{X}' \to \mathcal{X}_1$, $\pi_1 : \mathcal{X}_1 \to \mathcal{X}$ and $f_1 : \mathcal{X}_1 \to \mathcal{Y}'$ the obvious maps. By adjunction, as in §3.3, there is a morphism of functors $\mathrm{id} \to R\varepsilon_* \, L\varepsilon^*$ in the derived category of perfect complexes on $\mathcal{X}_1$. This gives a map

$$Rf_{1*} \, L\pi_1^* \, E_{\mathcal{X}} \to Rf'_* \, L\pi^* \, E_{\mathcal{X}} = Rf_{1*} \, R\varepsilon_* \, L\varepsilon^* \, L\pi_1^* \, E_{\mathcal{X}}.$$

The map (49) is the composite of this map with the base change morphism

$$Lp^* \, Rf_* \, E_{\mathcal{X}} \to Rf_{1*} \, L\pi_1^* \, E_{\mathcal{X}}$$

([SGA4] XVII 4.1.4). Let $K_{\mathcal{Y}'} \in D^b_{\mathcal{Y}'_0}(\mathcal{Y}')$ be a cone of the map (49). All such cones are isomorphic in the derived category, therefore the Chern character with supports $\mathrm{ch}_{\mathcal{Y}'_0}(K_{\mathcal{Y}'})$ is independent of choices (see §2.3). We define

$$\begin{aligned} \theta_{\mathcal{Y}'} := {} & \mathrm{ch}_{\mathcal{Y}'_0}(K_{\mathcal{Y}'}) \cdot \mathrm{Td}(p) \\ & + p^*(\mathrm{ch}(Rf_* \, E_{\mathcal{X}})) \, \mathrm{Td}_0(p) \\ & - f'_*[\pi^*(\mathrm{ch}(E_{\mathcal{X}}) \, \mathrm{Td}(f)) \cdot \mathrm{Td}_0(\pi)] \end{aligned} \tag{50}$$

in $\mathrm{CH}_\bullet(\mathcal{Y}'_0)_\mathbb{Q}$.

Note that, when $\mathcal{Y}' = \mathcal{Y}$, we have

$$\theta_\mathcal{Y} = 0. \tag{51}$$

PROPOSITION 4.

i) *When $f : \mathcal{X} \to \mathcal{Y}$ and $E_\mathcal{X}$ are fixed, and when $f' : \mathcal{X}' \to \mathcal{Y}'$ varies, the classes $\theta_{\mathcal{Y}'}$ are the components of a unique current $\theta \in \widetilde{D}(Y)_\mathbb{Q}$.*

ii) *The following identity holds in $\mathrm{CH}(\mathcal{Y}')_\mathbb{Q}$:*

$$i_*\,\theta_{\mathcal{Y}'} = f'_*\,\pi^*(\mathrm{ch}(E_\mathcal{X})\,\mathrm{Td}(f)) - p^*\,\mathrm{ch}(Rf_*\,E_\mathcal{X}). \tag{52}$$

*Proof.* — It follows from the axiom (M2) in §1.1. that, to check i), it is enough to prove the following. Consider a commutative diagram

$$\begin{array}{ccccc} \mathcal{X}'' & \xrightarrow{\rho} & \mathcal{X}' & \xrightarrow{\pi} & \mathcal{X} \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ \mathcal{Y}'' & \xrightarrow[r]{} & \mathcal{Y}' & \xrightarrow[p]{} & \mathcal{Y}, \end{array} \tag{53}$$

where $\rho$ and $\pi$ (resp. $r$ and $p$) are morphisms between models of $X$ (resp. $Y$) and let $E_\mathcal{X}$ be a vector bundle on $\mathcal{X}$. Then

$$r_*\,\theta_{\mathcal{Y}''} = \theta_{\mathcal{Y}'}, \tag{54}$$

where $\theta_{\mathcal{Y}'}$ is defined using $\pi$ and $p$, while $\theta_{\mathcal{Y}''}$ is defined with $\sigma = \pi\rho$ and $s = pr$.

Let $K_{\mathcal{Y}''}$ be a cone of the map

$$Ls^*\,Rf_*\,E_\mathcal{X} \to Rf''_*\,L\sigma^*\,E_\mathcal{X}.$$

Note that

$$Rr_*\,Ls^* = Rr_*\,Lr^*\,Lp^* = Lp^*.$$

Indeed, we know from Proposition 3.i) that $Rr^*\,Lr^* = \mathrm{id}$. Therefore

$$Rr_*\,Ls^*\,Rf_*\,E_\mathcal{X} = Lp^*\,Rf_*\,E_\mathcal{X}. \tag{55}$$

Similarly $R\rho_*\,L\rho^* = \mathrm{id}$, hence

$$Rr_*\,Rf''_*\,L\sigma^*\,E_\mathcal{X} = Rf'_*\,R\rho_*\,L\sigma^*\,E_\mathcal{X} = Rf'_*\,L\pi^*\,E_\mathcal{X}. \tag{56}$$

From (55) and (56) it follows that $Rr_*\,K_{\mathcal{Y}''}$ and $K_{\mathcal{Y}'}$ are isomorphic. The Riemann-Roch-Grothendieck theorem with supports ([F] Th. 18.2 and §20.1) implies that, since $\mathrm{Td}(s) = \mathrm{Td}(r)\,r^*\,\mathrm{Td}(p)$,

$$r_*(\mathrm{ch}_{\mathcal{Y}''_0}(K_{\mathcal{Y}''})\,\mathrm{Td}(s)) = \mathrm{ch}_{\mathcal{Y}'_0}(K_{\mathcal{Y}'})\,\mathrm{Td}(p). \tag{57}$$

Let us look at the other summands of $\theta_{\mathcal{Y}''}$ (see (50)). If we apply $r_*$ to the identity

$$\mathrm{Td}_0(s) = \mathrm{Td}_0(r)\, r^*\, \mathrm{Td}(p) + r^*\, \mathrm{Td}_0(p)$$

(see (46)) we obtain, by Proposition 3.ii) for $r$,

$$r_*\, \mathrm{Td}_0(s) = \mathrm{Td}_0(p).$$

It follows that

$$\begin{aligned}(58)\qquad r_*(s^*(\mathrm{ch}\, Rf_*\, E_{\mathcal{X}})\, \mathrm{Td}_0(s)) &= p^*\, \mathrm{ch}(Rf_*\, E_{\mathcal{X}})\, r_*\, \mathrm{Td}_0(s)\\ &= p^*\, \mathrm{ch}(Rf_*\, E_{\mathcal{X}})\, \mathrm{Td}_0(p).\end{aligned}$$

Similarly $r_*\, f''_* = f'_*\, \rho_*$ and $\rho_*\, \mathrm{Td}_0(\sigma) = \mathrm{Td}_0(\pi)$, therefore

$$\begin{aligned}(59)\; r_* f''_*(\sigma^*(\mathrm{ch}(E_{\mathcal{X}})\mathrm{Td}(f))\mathrm{Td}_0(\sigma)) &= f'_*\rho_*(\sigma^*(\mathrm{ch}(E_{\mathcal{X}})\mathrm{Td}(f))\mathrm{Td}_0(\sigma))\\ &= f'_*(\pi^*(\mathrm{ch}(E_{\mathcal{X}})\mathrm{Td}(f))\pi_*\mathrm{Td}_0(\sigma))\\ &= f'_*(\pi^*(\mathrm{ch}(E_{\mathcal{X}})\, \mathrm{Td}(f))\, \mathrm{Td}_0(\pi)).\end{aligned}$$

From (57), (58) and (59) we conclude that $r_*\, \theta_{\mathcal{Y}''} = \theta_{\mathcal{Y}'}$, as was to be shown.

To prove ii) we use (17) and (50) to get

$$\begin{aligned}(60)\qquad i_*\, \theta_{\mathcal{Y}'} = {} & (\mathrm{ch}(f'_*\, \pi^*\, E_{\mathcal{X}}) - \mathrm{ch}(p^*\, f_*\, E_{\mathcal{X}}))\, \mathrm{Td}(p)\\ & + p^*(\mathrm{ch}(f_*\, E_{\mathcal{X}}))\, (\mathrm{Td}(p) - 1)\\ & - f'_*[\pi^*(\mathrm{ch}(E_{\mathcal{X}})\, \mathrm{Td}(f))\, (\mathrm{Td}(\pi) - 1)]\\ = {} & \mathrm{ch}(f'_*\, \pi^*\, E_{\mathcal{X}})\, \mathrm{Td}(p)\\ & - p^*\, \mathrm{ch}(f_*\, E_{\mathcal{X}})\\ & + f'_*\, \pi^*(\mathrm{ch}(E_{\mathcal{X}})\, \mathrm{Td}(f))\\ & - f'_*[\pi^*(\mathrm{ch}(E_{\mathcal{X}})\, \mathrm{Td}(f))\, \mathrm{Td}(\pi)],\end{aligned}$$

where we wrote $f_*$ instead of $Rf_*$, $p^*$ instead of $Lp^*$ etc. The relative Todd class of $f \circ \pi = p \circ f'$ is

$$\pi^*(\mathrm{Td}(f))\, \mathrm{Td}(\pi) = \mathrm{Td}(f')\, f'^*\, (\mathrm{Td}(p)),$$

therefore the Riemann-Roch-Grothendieck theorem, when applied to $f'$, gives

$$\begin{aligned}(61)\; f'_*[\pi^*(\mathrm{ch}(E_{\mathcal{X}})\, \mathrm{Td}(f))\, \mathrm{Td}(\pi)] &= f'_*[\pi^*(\mathrm{ch}(E_{\mathcal{X}}))\, \mathrm{Td}(f')\, f'^*\, (\mathrm{Td}(p))]\\ &= \mathrm{ch}(f'_*\, \pi^*\, E_{\mathcal{X}})\, \mathrm{Td}(p).\end{aligned}$$

Combining (60) and (61) we get (52). q.e.d.

Let $\overline{E}$ (resp. $\overline{R\varphi_* E}$) be the bundle $E$ (resp. $R\varphi_* E$) equipped with the metric (resp. the virtual metric) defined by $E_{\mathcal{X}}$ (resp. $Rf_* E_{\mathcal{X}}$).

COROLLARY 2. — *The following identity of currents holds in* $D_{\text{closed}}(Y)_{\mathbb{Q}}$*:*

$$dd^c\,\theta = \varphi_*(\mathrm{ch}(\overline{E})\,\mathrm{Td}(f)) - \mathrm{ch}(\overline{R\varphi_* E}). \tag{62}$$

This corollary indicates that $\theta$ plays the role of the *higher analytic torsion* in Arakelov geometry ([GS3] or [BK]). Note that, when $f$ is flat, it follows from (62), Prop. 1 i) and [BGS] Th. 4.1.1, that $\theta$ lies in $\widetilde{A}(X)_{\mathbb{Q}}$.

**4.2.** THEOREM 1. — *Let $f : X \to Y$ be a map of varieties. Choose a virtual metric $h_f$ on the tangent complex $Tf$. There exists a unique direct image morphism*

$$f_* : \check{K}_0(X) \to \check{K}_0(Y)$$

*such that*

i) *When $x = \alpha(\eta)$ with $\eta \in \widetilde{D}(X)_{\mathbb{Q}}$, the following formula holds:*

$$f_*(\alpha(\eta)) = \alpha(f_*(\eta\,\mathrm{Td}(\overline{Tf}))). \tag{63}$$

ii) *Assume there is a map of models*

$$\underline{f} : \mathcal{X} \to \mathcal{Y}$$

*such that $h_f$ is defined by $T\underline{f}$, and that $x \in \check{K}_0(X)$ is the class of $(E, E_{\mathcal{X}}, 0)$, where $E_{\mathcal{X}}$ is a bundle on $\mathcal{X}$ with restriction $E$ to $X$. Let $\theta$ be the current defined in Proposition 4, i). Then $f_*(x)$ is the class of $(Rf_* E, R\underline{f}_* E_{\mathcal{X}}, \theta)$ in $\check{K}_0(Y)$.*

iii) *Suppose we choose two different virtual metrics $h_f$ and $h'_f$ on $Tf$ and let $f_*$, $f'_*$ be the corresponding direct image morphisms. Then, for any $x \in \check{K}_0(X)$, the following identity holds in $\check{K}_0(Y)$:*

$$f_*(x) - f'_*(x) = \alpha(f_*(\mathrm{ch}(x)\,\widetilde{\mathrm{Td}}(h_f, h'_f))). \tag{64}$$

*Furthermore, for any $x$ in $\check{K}_0(X)$ the following Riemann-Roch identity holds:*

$$\check{\mathrm{ch}}(f_*(x)) = f_*(\check{\mathrm{ch}}(x)\,\widehat{\mathrm{Td}}(\overline{Tf})). \tag{65}$$

**4.3**. To prove uniqueness in Theorem 1, first notice that the identity (63) fixes $f_*$ on the image of $\alpha$.

Next, if $h$ and $h'$ are two metrics on a vector bundle $E$ over $X$, the relation (37) in $\overset{\vee}{K}_0(X)$ together with (63) imply that if $x$ is the class of $(E,h,0)$ in $\overset{\vee}{K}_0(X)$ and $x'$ is the class of $(E,h',0)$, we must have

$$f_*(x)-f_*(x')=\alpha(f_*(\widetilde{\mathrm{ch}}(h,h')\,\mathrm{Td}(\overline{Tf}))). \tag{66}$$

On the other hand, given any map $f:X\to Y$ of varieties and any vector bundle $E$ on $X$, we may find a map of models $\underline{f}:\mathcal{X}\to\mathcal{Y}$ inducing $f$ on $X$, and a bundle $E_{\mathcal{X}}$ on $\mathcal{X}$ inducing $E$ on $X$ [RG]. If $Tf$ is equipped with the virtual metric defined by $T\underline{f}$, Theorem ii) will then specify the value of $f_*(x)$, where $x$ is the class of $(E,E_{\mathcal{X}},0)$. This, together with (63) and the anomaly formulae (64) and (66), proves the uniqueness of $f_*$.

To prove the existence of $f_*$ we have to show that the formula (63) and ii) are consistent with the anomaly formulae (64) and (66). This boils down to the following two facts. First, let (53) be the diagram considered in the proof of Proposition 4, and let $E_{\mathcal{X}}$ be a vector bundle on $\mathcal{X}$. Let $x\in\overset{\vee}{K}_0(X)$ be the class of $(E,E_{\mathcal{X}},0)$ and let $\varphi:X\to Y$ be the map of varieties induced by $f$, $f'$ and $f''$. We want to compare the direct images of $x$ when the virtual metric on $T\varphi$ is defined by $Tf$ or $Tf'$. Let $\theta$ be the current defined by $E_{\mathcal{X}}$ and the virtual metric $Tf$, and $\theta'$ the current defined by $\pi^*E_{\mathcal{X}}$ and $Tf'$ (Proposition 4). Let $h_0$ be the virtual metric $Rf_*\,E_{\mathcal{X}}$ on $R\varphi_*\,E$ and $h_1$ be the virtual metric $Rf'_*\,L\pi^*\,E_{\mathcal{X}}$ on the same complex. Combining (37) and (64), the identity

$$\theta-\theta'+\widetilde{\mathrm{ch}}(h_0,h_1)=\varphi_*(\mathrm{ch}(x)\,\widetilde{\mathrm{Td}}(Tf,Tf'))$$

must be true, at least after applying $\alpha$ to it. To prove this equality, writing $K$ for a cone of $Lp^*\,Rf_*\,E_{\mathcal{X}}\to Rf'_*\,L\pi^*\,E_{\mathcal{X}}$, all we need to check is the following identity in $\mathrm{CH}_\bullet(\mathcal{Y}''_0)_{\mathbb{Q}}$:

$$\theta_{\mathcal{Y}''}-\theta'_{\mathcal{Y}''}-r^*\,\mathrm{ch}_{\mathcal{Y}'_0}(K)=f''_*(\mathrm{ch}(\sigma^*\,E_{\mathcal{X}})\,\rho^*\,\widetilde{\mathrm{Td}}(f',f)_{\mathcal{Y}'_0}). \tag{67}$$

Conversely, by the cofinality axiom (M2), this identity will allow us to define $f_*(x)$ for any choice of virtual metrics on $Tf$, and (64) will hold always.

Second, to check that $f_*$ defines a map on $\overset{\vee}{K}_0(X)$, consider the diagram (48) in 4.1, let $S_{\mathcal{X}}$, $E_{\mathcal{X}}$, $Q_{\mathcal{X}}$ be bundles on $\mathcal{X}$, which restrict to $S$, $E$, $Q$ respectively on $X$. Assume there is a complex

$$\mathcal{E}_{\mathcal{X}}:0\to S_{\mathcal{X}}\to E_{\mathcal{X}}\to Q_{\mathcal{X}}\to 0$$

on $\mathcal{X}$ which restricts to an exact complex on $X$. Consider the associated Chern character with supports

$$\mathrm{ch}_{\mathcal{X}_0}(\mathcal{E}_{\mathcal{X}}) \in \mathrm{CH}_{\bullet}(\mathcal{X}_0)_{\mathbb{Q}}$$

(Proposition 2) with image $\widetilde{\mathrm{ch}}(\overline{\mathcal{E}})$ in $\widetilde{A}(X)_{\mathbb{Q}}$. On the other hand, the derived direct image of the exact sequence $\mathcal{E}_{\mathcal{X}}$ gives a distinguished triangle on $\mathcal{Y}$

$$Rf_*\, S_{\mathcal{X}} \to Rf_*\, E_{\mathcal{X}} \to Rf_*\, Q_{\mathcal{X}} \to Rf_*\, S_{\mathcal{X}}[1],$$

hence a metrized distinguished triangle in $D^b(Y)$ and, by the construction in §2.5, a Chern character with supports in $\mathcal{Y}_0$. We denote this class by $\mathrm{ch}_{\mathcal{Y}_0}(Rf_*\, \mathcal{E}_{\mathcal{X}}) \in \mathrm{CH}_{\bullet}(\mathcal{Y}_0)_{\mathbb{Q}}$. Finally, let $\theta_{\mathcal{Y}'}(E_{\mathcal{X}})$, $\theta_{\mathcal{Y}'}(S_{\mathcal{X}})$ and $\theta_{\mathcal{Y}'}(Q_{\mathcal{X}})$ be defined as in (50) from $E_{\mathcal{X}}$, $S_{\mathcal{X}}$ and $Q_{\mathcal{X}}$ respectively. From (37) and (63) we must have

(68)
$$\theta_{\mathcal{Y}'}(E_{\mathcal{X}})-\theta_{\mathcal{Y}'}(S_{\mathcal{X}})-\theta_{\mathcal{Y}'}(Q_{\mathcal{X}})+p^*\,\mathrm{ch}_{\mathcal{Y}_0}(Rf_*\,\mathcal{E}_{\mathcal{X}}) = f'_*\,\pi^*(\mathrm{ch}_{\mathcal{X}_0}(\mathcal{E}_{\mathcal{X}})\,\mathrm{Td}(Tf)).$$

Conversely, (67) and (68) will show that $f_* : \check{K}_0(X) \to \check{K}_0(Y)$ exists, satisfying i), ii) and iii) in Theorem 1. After that, to check (65), we are reduced to the situation considered in ii). For any diagram (48) as in 4.1, the element $\check{\mathrm{ch}}(\rho_*\, x)$ is then the projective system $(p^*\,\mathrm{ch}(Rf_*\, E_{\mathcal{X}}) + i_*\,\theta_{\mathcal{Y}'})$ in $\varinjlim \mathrm{CH}^{\bullet}(\mathcal{Y}')_{\mathbb{Q}}$. From Proposition 4, ii) we get

$$p^*\,\mathrm{ch}(Rf_*\, E_{\mathcal{X}}) + i_*\,\theta_{\mathcal{Y}'} = f'_*\,\pi^*(\mathrm{ch}(E_{\mathcal{X}})\,\mathrm{Td}(f)),$$

which is precisely the $\mathcal{Y}'$-component of

$$\varphi_*(\check{\mathrm{ch}}(x)\,\widehat{\mathrm{Td}}(\overline{T\varphi})).$$

We are thus left with checking (67) and (68).

**4.4**. Let us now check the equality (67). By the definition (50), if $K''$ is a cone of

$$Ls^*\, Rf_*\, E_{\mathcal{X}} \to Rf''_*\, L\sigma^*\, E_{\mathcal{X}}, \tag{69}$$

we have

$$\begin{aligned}\theta_{\mathcal{Y}''} = {} & \mathrm{ch}_{\mathcal{Y}''_0}(K'')\,\mathrm{Td}(s) + s^*(\mathrm{ch}(Rf_*\, E_{\mathcal{X}}))\,\mathrm{Td}_0(s) \\ & - f''_*[\sigma^*(\mathrm{ch}(E_{\mathcal{X}})\,\mathrm{Td}(f))\,\mathrm{Td}_0(\sigma)],\end{aligned} \tag{70}$$

and if $K'$ is a cone of

$$Lr^*\, Rf'_*\,\pi^*\, E_{\mathcal{X}} \to Rf''_*\, L\rho^*\,(\pi^*\, E_{\mathcal{X}})$$

we have

$$\theta'_{\mathcal{Y}''} = \mathrm{ch}_{\mathcal{Y}''_0}(K')\,\mathrm{Td}(r) + r^*(\mathrm{ch}(Rf'_*(\pi^*E_{\mathcal{X}})))\,\mathrm{Td}_0(r) - f''_*[\rho^*(\mathrm{ch}(\pi^*E_{\mathcal{X}})\,\mathrm{Td}(f'))\,\mathrm{Td}_0(\rho)]. \tag{71}$$

Since $E_{\mathcal{X}}$ is locally free we have

$$L\rho^*(\pi^*E_{\mathcal{X}}) = \rho^*\pi^*E_{\mathcal{X}} = \sigma^*E_{\mathcal{X}} = L\sigma^*E_{\mathcal{X}}.$$

Therefore the map (69) factors via $Lr^*Rf'_*(\pi^*E_{\mathcal{X}})$ and, by the octaedron axiom of triangulated categories, there exists a distinguished triangle in $D^b_{\mathcal{Y}''_0}(\mathcal{Y}'')$

$$Lr^*(K) \to K'' \to K' \to Lr^*(K)[1].$$

By (15) this implies

$$\mathrm{ch}_{\mathcal{Y}''_0}(K'') = r^*\,\mathrm{ch}_{\mathcal{Y}'_0}(K) + \mathrm{ch}_{\mathcal{Y}''_0}(K'). \tag{72}$$

Since $\mathrm{ch}_{\mathcal{Y}''_0}(K'')$ is supported on $\mathcal{Y}''_0$ we have (by (17))

$$\mathrm{ch}_{\mathcal{Y}''_0}(K'')\,\mathrm{Td}(s) = \mathrm{ch}_{\mathcal{Y}''_0}(K'')\,\mathrm{Td}_0(s) + \mathrm{ch}_{\mathcal{Y}''_0}(K''). \tag{73}$$

Similarly

$$\mathrm{ch}_{\mathcal{Y}''_0}(K')\,\mathrm{Td}(r) = \mathrm{ch}_{\mathcal{Y}''_0}(K')\,\mathrm{Td}_0(r) + \mathrm{ch}_{\mathcal{Y}''_0}(K'). \tag{74}$$

Furthermore

$$\begin{aligned} s^*(\mathrm{ch}(Rf_*E_{\mathcal{X}}))\,\mathrm{Td}_0(s) &= \mathrm{ch}(Ls^*Rf_*E_{\mathcal{X}})\,\mathrm{Td}_0(s) \\ &= \mathrm{ch}(Rf''_*L\sigma^*E_{\mathcal{X}})\,\mathrm{Td}_0(s) \\ &\quad - \mathrm{ch}_{\mathcal{Y}''_0}(K'')\,\mathrm{Td}_0(s) \end{aligned} \tag{75}$$

and

$$r^*(\mathrm{ch}(Rf'_*(\pi^*E_{\mathcal{X}})))\,\mathrm{Td}_0(r) = \mathrm{ch}(Rf''_*L\sigma^*E_{\mathcal{X}})\,\mathrm{Td}_0(r) - \mathrm{ch}_{\mathcal{Y}''_0}(K')\,\mathrm{Td}_0(r). \tag{76}$$

From (72)–(76) we conclude that

$$\begin{aligned} \theta_{\mathcal{Y}''} - \theta'_{\mathcal{Y}''} - r^*\,\mathrm{ch}_{\mathcal{Y}'_0}(K) = {} & \mathrm{ch}(Rf''_*L\sigma^*E_{\mathcal{X}})\,\mathrm{Td}_0(r) \\ & - f''_*[\sigma^*(\mathrm{ch}(E_{\mathcal{X}})\,\mathrm{Td}(f))\,\mathrm{Td}_0(\sigma)] \\ & - \mathrm{ch}(Rf''_*L\sigma^*E_{\mathcal{X}})\,\mathrm{Td}_0(r) \\ & + f''_*[\rho^*(\mathrm{ch}(\pi^*E_{\mathcal{X}})\,\mathrm{Td}(f'))\,\mathrm{Td}_0(\rho)]. \end{aligned} \tag{77}$$

Applying the Riemann-Roch-Grothendieck formula to $f''$ and $\sigma^*E_{\mathcal{X}}$, we get from (77) that

$$\theta_{\mathcal{Y}''} - \theta'_{\mathcal{Y}''} - r^*\,\mathrm{ch}_{\mathcal{Y}'_0}(K) = f''_*(\mathrm{ch}(\sigma^*E_{\mathcal{X}}) \cdot A) \tag{78}$$

with

$$A := \mathrm{Td}(f'')f''^*\,(\mathrm{Td}_0(r) - \mathrm{Td}_0(s)) \\ -\sigma^*(\mathrm{Td}(f))\,\mathrm{Td}_0(\sigma) + \rho^*(\mathrm{Td}(f'))\,\mathrm{Td}_0(\rho).$$

To compute $A$, we use the identities (44) and (45) from §3.1 to get

$$\begin{aligned} \mathrm{Td}(f'')\,f''^*\,\mathrm{Td}_0(r) &= \widetilde{\mathrm{Td}}(rf'', f''), \\ \mathrm{Td}(f'')\,f''^*\,\mathrm{Td}_0(s) &= \widetilde{\mathrm{Td}}(sf'', f''), \\ \sigma^*(\mathrm{Td}(f))\,\mathrm{Td}_0(\sigma) &= \widetilde{\mathrm{Td}}(f\sigma, f), \\ \rho^*(\mathrm{Td}(f'))\,\mathrm{Td}_0(\rho) &= \widetilde{\mathrm{Td}}(f'\rho, f'). \end{aligned}$$

Since $sf'' = f\sigma$ and $rf'' = f'\rho$, we get from this and (27)

$$\begin{aligned} (79) \qquad A &= \widetilde{\mathrm{Td}}(sf'', f'') - \widetilde{\mathrm{Td}}(rf'', f'') - \widetilde{\mathrm{Td}}(f\sigma, f) + \widetilde{\mathrm{Td}}(f'\rho, f') \\ &= \widetilde{\mathrm{Td}}(f'', f) - \widetilde{\mathrm{Td}}(f'', f') = \rho^*\,\widetilde{\mathrm{Td}}(f', f)_{\mathcal{Y}'_0}. \end{aligned}$$

From (78) and (79) the equality (67) follows.

**4.5.** To check (68) in §4.3 we let $K(E_{\mathcal{X}})$ be a cone of

$$Lp^*Rf_*E_{\mathcal{X}} \to Rf'_*L\pi^*E_{\mathcal{X}}$$

and we define $K(S_{\mathcal{X}})$ and $K(Q_{\mathcal{X}})$ similarly. Since these maps fit in a morphism of distinguished triangles

$$\begin{array}{ccccccc} Lp^*Rf_*S_{\mathcal{X}} & \longrightarrow & Lp^*Rf_*E_{\mathcal{X}} & \longrightarrow & Lp^*Rf_*Q_{\mathcal{X}} & \longrightarrow & Lp^*Rf_*S_{\mathcal{X}}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Rf'_*L\pi^*S_{\mathcal{X}} & \longrightarrow & Rf'_*L\pi^*E_{\mathcal{X}} & \longrightarrow & Rf'_*L\pi^*Q_{\mathcal{X}} & \longrightarrow & Rf'_*L\pi^*S_{\mathcal{X}}[1], \end{array}$$

where all lines and columns are acyclic on $Y$, we get from (32) in Lemma 1 ii) that

$$\begin{aligned} (80) \qquad & \mathrm{ch}_{\mathcal{Y}'_0}(K(E_{\mathcal{X}})) - \mathrm{ch}_{\mathcal{Y}'_0}(K(S_{\mathcal{X}})) - \mathrm{ch}_{\mathcal{Y}'_0}(K(Q_{\mathcal{X}})) \\ &= \mathrm{ch}_{\mathcal{Y}'_0}(Rf'_*\,L\pi^*\,\mathcal{E}_{\mathcal{X}}) - \mathrm{ch}_{\mathcal{Y}'_0}(Lp^*\,Rf_*\,\mathcal{E}_{\mathcal{X}}), \end{aligned}$$

where $Rf'_*\,L\pi^*\,\mathcal{E}_{\mathcal{X}}$ is the upper triangle and

$$(81) \qquad \mathrm{ch}_{\mathcal{Y}'_0}(Lp^*\,Rf_*\,\mathcal{E}_{\mathcal{X}}) = p^*\,\mathrm{ch}_{\mathcal{Y}_0}(Rf_*\,\mathcal{E}_{\mathcal{X}}).$$

On the other hand,

$$\begin{aligned} (82) \qquad & p^*\,[\mathrm{ch}(Rf_*\,E_{\mathcal{X}}) - \mathrm{ch}(Rf_*\,S_{\mathcal{X}}) - \mathrm{ch}(Rf_*\,Q_{\mathcal{X}})]\,\mathrm{Td}_0(p) \\ &= p^*\,\mathrm{ch}_{\mathcal{Y}_0}(Rf_*\,\mathcal{E}_{\mathcal{X}})\,\mathrm{Td}_0(p) \\ &= p^*\,\mathrm{ch}_{\mathcal{Y}_0}(Rf_*\,\mathcal{E}_{\mathcal{X}})\,\mathrm{Td}(p) - p^*\,\mathrm{ch}_{\mathcal{Y}_0}(Rf_*\,\mathcal{E}_{\mathcal{X}}). \end{aligned}$$

Since $f\pi = pf'$ we have

$$\pi^*(\mathrm{Td}(f))\,\mathrm{Td}(\pi) = \mathrm{Td}(f')\,f'^*(\mathrm{Td}(p)),$$

and this implies that

$$(83)\quad \begin{aligned} &f'_*[\pi^*((\mathrm{ch}(E_{\mathcal{X}}) - \mathrm{ch}(S_{\mathcal{X}}) - \mathrm{ch}(Q_{\mathcal{X}}))\,\mathrm{Td}(f))\,\mathrm{Td}_0(\pi)] \\ &= f'_*[\pi^*(\mathrm{ch}_{\mathcal{X}_0}(\mathcal{E}_{\mathcal{X}})\,\mathrm{Td}(f))\,\mathrm{Td}_0(\pi)] \\ &= f'_*[\pi^*(\mathrm{ch}_{\mathcal{X}_0}(\mathcal{E}_{\mathcal{X}})\,\pi^*\,\mathrm{Td}(f)\,\mathrm{Td}(\pi)] - f'_*\,\pi^*(\mathrm{ch}_{\mathcal{X}_0}(\mathcal{E}_{\mathcal{X}})\,\mathrm{Td}(f)) \\ &= f'_*[\pi^*(\mathrm{ch}_{\mathcal{X}_0}(\mathcal{E}_{\mathcal{X}}))\,\mathrm{Td}(f')\,f'^*\,\mathrm{Td}(p)] - f'_*\,\pi^*(\mathrm{ch}_{\mathcal{X}_0}(\mathcal{E}_{\mathcal{X}})\,\mathrm{Td}(f)) \\ &= \mathrm{ch}_{\mathcal{Y}'_0}(Rf'_*\,L\pi^*\,\mathcal{E}_{\mathcal{X}})\,\mathrm{Td}(p) - f'_*\,\pi^*(\mathrm{ch}_{\mathcal{X}_0}(\mathcal{E}_{\mathcal{X}})\,\mathrm{Td}(f)), \end{aligned}$$

where the last equality follows from the projection formula for $f'$ together with the Riemann-Roch-Grothendieck formula with supports. Putting (50), (80), (81), (82) and (83) together, we get

$$\theta_{\mathcal{Y}'}(E_{\mathcal{X}}) - \theta_{\mathcal{Y}'}(S_{\mathcal{X}}) - \theta_{\mathcal{Y}'}(Q_{\mathcal{X}}) + p^*\,\mathrm{ch}_{\mathcal{Y}_0}(Rf_*\,\mathcal{E}_{\mathcal{X}}) = f'_*\,\pi^*(\mathrm{ch}_{\mathcal{X}_0}(\mathcal{E}_{\mathcal{X}})\,\mathrm{Td}(f)),$$

i.e. (68) holds true. This ends the proof of Theorem 1.

**4.6**. Here are a few more properties of the direct image morphisms defined in Theorem 1.

THEOREM 2. — *Let $f : X \to Y$ be a map of varieties, equipped with an arbitrary virtual metric on $Tf$.*

i) *When $x \in \overset{\vee}{K}_0(X)$ and $y \in \overset{\vee}{K}_0(Y)$, we have*

$$f_*(xf^*(y)) = f_*(x)y.$$

ii) *If $f$ is flat, $f_*$ maps $\widehat{K}_0(X)$ into $\widehat{K}_0(Y)$.*

iii) *Let $g : Y \to Z$ be a map of varieties. Choose arbitrary virtual metrics on $Tg$ and $T(gf)$. Let $\widetilde{\mathrm{Td}}$ be the secondary Todd class of the metrized distinguished triangle on $X$*

$$\mathcal{T} : Tf \to T(gf) \to f^*Tg \to Tf[1].$$

*Then, for any $x \in \overset{\vee}{K}_0(X)$, the following identity holds:*

$$(84)\qquad (gf)_*(x) - g_*(f_*(x)) = -\alpha((gf)_*(\mathrm{ch}(x)\,\widetilde{\mathrm{Td}})).$$

**4.7**. To check Theorem 2 i) when $x$ or $y$ is in the image of $\alpha$, we just use the fact that the projection formula is true for forms and currents (§1.8 and [BGS] 1.5), together with (63).

We may then assume that $x$ is the class of $(E, h, 0)$ and $y$ is the class of $(F, h', 0)$. By the previous argument, and the anomaly formula (66), the difference

$$z = f_*(x\,f^*(y)) - f_*(x)\,y$$

does not depend on the choice of the metrics $h$ and $h'$. Furthermore, if we change the virtual metric on $Tf$ and if we denote by $\widetilde{\mathrm{Td}}$ the corresponding secondary Todd class, it follows from Theorem 1, iii) that $z$ gets replaced by

$$z + \alpha\, f_*(\mathrm{ch}(x\, f^*(y))\,\widetilde{\mathrm{Td}}) - \alpha(f^*(\mathrm{ch}(x)\,\widetilde{\mathrm{Td}}))\cdot y.$$

This is equal to $z$ since, by (39),

$$\alpha(f_*(\mathrm{ch}(x)\,\widetilde{\mathrm{Td}}))\cdot y = \alpha(f_*(\mathrm{ch}(x)\,\widetilde{\mathrm{Td}})\,\mathrm{ch}(y)) = \alpha\, f_*(\mathrm{ch}(x\, f^*(y))\,\widetilde{\mathrm{Td}}).$$

Consequently we may assume that there is a map of models $\underline{f} : \mathcal{X} \to \mathcal{Y}$ inducing $f$ on $X$ and defining the virtual metric on $Tf$, and that the metric on $E$ (resp. $F$) is defined by a bundle $E_{\mathcal{X}}$ (resp. $F_{\mathcal{Y}}$) on $\mathcal{X}$ (resp. $\mathcal{Y}$). Since $Rf_*(E_{\mathcal{X}} \otimes f^*F_{\mathcal{Y}})$ is isomorphic to $Rf_*E_{\mathcal{X}} \otimes F_{\mathcal{Y}}$ we know that $z$ can be written

$$z = \alpha(\eta), \quad \eta \in \widetilde{D}(Y)_{\mathbb{Q}}.$$

On the other hand, the Riemann-Roch formula (65) in Theorem 1, together with the projection formula for arithmetic Chow groups (§1.8) imply that $\widehat{\mathrm{ch}}(z) = 0$. Therefore

$$dd^c\eta = \omega(\widehat{\mathrm{ch}}(z)) = 0.$$

By Proposition 1, ii), if $\eta_{\mathcal{Y}}$ vanishes we can conclude that $\eta = 0$ and $z = 0$. But the analytic torsion $\theta_{\mathcal{Y}}$ for both $E_{\mathcal{X}}$ and $E_{\mathcal{X}} \otimes f^*F_{\mathcal{Y}}$ are zero by (51). Therefore $\eta_{\mathcal{Y}} = 0$.

**4.8.** Assume $f$ is flat and $x \in \widehat{K}_0(Y)$. Since $\mathrm{ch}(x)\,\mathrm{Td}(\overline{Tf})$ lies in $A_{\mathrm{closed}}(X)_{\mathbb{Q}}$ and $f_*$ maps forms into forms (§1.7, i.e. [BGS] (4.1.1) and (4.2.1)) $f_*(\mathrm{ch}(x)\,\mathrm{Td}(\overline{Tf}))$ lies in $A_{\mathrm{closed}}(Y)_{\mathbb{Q}}$. But it follows from the Riemann-Roch formula (65) that

$$f_*(\mathrm{ch}(x)\,\mathrm{Td}(\overline{Tf})) = \mathrm{ch}(f_*(x)).$$

Therefore $f_*(x)$ lies in $\widehat{K}_0(Y)$ (see 2.5).

**4.9.** The proof of Theorem 2, iii) is similar to Theorem 2, i). Namely, let

$$z = (gf)_*(x) - g_*(f_*(x)) + \alpha((gf)_*\,(\mathrm{ch}(x)\,\widetilde{\mathrm{Td}})).$$

When $x = \alpha(\eta)$, it follows from (63) that $z = 0$. Indeed, since, by (26) and (43),

$$\mathrm{Td}(\overline{T(gf)}) = \mathrm{Td}(\overline{Tf})f^*\,\mathrm{Td}(\overline{Tg}) - dd^c(\widetilde{\mathrm{Td}}),$$

we get

$$\begin{aligned}(gf)_*(x) &= \alpha(g_*(f_*(\eta\,\mathrm{Td}(\overline{Tf}))\,\mathrm{Td}(\overline{Tg})) \\ &\quad - \alpha((gf)_*(\eta\, dd^c(\widetilde{\mathrm{Td}}))) \\ &= g_*f_*(x) - (gf)_*(\alpha(\eta\, dd^c(\widetilde{\mathrm{Td}})))\end{aligned}$$

and $z = 0$ by the "Stokes formula"

$$\eta dd^c(\omega) = dd^c(\eta)\omega.$$

Therefore it is enough to check that $z = 0$ when $x$ is the class of $(E, E_{\mathcal{X}}, 0)$ for some extension of $E$ to a model of $X$. When the virtual metric $h_f$ on $Tf$ is replaced by $h_{f'}$, from Theorem 1, iii) and (63) we know that $z$ gets replaced by

$$z' = z - \alpha g_*[f_*(\mathrm{ch}(x)\,\widetilde{\mathrm{Td}}(h_f, h'_f))\,\mathrm{Td}(\overline{Tg})] - \alpha(gf)_*[\mathrm{ch}(x)(\widetilde{\mathrm{Td}} - \widetilde{\mathrm{Td}}')],$$

where $\widetilde{\mathrm{Td}}$ (resp. $\widetilde{\mathrm{Td}}'$) are the secondary Todd classes of the distinguished triangle $\mathcal{T}$, where the virtual metric on $Tf$ is $h_f$ (resp. $h'_f$), and we do not change the virtual metrics on $T(gf)$ and $f^*Tg$. From Lemma 1 iii) we know that

$$\widetilde{\mathrm{Td}} - \widetilde{\mathrm{Td}}' = \widetilde{\mathrm{Td}}(h_f, h'_f)\, f^*\,\mathrm{Td}(\overline{Tg}),$$

therefore $z' = z$.

One checks in a similar way (by shifting (31)) that $z$ does not depend on the virtual metrics on $Tg$ and $T(gf)$. Finally one may assume that there are maps of models $\underline{f} : \mathcal{X} \to \mathcal{Y}$ and $\underline{g} : \mathcal{Y} \to \mathcal{Z}$ which induce $f$ and $g$, that the metric on $E$ is given by a bundle $E_{\mathcal{X}}$ on $\mathcal{X}$, and that the virtual metric on $Tf$ (resp. $Tg$, resp. $T(gf)$) is given by $T\underline{f}$ (resp. $T\underline{g}$, resp. $T(\underline{gf})$). The distinguished triangle

$$T\underline{f} \to T(\underline{gf}) \to \underline{f}^*T\underline{g} \to T\underline{f}[1]$$

implies in that case that $\widetilde{\mathrm{Td}} = 0$. Since $R\underline{f}_*R\underline{g}_* = R(\underline{fg})_*$ we can write $z = \alpha(\eta)$, $\eta \in \widetilde{D}(Z)_{\mathbb{Q}}$, and from Riemann-Roch we deduce that

$$dd^c(\eta) = \mathrm{ch}(z) = (gf)_*(\mathrm{ch}(x)\mathrm{Td}(\overline{T(gf)})) - g_*(f_*(\mathrm{ch}(x)\mathrm{Td}(\overline{Tf}))\mathrm{Td}(\overline{Tg})) = 0.$$

To check that $\eta = 0$ (hence $z = 0$) it is enough that $\eta_{\mathcal{Z}} = 0$. But this follows from (51) applied to the maps $\underline{f}$, $\underline{g}$ and $\underline{gf}$. This ends the proof of Theorem 2.

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