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ALGEBRAIC AND SYMPLECTIC GROMOV-WITTEN INVARIANTS COINCIDE

by Bernd SIEBERT

Contents.

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Introduction.

Gromov-Witten invariants “count” (pseudo-)holomorphic curves on algebraic or symplectic manifolds. This amounts to intersection theory on moduli spaces of such curves. Because in general these are non-compact, singular and not of “expected dimension”, a rigorous mathematical definition is far from trivial. For a reasonably large class of manifolds including Fano and Calabi-Yau manifolds this has first been done using symplectic techniques by Ruan and Tian [Ru1], [RuTi1], [RuTi2]. The point in this approach is to restrict to sufficiently “generic” almost complex structures (tamed by the symplectic form). Then the moduli spaces are smooth of the expected dimension. This dimension is the index of the Fredholm operator describing the moduli space locally as map between appropriate Banach spaces. A certain positivity condition has to be imposed on the manifolds to assure the existence of a compactification by strata of lower dimensional manifolds.

The treatment of the general case required new techniques that would not rely on genericity and replace the fundamental class of the manifold by a homology class of the expected dimension, the *virtual fundamental class*. This has first been achieved in the algebraic context by Li and Tian by constructing a (bundle of) cone(s) inside a vector bundle over a compactified moduli space; the virtual fundamental class is then obtained by intersection with the zero section [LiTi1]. A similar approach, based on Li and Tian's idea of using cones inside vector bundles, but using the cotangent complex and stack-theoretic language is due to Behrend and Fantechi [Be], [BeFa].

The construction of virtual fundamental classes in the symplectic category has been carried out shortly later by a number of authors [FkOn], [LiTi2], [Ru2], [Si1]. The basic approach here is to write the moduli space as zero locus of a section of a finite rank (orbi-) vector bundle over a finite dimensional manifold (or rather orbifold). Locally this is not too hard. In references [FkOn], [LiTi2] the crucial globalization is achieved by allowing for jumps of dimension of the local models. This is still enough to construct a homology class as intersection of the “Euler class” (of the bundle) with the “fundamental class” (of the finite dimensional base). The other two references use the author's construction of a finite rank orbibundle over (a neighbourhood in an appropriate ambient space of) the moduli space. This leads to a global description of the moduli space as zero locus of a section of an orbibundle in an honest finite-dimensional orbifold.

It is natural to expect that for complex projective manifolds, to which both approaches apply, algebraic and symplectic virtual fundamental classes do agree (hence the Gromov-Witten invariants derived from them). As both approaches follow rather different tracks this is, however, far from obvious. The purpose of this paper is to confirm that the expectation is indeed correct. We prove

THEOREM 0.1. — *Let M be a complex projective manifold, $R \in H_2(M;\mathbb{Z})$, $g, k \geq 0$. Let $\mathcal{C}_{R,g,k}(M)$ be the moduli space of stable maps of genus g, with k marked points and representing class R (cf. Section 1.1).*

Then the homology class associated to the algebraic virtual fundamental class (a Chow class on $\mathcal{C}_{R,g,k}(M)$) as in [Be] and the symplectic virtual fundamental class as in [Si1] coincide.

For being specific we refer only to the constructions in [Be] and [Si1]. The equivalence of the two algebraic constructions and the four symplectic ones within their categories is another, fairly straightforward albeit tedious matter, that we rather leave to a more masochistic soul. For the symplectic case there are some comments in [Si3], §3.4.

Our proof has three central ingredients. First, we need a holomorphic version of the construction of the ambient space (denoted $\tilde{Z}$ in [Si3], §1.3, 1.4) into which $\mathcal{C}_{R,g,k}(M)$ embeds as zero locus of a section $\tilde{s}$ of a finite rank orbibundle F. This might not be possible globally, but we will gain analyticity up to some smooth factor and this will be enough to make the comparison work. The central point of the local construction is the fact that spaces of holomorphic maps from a Riemann surface with non-empty boundary (with continuous extension to the boundary) have a natural structure of complex Banach manifold (Proposition 2.1). The rest of the local construction, leading to "analytic Kuranishi models" for $\mathcal{C}_{R,g,k}(M)$ occupies Chapter 2.

The second part of the proof produces a cone $C(\tilde{s})$ inside F, as limit of the graphs of $t \cdot \Gamma_{\tilde{s}}$ as t tends to infinity. We will see (Chapter 3) that the fundamental classes of the graphs also convergence, to a well-defined $(\dim \tilde{Z})$-dimensional homology class living on $C(\tilde{s})$ (denoted $[C(\tilde{s})]$, by abuse of notation). Moreover, if $\tilde{s}$ "splits off a trivial factor" then both $C(\tilde{s})$ and $[C(\tilde{s})]$ split off a trivial factor too (a vector bundle with its fundamental class). Of course, if $\tilde{s}$ is holomorphic the cone is holomorphic too. It is in fact the cone that one obtains by applying [BeFa] to an obstruction theory naturally coming from the differential sequence associated to $\tilde{s}$. End second part.

Taken together this will be used in Chapter 4 to reduce the comparison of algebraic and symplectic virtual fundamental classes to a comparison of two morphisms of two-term complexes to the cotangent complex of $\mathcal{C}_{R,g,k}(M)$. One is coming from the description as zero set of a section as just mentioned, the other one is abstractly constructed in[BeFa] from the universal family and universal morphism to M. This comparison is the least obvious part of the proof. It requires an explicit study of the abstractly defined morphism (in derived categories) using Čech cochains. It is quite satisfying to see how the $\bar{\partial}$-equation (describing $\tilde{s}$) naturally arises by partial integration applied to fiber integrals coming from the explicit version of relative (Serre-) duality (see the proof of Lemma 4.8).

The first Chapter serves two purposes. First, it contains an account of the algebraic definition of virtual fundamental classes to fix notations and to make the paper more self-contained. We follow the elementary reformulation (avoiding Artin stacks) of [BeFa] previously given by the author [Si2]. Second we go over from algebraic to analytic spaces, or rather from Deligne-Mumford stacks to complex analytic orbispaces. The last Chapter 5 provides a GAGA-type result concerning push-forwards and relative duality in algebraic and analytic derived categories of sheaves. The main result is Proposition 5.4 which gives an explicit description of algebraic relative duality for algebraic families of prestable curves in terms of analytic Čech cochains and fiber integrals.

Our proof of the comparison theorem has been sketched in some detail in the survey [Si3], submitted in May 1997. Shortly afterwards we learned from J. Li and G. Tian that they were able to prove the same result, using their respective definitions of algebraic and symplectic virtual fundamental classes [LiTi3].

1. Complex analytic GW-theory.

To compare algebraic and symplectic definitions of GW-invariants, as a first, mostly trivial step, it is natural to translate the former into the category of complex (analytic) spaces. This will be the purpose of this chapter.

1.1. Analytic orbispaces versus Deligne-Mumford stacks.

Given a smooth projective scheme M over a field k the natural arena for GW-theory is the space $\mathcal{C}(M)$ of stable marked curves in M (Kontsevich's "stable maps"). So the k-rational points of $\mathcal{C}(M)$ are in one-to-one correspondence with isomorphism classes of triples $(C, \mathbf{x}, \varphi)$ with C a reduced, connected algebraic curve, proper over k and with at most ordinary double points, $\mathbf{x} = (x_1, \ldots, x_k)$ a tuple of k-rational points in the regular locus $C_{\text{reg}} \subset C$, and $\varphi : C \to M$ a k-morphism with the property that

$$\operatorname{Aut}(C, \mathbf{x}, \varphi) = \{\psi \in \operatorname{Aut}(C) \mid \psi(\mathbf{x}) = \mathbf{x}, \varphi \circ \psi = \varphi\}$$

is finite (*stability*). With the obvious notion of families of stable marked curves over (that is, parametrized by) k-schemes, $\mathcal{C}(M)$ (or rather the associated fibered groupoid) has been verified to be a Deligne-Mumford (DM-) stack [BeMa].

In the analytic category, i.e. $k = \mathbb{C}$ and M viewed as complex projective manifold, the DM-stack can be replaced by a notion of analytic orbispace that we now introduce. This will be a generalization of both complex orbifolds and complex spaces.

Local models for such spaces are tuples $(q : \widehat{U} \to U, G, \alpha)$ with

- G is a finite group, viewed as zero-dimensional reduced complex space
- $\widehat{U}$ is a (not necessarily reduced, but finite-dimensional) complex space
- $\alpha : G \times \widehat{U} \to \widehat{U}$ is a (not necessarily effective) holomorphic group action on $\widehat{U}$
- q is a quotient of $\widehat{U}$ by G in the category of complex spaces (or, equivalently, in the category of ringed spaces, cf. e.g. [KpKp], §49A.

We will often just write $U = \widehat{U}/G$ for such *local uniformizing system* (q, G, α). The definition of analytic orbispaces now runs completely analogous to the case of orbifolds as given in [Sa]: One first defines the notion of *morphisms*, and in particular *open embeddings* of local uniformizing systems. An *analytic orbispace structure* on a Hausdorff space X is then a covering by local uniformizing systems $\{U_i = \widehat{U}_i/G_i\}_{i \in I}$ (i.e. $X = \bigcup U_i$) compatible on overlaps $U_i \cap U_j \neq \emptyset$ by open embeddings. Finally, an *analytic orbispace* is an equivalence class of analytic orbispace structures. By abuse of notation we will also just write X for the analytic orbispace. Of course there is also a notion of morphisms of analytic orbispaces making the set of analytic orbispaces into a category, denoted $\mathfrak{AnOrb}$.

Similarly, we may introduce the notions of (topological or analytic) *orbibundles* and of *coherent orbisheaves* on X by requiring the associated linear fiber space over X (cf. e.g. [Fi], §1.4) to be analytic orbispaces over X with local uniformizers having a well-defined linear structure on the fibers.

This all being a trivial translation of [Si1], §1.1, which treats the case of topological Banach orbifolds, to the category of complex spaces we merely give these indications and refer to op.cit. for more details.

We claim that $\mathcal{C}(M)$ has naturally the structure of an analytic orbispace, in such a way that $\mathcal{C}(M)$ represents the functor

$$\mathfrak{C}(M) : \mathfrak{AnOrb} \longrightarrow \mathfrak{Sets}$$

$$T \longmapsto \left\{ \begin{array}{l} \text{analytic families of stable holomorphic} \\ \text{curves in } M \text{ parametrized by } T \end{array} \right\} \Big/ \text{iso.}$$

Here we use the following

DEFINITION 1.1. — *Let T and M be analytic orbispaces. An* analytic family of stable holomorphic curves in M *parametrized by T is a tuple of morphisms*

$$(q : X \to T, \underline{\mathbf{x}} : T \to X \times_T \ldots \times_T X, \Phi : X \to M)$$

of complex orbispaces with

- *q is flat*
- *for any $t \in T$, $(q^{-1}(t), \underline{\mathbf{x}}(t), \Phi|_{q^{-1}(t)})$ (with the induced analytic structure) is a stable marked holomorphic curve in M ([Si1], Def. 3.5).* □

The precise result is

PROPOSITION 1.2. — *For any complex space M, $\mathfrak{C}(M)$ is representable by an analytic orbi-space $\mathcal{C}(M)$.*

Proof. — To define a local uniformizing system at some stable holomorphic curve $(C, \mathbf{x}, \varphi)$ in M let $(q : \mathcal{C} \to S, \underline{\mathbf{x}})$ be an analytically semi-universal deformation of $(C, \mathbf{x})$, cf. e.g. [Si1], §2.2. It is well-known that $\mathrm{Hom}_S(\mathcal{C}, M)$, the space of morphisms from the fibers of q to M, is a complex space (representing the corresponding functor) [Do], [Po]. This is almost the space we want, but if $\mathrm{Aut}^0(C, \mathbf{x})$ is non-trivial we have to the equivalence relation on $\mathrm{Hom}_S(\mathcal{C}, M)$ generated by the germ of the action of $\mathrm{Aut}^0(C, \mathbf{x})$ on q. We refer to this process as "rigidification".

To this end let $\mathbf{y} = (y_1, \ldots, y_l)$ be a tuple of points in C such that

- $(C, \tilde{\mathbf{x}})$ is a (Deligne-Mumford) stable curve, where $\tilde{\mathbf{x}} = (x_1, \ldots, x_k, y_1, \ldots y_l)$
- for any i, φ is an immersion at y_i (possible by stability).

The first property requires in particular finiteness of $\{\Psi \in \mathrm{Aut}(C, \mathbf{x}) \mid \Psi(y_i) = y_i\}$, so a minimal choice requires the insertion of $l = \dim \mathrm{Aut}(C, \mathbf{x})$ points. By the second property we may choose local Cartier divisors $H_1, \ldots, H_l \subset M$ and open disks $U_i \subset C$ with $(\varphi|_{U_i})^{-1}(H_i) = y_i$ (y_i with reduced structure). Now let $(q : \mathcal{C} \to S, \underline{\tilde{\mathbf{x}}})$ be an analytically universal deformation of $(C, \tilde{\mathbf{x}})$ with $\tilde{\mathbf{x}} = (x_1, \ldots, x_k, y_1, \ldots, y_l)$. Extend U_i to open polycylinders (say) $\tilde{U}_i \subset \mathcal{C}$. By restriction to an open subspace $Z \subset \mathrm{Hom}_S(\mathcal{C}, M)$ we may assume that $(\psi|_{\tilde{U}_i})^{-1}(H_i)$ consists of exactly one reduced point for any $\psi \in Z$. Evaluation at the deformation of the ν-th marked point defines $k + l$ morphisms

$$\mathrm{ev}_\nu : Z \longrightarrow M.$$

Set

$$\widehat{U} := (\mathrm{ev}_{k+1}, \ldots, \mathrm{ev}_{k+l})^{-1}(H_1 \times \ldots \times H_l).$$

We claim that the restriction of the universal curve $\mathcal{C} \times_S \mathrm{Hom}_S(\mathcal{C}, M) \to \mathrm{Hom}_S(\mathcal{C}, M)$ to $\widehat{U}$ together with the universal (evaluation) morphism from the universal curve to M, is a universal deformation of $(C, \mathbf{x}, \varphi)$. So let $(X \to T, \underline{\mathbf{x}}', \Phi')$ be an analytic family of stable holomorphic curves together with an isomorphism of the fiber over some point $0 \in T$ with $(C, \mathbf{x}, \varphi)$. Since φ is transverse to H_i, local defining equations of H_i pull back to local holomorphic functions on X that restrict to local holomorphic coordinates for C near $\varphi^{-1}(H_i)$ on the central fiber. This shows that possibly after replacing T by a neighbourhood of $0 \in T$ the Cartier divisor ${\Phi'}^{-1}(H_i)$ is a section of $X \to T$. We denote this section by $\underline{y}'_i$ and write $\underline{\tilde{\mathbf{x}}}' = (\underline{\tilde{x}}'_1, \ldots, \underline{\tilde{x}}'_k, \underline{y}'_1, \ldots, \underline{y}'_l)$. By the universal property of $(\mathcal{C} \to S, \underline{\tilde{\mathbf{x}}})$ there exists a unique morphism $T \to S$ such that $(X \to T, \underline{\tilde{\mathbf{x}}}')$ is isomorphic to the pull-back of the universal family over S. Moreover, there is a unique such isomorphism inducing the given identification of X_0 with C.

In turn the universal property of the Hom-space produces a unique morphism $T \to \mathrm{Hom}_S(\mathcal{C}, M)$ such that Φ' is the composition of the evaluation map and the product morphism from X to $\mathcal{C} \times_S \mathrm{Hom}_S(\mathcal{C}, M)$. And if $w_i = 0$ is a defining equation for H_i then $(\Phi' \circ \underline{y}'_i)^* w_i = 0$ by definition of $\underline{y}'_i$. So the map from T to $\mathrm{Hom}_S(\mathcal{C}, M)$ indeed factors over $\widehat{U}$. Given any other map from T to $\widehat{U}$ with these properties, we can pull-back the sections $\underline{\tilde{\mathbf{x}}}$ to T to see that this map coincides with the one just constructed.

Since $\mathrm{Aut}(C,\mathbf{x},\varphi) \subset \mathrm{Aut}(C,\mathbf{x})$ acts on $(\mathcal{C} \to S, \tilde{\underline{\mathbf{x}}})$ ($(\mathcal{C} \to S, \underline{\mathbf{x}})$ is a semi-universal deformation of $(C,\mathbf{x})$!) the universal property now immediately implies an action of this group on $\widehat{U}$ and compatibility of this action with open embeddings of local uniformizing systems, the existence of which being itself provided by universality. A final remark concerns global existence of the universal curve and universal morphism. In fact, the universal curve is itself isomorphic to $\mathcal{C}(M)$, the evaluation morphism is evaluation at the last marked point and the morphism to $\mathcal{C}(M)$ is by forgetting the last marked point and stabilizing (i.e. successively contracting all unstable components on which the map is trivial). These are morphisms by work of Knudson [Kn]. □

1.2. Analytic global normal space, local construction.

By the analytic analog of [BeFa] or its elementary reformulation in [Si2] we need a (free) global normal space ([Si2], Def.3.1 and §4.1) for $\mathcal{C}(M)$ relative "the Artin stack of prestable curves" $\mathfrak{M} = \coprod_{g,k} \mathfrak{M}_{g,k}$ to construct the virtual fundamental class on $\mathcal{C}(M)$. Recall that a global normal space for $\mathcal{C}(M)$ relative $\mathfrak{M}$ would be a morphism in the derived category of coherent orbisheaves on $\mathcal{C}(M)$

$$\varphi^\bullet : \mathcal{F}^\bullet = [\mathcal{F}^{-1} \to \mathcal{F}^0] \longrightarrow \mathcal{L}^\bullet_{\mathcal{C}(M)/\mathfrak{M}}$$

with

- $\mathcal{F}^0$ and $\mathcal{F}^{-1}$ are locally free
- $\varphi^\bullet$ induces an isomorphism in H^0 and an epimorphism in H^{-1}.

Here $\mathcal{L}^\bullet_{\mathcal{C}(M)/\mathfrak{M}}$ is the analytic cotangent complex relative $\mathfrak{M}$. Invoking existing literature ([BiKo], [Fl], [Il]) for the case of analytic orbispaces (or an appropriate analog of DM-stacks in the analytic category) relative an Artin stack is however questionable. Instead of justifying this rather technical step we will give our construction on the level of local uniformizing systems and show in the next section that the corresponding quotients of cones in the complex orbibundle H with local uniformizer $F_1 = L(\mathcal{F}^{-1})$ globalize. It will be clear from the local construction of this cone that the result will be exactly the complex space associated to the DM-stack of cones in the stack-theoretic version of H in [BeFa].

We start with the "obstruction theory" for spaces of morphism as in [BeFa], Ch. 6, in the relevant relative formulation. The following discussion is literally valid both algebraically or analytically. For $(C, \mathbf{x}, \varphi) \in \mathcal{C}(M)$ let $(q : \mathcal{C} \to S, \underline{\tilde{\mathbf{x}}})$ be a universal deformation of a rigidification $(C, \tilde{\mathbf{x}})$ of $(C, \mathbf{x})$, $\tilde{\mathbf{x}} = (x_1, \ldots, x_k, y_1, \ldots, y_l)$, as in the proof of Proposition 1.2. Over the open subspace $Z \subset \mathrm{Hom}_S(\mathcal{C}, M)$ lives the universal curve $\pi : \Gamma := \mathcal{C} \times_S Z \to Z$ with the universal morphism $\Phi : \Gamma \to M$:

$$\begin{array}{ccccc} \mathcal{C} & \longleftarrow & \Gamma & \xrightarrow{\Phi} & M \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \\ S & \longleftarrow & Z & & \end{array}$$

From the functorial properties of the cotangent complex [Il], [Fl], I.2.17 we obtain morphisms

$$L\Phi^* \mathcal{L}^\bullet_M \longrightarrow \mathcal{L}^\bullet_\Gamma \longrightarrow \mathcal{L}^\bullet_{\Gamma/\mathcal{C}},$$

and, by flatness of π, an isomorphism [Fl], I.2.26, [Il], II.2.2.3

$$L\pi^* \mathcal{L}^\bullet_{Z/S} \longrightarrow \mathcal{L}^\bullet_{\Gamma/\mathcal{C}}.$$

All these morphisms are to be understood in $D^-(\mathcal{O}_\Gamma)$ the derived category of the category of $\mathcal{O}_\Gamma$-modules, bounded to the right, or rather the derived category $D^-_{\mathrm{coh}}(\mathcal{O}_\Gamma)$ of complexes with coherent cohomology. The resulting morphism $L\Phi^* \mathcal{L}^\bullet_M \to L\pi^* \mathcal{L}^\bullet_{Z/S}$ is tensored (in the left-derived sense) with ω_π, the relative dualizing sheaf of $\pi : \Gamma \to Z$. Applying $R\pi_*$ we get

$$R\pi_*(L\Phi^* \mathcal{L}^\bullet_M \overset{L}{\otimes} \omega_\pi) \longrightarrow R\pi_*(L\pi^* \mathcal{L}^\bullet_{Z/S} \overset{L}{\otimes} \omega_\pi) \;\simeq\; \mathcal{L}^\bullet_{Z/S} \overset{L}{\otimes} R\pi_* \omega_\pi.$$

Now by smoothness of M, $\mathcal{L}^\bullet_M = \Omega_M$ (viewed as complex concentrated in degree 0), hence $L\Phi^* \mathcal{L}^\bullet_M \overset{L}{\otimes} \omega_\pi = \Phi^* \Omega_M \otimes \omega_\pi$, and by relative duality [Ha], [RaRuVe]

$$R\pi_*(\Phi^* \Omega_M \otimes \omega_\pi) \;\simeq\; [R\pi_* \Phi^* \mathcal{T}_M]^\vee, \quad R\pi_* \omega_\pi \;\simeq\; \mathcal{O}_Z,$$

where $\mathcal{T}_M = \Omega_M^\vee$ is the locally free sheaf of holomorphic vector fields on M. For any complex $\mathcal{E}^\bullet$ of coherent sheaves, bounded to the left, we use the notation $(\mathcal{E}^\bullet)^\vee = \mathrm{Ext}(\mathcal{E}^\bullet, \mathcal{O})$ for the dual complex in the derived sense. In particular, $\mathcal{F}^\vee = \mathrm{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{O})$ for locally free coherent sheaves. We have thus produced a morphism (which is in fact an "obstruction theory for Z relative S" in the notation of [BeFa])

$$[R\pi_* \Phi^* \mathcal{T}_M]^\vee \longrightarrow \mathcal{L}^\bullet_{Z/S}. \tag{1.1}$$

To represent the left-hand side by a morphism of locally free sheaves one needs to assume M projective. Let $\mathcal{H}_M$ be an ample line bundle on M. Then by stability

$$\mathcal{L} \;:=\; \omega_\pi(\underline{x}_1 \ldots + \underline{x}_k) \otimes \Phi^* \mathcal{H}_M^{\otimes 3}$$

has positive degree on each irreducible component of any fiber of π, where we wrote $\underline{x}_i$ for the Cartier divisor corresponding to the i-th entry of $\underline{\tilde{\mathbf{x}}}$. It is not hard to see that choosing ν large enough the natural morphism

$$\mathcal{N} \;:=\; \pi^*\pi_*(\Phi^*\mathcal{T}_M \otimes \mathcal{L}^{\otimes\nu}) \otimes \mathcal{L}^{\otimes -\nu} \longrightarrow \Phi^*\mathcal{T}_M$$

is surjective and $\pi_*\mathcal{N} = \pi_*\mathcal{K} = 0$, $\mathcal{K} := \ker(\mathcal{N} \to \Phi^*\mathcal{T}_M)$ (cf. [Be], Prop. 5). Pushing forward the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{N} \longrightarrow \Phi^*\mathcal{T}_M \longrightarrow 0 \tag{1.2}$$

by π in the derived sense we obtain an exact triangle

$$R\pi_*\Phi^*\mathcal{T}_M[-1] \longrightarrow [0 \to R^1\pi_*\mathcal{K}] \longrightarrow [0 \to R^1\pi_*\mathcal{N}] \longrightarrow R\pi_*\Phi^*\mathcal{T}_M,$$

hence an isomorphism of the mapping cone of the middle morphism with $R\pi_*\Phi^*\mathcal{T}_M$:

$$[R^1\pi_*\mathcal{K} \to R^1\pi_*\mathcal{N}] \xrightarrow{\simeq} R\pi_*\Phi^*\mathcal{T}_M.$$

Note that $R^1\pi_*\mathcal{K}$, $R^1\pi_*\mathcal{N}$ are locally free for $\pi_*\mathcal{K} = \pi_*\mathcal{N} = 0$, The derived dual of $[R^1\pi_*\mathcal{K} \to R^1\pi_*\mathcal{N}]$ may thus be taken entrywise. Write $\mathcal{G} := (R^1\pi_*\mathcal{K})^\vee$, $\mathcal{H} := (R^1\pi_*\mathcal{N})^\vee$, and G, H for the corresponding vector bundles: $\mathcal{G} = \mathcal{O}(G^\vee)$, $\mathcal{H} = \mathcal{O}(H^\vee)$. Together with (1.1) we arrive at a morphism in the derived category

$$\varphi^\bullet : [\mathcal{H} \to \mathcal{G}] \xrightarrow{\simeq} (R\pi_*\Phi^*\mathcal{T}_M)^\vee \longrightarrow \mathcal{L}^\bullet_{Z/S}.$$

This is the local version of the free global normal space $\mathcal{F}^\bullet \to \mathcal{L}^\bullet_{\mathcal{C}(M)/\mathfrak{M}}$. In fact, by [BeFa], Prop. 6.2.

PROPOSITION 1.3. — *$\varphi^\bullet$ induces an isomorphism in H^0 and an epimorphism in H^{-1}.* □

1.3. Analytic global normal cone.

We now review how to produce a cone (over Z) from a global normal space $\phi^\bullet : [\mathcal{H} \to \mathcal{G}] \to \mathcal{L}^\bullet_{Z/S}$. It is convenient to go over to linear fiber spaces over Z. In a hopefully self-explanatory notation $\phi^\bullet$ thus becomes $\Phi^\bullet : (L_{Z/S})_\bullet \to [G \to H]$. Let $Z \hookrightarrow W$ be an embedding of Z into a complex space W that is smooth over S. This is possible at least locally. Let $C_{Z|W}$ be the normal cone of Z in W, which is a closed subspace of the normal space $N_{Z|W}$, the linear space associated to the conormal sheaf $\mathcal{I}/\mathcal{I}^2$, $\mathcal{I} \subset \mathcal{O}_W$ the ideal defining Z. There is a natural morphism (in the derived sense) of linear fiber spaces over Z [Il], Cor. 3.1.3

$$[T_{W|S}|_Z \to N_{Z|W}] \longrightarrow (L_{Z|S})_\bullet$$

inducing isomorphism in H^0 and H^1. We are now in position to produce a cone $C(\Phi^\bullet) \subset H$ by pushing forward the $T_{W|S}|_Z$-cone $C_{Z|W} \subset N_{Z|W}$ under the composition of this morphism with $\Phi^\bullet$ in the way defined in [Si2], Ch. 2. $C(\Phi^\bullet)$ depends only on the map in cohomology induced by $\Phi^\bullet$ [Si2], Thm. 3.3.

From the free global normal space for unrigidified local models of $C(M)$ we thus get a well-defined closed analytic cone $C^H \subset H$. To globalize first note that G and H have straightforward analogs on $\mathcal{C}(M)$. In fact, if $(\pi : \Gamma \to \mathcal{C}(M), \tilde{\mathbf{x}})$ is the universal (marked) curve with universal morphism $\Phi : \Gamma \to M$ we put

$$\mathcal{L} := \omega_\pi(\underline{x_1} + \ldots + \underline{x_k}) \otimes \Phi^*\mathcal{H}_M^{\otimes 3}, \quad \mathcal{N} := \pi^*\pi_*(\Phi^*\mathcal{T}_M \otimes \mathcal{L}^{\otimes 3}) \otimes \mathcal{L}^{\otimes -3},$$
$$\mathcal{K} := \ker(\mathcal{N} \to \Phi^*\mathcal{T}_M).$$

Then G, H are the orbibundles belonging to the orbisheaves $R^1\pi_*\mathcal{K}$, $R^1\pi_*\mathcal{N}$. To keep the notation within limits we stay with our previous symbols G, H, Γ etc. If we want to explicitely refer to uniformizing objects over a local uniformizer $\widehat{U}$ say, we will use the notation $H|_{\widehat{U}}$ etc.

Now let $(C, \mathbf{x}, \varphi) \in \mathcal{C}(M)$. A chart $\widehat{U}$ for the analytic orbispace $\mathcal{C}(M)$ was given by imposing incidence conditions on an open $Z \subset \mathrm{Hom}_S(\mathcal{C}, M)$, cf. the proof of Proposition 1.2:

$$\widehat{U} = (\mathrm{ev}_{k+1}, \ldots, \mathrm{ev}_{k+l})^{-1}(H_1 \times \ldots H_l).$$

Letting $\Phi^\bullet$ be associated to the universal objects on the unrigidified chart Z we define

$$\widehat{C}^H := C(\Phi^\bullet)|_{\widehat{U}},$$

that is, the intersection of $C(\Phi^\bullet)$ with $H|_{\widehat{U}}$ as closed analytic subspaces of H (meaning the bundle over Z). Since the automorphism group of $(C, \mathbf{x}, \varphi)$ acts naturally on the whole construction, $\widehat{C}^H$ is $\mathrm{Aut}(C, \mathbf{x}, \varphi)$-invariant. With the identification of $H|_{\widehat{U}}$ with a local uniformizer of H we may now view $\widehat{C}^H / \mathrm{Aut}(C, \mathbf{x}, \varphi)$ as locally closed complex subspace of H.

PROPOSITION 1.4. — *The germ of $\widehat{C}^H / \mathrm{Aut}(C, \mathbf{x}, \varphi) \subset H$ at any point of its support is independent of choices.*

Proof. — To compare two local uniformizing systems $\widehat{U}' \subset Z' \subset \mathrm{Hom}_{S'}(\mathcal{C}', M)$, $\widehat{U}'' \subset Z'' \subset \mathrm{Hom}_{S''}(\mathcal{C}'', M)$, constructed from the insertion of l' respectively l'' points y_i', y_i'' into C (cf. the proof of Proposition 1.2) we may consider (appropriate shrinkings of) Z' and Z'' as subspaces of a common bigger space $Z'' \subset \mathrm{Hom}_S(\mathcal{C}, M)$. The latter space is simply constructed by inserting the union of $\{y_i'\}$ and $\{y_i''\}$, which is a tuple of $l \leq l' + l''$ points. The point is that for any increase of numbers of inserted points there is a forgetful map (forgetting the additional points), and this induces morphisms $Z \to Z'$, $Z \to Z''$. In fact, Z is just a product of Z' with an open set D in $\mathbb{C}^{l-l'}$ that rules the deformation of the additionally inserted points. A similar statement is true for the universal curve over Z, and this is obviously compatible with the evaluation map. So the cone constructed from Z is just a product of the cone constructed from Z' with D. Now the composition

$$Z' \hookrightarrow Z \longrightarrow Z''$$

induces the unique isomorphisms of the corresponding universal deformations U', U, U'' of $(C, \mathbf{x}, \varphi)$ and this shows the claim. □

The locally closed subcones of (the underlying complex space of) H thus glue to a well-defined closed subcone $C^H \subset H$ that is locally pure dimensional of dimension $\operatorname{rk} G$ plus expected dimension.

The comparison with the symplectic treatment will happen on the level of underlying spaces to which all relevant objects descend. In the symplectic case this is the set of isomorphism classes of stable holomorphic curves, whereas from the complex-analytic treatment it also inherits the structure of a locally ringed space. In case M is projective it follows either by construction or by using the universal property that the latter space is exactly the analytic space corresponding to the coarse moduli space underlying the DM-stack of stable curves in M (which is in fact projective algebraic as shown in [FuPa]).

Now the (associated analytic) map from local étale covers of the stack $C(M)_{\mathrm{alg}}$ to the coarse moduli space ([BeMa], after Prop. 4.7) factors (locally analytically) through a smooth map to our (even unrigidified) charts. Moreover, the GAGA-theorems from Chapter 5 show compatibility of the stack-theoretic global normal space and our analytic global normal space $\varphi^\bullet$. This proves

PROPOSITION 1.5. — *The analytic cone $C \subset H$ over the coarse moduli space associated to the stack-theoretic cone from [Be] coincides with the complex space underlying the complex orbispace C^H.* □

To obtain the analytic analog of Behrend's virtual fundamental class $J(M)$ (an element in the Chow group $A_*(\mathcal{C}(M))$) we finally need to intersect the fundamental class $[C^H] \in A_*(C^H)$ with the zero section of H. Algebraically this is done by applying a bivariant class $\sigma^! \in A^*(\mathcal{C}(M) \hookrightarrow H) \otimes_{\mathbb{Z}} \mathbb{Q}$. The existence of the latter in the category of DM-stacks follows from the work of Vistoli. It results from the compatibility of algebraic intersection theory with homology theory ([Fu], Ch. 19) that the image of $J(M)$ in $H_*(\mathcal{C}(M))$ (that we will also denote $J(M)$) is nothing but

$$J(M) \;=\; [C^H] \cap \Theta_H \;\in\; H_*(\mathcal{C}(M)), \tag{1.3}$$

where $[C^H]$ is the fundamental class (of the underlying coarse modulis spaces), and $\Theta_H \in H^{\mathrm{rk}\,H}_{\mathcal{C}(M)}(H,\mathbb{Q})$ is the Thom class of the orbibundle H (taking into account multiplicities coming from the stabilizers of the local groups, cf. [Si1], §1.2) (1).

2. Analytic Kuranishi models.

A Kuranishi model for $\mathcal{C}(M)$ is a locally closed embedding of a local uniformizer of $\mathcal{C}(M)$ into some $\mathbb{C}^N$ won from a holomorphic Fredholm map between complex Banach spaces having $\mathcal{C}(M)$ as one of its fibers. Finding Kuranishi models in our situation of an integrable complex structure on M is easier than in the general symplectic case, because we may restrict to holomorphic maps on a large part of the curve, notably near the singularities. This is due to the Stein property of open Riemann surfaces. We thus begin with a study of moduli spaces of holomorphic maps from an open Riemann surface.

2.1. Spaces of holomorphic maps from open Riemann surfaces.

Throughout this section we fix an *open* Riemann surface Σ, whose ideal boundary (denoted $\partial\Sigma$ by abuse of notation) we assume to consist of circles only (no punctures). Denote by $\bar{\Sigma} = \Sigma \cup \partial\Sigma$ the corresponding Riemann surface with boundary. Sobolev spaces for maps or functions on Σ will be understood with respect to a Riemannian metric on Σ extending over a neighbourhood of the boundary.

As a preparation let us generalize the decomposition

$$L^p_1(\Sigma;\mathbb{C}) \;=\; L^p(\Sigma;\mathbb{C}) \times \mathcal{O}^{1,p}(\Sigma), \quad \mathcal{O}^{1,p}(\Sigma) := \mathcal{O}(\Sigma) \cap L^p_1(\Sigma;\mathbb{C}),$$

(1) We could also define $J(M)$ as analytic Chow class, but since we want to compare $J(M)$ with the homologically defined $\mathcal{GW}^M \in H_*(\mathcal{C}(M))$ we prefer to work in homology already at this point.

that we used implicitely in [Si1], §4.3 for plane circular domains, to arbitrary open Riemann surfaces. To that end let $K \in \Omega^1(\Sigma \times \Sigma \setminus \Delta)$ be a meromorphic 1-form with a simple pole of constant residue 1 along the diagonal Δ extending to a neighbourhood of the boundary and without zeros. K exists by the Stein property of Σ. Then the singular integral operator

$$T\gamma(z) := \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{\Sigma \setminus B_\varepsilon(z)} K(z,w) \wedge \gamma(w)$$

is a right-inverse for $\bar\partial : L_1^p(\Sigma; \mathbb{C}) \to L^p(\Sigma; \bar\Omega_\Sigma)$. Moreover, for any $f \in L_1^p(\Sigma; \mathbb{C})$ it holds

$$f = T\bar\partial f + Hf, \quad Hf(z) = \frac{1}{2\pi i} \int_{\partial\Sigma} K(z,w) \cdot f(w).$$

The proofs of these statements run as in the case $\Sigma = \Delta$. This establishes the claimed decomposition of $L_1^p(\Sigma; \mathbb{C})$.

The rest of this section is devoted to the following result.

PROPOSITION 2.1. — *The space* $\mathrm{Hom}^{1,p}(\Sigma, M)$ *of holomorphic maps* $\varphi : \Sigma \to M$ *of Sobolev class* L_1^p *has naturally a structure of complex Banach manifold. The evaluation map*

$$\mathrm{ev} : \mathrm{Hom}^{1,p}(\Sigma, M) \times \Sigma \longrightarrow M, \quad (\varphi, z) \longmapsto \varphi(z)$$

is holomorphic with respect to this complex structure. The tangent space at some $\varphi \in \mathrm{Hom}^{1,p}(\Sigma, M)$ *can naturally be identified with* $\mathcal{O}^{1,p}(\Sigma, \varphi^* T_M)$.

Moreover, with this complex structure, $\mathrm{Hom}^{1,p}(\Sigma, M)$ *has the following "universal property": Let* T *be a complex space and* $\Phi : T \times \Sigma \to M$ *be a holomorphic map inducing a continuous map* $\rho : T \to L_1^p(\Sigma, M)$. *Then* ρ *factors over a unique holomorphic map* $T \to \mathrm{Hom}^{1,p}(\Sigma, M)$.

We put "universal property" in quotation marks because $\mathrm{Hom}^{1,p}(\Sigma, M)$ as an infinite dimensional space does not belong to the category of complex spaces for which the property is tested (we would have to use Douady's more involved notion of Banach analytic spaces to remedy this).

As a first step we observe that we can build up Σ inductively starting from a disk by adding arbitrarily "thin" annuli or pairs of pants. We are thus basically reduced to the case of arbitrarily thin 1- or 2-connected plane domains by means of

LEMMA 2.2. — *Let* $\Sigma = \Sigma_1 \cup \Sigma_2$ *with* $\partial\Sigma_1 \cap \partial\Sigma_2 = \emptyset$ *and* $\varphi \in \mathrm{Hom}^{1,p}(\Sigma, M)$. *Put* $\Sigma_{12} = \Sigma_1 \cap \Sigma_2$ *and* $\varphi_i = \varphi|_{\Sigma_i}$, $\varphi_{12} = \varphi|_{\Sigma_{12}}$.

Assume the proposition is true for Σ_1, Σ_2 and Σ_{12} locally around φ_1, φ_2 and φ_{12} respectively. Then the proposition is true for Σ locally around φ.

Proof. — Let θ be an isomorphism of a neighbourhood of φ_{12} in $\mathrm{Hom}^{1,p}(\Sigma_{12}, M)$ with a neighbourhood of the origin in $\mathcal{O}^{1,p}(\Sigma_{12}; \varphi_{12}^* T_M)$ whose differential at φ_{12} we assume to be the identity. We consider the map

$$\Xi : (\psi_1, \psi_2) \longmapsto \theta(\psi_1|_{\Sigma_{12}}) - \theta(\psi_2|_{\Sigma_{12}})$$

from a neighbourhood of (φ_1, φ_2) in $\mathrm{Hom}^{1,p}(\Sigma_1, M) \times \mathrm{Hom}^{1,p}(\Sigma_2, M)$ to $\mathcal{O}^{1,p}(\Sigma_{12}; \varphi_{12}^* T_M)$. The derivative of this map is nothing but the Čech-differential on sections of $\varphi^* T_M$ associated to the covering $\Sigma = \Sigma_1 \cup \Sigma_2$. We claim that the latter is a split submersion. To this end let T^1 and T^2 be right-inverses to $\bar\partial$ on $\varphi_1^* T_M$ and $\varphi_2^* T_M$ as above. For simplicity we assume the meromorphic 1-forms (on $\Sigma_i \times \Sigma_i$) involved in the definition to be restrictions of a meromorphic 1-form K on $\Sigma \times \Sigma$. Note that the pull-backs of T_M are trivial as holomorphic vector bundles, so the definition of a right-inverse to $\bar\partial$ extends to this case by use of a vector valued analog with respect to a holomorphic trivialization of $\varphi^* T_M$ (which we may take to extend continuously to $\bar\Sigma$). Let $1 \equiv \rho_1 + \rho_2$ be a partition of unity subordinate to Σ_i. A splitting of the Čech-complex is given by

$$\begin{aligned} \mathcal{O}^{1,p}(\Sigma_{12}; \varphi_{12}^* T_M) &\longrightarrow \mathcal{O}^{1,p}(\Sigma_1; \varphi_1^* T_M) \times \mathcal{O}^{1,p}(\Sigma_1; \varphi_1^* T_M) \\ v &\longmapsto (\rho_2 \cdot v - T^1(v \bar\partial \rho_2), -\rho_1 \cdot v + T^2(v \bar\partial \rho_1)). \end{aligned}$$

In fact, letting T be the singular integral operator on Σ belonging to K, we obtain

$$\rho_2 \cdot v - T^1(v\bar\partial\rho_2) + \rho_1 \cdot v - T^2(v\bar\partial\rho_1)) \;=\; v - T(v\bar\partial(\rho_1 + \rho_2)) \;=\; v.$$

In view of the holomorphic implicit function theorem this shows the statement about the structure of complex Banach manifold and the description of its tangent spaces. The "universal property" is obvious because for any complex space T a holomorphic map $T \times \Sigma \to M$ is equivalent to the giving of a pair of holomorphic maps $T \times \Sigma_i \to M$ coinciding on $T \times \Sigma_{12}$. $\square$

Next we treat the case that the map φ is "small" compared to the modulus of Σ, as made precise in the following lemma.

LEMMA 2.3. — *Let $\varphi \in \mathrm{Hom}^{1,p}(\Sigma, M)$ and assume there is a covering U_i of Σ by finitely many open sets having piecewise smooth boundary in $\bar\Sigma$ and such that (1) there exists a smooth partition of unity $\{\rho_i\}$ subordinate to U_i (2) $U_i \cap U_j \cap U_k = \emptyset$ for any three pairwise different* *indices i, j, k (3) for any i the closure of $\varphi(U_i)$ is contained in a holomorphic coordinate chart of M. Then the assertions of the proposition hold locally around φ.*

Assumption (1) means that a shrinking of $\{U_i\}$ is still a covering of Σ. This requires the closures of U_i in $\bar{\Sigma}$ to meet in one-dimensional subsets on $\partial\Sigma$.

Proof. — Let $\gamma_i : M \supset W_i \to \mathbb{C}^n$ denote holomorphic coordinates on M with $\mathrm{cl}\,\varphi(U_i) \subset W_i$. We consider the gluing map

$$\Xi : \prod_i \mathrm{Hom}^{1,p}(U_i, W_i) \longrightarrow \prod_{i<j} \mathrm{Hom}^{1,p}(U_i \cap U_j, \mathbb{C}^n),$$
$$(\psi_i)_i \longmapsto \Big(\gamma_i \circ \psi_i|_{U_j} - \gamma_i \circ \psi_j|_{U_i}\Big)_{ij}.$$

Clearly, the spaces $\mathrm{Hom}^{1,p}(U_i, W_i)$ are open sets in complex Banach spaces via γ_i. As in the previous lemma we want to show that the differential of Ξ at $(\varphi|_{U_i})_i$ is a split submersion. This differential is the map

$$\prod_i \mathcal{O}^{1,p}(U_i; \varphi^* T_M) \longrightarrow \prod_{i<j} \mathcal{O}^{1,p}(U_i \cap U_j, ; \mathbb{C}^n), \quad (v_i)_i \longmapsto (D\gamma_i)(v_i - v_j).$$

Note that the differential of γ_i is an isomorphism for any i. The requested splitting can thus be defined by

$$(D\gamma_i(v_{ij}))_{ij} \longmapsto \Big(\sum_j \rho_j v_{ij} - T^i(v_{ij}\bar{\partial}\rho_j)\Big)_i.$$

As in the proof of the previous lemma we use a holomorphic trivialization of $\varphi^* T_M$ to construct a right-inverse T to $\bar{\partial}$, the restriction of which to sections with support in U_i is T^i. The verification that this is indeed a splitting of the differential and the "universal property" run also as above. We conclude that $\Xi^{-1}(0)$ is a chart for $\mathrm{Hom}^{1,p}(\Sigma, M)$ at φ with tangent space $\mathcal{O}^{1,p}(\Sigma, \psi^* T_M)$ at ψ. The universal property is again obvious. □

Proof of Proposition 2.1. — Locally around some $\varphi \in \mathrm{Hom}^{1,p}(\Sigma, M)$ we construct the complex Banach manifold structure in the way already indicated: Write Σ as union of open Riemann surfaces Σ_i with the following properties:

(1) Each Σ_i is either a disk, an annulus, or a pair of pants (a two-connected plane domain).

(2) For any $i \neq j$ the intersection $\Sigma_i \cap \Sigma_j$ is a union of (one or two) annuli.

(3) Each Σ_i has a covering by open sets as required by Lemma 2.3 (the same holds then also true for the pairwise intersections). Applying the two lemmata we obtain the statement of the proposition locally around φ.

It remains to show independence of choices (in particular of the decomposition $\Sigma = \bigcup \Sigma_i$) and biholomorphicity of changes of coordinates. So let $V = \Xi^{-1}(0)$, $V' = (\Xi')^{-1}(0)$ be two such charts with $V \cap V' \neq \emptyset$ as subsets of $\mathrm{Hom}^{1,p}(\Sigma, M)$. From the holomorphic evaluation map $V' \times \Sigma \to M$, upon restriction to Σ_i, we obtain maps

$$V' \longrightarrow \mathrm{Hom}^{1,p}(\Sigma_i, M).$$

These are holomorphic maps of Banach manifolds and compatible on intersections $\Sigma_i \cap \Sigma_j$. We thus obtain a holomorphic map of Banach manifolds $V' \to V$. The same reasoning with V, V' exchanged shows that this map is invertible. $\square$

2.2. Localization of variations of complex structure.

As a further preliminary to the construction of analytic Kuranishi models we want to show that any deformation $q : X \to S$ of a closed Riemann surface $\Sigma = X_0$ is obtained by changing the complex structure on *arbitrarily small* open sets. To see this take a meromorphic function f on (a neighbourhood of the central fiber of) X exhibiting Σ as a (say d-sheeted) branched covering of $\mathbb{P}^1$ with only simple branch points. Then nearby fibers are also simply branched coverings of $\mathbb{P}^1$, and the set of branch points varies holomorphically with s. Let $V_\varepsilon \subset \mathbb{P}^1$ be an ε-neighbourhood (in any metric on $\mathbb{P}^1$) of the branch locus of $\Sigma \to \mathbb{P}^1$. For sufficiently small s the branch locus of $X_s \to \mathbb{P}^1$ is still contained in V_ε. For any such s the map f induces an isomorphism $X_s \setminus f^{-1}(V_\varepsilon) \simeq \Sigma \setminus f^{-1}(V_\varepsilon)$. Since we are only interested in the germ of the deformation we may take ε arbitrarily small.

Let $\Delta \simeq V_j \subset V_\varepsilon$ be a connected component containing the j-th branch point. Then for ε sufficiently small $f^{-1}(V_j)$ is the disjoint union of $d-2$ copies of $S \times \Delta$ and a two-fold cover of an open set $W_j \subset X$ over $S \times \Delta$ branched along $z = b(s)$. So the holomorphic function $b(s)$ describes the variation of the j-th branch point. Now the transformation

$$(s, z) \longmapsto \left(s, \frac{z - h(s)}{1 - h(s) z}\right)$$

exhibits a biholomorphism of $S \times \Delta$ mapping the branch locus to $S \times \{0\}$. This shows that W_j is a product $S \times \Delta$.

Taking into account also deformations of nodes ($Z_t = \{(z,w) \in \Delta \times \Delta \mid zw = t\}$) we obtain:

LEMMA 2.4. — *Let $p : X \to S$ be a deformation of a prestable curve C. Then, possibly after shrinking S there exists a decomposition $X = U_0 \cup \bigcup_{i=1}^{l} U_i \cup \bigcup_{j=1}^{m} W_j$ with*

- *$U_0 = S \times \Sigma$, where Σ is a non-compact Riemann surface*
- *$U_i(s) = Z_{t_i(s)}$ for some $t_i \in \mathcal{O}(S)$ with $t_i(0) = 0$*
- *$W_j \simeq S \times \Delta$ is a product*
- *$U_i \cap U_{i'} = U_i \cap W_j = W_j \cap W_{j'} = \emptyset$ for any $i, i', j, j' \geq 1$, $i \neq i'$, $j \neq j'$; $U_0 \cap U_i \simeq S \times (A_i \cup A_i')$; $U_0 \cap W_j \simeq S \times A_j''$ for some annuli A_i, A_i', A_j''.*

Moreover, the U_i ($i > 0$) and W_j can be chosen arbitrarily small. □

The essential point of this decomposition is that all the open sets together with their intersections are constant in the family. This implies immediate holomorphic trivializations of Banach bundles of holomorphic sections to be used below.

2.3. Construction of analytic Kuranishi models.

We now want to make the construction of (local) Kuranishi models in [Si1] holomorphic. At $(C, \mathbf{x}, \varphi) \in \mathcal{C}(M)$ the construction worked as follows: Let

$$(q : \mathcal{C} \to S, \underline{\mathbf{x}} : S \to \mathcal{C} \times_S \ldots \times_S \mathcal{C})$$

be an analytically semiuniversal deformation of $(C, \mathbf{x})$, $C = q^{-1}(0)$. We showed in op. cit. that the space of Sobolev maps (with weights at the nodes) from the $C_s = q^{-1}(s)$ to M

$$\check{L}_1^p(\mathcal{C}/S; M) = \coprod_{s \in S} \check{L}_1^p(C_s; M)$$

is a Banach manifold, locally isomorphic to $S \times V$, $V \subset \check{L}_1^p(C; \varphi^* T_M)$ open. The map $\varphi \mapsto \bar{\partial}\varphi$ is locally a family of differentiable Fredholm maps

$$H_s = H|_{\{s\} \times V}, \quad H : S \times V \longrightarrow E := \check{L}^p(C; \varphi^* T_M \otimes \bar{\Omega}).$$

Choose a linear projection $\mathrm{pr}_Q : E \to Q$ with finite-dimensional kernel and with $\mathrm{pr}_Q \circ DH_0$ surjective. The implicit function theorem with parameter

s then shows that $\tilde{Z} = (\mathrm{pr}_Q \circ H)^{-1}(0)$ is a finite-dimensional manifold near $(0,0) \in S \times V$. Provided $(C, \mathbf{x})$ is stable, a local uniformizer of $\mathcal{C}(M)$ is locally given as zero locus of $\kappa|_{\tilde{Z}}$, $\kappa = H - \mathrm{pr}_Q \circ H = \mathrm{pr}_F \circ H$, $\mathrm{pr}_F : E \to F$ the projection with kernel Q onto a finite dimensional subspace $F \subset E$ with $E = F + \mathrm{im}\, DH_0$. So F spans the cokernel of DH_0. If $(C, \mathbf{x})$ is not stable one finally has to choose a slice to the action of $\mathrm{Aut}^0(C, \mathbf{x})$ on $S \times V$.

Instead of making this construction holomorphic in each step we restrict to a set of maps in $\check{L}_1^p(\mathcal{C}/S; M)$ that are already largely holomorphic. We take a decomposition of C as provided by Lemma 2.4 applied to $q : \mathcal{C} \to S$. Without loss of generality we may assume that each irreducible component of C contains at least one W_j that remains constant under the deformation, say $j \in \{1, \ldots, m'\}$. We assume that the closure of the image of U_i $(i > 0)$ and of W_j under φ map into holomorphic coordinate charts

$$\gamma_i : M \supset M_i \to \mathbb{C}^n, \quad i = 1, \ldots, l + m.$$

So M_i are open sets in M containing $\mathrm{cl}\, \varphi(U_i(0))$ respectively $\mathrm{cl}\, \varphi(W_{i-l}(0))$ (for $i > l$). Our ambient Banach manifold will be the subset of $\check{L}_1^p(\mathcal{C}/S; M)$ of maps that are holomorphic on $\bigcup_{i \geq 0} U_i \cup \bigcup_{j=m'+1}^{m} W_j$. So the flexibility provided by L_1^p-maps survives only on the part of $W_1, \ldots, W_{m'}$ not meeting $\Sigma = U_0$. The space of such maps can be described as fiber over 0 of a holomorphic map $\Xi : B \to B'$ of complex Banach manifolds. The domain B is the fibered product over S of the spaces

$$S \times \mathrm{Hom}^{1,p}(\Sigma, M); \quad \mathrm{Hom}_S^{1,p}(U_i, M_i), i = 1, \ldots, l;$$
$$S \times L_1^p(W_j(0); M_{l+j}), j = 1, \ldots, m'; \quad \mathrm{Hom}_S^{1,p}(W_j, M_{l+j}), j = m'+1, \ldots, m,$$

(we use Proposition 2.1) and B' will be

$$\prod_{i=1}^{l} \mathrm{Hom}^{1,p}(A_i \cup A_i', \mathbb{C}^n) \times \prod_{j=1}^{m'} L_1^p(A_j''; \mathbb{C}^n) \times \prod_{j=m'+1}^{m} \mathrm{Hom}^{1,p}(A_j'', \mathbb{C}^n).$$

Here we wrote $\mathrm{Hom}_S^{1,p}(U_i, M_i)$ for $\coprod_{s \in S} \mathrm{Hom}^{1,p}(U_i(s), M_i)$ that in view of the identification of holomorphic function spaces on deformation spaces of nodes given in [Si1], §4.2 is an open set in the product of S with a complex Banach space via γ_i. By the product structure of W_j we also have a trivialization of $\mathrm{Hom}_S^{1,p}(W_j, M_{l+j}) = \coprod_{s \in S} \mathrm{Hom}^{1,p}(W_j(s), M_{l+j})$. The map Ξ is the obvious nonlinear version of the first Čech differential. It sends $(\psi_0, \psi_i, \psi_j')$ to

$$\left(\gamma_i \circ \psi_i - \gamma_i \circ \psi_0, \gamma_j \circ \psi_j' - \gamma_j \circ \psi_0\right),$$

and this is clearly a holomorphic map.

PROPOSITION 2.5. — *In a neighbourhood of* $(C, \mathbf{x}, \varphi)$, $\mathcal{B} := \Xi^{-1}(0)$ *is a complex Banach manifold lying smoothly over* S.

Proof. — In view of the implicit function theorem we have to show that the differential $D\Xi_0$ is a split submersion, where Ξ_0 is the restriction of Ξ to $s = 0$. With the natural identification of the tangent space relative S of B at φ with

$$\mathcal{O}^{1,p}(\Sigma; \varphi^* T_M) \times \prod_{i=1}^{l} \mathcal{O}^{1,p}(U_i, \varphi^* T_M) \times \prod_{j=1}^{m'} L_1^p(W_j(0); \varphi^* T_M) \times \prod_{j=m'+1}^{m} \mathcal{O}^{1,p}(W_j(0), \varphi^* T_M)$$

the differential is isomorphic to the linear Čech differential, mapping (v_0, v_i, v_j') to $(v_i - v_0, v_j' - v_0)$, as element in

$$\prod_{i=1}^{l} \mathcal{O}^{1,p}(A_i \cup A_i'; \varphi^* T_M) \times \prod_{j=1}^{m'} L_1^p(A_j''; \varphi^* T_M) \times \prod_{j=m'+1}^{m} \mathcal{O}^{1,p}(A_j'', \varphi^* T_M).$$

To define the splitting let T be a right-inverse to the $\bar\partial$-operator on $\varphi^* T_M$ restricted to $C^0 = C_0^0$, where

$$C_s^0 := \bigcup_{i=0}^{l} U_i(s) \cup \bigcup_{j=m'+1}^{m} W_j(s),$$

constructed as singular integral operator via a holomorphic trivialization as in the last section. Let T^i be the restriction of T to $U_i(0)$ (for $i = 0, \ldots, l$) and to $W_{m'+i-l}(0)$ respectively (for $i = l+1, \ldots, l+m-m'$). For brevity we put $H^i = \mathrm{id} - T^i \circ \bar\partial$. For $j = 1, \ldots, m'$ we also need a (complex linear) extension map

$$\Omega^j : L_1^p(U_0(0) \cap W_j(0); \varphi^* T_M) \longrightarrow L_1^p(W_j(0); \varphi^* T_M),$$

that is, a right-inverse to the corresponding restriction map. The existence of Ω^j is a standard fact of Sobolev theory. Finally let ρ_i, ρ_j' be a partition of unity on C subordinate to our covering. A right inverse to $D\Xi_0$ is now given by sending (w_i, w_j') to (v_0, v_i, v_j') with

$$\begin{aligned}
v_0 &= -H^0\Big(\sum_{i=1}^{l} \rho_i w_i + \sum_{j=m'+1}^{m} \rho_j w_j'\Big) && \\
v_i &= H^i(\rho_0 w_i) && i = 1, \ldots, l\\
v_j' &= \Omega^j(w_j' + (v_0|_{A_j''})) && j = 1, \ldots, m'\\
v_j' &= H^{l+j-m'}(\rho_0' w_j') && j = m'+1, \ldots, m.
\end{aligned}$$

The verification that this is indeed a splitting is straightforward (as in the proofs of the lemmata in the previous section). An application of the implicit function theorem completes the proof. □

In the sequel we identify $(\psi_0, \psi_i, \psi'_j) \in \mathcal{B}$ with the induced L^p_1-map $\psi : C_s \to M$ that is holomorphic on C^0_s. Choosing a biholomorphism $z_j : \Delta^j \simeq \Delta$ for the complement Δ^j of $\mathrm{cl}\, A''_j$ in $W_j(0)$ $(j = 1, \dots, m')$, the (non-linear) $\bar\partial$-equation $\varphi \mapsto \bar\partial\varphi$ can now be viewed as holomorphic map

$$\Theta : \mathcal{B} \longrightarrow \prod_{j=1}^{m'} L^p(\Delta^j; \mathbb{C}^n), \quad (\psi_0, \psi_i, \psi'_j) \longmapsto \Big(\frac{\partial}{\partial \bar z_j} \gamma_j \circ \psi'_j\Big)_{j=1,\dots,m'}.$$

PROPOSITION 2.6. — *There is a natural map from the differential at $(C, \mathbf{x}, \varphi)$ of Θ relative S to*

$$\bar\partial : \check L^p_1(C; \varphi^* T_M) \longrightarrow \check L^p(C; \varphi^* T_M \otimes \bar\Omega)$$

inducing isomorphisms of kernels and cokernels. In particular Θ is Fredholm at $(C, \mathbf{x}, \varphi)$.

Proof. — Let us write Θ_0 for the restriction of Θ to the central fiber $\mathcal{B}_0$ of $\mathcal{B}$ over $0 \in S$. From the proof of the last proposition the tangent space of $\mathcal{B}_0$ at $\psi : C_s \to M$ can be identified with

$$V := \{v \in \check L^p_1(C; \varphi^* T_M) \mid v|_{C^0} \in \mathcal{O}(C^0; \varphi^* T_M)\}.$$

Thus

$$D\Theta_0 : V \longrightarrow \prod_{j \le m'} L^p(\Delta^j; \mathbb{C}^n), \quad v \longmapsto \Big(D\gamma_j(\frac{\partial v}{\partial \bar z_j})\Big)_j.$$

Multiplying the components on the right-hand side by $d\bar z_j$ and extending trivially by zero as section of $\varphi^* T_M \otimes \bar\Omega$ we obtain a commutative square

$$\begin{array}{ccc} V & \xrightarrow{D\Theta_0} & \prod_{j \le m'} L^p(\Delta^j; \mathbb{C}^n) \\ \downarrow & & \downarrow \\ \check L^p_1(C; \varphi^* T_M) & \xrightarrow{\bar\partial} & \check L^p(C; \varphi^* T_M \otimes \bar\Omega) \end{array} \tag{2.4}$$

that we claim to induce isomorphisms of kernels and cokernels of the horizontal arrows. The diagram certainly gives rise to isomorphisms on the kernels, both being equal to $H^0(C; \varphi^* T_M)$. To prove injectivity on cokernels let $\gamma \in \prod_{j \le m'} L^p(\Delta^j, \mathbb{C}^n)$ and assume that its trivial extension $\tilde\gamma$ can be written as $\bar\partial v$ with $v \in \check L^p_1(C; \varphi^* T_M)$. Then $\bar\partial v = 0$ away from $\bigcup \Delta^j$ and so $v \in V$.

As for surjectivity, the soft resolution of $\mathcal{O}^{1,p}(\varphi^* T_M|_{C^0})$ by sheaves of Sobolev sections (restriction of [Si1], Prop. 4.5 to C^0)

$$0 \longrightarrow \mathcal{O}^{1,p}(\varphi^* T_M|_{C^0}) \longrightarrow \check{\mathcal{L}}_1^p(\varphi^* T_M|_{C^0}) \xrightarrow{\bar\partial} \check{\mathcal{L}}^p(\varphi^* T_M|_{C^0}) \longrightarrow 0$$

together with $H^1(C^0; \varphi^* T_M) = 0$ (C^0 is Stein!) shows the surjectivity of

$$\bar\partial : \ \check{L}_1^p(C^0; \varphi^* T_M) \longrightarrow \check{L}^p(C^0; \varphi^* T_M \otimes \bar\Omega).$$

So for any $\gamma \in \check{L}^p(C^0; \varphi^* T_M \otimes \bar\Omega)$ there is a solution to the equation

$$\bar\partial v_0 \ = \ \gamma|_{C^0}.$$

Let $v \in \check{L}_1^p(C; \varphi^* T_M)$ be an extension of v_0 to all of C. Then $\gamma - \bar\partial v$ has support in $\bigcup_{j \le m'} \Delta^j$, hence is in the image of the right-hand vertical map of (2.4). This shows surjectivity of (2.4) on cokernels. □

By the proposition, $\Theta^{-1}(0)$ is thus given as the fiber of a holomorphic Fredholm map between complex Banach manifolds, hence has naturally the structure of a (finite dimensional) complex space.

THEOREM 2.7. — *The germs of $\Theta^{-1}(0)$ and of $\mathrm{Hom}_S(\mathcal{C}; M)$ at $(C, \mathbf{x}, \varphi)$ are canonically isomorphic.*

Proof. — We have to check the universal property of the hom-functor for $\Theta^{-1}(0)$. So let $T \to S$ be a morphism of complex spaces mapping a distinguished point $0 \in T$ to $0 \in S$ and write $\mathcal{C}_T = T \times_S \mathcal{C}$. To any morphism $\Phi : \mathcal{C}_T \to M$ inducing φ on the central fiber we claim the existence of a unique (germ of) morphism $\Lambda : T \to \Theta^{-1}(0)$ such that Φ factors over the evaluation morphism $\Theta^{-1}(0) \times_S \mathcal{C} \to M$ via $\Lambda \times \mathrm{id}_{\mathcal{C}}$.

First we observe that the morphism Φ from $\mathcal{C}_T$ to M is equivalent to the giving of a tuple of S-morphisms from T to the following spaces: $S \times \mathrm{Hom}^{1,p}(\Sigma, M)$, $\mathrm{Hom}_S^{1,p}(U_i, M_i)$ ($i = 1, \ldots, l$), $\mathrm{Hom}_S^{1,p}(W_j, M_{l+j})$ ($j = 1, \ldots, m$), such that the composition with the analog Ξ_h of Ξ from the product B_h ("h" for "holomorphic") of these spaces to

$$B_h' := \prod_{i=1}^{l} \mathrm{Hom}^{1,p}(A_i \cup A_i', \mathbb{C}^n) \times \prod_{j=1}^{m} \mathrm{Hom}^{1,p}(A_j'', \mathbb{C}^n)$$

is the zero morphism. Recall that the term "S-morphism" means compatibility with the morphisms to S that the relevant spaces do possess (including T). Now consider the following diagram of complex Banach manifolds:

$$\begin{tikzcd}
 & B_h \arrow[r, "\Xi_h"] \arrow[d] & B'_h \arrow[d] \\
\mathcal{B} \arrow[r] \arrow[d, "\Theta"] & B \arrow[r, "\Xi"] \arrow[d, "\bar\partial"] & B' \arrow[d, "\bar\partial"] \\
\prod_{j=1}^{m'} L^p(\Delta^j;\mathbb{C}^n) \arrow[r] & \prod_{j=1}^{m'} L^p(W_j(0);\mathbb{C}^n) \arrow[r] & \prod_{j=1}^{m'} L^p(A''_j;\mathbb{C}^n)
\end{tikzcd}$$

The right-hand horizontal maps are all split submersions, the left-hand horizontal and upper vertical maps are closed embeddings (the kernels of the following maps) and the lower vertical maps are given by $\bar\partial$-operators. What we have just said about families amounts to saying that the fiber over 0 (as ringed space) of the non-linear Fredholm map $B_h \to B'_h$ represents the Hom-functor under consideration. Now a morphism to this fiber is equivalent to a morphism to B which composed with each of the maps to B' and to $\prod L^p(W_j(0);\mathbb{C}^n)$ is the zero morphism. So this is nothing but a morphism to $\mathcal{B}$ whose composition with Θ to $\prod L^p(\Delta^j;\mathbb{C}^n)$ is zero, which in turn is the same as a morphism to the fiber of Θ over 0. All these morphisms are to be understood in the category of ringed spaces. This shows the universal property of the hom-functor for $\Theta^{-1}(0)$. □

We do not bother about explicit local analytic Kuranishi models or analytic rigidification here, because we will never need this. The former can be easily established by the usual procedure of choosing a finite-dimensional complex linear subspace in $\prod_{j\le m'} L^p(\Delta^j;\mathbb{C}^n)$ spanning $\operatorname{coker} D\Theta_0$; a holomorphic rigidification on the other hand can be provided by requiring incidence of additional marked points with local analytic divisors in M as in Proposition 1.2. Instead we will show in Section 4.1 how to incorporate the local construction given here in the global construction of [Si1].

3. Limit cones.

3.1. Cone classes.

This section deals with a purely topological question in finite dimensions. Let N be a finite dimensional, oriented, topological orbifold,

$q : F \to N$ an orbibundle and s a continuous section of F ([2]). For any $l > 0$ let $p_l : N \times \mathbb{R}^l \to N$ be the projection and let $s_l \in \Gamma(N \times (\mathbb{R}^l \setminus \{0\}), p_l^* F)$ be defined by

$$s_l(x, v) \;=\; |v|^{-1} \cdot s(x).$$

For any $v \in \mathbb{R}^l \setminus \{0\}$ the restriction of the graph Γ_{s_l} of s_l to $N \times \{v\}$ is $\Gamma_{|v|^{-1} \cdot s}$.

DEFINITION 3.1. — *Let $\overline{\Gamma_{s_l}}$ be the closure of Γ_{s_l} in $p_l^* F$. The* cone $C(s) \subset F$ *associated to s is defined by*

$$C(s) \;:=\; \overline{\Gamma_{s_l}} \cap (F \times \{0\}). \qquad \square$$

$C(s)$ is independent of l and lies over the zero locus of s:

PROPOSITION 3.2.

1) $C(s) = \lim\limits_{t \to \infty} \Gamma_{t \cdot s}$ *in the sense of Hausdorff convergence of closed sets (hence $C(s)$ is independent of l)*

2) $q(C(s)) = Z(s)$.

Proof.

1) Recall that convergence $\Gamma_{t \cdot s} \overset{t \to \infty}{\longrightarrow} C(s)$ means two things:

• $C(s) = \bigcap\limits_{t_0} \mathrm{cl}\Big(\bigcup\limits_{t \geq t_0} \Gamma_{t \cdot s} \Big)$.

• For any compactum $K \subset F$ and any neighbourhood U of $C(s) \subset F$ there exists a t_0 with

$$\Gamma_{t \cdot s} \cap K \subset U \quad \text{for any} \quad t > t_0.$$

Both properties follow easily from the corresponding facts for $\overline{\Gamma_l}$: Since for any $t_0 > 0$ the restriction of the projection $P_l : p_l^* F = F \times \mathbb{R}^l \to F$ to $F \times \bar{B}_{t_0^{-1}}(0)$ is proper and F is locally compact we have

$$\begin{aligned} \mathrm{cl}_F \Big(\bigcup_{t \geq t_0} \Gamma_{t \cdot s} \Big) &= P_l \Big[\mathrm{cl}_{F \times \mathbb{R}^l} (\Gamma_{s_l} \cap (F \times \bar{B}_{t_0^{-1}}(0)) \Big] \\ &= P_l \Big[\overline{\Gamma_{s_l}} \cap (F \times \bar{B}_{t_0^{-1}}(0)) \Big], \end{aligned}$$

([2]) More generally, we may take N to be an oriented, $\mathbb{Q}$-homology manifold and $F \to N$ a not necessarily locally trivial cone over S, i.e. F have a continuous, fiber-preserving action of the multiplicative semigroup $\mathbb{R}_{\geq 0}$ that is proper away from the zero section $0 \cdot F \simeq N$.

where we indicated with subscripts in which spaces closures are being taken. The intersection of these sets over all t_0 yields

$$\bigcap_{t_0} \mathrm{cl}\Big(\bigcup_{t \geq t_0} \Gamma_{t\cdot s}\Big) \;=\; P_l\Big(\overline{\Gamma_{s_l}} \cap (F \times \{0\})\Big) \;=\; C(s).$$

For the second point let $K, U \subset F$ be as stated in the hypothesis. Consider the other projection $Q_l : F \times \mathbb{R}^l \to \mathbb{R}^l$. The restriction of Q_l to $\overline{\Gamma_{s_l}} \cap P_l^{-1}(K)$ is proper, and for the fiber over $0 \in \mathbb{R}^l$ the following inclusion holds:

$$\overline{\Gamma_{s_l}} \cap (K \times \{0\}) \subset P_l^{-1}(U).$$

Then the same inclusion holds for $\overline{\Gamma_{s_l}} \cap (K \times \{v\})$, $v \in B_{t_0^{-1}}(0)$, for some $t_0 > 0$. But this means

$$\Gamma_{t\cdot s} \cap K \subset U \quad \forall\, t > t_0.$$

2) One inclusion follows from $\Gamma_{s_l} \supset Z(s) \times (\mathbb{R}^l \setminus \{0\})$. Conversely, let $f \in F_x$ and $s(x) \neq 0$. Let K be a compact neighbourhood of f in F such that s has no zeros on $q(K)$. Then

$$A \;=\; \{(t, y) \in \mathbb{R}_{\geq 0} \times N \mid t \cdot s(y) \in K\}$$

is compact. Choosing $c > 0$ with $\mathrm{pr}_1(A) \subset [0, c]$ we obtain

$$\Gamma_{t\cdot s} \cap K \;=\; \emptyset \quad \text{for any } t > c.$$

Hence $f \notin C(s)$. □

The reason for introducing the factor $\mathbb{R}^l$ with $l > 1$ is the long exact sequence of homology groups (of the second kind, that is with locally finite singular chains)

$$H_{n+l}(C(s)) \longrightarrow H_{n+l}(\overline{\Gamma_{s_l}}) \longrightarrow H_{n+l}(\Gamma_{s_l}) \longrightarrow H_{n+l-1}(C(s)).$$

Provided $n + l - 1 > \dim C(s)$ (e.g. $l > \dim F - n + 1$) the groups on the left and right vanish by the general vanishing theorem of homology, cf. e.g. [Iv], IX.1, Prop. 1.6. So the (orbifold!) fundamental class

$$[\Gamma_{s_l}] \;=\; (s_l)_*[N \times \mathbb{R}^l]$$

extends uniquely to an $(n+l)$-dimensional homology class on $\overline{\Gamma_{s_l}}$, that will be conveniently denoted $[\overline{\Gamma_{s_l}}]$ (slight abuse of notation). Here we assumed (without loss of generality) N to be pure n-dimensional and we chose an orientation of $\mathbb{R}^l$ that will finally drop out. We are now in position to define the limit of $[\Gamma_{t_s}]$ as t tends to ∞.

Definition 3.3. — *Let $\delta_0 \in H^l_{\{0\}}(\mathbb{R}^l)$ be Poincaré-dual to $\{0\} \subset \mathbb{R}^l$ with respect to the chosen orientation. Let $Q_l : p_l^* F \to \mathbb{R}^l$ be the projection. Noticing that $C(s) = \overline{\Gamma_{s_l}} \cap Q_l^{-1}(0)$ we define*

$$[C(s)] \;:=\; [\overline{\Gamma_{s_l}}] \cap Q_l^* \delta_0 \in H_n(C(s)).$$

□

Implicit in the notation is the first statement in

PROPOSITION 3.4. — *$[C(s)]$ is independent of l and homologous to $[\Gamma_{t\cdot s}]$ for any $t \neq 0$ as class on F.*

Proof. — We identify $\mathbb{R}^l$ with a linear subspace in $\mathbb{R}^{l+1}$. Let $\eta \in H^1_{\mathbb{R}^l}(\mathbb{R}^{l+1})$ be the corresponding cohomology class with supports. We write δ^k for the integral generator of $H^k_{\{0\}}(\mathbb{R}^k)$ (previously denoted δ_0). Then

$$[\Gamma_{s_l}] = [\Gamma_{s_{l+1}}] \cap Q^*_{l+1}\eta, \quad \delta^{l+1} = \eta \cup \delta^l,$$

hence

$$\begin{aligned}[\overline{\Gamma_{s_l}}] \cap Q^*_l \delta^l &= \left([\overline{\Gamma_{s_{l+1}}}] \cap Q^*_{l+1}\eta\right) \cap Q^*_l \delta^l \\ &= [\overline{\Gamma_{s_{l+1}}}] \cap Q^*_{l+1}(\eta \cup \delta^l) = [\overline{\Gamma_{s_{l+1}}}] \cap Q^*_{l+1}\delta^{l+1}.\end{aligned}$$

And for $t > 0$ the Poincaré-dual δ_t to $\{t\} \subset \mathbb{R}^l$ is cohomologous to δ^l as class on $\mathbb{R}^l$. Thus

$$[\Gamma_{t\cdot s}] = [\Gamma_{s_l}] \cap Q^*_l \delta_t = [\overline{\Gamma_{s_l}}] \cap Q^*_l \delta_t = [\overline{\Gamma_{s_l}}] \cap Q^*_l \delta^l = [C(s)]$$

in $H_n(F)$. □

In a holomorphic situation we retrieve the following familiar picture [Fu], §14.1: Let F be a holomorphic vector bundle over a complex manifold N, and let Z be the zero locus of a holomorphic section s of F. The differential of s induces a closed embedding ι of the *normal bundle* $N_{Z|N}$ of Z in N into F. $N_{Z|N}$ is the linear fiber space over Z associated to the conormal sheaf $\mathcal{I}/\mathcal{I}^2$, $\mathcal{I}$ the ideal sheaf of Z in N. The *normal cone* $C_{Z|N}$ (the analytic analog of $\mathrm{Spec}_Z \oplus_{d\geq 0}\mathcal{I}^d/\mathcal{I}^{d+1}$) is a closed subspace of $N_{Z|N}$. Thus $\iota(C_{Z|N})$ is a closed subcone of F.

PROPOSITION 3.5. — *$C(s) = \iota(C_{Z|N})$ and $[C(s)] = \iota_*[C_{Z|N}] = [\iota(C_{Z|N})]$ (where $[C_{Z|N}]$, $[\iota(C_{Z|N})]$ are the fundamental classes of the correponding complex spaces).*

Proof. — Our construction of $C(s)$ and $[C(s)]$ is nothing but (a real version of) the "deformation to the normal cone", which in this case states that $\iota(C_{Z|N})$ can be obtained as analytic limit of $\Gamma_{t\cdot s}$, $t \in \mathbb{C}$, $|t| \to \infty$ [Fu], Rem. 5.1.1. From this the compatibility of the two limits can be easily deduced. □

It should be clear that the same conclusions hold in the category of analytic orbifolds, but we will not need this.

For later reference we also observe here two simple lemmata:

LEMMA 3.6. — *Let $q : \widehat{F}_U = \widehat{U} \times \mathbb{R}^r \overset{/G^F}{\longrightarrow} F|_U$ be a local uniformizing trivialization of a topological orbibundle F over a local uniformizing system $U = \widehat{U}/G$ of the base orbifold N, and let s be a section of F uniformized by $\hat{s}_U : \widehat{U} \to \widehat{F}_U$. Let b be the generic covering degree of q. Then*

$$C(s) = q(C(\hat{s})), \quad [C(s)] = \frac{1}{b} q_*[C(\hat{s})].$$

Proof. — This follows immediately by the corresponding identities for $\Gamma_{t\cdot s}$ and $\Gamma_{t\cdot\hat{s}}$. □

3.2. Local decomposition of cone classes.

Returning to GW-theory we will choose in Section 4.1 a morphism $\tau : F \to \mathcal{E}$ such that locally there is complex subbundle $F^h \subset F$ with $\tau^h = \tau|_{F^h}$ (essentially) holomorphically spanning the cokernel of the linearization of $s = s_{\bar{\partial}}$, cf. Proposition 4.2. We obtain two finite-dimensional oriented orbifolds $\tilde{Z} \subset F$ and $\tilde{Z}^h \subset F^h$ as zero loci of $\tilde{s} = q^* s + \tau$ and $\tilde{s}^h = (q^h)^* s + \tau^h$ respectively, $q : F \to \mathcal{C}(M,p)$, $q^h : F^h \to \mathcal{C}(M,p)$ the projections. The tautological sections s_{can} and s^h_{can} of $q^* F$ and $(q^h)^* F$ both have zero locus $\mathcal{C}(M)$. The associated cones and cone classes defined in the last section will be written

$$C(\tau) = C(s_{\text{can}}) \subset F|_Z, \; [C(\tau)]$$
$$C(\tau^h) = C(s^h_{\text{can}}) \subset F^h|_Z, \; [C(\tau^h)].$$

For a decomposition $F = F^h \oplus \bar{F}$ and homology classes α, β supported on closed subsets $A \subset F^h$, $B \subset \bar{F}$ let us write

$$\alpha \oplus \beta \;:=\; (\alpha \times \beta) \cap (q^h \times \bar{q})^* \delta_\Delta$$

for their *direct sum*, where $q^h \times \bar{q} : F^h \times \bar{F} \to N \times N$ is the product of the bundle projections and $\delta_\Delta \in H^*_\Delta(N \times N)$ is Poincaré-dual to the diagonal $\Delta \subset N \times N$.

The object of this section is to show that $C(\tau)$ is already determined by $C(\tau^h)$.

PROPOSITION 3.7. — *Let $F = F^h \oplus \bar{F}$ be a decomposition into complex orbibundles in such a way that*

• $\tau^h := \tau|_{F^h}$ *is injective and spans the cokernel of the linearization* σ *relative a local, finite dimensional parameter space along* $\mathcal{C}(M) = Z(s)$ *and has the regularity properties of* τ *(cf. [Si1], Def. 1.15)*

• $\bar{\tau} := \tau|_{\bar{F}}$ *maps to* $\operatorname{im}\sigma$ *along* $\mathcal{C}(M)$. *Then* $C(\tau) = C(\tau^h) \oplus \bar{F}$ *and* $[C(\tau)] = [C(\tau^h)] \oplus [\bar{F}]$.

Before turning to the proof three remarks are in order: First, while $s = s_{\bar{\partial}}$ is not in general globally differentiable, locally it is so relative to a parameter space S of a semiuniversal deformation of the curve. The corresponding relative differential is nothing but the linear $\bar{\partial}$-operator from $\check{L}^p_1(C, \varphi^* T_M)$ to $\check{L}^p(C, \varphi^* T_M \otimes \bar{\Omega})$ (restricted to a complement of the kernel in case $(C, \mathbf{x})$ is not stable). Second, it will be crucial that differentiability properties are imposed only on τ^h. $\bar{F}$ will indeed only be constructed as topological subbundle. And third, the proposition together with Proposition 3.5 shows that, locally, $C(\tau)$ is the product of an analytic cone over $\mathcal{C}(M)$, pure-dimensional of dimension equal to expected dimension (index of relative differential plus $\dim S$ minus $\dim \operatorname{Aut}(C, \mathbf{x})$) plus $\operatorname{rk} F^h$, and a complex vector space of dimension $\operatorname{rk} \bar{F}$.

Proof. — Let $\bar{s}_{\mathrm{can}}$ be the tautological section of $q^* \bar{F}$. Consider the section θ of $q^* F \times \mathbb{R}^l \times \mathbb{R} \to \tilde{Z} \times \mathbb{R}^l \times \mathbb{R}$ over $F \times (\mathbb{R}^l \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ defined by

$$\theta(z, v, u) \;:=\; |v|^{-1} \cdot (s^h_{\mathrm{can}} + u \cdot \bar{s}_{\mathrm{can}}).$$

This should be viewed as a two-parameter family of sections of $q^* F$ interpolating between $(s_{\mathrm{can}})_l = |v|^{-1} \cdot s_{\mathrm{can}}$ $(u = 1)$ and $(s^h_{\mathrm{can}})_l \oplus 0$ $(u = 0)$. Write θ_u for the restriction of θ to $q^* F \times \mathbb{R}^l \times \{u\}$ and Q_F, Q_l, Q for the projections from $q^* F \times \mathbb{R}^l \times \mathbb{R}$ to the three factors. We denote by $\delta_0^{\mathbb{R}^l} \in H^l_{\{0\}}(\mathbb{R}^l)$ and $\delta_u^{\mathbb{R}} \in H^1_{\{u\}}(\mathbb{R})$ the Poincaré duals to $\{0\} \subset \mathbb{R}^l$, $\{u\} \subset \mathbb{R}$. By definition

$$[C(\tau)] \;=\; [\overline{\Gamma_{\theta_1}}] \cap Q_l^* \delta_0^{\mathbb{R}^l},$$

as classes supported on $C(\tau) \times \{0\} \times \{1\} \subset q^* F \times \{0\} \times \{1\}$. $[\overline{\Gamma_{\theta_1}}]$ is the unique class extending the fundamental class of the oriented orbifold Γ_{θ_1}. In the construction of local uniformizers for Γ_θ we may take u as parameter in the application of the implicit function theorem, cf. the proof of [Si1], Thm. 1.21. This shows

$$[\Gamma_{\theta_1}] \;=\; [\Gamma_\theta] \cap Q^* \delta_1^{\mathbb{R}}$$

as classes supported on Γ_{θ_1}. Since by Lemma 3.8 below $\overline{\Gamma_{\theta_u}} = \overline{\Gamma_\theta} \cap Q^{-1}(u)$ (the disjoint union of Γ_{θ_u} and $C(\tau) \times \{0\} \times \{u\}$), we get at $u = 1$

$$A_1 \;:=\; [C(\tau)] \times \{0\} \times \{1\} \;=\; [\overline{\Gamma_\theta}] \cap (Q_l \times Q)^* (\delta_0^{\mathbb{R}^l} \times \delta_1^{\mathbb{R}}).$$

And by the same lemma

$$A_u \;:=\; [\overline{\Gamma_\theta}] \cap (Q_l \times Q)^*(\delta_0^{\mathbb{R}^l} \times \delta_u^{\mathbb{R}}), \quad u \in \mathbb{R}$$

is a family of classes on $C(\tau) \times \{0\} \times \mathbb{R}$, and these are mutually homologic because the $\delta_u^{\mathbb{R}}$ are cohomologous classes on $\mathbb{R}$.

Together with $\overline{\Gamma_{\theta_0}} \cap Q_l^{-1}(0) = (C(\tau) \oplus \bar{F}) \times \{0\} \times \{0\}$ we obtain

$$[C(\tau)] \;=\; (Q_F)_* A_1 \;=\; (Q_F)_* A_0 \;=\; [C(\tau^h)] \oplus [\bar{F}].$$

The set-theoretic part of the claim is also proved in the lemma below. □

We still owe the set-theoretic part of the lemma:

LEMMA 3.8.

1) $\overline{\Gamma_\theta} \cap (q^* F \times \{0\} \times \mathbb{R}) \;=\; C(\tau) \times \{0\} \times \mathbb{R}$.

2) $C(\tau) = C(\tau^h) \oplus \bar{F}$ *(set-theoretically).*

Proof. — We will show the inclusions

$$\overline{\Gamma_\theta} \cap Q_l^{-1}(0) \subset (C(\tau^h) \oplus \bar{F}) \times \{0\} \times \mathbb{R}, \quad C(\tau^h) \oplus \bar{F} \subset C(\tau).$$

The lemma will then be finished with $C(\tau) = \overline{\Gamma_{\theta_1}} \cap Q_l^{-1}(0)$ (hence (2)) and $C(\tau^h) \oplus \bar{F} = \overline{\Gamma_{\theta_0}} \cap Q_l^{-1}(0)$, together with the observation that rescaling $(f^h, \bar{f}, v, u) \mapsto (f^h, u \cdot \bar{f}, v, 1)$ gives

$$\overline{\Gamma_\theta} \cap \Big(q^* F \times \{0\} \times (\mathbb{R} \setminus \{0\})\Big) \;=\; (C(\tau^h) \oplus \bar{F}) \times \{0\} \times (\mathbb{R} \setminus \{0\}).$$

In view of Proposition 3.2 we have to prove the following: Let $t_\nu \in \mathbb{R}$, $u_\nu \in \mathbb{R}$, $f_\nu \in F_{x_\nu}$ be sequences with

$$s_{\bar{\partial}}(x_\nu) - t_\nu \cdot \Big(\tau^h_{x_\nu}(f^h_\nu) - u_\nu \cdot \bar{\tau}_{x_\nu}(\bar{f}_\nu)\Big) = 0 \tag{3.5}$$

(cf. the definition of θ together with that of $\tilde{Z} \subset F$ as zero locus of $q^* s_{\bar{\partial}} - \tau$) and

$$t_\nu \longrightarrow 0, \quad u_\nu \longrightarrow u, \quad f_\nu \longrightarrow f.$$

As before we use superscript "h" and a bar to denote components in F^h and $\bar{F}$ respectively and τ_y (etc.) to denote the restriction of τ to F_y . Any $(f, 0, u) \in \overline{\Gamma_\theta} \cap Q_l^{-1}(0)$ is of this form. We claim $f^h \in C(\tau^h)$.

To this end we want to work on adapted charts. Recall that at $z = (C, \mathbf{x}, \varphi)$, $s_{\bar{\partial}}$ was locally uniformized by

$$\hat{s}_{\bar{\partial}} : \; S \times V \longrightarrow E_0,$$

S the parameter space of deformations of the domain of the curve, $V \subset \check{L}_1^p(C;\varphi^*T_M)$ (an open set in a linear subspace) of finite codimension and $E_0 = \check{L}^p(C;\varphi^*T_M \otimes \bar{\Omega})$ uniformizing $\mathcal{E}_z$. This map was differentiable relative S (i.e., for fixed $s \in S$) with relative differential $D_V \hat{s}_{\bar{\partial}}$ uniformly continuous at the center $(0,0) \in S \times V$. Write $\sigma = D_V \hat{s}_{\bar{\partial}}(0,0)$. By the regularity properties of $\hat{\tau}^h$ and since $\hat{\tau}_z^h$ is injective and spans $\operatorname{coker} \sigma$ we may change the trivialization of $\mathcal{E}$ in such a way that $\widehat{F}^h$ is identified via $\hat{\tau}^h$ with its image C on the central fiber E_0.

Choose a complementary subspace $P \subset V$ to $K := \ker \sigma$, set $Q = \sigma(P)$, and write $\mathrm{pr}_Q : E_0 \to Q$, $\mathrm{pr}_C : E_0 \to C$ for the projections with kernel C respectively Q. We can now apply the implicit function theorem to $\mathrm{pr}_Q \circ \hat{s}_{\bar{\partial}} : S \times K \times P \to Q$ with parameter space $W = S \times K$. We can thus change coordinates on $S \times V$ in such a way that

$$\begin{aligned} \hat{s}_{\bar{\partial}}(w,p) &= (p, \kappa(w,p)) \ \in \ Q \times C \\ \hat{\tau}^h_{(w,p)}(f^h) &= (0, f^h) \\ \hat{\bar{\tau}}_{(w,p)}(\bar{f}) &= (\mathrm{pr}_Q \hat{\bar{\tau}}_{(w,p)}(\bar{f}), \mathrm{pr}_C \hat{\bar{\tau}}_{(w,p)}(\bar{f})) \end{aligned}$$

with $\kappa : W \times P \to C$ differentiable relative W, $D_P\kappa(0,0) = 0$, $D_P\kappa$ uniformly continuous at $(0,0)$, and $\mathrm{pr}_C \bar{\tau}_{(0,0)} = 0$ ($\operatorname{im} \hat{\bar{\tau}}_{(0,0)} \subset \operatorname{im} \sigma$ by hypothesis).

Since the structure map $S \times V \to \mathcal{C}(M;p)$ is locally proper, it suffices to prove the claim about the limit on local uniformizers. For readabilities sake we will drop the hats that usually indicate local uniformizers. Write $x_\nu = (w_\nu, p_\nu) \in W \times P$, $x = (0,0)$. Equation 3.5 now splits into the two equations (in Q and C respectively)

$$\begin{aligned} p_\nu - t_\nu u_\nu \, \mathrm{pr}_Q \bar{\tau}_{(w_\nu,p_\nu)}(\bar{f}_\nu) &= 0 \\ \kappa(w_\nu,p_\nu) - t_\nu f_\nu^h - t_\nu u_\nu \, \mathrm{pr}_C \bar{\tau}_{(w_\nu,p_\nu)}(\bar{f}_\nu) &= 0. \end{aligned}$$

We claim that

$$f'_\nu \ := \ \left(\frac{1}{t_\nu}\kappa(w_\nu,0), \bar{f}_\nu\right)$$

has the same limit $(f^h, \bar{f})$ as f_ν. From the first equation we see that ($t_\nu \neq 0$)

$$\frac{p_\nu}{t_\nu} \longrightarrow u \cdot \mathrm{pr}_Q \bar{\tau}_{(0,0)}(\bar{f}) \ = \ u \cdot \bar{\tau}_{(0,0)}(\bar{f}).$$

By uniform continuity of $D_P\kappa$ and because $D_P\kappa(0,0) = 0$

$$\frac{1}{|t_\nu|}\Big|\kappa(w_\nu,p_\nu) - \kappa(w_\nu,0)\Big| \ \leq \ \|D_P\kappa\|_{B_{|p_\nu|}(0)} \cdot \left|\frac{p_\nu}{t_\nu}\right| \ \longrightarrow \ 0.$$

Together with

$$\frac{1}{t_\nu}\kappa(w_\nu,p_\nu) \ = \ f_\nu^h + u_\nu \cdot \mathrm{pr}_C \bar{\tau}_{(w_\nu,p_\nu)}(\bar{f}_\nu) \ \longrightarrow \ f^h$$

(which is where the assumption $\operatorname{im}\bar\tau \subset \operatorname{im}\sigma$ comes in) this establishes the claim and hence the first inclusion, for (f'_ν, t_ν) are in Γ_{θ_0}.

Turning to the second inclusion $C(\tau^h) \oplus \bar F \subset C(\tau)$ we replace $\bar F$ by a subbundle $\bar F' \subset F$ with $\bar F'_0 = \bar F_0$ and such that $\bar\tau' = \tau|_{\bar F'}$ has the regularity properties of τ. To $f^h \in C(\tau^h)$ choose sequences $0 \neq t_\nu \to 0$, $F_{x_\nu=(w_\nu,0)} \ni f^h_\nu \to f^h$ with

$$\kappa(w_\nu, 0) - t_\nu f^h_\nu \;=\; 0.$$

For any $\bar f \in \bar F_0 = \bar F'_0$ we want to find $f^{h\prime}_\nu \to f^h$ with

$$\begin{aligned} p_\nu - t_\nu \operatorname{pr}_Q \bar\tau'_{(w_\nu,p_\nu)}(\bar f) &= 0\\ \kappa(w_\nu, p_\nu) - t_\nu f^{h\prime}_\nu - t_\nu \operatorname{pr}_C \bar\tau'_{(w_\nu,p_\nu)}(\bar f) &= 0. \end{aligned}$$

To the first equation we may apply the implicit function theorem with parameters t_ν, w_ν to conclude unique existence of $p_\nu \to 0$ for ν large from the solution $p = 0$, $t = 0$, $w = 0$ in the limit $\nu \to \infty$. The second equation in turn forces

$$f^{h\prime}_\nu \;=\; \frac{1}{t_\nu}\kappa(w_\nu, p_\nu) - \operatorname{pr}_C \bar\tau'_{(w_\nu,p_\nu)}(\bar f).$$

Since as above

$$f^{h\prime}_\nu - f^h_\nu \;=\; \frac{1}{t_\nu}\Big(\kappa(w_\nu, p_\nu) - \kappa(w_\nu, 0)\Big) \;\longrightarrow\; 0$$

we deduce $f^{h\prime}_\nu \to f^h$ as claimed. □

4. Comparison of algebraic and limit cone.

4.1. Choice of Kuranishi structure.

Let (M,ω) be a Kähler manifold. Recall the construction of [Si1] applied to (M,ω) viewed as symplectic manifold with almost complex structure the integrable one: The space $\mathcal{C}(M;p) = \bigcup_{R,g,k} \mathcal{C}_{R,g,k}(M;p)$ of stable marked complex curves in M of Sobolev class L^p_1 is a Banach orbifold. The map $(C, \mathbf{x}, \varphi) \mapsto \bar\partial\varphi$ is a section $s_{\bar\partial}$ of the Banach orbibundle $\mathcal{E}$ with fibers $\mathcal{E}_{(C,\mathbf{x},\varphi)} = \check L^p(C; \varphi^* T_M \otimes \bar\Omega_C)$. Local uniformizing systems at $(C, \mathbf{x}, \varphi)$ are of the form $S \times V$ with S the base of an analytically semiuniversal deformation of $(C, \mathbf{x})$ and V an open subset of a linear subspace of $\check L^p_1(C; \varphi^* T_M)$ of codimension equal to $\dim \operatorname{Aut}(C, \mathbf{x})$. In such a chart $\hat s_{\bar\partial}$ is differentiable relative S, with differential σ relative S a family of Fredholm operators that is uniformly continuous at $(0,0) \in S \times V$. By spanning the cokernel of σ along some compact part of $\mathcal{C}(M)$, say $\mathcal{C}_{R,g,k}(M) = \mathcal{C}(M) \cap \mathcal{C}_{R,g,k}(M;p)$, by sections of $\varphi^* T_M \otimes \bar{\Omega}_C$ supported away from the singularities of C, parallel transport by local trivializations of $\mathcal{E}$ and multiplication with a bump function in a neighbourhood of $\mathcal{C}_{R,g,k}(M;p)$ in $\mathcal{C}(M;p)$ we constructed a morphism ("Kuranishi structure")

$$\tau : F \longrightarrow \mathcal{E}$$

from a finite rank complex orbibundle F living on a neighbourhood of $\mathcal{C}_{R,g,k}(M;p)$ in $\mathcal{C}(M;p)$ with the following properties:

- τ has the same differentiability properties as $s_{\bar\partial}$
- τ spans the cokernel of the linearization of $s_{\bar\partial}$ along $Z(s_{\bar\partial})$, i.e. for any $(C, \mathbf{x}, \varphi) \in \mathcal{C}(M)$

$$\operatorname{im} \hat{\tau}_{(C,\mathbf{x},\varphi)} + \operatorname{im} \sigma_{(C,\mathbf{x},\varphi)} \;=\; \widehat{\mathcal{E}}_{(C,\mathbf{x},\varphi)}.$$

The section $\tilde{s} := q^* s - \tau$, $q : F \to \mathcal{C}(M;p)$ the bundle projection, is then a transverse (locally relative S) section of the Banach orbibundle $q^*\mathcal{E}$ over the total space of F. $\tilde{Z} = Z(\tilde{s}) \subset F$ is thus an oriented, finite-dimensional orbifold. Let $\Theta_F \in H^{\operatorname{rk} F}_{\mathcal{C}(M;p)}(F)$ be the Thom class of F ($\mathcal{C}(M;p)$ identified with the zero section of F). The virtual fundamental class of $\mathcal{C}_{R,g,k}(M)$ was defined by

$$\mathcal{GW}^M_{R,g,k} \;:=\; [\tilde{Z}] \cap \Theta_F.$$

Note that the restriction of Θ_F to $\tilde{Z}$ can also be written $s^*_{\mathrm{can}} \Theta_{q^*F}$, so this definition is essentially finite dimensional.

To compare with the complex analytic definition of virtual fundamental classes, as a first try one might want to make the whole symplectic construction Banach analytic. This seems to be hard if not impossible. It is however easy to gain enough analyticity locally to make the comparison with the analytic treatment given in the first chapter work.

At holomorphic $(C, \mathbf{x}, \varphi)$ let $\Theta : \mathcal{B} \to \prod_{j \le m'} L^p(\Delta^j; \mathbb{C}^n)$ be a holomorphic Kuranishi model as constructed in Section 2.3. Let $\breve{\pi} : \breve{\Gamma} := \mathcal{B} \times_S C$ be the universal curve over $\mathcal{B}$ and $\breve{\Phi} : \breve{\Gamma} \to M$ be the evaluation map (generally we will use the accent $\breve{}$ for objects on $\mathcal{B}$). We will write $U_0 \subset \breve{\Gamma}$ for the union of the open sets formerly denoted Σ, U_i and W_j for $j > m'$, and U_i for W_i, $i = 1, \ldots, m'$. So $\{U_i\}$ is an open covering of $\breve{\Gamma}$ that is Stein relative $\mathcal{B}$. Let $\breve{\mathcal{T}}$ be the tangent bundle of $\mathcal{B}$ relative S and $\breve{\mathcal{E}} := \mathcal{B} \times \prod_{j \le m'} L^p(\Delta^j; \mathbb{C}^n)$. The former has fiber $\check{L}^p_1(C';\varphi'^*T_M)\cap\mathcal{O}^{1,p}(U_0;\varphi^*\mathcal{T}_M)$ at $(C',\mathbf{x}',\varphi')\in\mathcal{B}$ while the latter should be viewed as a version of the Banach bundle $\mathcal{E}\downarrow\mathcal{C}(M;p)$ on $\mathcal{B}$. Note that while $\check{\mathcal{T}}$ and $\check{\mathcal{E}}$ parametrize non-holomorphic objects they are *holomorphic* Banach bundles over $\mathcal{B}$. The Fredholm map Θ exhibiting $\mathrm{Hom}_S(\mathcal{C},M)$ as fiber over 0 can now be viewed as holomorphic section of $\check{\mathcal{E}}$.

Similarly, while the evaluation map is not holomorphic (along the fibers of $\check{\pi}$) $\check{\Phi}^*T_M$ is a holomorphic vector bundle over $\check{\Gamma}$, local holomorphic trivializations being given by pull-back of a frame of local holomorphic vector fields on M.

While U_0 might now have singularities a straightforward modification of the arguments in [Si1], §4.2 shows that the spaces of relative holomorphic Čech cochains $\check{\pi}^{(i)}_*\check{\Phi}^*\mathcal{T}_M$ are still holomorphic Banach bundles over $\mathcal{B}$ (and the same holds true by replacing $\check{\Phi}^*\mathcal{T}$ by any finite rank holomorphic vector bundle over $\check{\Gamma}$). Recall also from op.cit. that (the restrictions to $\mathcal{B}\subset\mathcal{C}(M;p)$) of the tangent bundle of $\mathcal{C}(M;p)$ relative S and the Banach bundle $\mathcal{E}$ can be written

$$\mathcal{T} \;=\; \check{\pi}^p_{1*}\check{\Phi}^*\mathcal{T}_M,\qquad \mathcal{E}\;=\;\check{\pi}^p_*(\check{\Phi}^*\mathcal{T}_M\otimes_{\mathbb{C}}\bar{\Omega}_{\check{\Gamma}/\mathcal{B}}).$$

We will also need to extend the holomorphic bundles G and H from Section 1.2 to $\mathcal{B}$. The sheaf of sections of H was $R^1\pi_*\mathcal{N}$, where $\mathcal{N}=\pi^*\pi_*(\Phi^*\mathcal{T}_M\otimes\mathcal{L}^\nu)\otimes\mathcal{L}^{\otimes-\nu}$, $\mathcal{L}=\omega(x_1+\ldots+x_k)\otimes\Phi^*\mathcal{H}_M^{\otimes 3}$, fits into an exact sequence of locally free sheaves

$$0\longrightarrow\mathcal{K}\xrightarrow{\kappa}\mathcal{N}\xrightarrow{\nu}\Phi^*\mathcal{T}_M\longrightarrow 0.$$

Now Φ^*T_M and $\mathcal{L}$ extend naturally holomorphically to $\check{\Gamma}$ and so do $\mathcal{N}$ and $\mathcal{K}$ and the above sequence. Let $\check{\mathcal{N}}$ and $\check{\mathcal{K}}$ denote these extensions. We thus obtain a commutative diagram of holomorphic Banach bundles over $\mathcal{B}$ with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \check{\pi}^{(0)}_*\check{\mathcal{K}} & \xrightarrow{\kappa^{(0)}} & \check{\pi}^{(0)}_*\check{\mathcal{N}} & \xrightarrow{\nu^{(0)}} & \check{\pi}^{(0)}_*\check{\Phi}^*\mathcal{T}_M & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{\pi}^{(1)}_*\check{\mathcal{K}} & \xrightarrow{\kappa^{(1)}} & \check{\pi}^{(1)}_*\check{\mathcal{N}} & \xrightarrow{\nu^{(1)}} & \check{\pi}^{(1)}_*\check{\Phi}^*\mathcal{T}_M & \longrightarrow & 0\\
 & & {\scriptstyle q_{\check{G}}}\downarrow & & \downarrow{\scriptstyle q_{\check{H}}} & & & & \\
 & & \check{G} & \xrightarrow{R^1\check{\pi}_*\kappa} & \check{H} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}\tag{4.6}$$

where we *define* $\breve{G}$ and $\breve{H}$ as cokernels of the first two columns. These are holomorphic extensions of G and H to a neighbourhood of $(C, \mathbf{x}, \varphi)$ in $\mathcal{B}$.

Recall that we called a commutative square of Banach bundles

$$\begin{array}{ccc} E & \xrightarrow{\gamma} & G \\ \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} \\ F & \xrightarrow{\delta} & H \end{array}$$

a *quasi-isomorphism* (between α and β, and between γ and δ) iff the sequence

$$0 \longrightarrow E \xrightarrow{(\alpha,\gamma)} F \oplus G \xrightarrow{\delta-\beta} H \longrightarrow 0$$

is exact [Si1], Def. 4.8. Equivalently, (α, β) induces fiberwise isomorphisms between kernels and cokernels of γ and δ (or the other way around). In op.cit. we also required this sequence to be split in the case of which the square is not only cartesian ($E \simeq F \oplus_H G$) but also cocartesian ($H \simeq (F \oplus G)/E$, as Banach bundles!). All our quasi-isomorphisms will in fact be split but since we will never need this property we will not verify it.

Now a diagram like (4.6) above always induces a quasi-isomorphism between $[\breve{G} \to \breve{H}]$ and $[\breve{\pi}_*^{(0)} \breve{\Phi}^* \mathcal{T}_M \to \breve{\pi}_*^{(1)} \breve{\Phi}^* \mathcal{T}_M]$ (unique up to homotopy) locally as follows: Let η and θ be (local) holomorphic right-inverses to $q_{\breve{G}}$ and $q_{\breve{H}}$. Then

$$q_{\breve{H}}(\theta \circ R^1 \breve{\pi}_* \kappa - \kappa^{(1)} \circ \eta) \;=\; 0,$$

so the term in the bracket lifts (uniquely) to a map

$$\zeta : \; \breve{G} \longrightarrow \breve{\pi}_*^{(0)} \mathcal{N}.$$

Define

$$\alpha^{(0)} \;:=\; \nu^{(0)} \circ \zeta, \quad \alpha^{(1)} \;:=\; \nu^{(1)} \circ \theta.$$

Recall also the morphisms [Si1], §4.3

$$\begin{array}{cccccc} \breve{\pi}_*^{(0)} \breve{\Phi}^* \mathcal{T}_M & \longrightarrow & \mathcal{T}, & (v_i) & \longmapsto & \sum_i \rho_i v_i \\ \breve{\pi}_*^{(1)} \breve{\Phi}^* \mathcal{T}_M & \longrightarrow & \mathcal{E}, & (v_{ij}) & \longmapsto & \frac{1}{2} \sum_{i,j} v_{ij} \cdot \bar{\partial} \rho_i. \end{array}$$

Here ρ_i is a partition of unity subordinate to U_i, so these maps indeed factor over the inclusions $\breve{\mathcal{T}} \hookrightarrow \mathcal{T}$ and $\breve{\mathcal{E}} \hookrightarrow \mathcal{E}$.

LEMMA 4.1. — *The squares in the following diagram are quasi-isomorphisms:*

$$\begin{array}{ccccccc} \breve{G} & \xrightarrow{\alpha^{(0)}} & \breve{\pi}_*^{(0)} \breve{\Phi}^* \mathcal{T}_M & \xrightarrow{\beta^{(0)}} & \breve{\mathcal{T}} & \longrightarrow & \mathcal{T} \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\sigma=\bar{\partial}} \\ \breve{H} & \xrightarrow{\alpha^{(1)}} & \breve{\pi}_*^{(1)} \breve{\Phi}^* \mathcal{T}_M & \xrightarrow{\beta^{(1)}} & \breve{\mathcal{E}} & \longrightarrow & \mathcal{E} \end{array}$$

Proof. — The claim for the first square follows from chasing Diagram 4.6. That the last square induces isomorphisms on kernels and cokernels of the vertical maps has been proven in Proposition 2.6. The composition of the right two squares is a quasi-isomorphism essentially by [Si1], Cor. 4.11, observing that there is a (family of) right-inverses T to the $\bar{\partial}$-operator on U_0 (cf. Section 2.1). This shows that also the middle square is a quasi-isomorphism. □

Notice that F extends naturally as finite rank orbibundle to a neighbourhood of $\mathcal{C}_{R,g,k}(M)$ in $\mathcal{C}(M;p)$. In fact, the finite rank orbibundle in [Si1], §6.4 was a direct sum of bundles of this form. Locally we may thus take for τ an extension of $\beta^{(1)} \circ \alpha^{(1)}$ to a local uniformizing system of $\mathcal{C}(M;p)$, multiplied by a bump function, as specified in [Si1], §6.5.

PROPOSITION 4.2. — *For any $R \in H_2(M;\mathbb{Z})$, $g, k \geq 0$, the Kuranishi structure $\tau : F \to \mathcal{E}$ for $s_{\bar{\partial}}$ may be chosen in such a way that*

- *$F = \bigoplus_{\nu=1}^{l} F_\nu$ with each F_ν restricting to the orbibundle H along Z*

- *for any $(C, \mathbf{x}, \varphi) \in \mathcal{C}_{R,g,k}(M)$ there exists an open neighbourhood $U \subset \mathcal{C}(M;p)$ and a ν with $\tau_\nu = \tau|_{F_\nu}$ an extension of $\beta^{(1)} \circ \alpha^{(1)}$. In particular,*

$$\tilde{Z}_\nu \;:=\; Z(q_\nu^* s_{\bar{\partial}} + \tau_\nu) \;\subset\; F_\nu$$

is a complex orbifold at $(C, \mathbf{x}, \varphi)$.

Proof. — $\tilde{Z}_\nu$ is the set of pairs $\big((C, \mathbf{x}, \varphi), f \in F_{\nu,(C,\mathbf{x},\varphi)}\big)$ obeying

$$\bar{\partial}\varphi \;=\; \tau_\nu(f).$$

By construction, τ_ν has support away from U_0. So φ is holomorphic on $U_0(s)$ and the above equation can actually be viewed on a local uniformizer as map

$$\bar{\partial} - \hat{\tau} : \mathcal{B} \times \widehat{F}_0 \;\longrightarrow\; \prod_{j \leq m'} L^p(\Delta, \mathbb{C}^n),$$

$\mathcal{B}$ a complex Banach manifold of the form given in Section 2.3. This map is holomorphic with differential relative S an epimorphism with finite dimensional kernel. An application of the holomorphic implicit function theorem with parameter space S shows that $\tilde{Z}_\nu$ is locally uniformized by a complex manifold. If $(C, \mathbf{x})$ is not stable we also have to take the quotient by the germ of the action of $\mathrm{Aut}^0(C, \mathbf{x})$. In [Si1], §5.3 this has been achieved by imposing an averaged version (involving integrals over bump functions in M) of the rigidification procedure given in Proposition 1.2. The reason was that transversality is not on open condition in L_1^p-spaces over two-dimensional domains. On the subspace $\mathcal{B} \subset S \times \check{L}_1^p(C; \varphi^* T_M)$ we can however use rigidification by incidence conditions with local transversal divisors as in Proposition 1.2, as long as the intersection is on the holomorphic part of φ. The proof that this is in fact a quotient is a simple application of the implicit function theorem, parallel to the discussion in [Si1], §5.3. Obviously, this slice by incidence with transversal divisors is holomorphic, hence a complex manifold. And the unrigidified $Z(q_\nu^* s_{\bar\partial} + \tau_\nu)$ is just a product of the slice and an open set in $\mathbb{C}^l$, $l = \dim \mathrm{Aut}(C, \mathbf{x})$. □

4.2. Reduction to local, holomorphic situation.

From the Kuranishi structure $\tau : F \to \mathcal{E}$ (Proposition 4.2) we obtain a cone $C^F \subset F$ supporting a class $[C^F]$ of dimension $d + \mathrm{rk}\, F$ as cone and class associated to the tautological section of $q^* F|_{\tilde Z}$, $d = d(M, R, g, k)$ the expected dimension of $\mathcal{C}_{R,g,k}(M)$, $q : F \to \mathcal{C}(M; p)$ the bundle projection, $\tilde Z = Z(q^* s + \tau)$. Recall that the restriction of F to Z decomposes into a direct sum of copies of H. We define

$$\mu : \; H \hookrightarrow F|_Z$$

to be the diagonal embedding. Let $\Theta_{F/H}$ be the Thom class of the orbibundle $F/\mu(H)$. Pulling back to F yields a class $\Theta \in H_H^{\mathrm{rk}\, F - \mathrm{rk}\, H}(F|_Z)$. Then $[C^F] \cap \Theta \in H_{d+\mathrm{rk}\, H}(C^F \cap H)$ is the intersection of $[C^H]$ with H. The comparison theorem will readily follow from

Proposition 4.3. — *$C^H = C^F \cap H$, $[C^H] = [C^F] \cap \Theta$ in $H_{d+\mathrm{rk}\, H}(C^F \cap H)$.*

The proof of this proposition occupies the rest of this chapter. Notice that since C^H is a complex space of pure dimension $d + \mathrm{rk}\, H$, the homological statement concerns classes of top dimension and can thus be checked locally together with the set-theoretic part of the claim.

By Lemma 3.6 and the definition of C^H it suffices to check the claims of the proposition on the level of local uniformizers. Since the local uniformizers in the complex analytic (Proposition 1.2) and in the complex Banach manifold treatment (Proposition 4.2) can be obtained by incidence conditions with the same set of divisors $H_1, \ldots, H_l$, and all objects are just trivial products of the restriction to the slice with an l-dimensional smooth space, it even suffices to work on unrigidified charts.

For $(C, \mathbf{x}, \varphi) \in \mathcal{C}_{R,g,k}(M)$ let τ_ν be as in Proposition 4.2. As usual we mark liftings to the local uniformizing systems under study by a hat and with subscript 0 the restriction to the center of a local uniformizing system.

LEMMA 4.4. — *Possibly after shrinking U to a smaller neighbourhood of $(C, \mathbf{x}, \varphi)$ there exists a topological decomposition*

$$\widehat{F} = \bar{F} \oplus \widehat{F}_\nu$$

with $\tau(\bar{F}) \subset \operatorname{im} \sigma$ and such that

$$\widehat{C}^F = \bar{F} \oplus \widehat{C}^h, \quad [\widehat{C}^F] = [\bar{F}] \oplus [\widehat{C}^h],$$

where $\widehat{C}^h$, $[\widehat{C}^h]$ are the cone and cone class obtained from $\hat{\tau}_\nu$.

Proof. — By invoking Proposition 3.7 we just have to define $\bar{F}$. By the Fredholm property of $\sigma = \bar{\partial}$ and since $\hat{\tau}$ spans the cokernel of $\hat{\sigma}$ the family (over S) of linear maps

$$S \times \Big(F_0 \oplus \check{L}_1^p(C; \varphi^* T_M)\Big) \longrightarrow \check{L}^p(C; \varphi^* T_M), \quad (s, f, v) \longmapsto \hat{\tau}_{s,v}(f) - \bar{\partial} v$$

consists of split epimorphisms. An application of the implicit function theorem thus shows that $\widehat{T}_\tau := \widehat{F} \oplus_{\widehat{\mathcal{E}}} \widehat{T}$ is a topological vector bundle on $\widehat{U}$ of rank $\operatorname{rk} F + d$. It fits into a quasi-isomorphism

$$\begin{array}{ccc} \widehat{T}_\tau & \longrightarrow & \widehat{T} \\ {\scriptstyle\hat{\rho}}\big\downarrow & & \big\downarrow{\scriptstyle\hat{\sigma}} \\ \widehat{F} & \xrightarrow{\hat{\tau}} & \widehat{\mathcal{E}} \end{array}$$

Therefore $\hat{\rho}_0^{-1}(\widehat{F}_\nu)$ is a linear subspace of $\widehat{T}_{\tau,0}$ of dimension $\operatorname{rk} F_\nu + d$. Let $P \subset \widehat{T}_\tau$ be a subbundle restricting to a complementary subspace to $\hat{\rho}_0^{-1}(\widehat{F}_\nu)$ in $\widehat{T}_{\tau,0}$. Then, possibly after going over to a smaller local uniformizing system, we may set $\bar{F} := \hat{\rho}(P)$. □

LEMMA 4.5. — *With the identification $H = F_\nu$ it holds*

$$\hat{\mu}^{-1}(C^F) = \widehat{C}^h, \quad [C^F] \cap \Theta = [\widehat{C}^h].$$

Proof. — Consider the family of morphisms

$$\hat{\mu}_t = (t \cdot \mathrm{id}, \ldots, t \cdot \mathrm{id}, \mathrm{id}, t \cdot \mathrm{id}, \ldots, t \cdot \mathrm{id}) : H \longrightarrow F = \oplus_\lambda F_\lambda$$

with "id" at the ν-th entry. By the previous lemma the claim holds with $\hat{\mu}_0$ replacing $\hat{\mu}$. Since $\hat{\mu}_t$ is a proper homotopy between $\hat{\mu}_0$ and $\hat{\mu} = \hat{\mu}_1$ we just have to show

$$\mu_t^{-1}(\widehat{C}^F) = \widehat{C}^h$$

for any t. To verify this on the fiber over some $\hat{z} \in \widehat{U}$ let $R^F \subset \widehat{F}_{\nu,\hat{z}}$ map isomorphically to $R := \operatorname{coker} \hat{\sigma}_{\hat{z}}$. Another application of Proposition 3.7 with a larger complementary space $\bar{F}$ shows

$$\widehat{C}^F \cap \widehat{F_{\hat{z}}} = q^{-1}(C^R),$$

for some cone $C^R \subset R$ where $q : R^F \to R$ is the quotient map. Letting $q^H : \widehat{H}_{\hat{z}} \to R^H$ be the cokernel of $\widehat{G}_{\hat{z}} \to \widehat{H}_{\hat{z}}$ and $\bar{\mu}_t : R^H \to R$ be the map induced by μ_t, we obtain

$$\mu_t^{-1}(\widehat{C}^F) = (q^H)^{-1}\bar{\mu}_t^{-1}(\widehat{C}^R).$$

But $\bar{\mu}_t = \lambda_t \cdot \bar{\mu}_0$ for some $\lambda_t \in \mathbb{R}_{>0}$, for the maps from R^H to $\operatorname{coker}(\bar{\partial} : \widehat{\mathcal{T}}_{\hat{z}} \to \widehat{\mathcal{E}}_{\hat{z}})$ induced by any of the τ_ν all coincide. Hence

$$\mu_t^{-1}(\widehat{C}^F) = (q^H)^{-1}\bar{\mu}_0^{-1}(\widehat{C}^R) = \hat{\mu}_0^{-1}(\widehat{C}^F) = \widehat{C}^h. \qquad \square$$

To prove Proposition 4.3 it remains to compare $\widehat{C}^h$ and $\widehat{C^H}$, which will be the concern of the next section.

4.3. Comparison of holomorphic normal spaces.

We consider the following situation: Let $q : \mathcal{C} \to S$ be a prestable curve over a smooth parameter space S (this will be applied to an analytically semiuniversal deformation of $(C, \mathbf{x})$), $Z = \operatorname{Hom}_S(\mathcal{C}, M)$ with universal curve and universal morphism

$$\pi : \Gamma \longrightarrow Z, \quad \Phi : \Gamma \to M.$$

We embed Z into a complex manifold $\tilde{Z}$ as in Proposition 4.2 (where the present $\tilde{Z}$ is denoted $\tilde{Z}_\nu$). Explicitely, possibly after shrinking S, we work on the complex Banach orbifold $\mathcal{B}$ of L^p_1-maps from fibers C_s of q to M, holomorphic away from a union of small disks $\bigcup \Delta^j$, as constructed in Section 2.3. According to Theorem 2.7, Z is the fiber over 0 of the holomorphic Fredholm map

$$\Theta : \mathcal{B} \longrightarrow \prod_{j \le m'} L^p(\Delta^{1/2}; \mathbb{C}^n) =: \mathcal{E}', \quad \psi \longmapsto \left(\bar{\partial}\psi|_{\Delta^j}\right),$$

in appropriate holomorphic coordinates on M and U_i. $\tilde{Z}$ on the other hand is obtained from a holomorphic morphism $\tau : \mathcal{B} \times \mathbb{C}^r \to \mathcal{E}'$, spanning the cokernel of G at any $z \in Z$, as fiber over 0 of

$$\tilde{\Theta} : \mathcal{B} \times \mathbb{C}^r \longrightarrow \mathcal{E}', \quad (\psi, a) \longmapsto \tau_\psi(a) + G(\psi).$$

Let $\tilde{\pi} : \tilde{\Gamma} \to \tilde{Z}$ be the universal curve over $\tilde{Z}$ (this is holomorphic) and $\tilde{\Phi} : \tilde{\Gamma} \to M$ the (usually non-holomorphic) evaluation map. The maps defined so far fit into the following diagram:

$$\begin{array}{ccccc} \Gamma & \xhookrightarrow{j} & \tilde{\Gamma} & \xrightarrow{\tilde{\Phi}} & M \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} & & \\ Z & \xhookrightarrow{i} & \tilde{Z} & & \end{array}$$

Z can also be viewed as zero locus of the tautological section s_{can} of the (trivial) bundle $F = \tilde{Z} \times \mathbb{C}^r \downarrow \tilde{Z} \subset \mathcal{B} \times \mathbb{C}^r$ (in the notation of Proposition 4.2, F corresponds to $q_\nu^* F_\nu$, $q_\nu : \tilde{Z}_\nu \to \mathcal{B}$ the projection); or, using the given trivialization, as fiber over 0 of the projection

$$\mathrm{pr}_2 : \tilde{Z} \subset \mathcal{B} \times \mathbb{C}^r \longrightarrow \mathbb{C}^r.$$

To such a description belongs a global normal space for Z as follows: let $\mathcal{F} = \mathcal{O}(F^\vee)$. Evaluation at s_{can} yields an epimorphism

$$\mathcal{F} \longrightarrow \mathcal{I}$$

to the ideal sheaf of Z in $\tilde{Z}$. We define ψ^{-1} to be the composition with the map $\mathcal{I} \to \mathcal{I}/\mathcal{I}^2$ to the conormal sheaf. Put for ψ^0 the identity morphism on $\Omega_{\tilde{Z}/S}|_Z$ and $d : \mathcal{I}/\mathcal{I}^2 \to \Omega_{\tilde{Z}/S}|_Z$ the differential. Then

$$\psi^\bullet : [\mathcal{F} \overset{d\circ\psi^{-1}}{\to} \Omega_{\tilde{Z}|S}|_Z] \longrightarrow [\mathcal{I}/\mathcal{I}^2 \overset{d}{\to} \Omega_{\tilde{Z}/S}/_Z] = \tau_{\geq 1}\mathcal{L}^\bullet_{Z|S}$$

is a global normal space for Z relative S. The corresponding cone $C(\psi^\bullet) \subset F$ is by definition the image of the normal cone $C_{Z|\tilde{Z}}$ of Z in $\tilde{Z}$ under the embedding $N_{Z|\tilde{Z}} \hookrightarrow F$. By Proposition 3.5

$$C(\Psi^\bullet) = C(s_{\mathrm{can}}), \quad [C(\psi^\bullet)] = [C(s_{\mathrm{can}})],$$

set-theoretically and in homology respectively. Note that in the notation of the previous section $C(s_{\mathrm{can}})$ and $[C(s_{\mathrm{can}})]$ are (non-rigidified versions of) C^h and $[C^h]$.

On the other hand we have the global normal space

$$\varphi^\bullet : [\mathcal{H} \to \mathcal{G}] \longrightarrow \mathcal{L}^\bullet_{Z|S}$$

constructed from π, Φ (Section 1.2). By construction $\mathcal{H} = \mathcal{F}$ and $C(\varphi^\bullet)$ is the non-rigidified version of C^H. To prove Proposition 4.3 (non-rigidified) we have to show $C(\varphi^\bullet) = C(\psi^\bullet)$. Since global normal spaces depend only on the (ray of) map induced in cohomology [Si2], Thm. 3.3 this will follow from

LEMMA 4.6. — *Locally, there exists an invertible holomorphic function χ and an isomorphism*

$$\lambda^\bullet : [\mathcal{H} \to \mathcal{G}] \longrightarrow [\mathcal{H} \to \Omega_{\tilde{Z}/S}|_Z]$$

with $\lambda^{-1} = \mathrm{id}$ and $H^i(\psi^\bullet \circ \lambda^\bullet) = \chi \cdot H^i(\varphi^\bullet)$, $i = -1, 0$.

The proof will occupy the rest of this section. The maps in the lemma fit into the following diagram, which is claimed to commute in cohomology up to multiplication by χ:

$$\begin{tikzcd}
{[\mathcal{H} \to \Omega_{\tilde{Z}/S}|_Z]} \arrow[dr, "\psi^\bullet"] & \\
& {[\mathcal{I}/\mathcal{I}^2 \to \Omega_{\tilde{Z}/S}|_Z]} \\
{[\mathcal{H} \to \mathcal{G}]} \arrow[uu, "\lambda^\bullet"] \arrow[ur, "\varphi^\bullet"'] &
\end{tikzcd}$$

DEFINITION OF $\lambda^\bullet$. — *To define λ^0 we observe that since $\tilde{Z}$ is solution to the equation $\bar{\partial}\varphi = \tau_\nu(C, \mathbf{x}, \varphi)(h)$ the tangent bundle $T_{\tilde{Z}/S}$ is canonically isomorphic to $H \oplus_{\check{\mathcal{E}}} \check{\mathcal{T}}$. By Lemma 4.1, this fibered product is in turn canonically isomorphic to G. We define λ^0 as the dual of the composition*

$$T_{\tilde{Z}/S}|_Z \simeq H \oplus_{\check{\mathcal{E}}} \check{\mathcal{T}} \simeq G.$$

It is clear from the construction that $\lambda^\bullet : [\mathcal{H} \to \mathcal{G}] \to [\mathcal{H} \to \Omega_{\tilde{Z}/S}|_Z]$ is a commutative square.

REPLACING $[\mathcal{H} \to \mathcal{G}]$ BY A ČECH COMPLEX. — To begin with we simplify the problem by dropping a common part from the definition of $\lambda^\bullet$ and $\varphi^\bullet$ as follows. First note that $[\mathcal{H} \to \mathcal{G}]$ represents $[R\pi_*\Phi^*\mathcal{T}_M]^\vee$ in the derived sense. Derived objects such as $[R\pi_*\Phi^*\mathcal{T}_M]^\vee$ are unique up to unique isomorphism in the derived category and the existence of such an isomorphism is what we mean by "represent". But the last steps in the construction of $\varphi^\bullet$ consisted of the composition of this isomorphism with duality

$$[\mathcal{H} \to \mathcal{G}] \simeq [R\pi_*\Phi^*\mathcal{T}_M]^\vee \simeq R\pi_*(\Phi^*\Omega_M \otimes \omega),$$

so we may as well drop this composition and work with $R\pi_*(\Phi^*\Omega_M \otimes \omega)$ directly. The latter in turn can be represented by the Čech complex

$$[\pi_*^{(0)}(\Phi^*\Omega_M \otimes \omega) \to \pi_*^{(1)}(\Phi^*\Omega_M \otimes \omega)],$$

and this gives an explicit identification of the cohomology of $\mathcal{H} \to \mathcal{G}$ with $R^i\pi_*(\Phi^*\Omega_M \otimes \omega)$, $i = 0, 1$.

Similarly we may factor $\lambda^\bullet$ over the morphism of complexes of holomorphic Banach bundles representing relative duality (Proposition 5.4)

$$\begin{array}{ccc} \pi_*^{(0)}(\Phi^*\Omega_M\otimes\omega)\longrightarrow(\pi_*^{(1)}\Phi^*\mathcal{T}_M)^\vee & & (\alpha_i)\longmapsto\Big((v_{ij})\mapsto\sum_{i,j}\int_{\Gamma/Z}\Big(\frac{\alpha_i+\alpha_j}{2})(v_{ij}\Big)\wedge\bar\partial\rho_i\Big)\\ \downarrow\qquad\qquad\qquad\downarrow & & \\ \pi_*^{(1)}(\Phi^*\Omega_M\otimes\omega)\longrightarrow(\pi_*^{(0)}\Phi^*\mathcal{T}_M)^\vee & & (\alpha_{ij})\longmapsto\Big((v_k)\mapsto\sum_{i,j}\int_{\Gamma/Z}\alpha_{ij}\Big(\frac{v_i+v_j}{2}\Big)\wedge\bar\partial\rho_i\Big). \end{array}$$

In fact, this duality morphism is the composition of the (topological) dual of

$$[\beta^{(0)},\beta^{(1)}]:\ [\pi_*^{(0)}\Phi^*\mathcal{T}_M\to\pi_*^{(1)}\Phi^*\mathcal{T}_M]\ \longrightarrow\ [\check{\mathcal{T}}\to\check{\mathcal{E}}]$$

from Lemma 4.1 (restricted to Z) and

$$\begin{array}{ccccc} \pi_*^{(0)}(\Phi^*\Omega_M\otimes\omega) & \longrightarrow & (\check{\mathcal{E}})^\vee & (\alpha_i)\ \longmapsto & \Big(\gamma\mapsto\sum_i\int_{\Gamma/Z}\rho_i\alpha_i\wedge\gamma\Big)\\ \downarrow & & \downarrow & & \\ \pi_*^{(1)}(\Phi^*\Omega_M\otimes\omega) & \longrightarrow & (\check{\mathcal{T}})^\vee & (\alpha_{ij})\ \longmapsto & \Big(v\mapsto\sum_{i,j}\int_{\Gamma/Z}\alpha_{ij}(v)\wedge\bar\partial\rho_i\Big). \end{array}$$

To verify this one needs a little computation. The composition of the two upper horizontal arrows applied to a local holomorphic section (α_i) of $\pi_*^{(0)}(\Phi^*\Omega_M\otimes\omega)$ and evaluated at a section (v_{jk}) of $\pi_*^{(1)}\Phi^*\mathcal{T}_M$ leads to the fiber integral

$$\frac{1}{2}\sum_{i,j,k}\int_{\Gamma/Z}\rho_i\alpha_i(v_{jk}\bar\partial\rho_j).$$

This indeed agrees with the upper horizontal arrow of the duality morphism by noting that a partial integration computation shows

$$\int_{\Gamma/Z}\alpha_i(v_{ij}\bar\partial\rho_i)\ =\ 2\int\rho_i\alpha_i(v_{ij}\bar\partial\rho_i.$$

Similarly for the lower horizontal arrows.

We may therefore draw a commutative diagram

$$\begin{array}{ccccccccc} H^\vee & \longleftarrow & (\pi_*^{(1)}\Phi^*\mathcal{T}_M)^\vee & \longleftarrow & \pi_*^{(0)}(\Phi^*\Omega_M\otimes\omega) & \longrightarrow & (\check{\mathcal{E}})^\vee & \longrightarrow & H^\vee\\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\ G^\vee & \longleftarrow & (\pi_*^{(0)}\Phi^*\mathcal{T}_M)^\vee & \longleftarrow & \pi_*^{(1)}(\Phi^*\Omega_M\otimes\omega) & \longrightarrow & (\check{\mathcal{T}})^\vee & \longrightarrow & T^\vee_{\check{Z}/S}|_Z\\ & & & & \downarrow & & & & \downarrow\\ & & & & R^1\pi_*(\Phi^*\Omega_M\otimes\omega) & & \xrightarrow{H^0(\lambda^\bullet)} & & \Omega_{Z/S} \end{array}\tag{4.7}$$

The first three squares are compatible with the dual of $[G \to H] \to [\check{\mathcal{T}} \to \check{\mathcal{E}}]$. It thus suffices to prove the claim with $\lambda^\bullet$ replaced by the composition of the right-hand squares in Diagram 4.7.

COMPUTATION OF $H^0(\psi^\bullet \circ \lambda^\bullet) = H^0(\lambda^\bullet)$. — To describe $H^0(\lambda^\bullet)$ we now follow the bold printed part of the diagram. Let $(\alpha_{ij}) \in \pi_*^{(1)}(\Phi^*\Omega_M \otimes \omega)$ be a holomorphic family of Čech cochains representing a local holomorphic section α of $R^1\pi_*(\Phi^*\Omega_M \otimes \omega)$. The associated local holomorphic section of $(\check{\mathcal{T}})^\vee$ is

$$\check{L}_1^p(C;\varphi^*T_M)\cap\mathcal{O}(\varphi^*T_M|_{U_0}) = \check{\mathcal{T}}_{(C,\mathbf{x},\varphi)} \ni v \longmapsto \sum_{i,j}\int_{\Gamma/Z}\alpha_{ij}(v)\wedge\bar{\partial}\rho_i. \tag{4.8}$$

The restriction of this section to $Z \subset \mathcal{B}$ is $H^0(\lambda^\bullet)(\alpha)$.

COMPARISON WITH $H^0(\varphi^\bullet)$. — On the zeroth cohomology $\varphi^\bullet$ is described by the following diagram:

$$\begin{array}{ccccc}
\pi_*^{(1)}(\Phi^*\Omega_M \otimes \omega) & \longrightarrow & \pi_*^{(1)}(\Omega_{\Gamma/C} \otimes \omega) & \xrightarrow{\simeq} & \pi_*^{(1)}(\pi^*\Omega_{Z/S} \otimes \omega) \\
\downarrow & & & & \downarrow \\
R^1\pi_*(\Phi^*\Omega_M \otimes \omega) & & & & \Omega_{Z/S}
\end{array}$$

The first horizontal arrow is by pull-back with $\Phi^* : \Phi^*\Omega_M \to \Omega_\Gamma$ composed with the quotient $\Omega_\Gamma \to \Omega_{\Gamma/C}$; the second horizontal arrow is by the isomorphism $\Omega_{\Gamma/C} \simeq \pi^*\Omega_{Z/S}$; and the right-hand vertical arrow is by projection formula composed with the trace morphism $R^1\pi_*\omega \simeq \mathcal{O}_Z$. Explicitely, the map is (up to multiplication by an invertible holomorphic function χ that we suppress here and in the sequel)

$$(\alpha_{ij}) \longmapsto \sum_{i,j}\int(\Phi^*\alpha_{ij}) \wedge \bar{\partial}\rho_i, \tag{4.9}$$

interpreted as section of $\Omega_{Z/S}$ (that is, Φ^* taken as pull-back of differential forms). Again we used the explicit version of relative duality given in Proposition 5.4.

LEMMA 4.7. — $H^0(\varphi^\bullet) = H^0(\lambda^\bullet)$.

Proof. — We evaluate $H^0(\varphi^\bullet)(\alpha)$ at a local holomorphic section v of $\check{\mathcal{T}}$. Since $\operatorname{supp}\bar{\partial}\rho_i \subset \bigcup_{j\le m'}\Delta^j$ we may restrict attention to α with only non-zero component α_{ij}. With local holomorphic coordinates t on Δ, s on S and w^μ on M we may write

$$\alpha_{ij} = \sum_\mu a_\mu(t,s,\varphi)dw^\mu \otimes dt, \quad v = \sum_\mu v^\mu(t,s,\varphi)\partial_{w^\mu}$$

for the relevant local parts. We obtain

$$\begin{aligned}
H^0(\varphi^\bullet)(\alpha)(v) &= \Big(\int_{\Gamma/Z} \Phi^*\alpha_{ij} \wedge \bar\partial\rho_i\Big)(v) \\
&= \int_\Delta \sum_\mu a_\mu(t,s,\varphi) v^\mu(t,s,\varphi)\, dt \wedge \bar\partial\rho_i \\
&= \int_\Delta \alpha_{ij}(v)\bar\partial\rho_i \\
&= H^0(\lambda^\bullet)(\alpha)(v).
\end{aligned}$$

□

COMPUTATION OF $H^{-1}(\psi^\bullet \circ \lambda^\bullet) = H^{-1}(\psi^\bullet)$. — For holomorphic sections of $H^\vee$ coming from $(\check{\mathcal{E}})^\vee$ (cf. Diagram 4.7), pairing with the tautological section of H is nothing but pairing with the section $\check s_{\bar\partial}$ of $\check{\mathcal{E}}$. Given a local holomorphic section α of $\pi_*(\Phi^*\Omega_M \otimes \omega)$ let $(\tilde\alpha_i)$ be a local holomorphic section of $\tilde\pi_*^{(0)}(\tilde\Phi^*\Omega_M \otimes \omega)$ extending $\alpha|_{U_i}$. Then as map to the ideal sheaf $\mathcal{I}$ of Z in $\tilde Z$,

$$H^{-1}(\psi^\bullet)(\alpha) = \sum_i \int \rho_i\tilde\alpha_i(\tilde s_{\bar\partial}) = \Big(\mathcal{B} \ni (s,\varphi) \mapsto \sum_i \int_{\tilde\Gamma/\tilde Z} \rho_i\tilde\alpha_i(\bar\partial\varphi)\Big)\Big|_Z.$$

The induced section of $\mathcal{I}/\mathcal{I}^2$ depends only on α, not on the choice of extension $(\tilde\alpha_i)$.

COMPUTATION OF $H^{-1}(\varphi^\bullet)$. — This step is a little harder and the most interesting part of the proof. It will show that the $\bar\partial$-operator naturally turns up by an integration by parts. Recall that $\varphi^\bullet$ was defined in Section 1.2 by $R\pi_*(.\otimes\omega)$ of

$$L\Phi^*\mathcal{L}_M^\bullet \xrightarrow{L\Phi^*} \mathcal{L}_{\Gamma/C}^\bullet \simeq L\pi^*\mathcal{L}_{Z/S}^\bullet. \tag{4.10}$$

All this is compatible with truncation $\tau_{\geq -1}$. Let $\mathcal{I} \subset \mathcal{O}_{\tilde Z}$ and $\mathcal{J} \subset \mathcal{O}_{\tilde\Gamma}$ be the ideal sheaves defining $Z \subset \tilde Z$ and $\Gamma \subset \tilde\Gamma$ respectively. The following parts of the truncation of the previous sequence of complexes are immediate:

$$\begin{aligned}
L\Phi^*\mathcal{L}_M^\bullet &= [\Phi^*\Omega_M] \quad \text{(one term in degree zero)} \\
\tau_{\geq -1}\mathcal{L}_\Gamma^\bullet &= [\mathcal{J}/\mathcal{J}^2 \to \Omega_{\tilde\Gamma}|_\Gamma] \to \tau_{\geq -1}\mathcal{L}_{\Gamma/C}^\bullet = [\mathcal{J}/\mathcal{J}^2 \to \Omega_{\tilde\Gamma/C}|_\Gamma] \\
\tau_{\geq -1}\mathcal{L}_{Z/S}^\bullet &= [\mathcal{I}/\mathcal{I}^2 \to \Omega_{\tilde Z/S}|_Z] \\
\tau_{\geq -1}L\pi^*\mathcal{L}_{Z/S}^\bullet &= [\pi^*(\mathcal{I}/\mathcal{I}^2) \to \pi^*\Omega_{\tilde Z/S}] = [\mathcal{J}/\mathcal{J}^2 \to \Omega_{\tilde\Gamma/C}|_\Gamma],
\end{aligned}$$

the last line by flatness of π. To work out $L\Phi^*$ we decompose Φ as a closed embedding into a smooth space followed by a projection

$$\Phi:\ \Gamma \overset{(\iota,\Phi)}{\hookrightarrow} \tilde\Gamma \times M \xrightarrow{p} M.$$

Let $\tilde{\mathcal{J}}$ be the ideal sheaf of Γ in $\tilde{\Gamma}\times M$. Writing $\tau_{\geq -1}\mathcal{L}^\bullet_\Gamma = [\tilde{\mathcal{J}}/\tilde{\mathcal{J}}^2 \to \Omega_{\tilde{\Gamma}\times M}|_\Gamma]$,

$$\tau_{\geq -1}L\Phi^* : \ [0\to \Phi^*\Omega_M] \longrightarrow [\tilde{\mathcal{J}}/\tilde{\mathcal{J}}^2 \to \Omega_{\tilde{\Gamma}}|_\Gamma \oplus \Phi^*\Omega_M]$$

is nothing but the inclusion on the degree zero term. To go further we need the explicit quasi-isomorphism between the two representations of $\tau_{\geq -1}\mathcal{L}^\bullet_\Gamma$ obtained from the embeddings into $\tilde{\Gamma}$ and $\tilde{\Gamma}\times M$ respectively: It is given by the natural inclusion

$$[\tilde{\mathcal{J}}/\tilde{\mathcal{J}}^2 \to \Omega_{\tilde{\Gamma}}|_\Gamma] \longrightarrow [\tilde{\mathcal{J}}/\tilde{\mathcal{J}}^2 \to \Omega_{\tilde{\Gamma}}|_\Gamma \oplus \Phi^*\Omega_M].$$

The truncation of sequence (4.10) can thus explicitely be written

$$\begin{array}{ccccccc}
0 & \longrightarrow & \tilde{\mathcal{J}}/\tilde{\mathcal{J}}^2 & \longleftarrow & \mathcal{J}/\mathcal{J}^2 & = & \mathcal{J}/\mathcal{J}^2 \simeq \pi^*(\mathcal{I}/\mathcal{I}^2)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\Phi^*\Omega_M & \longrightarrow & \Omega_{\tilde{\Gamma}}|_\Gamma \oplus \Phi^*\Omega_M & \longleftarrow & \Omega_{\tilde{\Gamma}}|_\Gamma & \longrightarrow & \Omega_{\tilde{\Gamma}/C}|_\Gamma \simeq \pi^*(\Omega_{\tilde{Z}/S}|_Z)
\end{array}$$

To write down $R\pi_*(.\otimes\omega)$ of this diagram we abbreviate

$$\mathrm{Ev} := \Phi^*\Omega_M, \quad J := \mathcal{J}/\mathcal{J}^2, \quad \tilde{J} := \tilde{\mathcal{J}}/\tilde{\mathcal{J}}^2,$$

and, for a sheaf $\mathcal{F}$ on Γ

$$\mathcal{F}^{(\nu)} := \pi_*^{(\nu)}(\mathcal{F}\otimes\omega).$$

We also omit restrictions to Γ. For example,

$$(\Omega_{\tilde{\Gamma}/C}\oplus \mathrm{Ev})^{(1)} = \pi_*^{(1)}\big((\Omega_{\tilde{\Gamma}/C}|_\Gamma \oplus \Phi^*\Omega_M)\otimes\omega\big).$$

A minor technical point arises when $\mathcal{F}$ is not locally free. Then $\mathcal{F}^{(\nu)}$ is not a holomorphic Banach bundle over Z. Rather there will be a holomorphic Banach bundle E and a closed (ringed) subspace $F^{(\nu)}\subset E$ given by finitely many holomorphic functions that are linear in the fiber directions (i.e. $F^{(\nu)}$ is a Banach version of a linear fiber space over Z in complex analysis), and $\mathcal{F}^{(\nu)}$ is the sheaf of germs of holomorphic morphisms to the trivial fiber space $Z\times\mathbb{C}$. For our purposes the knowledge of what a (holomorphic) section of $\mathcal{F}^{(\nu)}$ is together with the fact that kernel and cokernel of the Čech differential $\mathcal{F}^{(0)}\to\mathcal{F}^{(1)}$ are the coherent sheaves $\pi_*(\mathcal{F}\otimes\omega)$ and $R^1\pi_*(\mathcal{F}\otimes\omega)$ will suffice.

With these conventions the truncation of $R\pi_*(\cdot\otimes\omega)$ applied to sequence (4.10) is

$$\begin{array}{ccccccc}
(\mathrm{Ev})^{(0)} & \longrightarrow & \Big((\Omega_{\tilde{\Gamma}/C}\oplus\mathrm{Ev})^{(0)}\oplus\tilde{J}^{(1)}\Big)/\tilde{J}^{(0)} & \longleftarrow & (\Omega^{(0)}_{\tilde{\Gamma}/C}\oplus J^{(1)})/J^{(0)} & \longrightarrow & \mathcal{I}/\mathcal{I}^2\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
(\mathrm{Ev})^{(1)} & \longrightarrow & (\Omega_{\tilde{\Gamma}/C}\oplus\mathrm{Ev})^{(1)} & \longleftarrow & \Omega^{(1)}_{\tilde{\Gamma}/C} & \longrightarrow & \Omega_{\tilde{Z}/S}|_Z
\end{array}$$

Here $\tilde{J}^{(0)}$, $J^{(0)}$ map to $(\Omega_{\tilde{\Gamma}/C} \oplus \mathrm{Ev})^{(0)}$ and to $\Omega^{(0)}_{\tilde{\Gamma}/C}$ respectively by the Kähler differentials

$$d\colon\ \tilde{\mathcal{J}}/\tilde{\mathcal{J}}^2 \longrightarrow \Omega_{\tilde{\Gamma}\times M}|_\Gamma \simeq \Omega_{\tilde{\Gamma}}|_\Gamma \oplus \Phi^*\Omega_M, \quad d: \mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{\tilde{\Gamma}}|_\Gamma,$$

while the maps to $\tilde{J}^{(1)}$ and $J^{(1)}$ are by Čech differentials. Similarly for the maps on Čech 1-cycles, but with one negative sign. The two horizontal maps on the right are induced by the projection formula composed with the trace morphism $R^1\pi_*\omega \simeq \mathcal{O}_Z$ (Lemma 5.3).

Now let us chase some $\alpha \in \pi_*(\Phi^*\Omega_M \otimes \omega)$ through the upper horizontal sequence. Let $\alpha_i = \alpha|_{U_i}$. We claim that $((0 \oplus \alpha_i), 0) \in ((\Omega_{\tilde{\Gamma}/C})^{(0)} \oplus \tilde{J}^{(1)})$ lies in $\Omega^{(0)}_{\tilde{\Gamma}/C} \oplus J^{(1)}$ modulo $\tilde{J}^{(0)}$. From the exact sequence

$$0 \longrightarrow \mathcal{J}/\mathcal{J}^2 \longrightarrow \tilde{\mathcal{J}}/\tilde{\mathcal{J}}^2 \longrightarrow \Phi^*\Omega_M \longrightarrow 0$$

we obtain

$$0 \longrightarrow J^{(\nu)} \xrightarrow{p_1^*} \tilde{J}^{(\nu)} \xrightarrow{d_M} \mathrm{Ev}^{(\nu)} \longrightarrow 0.$$

There thus exists a local holomorphic section $(\tilde{g}_i)$ of $(\tilde{J})^{(0)}$ with

$$d_M\tilde{g}_i = -\alpha_i.$$

But then, if $U_i \cap U_j \neq \emptyset$ (i.e. $i = 0$ or $j = 0$)

$$d_M(\tilde{g}_i - \tilde{g}_j) = \alpha_i - \alpha_j = 0,$$

as section of Ev. So the local section $(\tilde{g}_i - \tilde{g}_j)$ of $\tilde{J}^{(1)}$ actually comes from a section g_{ij} of $J^{(1)}$. Therefore, $((0 \oplus \alpha_i), 0)$ lifts modulo $\tilde{J}^{(0)}$ to the local section

$$(d_{\tilde{\Gamma}}\tilde{g}_i, g_{ij})$$

of $\Omega^{(0)}_{\tilde{\Gamma}/C} \oplus J^{(1)}$. This maps to $\mathcal{I}/\mathcal{I}^2$ by fiber integration

$$\frac{1}{2}\sum_{i,j}\int_{\Gamma/Z} g_{ij} \wedge \bar{\partial}\rho_i. \tag{4.11}$$

In view of the form of $H^{-1}(\psi^\bullet)$ discussed above it remains to be shown:

LEMMA 4.8. — *As local section of $\mathcal{I}/\mathcal{I}^2$ this integral equals*

$$\tilde{Z} \ni z \longmapsto \sum_i \int_{\tilde{\Gamma}_z} \rho_i\tilde{\alpha}_i(\bar{\partial}\varphi_z),$$

where $(\tilde{\alpha}_i)$ extends (α_i) as section of $\tilde{\pi}_*(\tilde{\Phi}^*\Omega_M \otimes \omega)$, and we wrote $\varphi_z := \tilde{\Phi}|_{\tilde{\Gamma}/\tilde{Z}}$.

Proof. — $(\mathrm{id}, \tilde{\Phi}) : \tilde{\Gamma} \to \tilde{\Gamma} \times M$ is a (usually non-holomorphic) extension of the closed embedding $\Gamma \hookrightarrow \tilde{\Gamma} \times M$. The functions $(\mathrm{id}, \Phi)^* \tilde{g}_i$ will therefore not in general be holomorphic. But the fiber integrals

$$(4.12)\quad z \longmapsto \frac{1}{2} \sum_{i,j} \int_{\tilde{\Gamma}/\tilde{Z}} (\mathrm{id}, \Phi)^* (\tilde{g}_i - \tilde{g}_j) \wedge \bar{\partial} \rho_i = \sum_i \int_{\tilde{\Gamma}/\tilde{Z}} (\mathrm{id}, \Phi)^* \tilde{g}_i \wedge \bar{\partial} \rho_i$$

will be local holomorphic functions on $\tilde{Z}$ as one easily sees in local coordinates. Since $\tilde{g}_i - \tilde{g}_j$ induces the holomorphic section g_{ij} of $J^{(1)}$ the fact that $\bar{\partial}\rho_i = -\bar{\partial}\rho_j$ on $U_i \cap U_j$ shows that this holomorphic function induces the Section (4.11) of $\mathcal{I}/\mathcal{I}^2$.

On the other hand, partial integration applied to (4.12) results in

$$-\sum_i \int_{\tilde{\Gamma}_z} \rho_i \bar{\partial} (\mathrm{id}, \Phi)^* \tilde{g}_i = -\sum_i \int_{\tilde{\Gamma}_z} \rho_i d_M \tilde{g}_i (\bar{\partial} \varphi_z).$$

Here we wrote $\varphi_z = \Phi|_{\tilde{\Gamma}_z}$ and d_M to denote the composition

$$\mathcal{O}_{\tilde{\Gamma} \times M} \xrightarrow{d} \Omega_{\tilde{\Gamma} \times M} \simeq \Omega_{\tilde{\Gamma}} \boxplus \Omega_M \longrightarrow p_2^* \Omega_M.$$

Putting $\tilde{\alpha}_i := -d_M \tilde{g}_i$ with $d_M \tilde{g}_i$ viewed as holomorphic section of $\tilde{\pi}_*^{(0)}(\tilde{\Phi}^* \Omega_M \otimes \omega)$ finishes the proof. □

5. Relative duality.

A (say algebraic) family of prestable curves $\pi : X \to S$ is Gorenstein: It has an invertible relative dualizing sheaf $\omega_{X/S}$. For any coherent sheaf $\mathcal{F}$ on X the theory of duality in derived categories developed in [Ha] then takes the following form. (We basically adopt the terminology of loc.cit. except that we drop any underlining and we write Ext for R Hom.) It provides a *trace isomorphism*

$$(5.13)\qquad \mathrm{tr}_\pi : R\pi_* \omega_{X/S}[1] \simeq R^1 \pi_* \omega_{X/S} \longrightarrow \mathcal{O}_S$$

and a *Yoneda morphism* (our notation)

$$(5.14)\qquad R\pi_* \operatorname{Ext}_X(\mathcal{F}, \omega_{X/S}) \longrightarrow \operatorname{Ext}_S(R\pi_* \mathcal{F}, R\pi_* \omega_{X/S}).$$

Composing we obtain the *duality morphism*

$$(5.15)\qquad R\pi_* \operatorname{Ext}_X(\mathcal{F}, \omega_{X/S}) \longrightarrow \operatorname{Ext}_S(R\pi_* \mathcal{F}, \mathcal{O}_S) = (R\pi_* \mathcal{F})^\vee,$$

and the content of the duality theorem is that this is an isomorphism.

The morphisms are in $D^+_{\mathrm{coh}}(S)$, the derived category of complexes of $\mathcal{O}_S$-modules bounded below and with coherent cohomology. Sheaves are identified with complexes concentrated in degree 0. For applicability of these statements in the Gorenstein rather than the smooth case see the remark at the beginning of [Ha], VII.4. A useful reference for all this is also [Lp]. We will not use duality theory for complex spaces, involving Fréchet sheaves [RaRuVe].

The purpose of this chapter is to (a) make the transition from algebraic sheaves to associated analytic sheaves (GAGA) and (b) to give an explicit formulation of duality in terms of (analytic) Čech cochains and fiber integrals, at least locally analytically over S.

Let us first comment on a subtlety that may be a source of confusion: In our application it is crucial that all morphisms are unique up to unique isomorphism (in the derived sense, algebraically or analytically). Explicitely, this means that whenever we choose two representatives of any of the objects (such as $R\pi_*\mathcal{F}$) in terms of complexes of $\mathcal{O}_X$-modules there is a (sequence of) quasi-isomorphism(s) between them that is unique up to homotopy. The same is true for morphisms. In particular the maps induced in cohomology are indeed unique up to unique isomorphism. In view of [Si2], Thm. 3.3, this is enough to assure that the associated (analytic or algebraic) cones are unique up to unique isomorphism and hence are compatible with changes of local uniformizing systems.

Our plan is (1) to express the Yoneda morphism in terms of algebraic Čech-cochains (2) to go over to analytic sheaves and to admit refinements of the covering, and finally (3) to give the trace isomorphism (5.13) analytically. We do not indicate in our notation if we are working algebraically or analytically. For example, X will denote either the scheme or its associated analytic space, but the meaning will always be clear from the context.

Let V_i be an affine open cover of X. Then (the complex associated to) a coherent sheaf $\mathcal{G}$ on X is quasi-isomorphic to the complex of Čech-sheaves $[\mathcal{G}^{(0)} \to \mathcal{G}^{(1)}]$. Recall that for any open $U \subset X$ the space of sections of $\mathcal{G}^{(0)}$ over U is $\prod_i \mathcal{G}(V_i \cap U)$, and similarly for $\mathcal{G}^{(1)}$. The corresponding Čech cochains relative π are then just $\pi_*^{(i)}\mathcal{G} := \pi_*\mathcal{G}^{(i)}$. Since the sheaves $\mathcal{G}^{(i)}$ are π_*-acyclic we can represent $R\pi_*\mathcal{G}$ by $[\pi_*^{(0)}\mathcal{G} \to \pi_*^{(1)}\mathcal{G}]$.

We claim that if $\mathcal{G}$ is locally free then (possibly after shrinking S) we can choose V_i in such a way that $\pi_*^{(i)}\mathcal{G}$ are projective (i.e., locally free) $\mathcal{O}_S$-modules. To this end we factor $X \to S$ into a finite flat morphism $\kappa : X \to S \times \mathbb{P}^1$ and the projection $S \times \mathbb{P}^1 \to \mathbb{P}^1$. This is always possible after shrinking S. We put $\bar{V}_0 := S \times (\mathbb{P}^1 \setminus \{\infty\})$, $\bar{V}_1 := S \times (\mathbb{P}^1 \setminus \{0\})$ and $V_\nu = \kappa^{-1}(\bar{V}_\nu)$. Denote the projections $\bar{V}_\nu \to S$ by p_ν. Now $\kappa_*\mathcal{G}$ is locally free by flatness of κ. Writing $S = \operatorname{Spec} A$, $\bar{V}_\nu = \operatorname{Spec} A[T]$ this means that the $A[T]$-module M associated to $\kappa_*\mathcal{G}|_{\bar{V}_\nu}$ is projective. Equivalently, it is a direct summand of a free $A[T]$-module. Viewed as an A-module, $A[T]$ being a free A-module, M is a direct summand of a free A-module, hence projective. But the A-module M is just the module associated to $p_{\nu *}\kappa_*\mathcal{G}$, so this shows projectivity of $\pi_*^{(0)}\mathcal{G}$. A similar argument with $A[T]$ replaced by the ring of Laurent series $A[T]_{(T)}$ establishes the claim for $\pi_*^{(1)}\mathcal{G}$.

We now restrict to locally free sheaves $\mathcal{F}$ (slightly easier and sufficient for our purposes). By the digression we can then represent $\operatorname{Ext}_S(R\pi_*\mathcal{F}, R\pi_*\omega_{X/S})$ by $\operatorname{Hom}_S^\bullet(\pi_*\mathcal{F}^\bullet, \pi_*\omega^\bullet)$. Written out this complex takes the form

$$\begin{aligned}\operatorname{Hom}_S^\bullet&\left([\pi_*^{(0)}\mathcal{F} \to \pi_*^{(1)}\mathcal{F}], [\pi_*^{(0)}\omega_{X/S} \to \pi_*^{(1)}\omega_{X/S}]\right)\\ &= \Big[\operatorname{Hom}_S(\pi_*^{(0)}\mathcal{F}, \pi_*^{(0)}\omega_{X/S}) \times \operatorname{Hom}_S(\pi_*^{(1)}\mathcal{F}, \pi_*^{(1)}\omega_{X/S})\\ &\qquad \to \operatorname{Hom}_S(\pi_*^{(0)}\mathcal{F}, \pi_*^{(1)}\omega_{X/S})\Big].\end{aligned}$$

LEMMA 5.1. — *The Yoneda morphism*

$$R\pi_*[\mathcal{F}^\vee \otimes \omega_{X/S}] \to \operatorname{Ext}_S(R\pi_*\mathcal{F}, R\pi_*\omega_{X/S})$$

can be represented by the pair of morphisms (φ^0, φ^1) *with*

$$\begin{aligned}\varphi^0 : \pi_*^{(0)}(\mathcal{F}^\vee \otimes \omega_{X/S}) &\longrightarrow \operatorname{Hom}(\pi_*^{(0)}\mathcal{F}, \pi_*^{(0)}\omega_{X/S}) \times \operatorname{Hom}(\pi_*^{(1)}\mathcal{F}, \pi_*^{(1)}\omega_{X/S})\\ (\alpha_i)_i &\longmapsto \left((\beta_i)_i \mapsto (\alpha_i(\beta_i))_i, (\beta_{ij})_{ij} \mapsto \left(\left(\frac{\alpha_i + \alpha_j}{2}\right)(\beta_{ij})\right)_{ij}\right)\end{aligned}$$

and

$$\begin{aligned}\varphi^1 : \pi_*^{(1)}(\mathcal{F}^\vee \otimes \omega_{X/S}) &\longrightarrow \operatorname{Hom}(\pi_*^{(0)}\mathcal{F}, \pi_*^{(1)}\omega_{X/S})\\ (\alpha_{ij})_{ij} &\longmapsto \left((\beta_i)_i \mapsto \left(\alpha_{ij}\left(\frac{\beta_i + \beta_j}{2}\right)\right)_{ij}\right).\end{aligned}$$

Proof. — The Yoneda morphism is defined in [Ha], II.5.5. The recipe is to first represent $\mathcal{F}$ and $\omega_{X/S}$ by a π_*-acyclic resolution $\mathcal{F}^\bullet$ and by an injective resolution $\omega^\bullet$ respectively. Then $\operatorname{Hom}_X^\bullet(\mathcal{F}^\bullet, \omega^\bullet)$ represents $\operatorname{Ext}_X([\mathcal{F}], [\omega])$ and consists of flasque sheaves. Thus $R\pi_* \operatorname{Ext}_X([\mathcal{F}], [\omega])$ can be represented by $\pi_* \operatorname{Hom}_X^\bullet(\mathcal{F}^\bullet, \omega^\bullet)$. Similarly, $\operatorname{Ext}_S(R\pi_*[\mathcal{F}], R\pi_*[\omega])$ is represented by $\operatorname{Ext}_S(\pi_*\mathcal{F}^\bullet, \pi_*\omega^\bullet)$. With these representatives the Yoneda morphism is simply the composition of natural morphisms

$$\pi_* \operatorname{Hom}_X^\bullet(\mathcal{F}^\bullet, \omega^\bullet) \longrightarrow \operatorname{Hom}_S^\bullet(\pi_*\mathcal{F}^\bullet, \pi_*\omega^\bullet) \longrightarrow \operatorname{Ext}_S(\pi_*\mathcal{F}^\bullet, \pi_*\omega^\bullet).$$

For $\mathcal{F}^\bullet$ we may take the Čech complex $\mathcal{F}^{(0)} \to \mathcal{F}^{(1)}$. And for our special choice of affine covering the second arrow becomes an isomorphism and is thus understood. We claim that we may take for $\omega^\bullet$ the Čech resolution $\omega^{(0)} \to \omega^{(1)}$ as well, instead of an injective one. In fact, $\mathrm{Hom}_X(\mathcal{F}^{(i)}, \omega^{(i)})$ consists of a direct sum of coherent sheaves supported on affine sets and is thus π_*-acyclic. By injectivity of $\omega^\bullet$ there exists a map of complexes $[\omega^{(0)} \to \omega^{(1)}] \to \omega^\bullet$. This map induces a commutative diagram of complexes

$$\begin{array}{ccc}
\pi_* \mathrm{Hom}^\bullet_X([\mathcal{F}^{(0)} \to \mathcal{F}^{(1)}], [\omega^{(0)} \to \omega^{(1)}]) & \longrightarrow & \mathrm{Hom}^\bullet_S([\pi^{(0)}_* \mathcal{F} \to \pi^{(1)}_* \mathcal{F}], [\pi^{(0)}_* \omega \to \pi^{(1)}_* \omega]) \\
\downarrow & & \downarrow \\
\pi_* \mathrm{Hom}^\bullet_X([\mathcal{F}^{(0)} \to \mathcal{F}^{(1)}], [\omega^0 \to \omega^1) & \longrightarrow & \mathrm{Hom}^\bullet_S([\pi^{(0)}_* \mathcal{F} \to \pi^{(1)}_* \mathcal{F}], [\pi_* \omega^0 \to \pi_* \omega^1])
\end{array}$$

in which *the vertical morphisms are quasi-isomorphisms.* End proof of claim.

It remains to replace the resolution $\mathrm{Hom}^\bullet_X(\mathcal{F}^\bullet, \omega^\bullet)$ of $\mathrm{Hom}_X(\mathcal{F}, \omega)$ by its Čech resolution $\mathrm{Hom}_X(\mathcal{F}, \omega)^{(0)} \to \mathrm{Hom}_X(\mathcal{F}, \omega)^{(1)}$. Explicitely, we have the following quasi-isomorphism:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(\mathcal{F}, \omega) & \longrightarrow & \mathrm{Hom}^{(0)}(\mathcal{F}, \omega) & \xrightarrow{\check{d}_{\mathrm{Hom}}} & \mathrm{Hom}^{(1)}(\mathcal{F}, \omega) \\
 & & {\scriptstyle \mathrm{Id}}\big\downarrow & & \big\downarrow{\scriptstyle F} & & \big\downarrow{\scriptstyle G} \\
 & & & & \mathrm{Hom}(\mathcal{F}^{(0)}, \omega^{(0)}) & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathcal{F}, \omega) & \longrightarrow & \oplus & \xrightarrow{\delta} & \mathrm{Hom}(\mathcal{F}^{(0)}, \omega^{(1)}) \\
 & & & & \mathrm{Hom}(\mathcal{F}^{(1)}, \omega^{(1)}) & &
\end{array}$$

The maps are $\delta(a; b) = \check{d}_\omega \circ a - b \circ \check{d}_{\mathcal{F}}$,

$$F: (\varphi_i)_i \longmapsto \Big((f_i)_i \mapsto (\varphi_i(f_i))_i; (f_{ij})_{ij} \mapsto \Big(\big(\frac{\varphi_i + \varphi_j}{2}\big)(f_{ij})\Big)\Big)$$

$$G: (\varphi_{ij})_{ij} \longmapsto \Big((f_i)_i \mapsto \varphi_{ij}\Big(\frac{f_i + f_j}{2}\Big)_{ij}\Big),$$

and $\check{d}_{\mathrm{Hom}}$, $\check{d}_\omega$, $\check{d}_{\mathcal{F}}$ are Čech differentials. Composing these maps with the natural map to $\mathrm{Hom}^\bullet_S(\pi_* \mathcal{F}^\bullet, \pi_* \omega^\bullet)$ from above gives the stated result. □

For the next step (2) we view $X \to S$ as a morphism of complex spaces. The analytic sheaves associated to the algebraic sheaves $\pi^{(i)}_*(\mathcal{F}^\vee \otimes \omega_{X/S})$ etc. are given by pull-back under the morphism of ringed spaces from the complex space $(S^{\mathrm{an}}, \mathcal{O}_{S^{\mathrm{an}}})$ underlying S to the scheme $(S, \mathcal{O}_S)$. This amounts to going over to analytic topology and tensoring with the structure sheaf of S^{an} over the pull-back of $\mathcal{O}_S$. The effect is that one considers analytic Čech cochains that are algebraic along the fibers of π.

Since $\mathcal{F}$ was supposed to be locally free the dualization of a sequence analogous to (1.2) provides a short exact sequence of algebraic sheaves on X

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0 \tag{5.16}$$

with $R^1\pi_*\mathcal{G} = R^1\pi_*\mathcal{H} = 0$. Taking the associated π_*-acyclic resolution by (algebraic) Čech sheaves $0 \to \mathcal{F} \to \mathcal{F}^{(0)} \to \mathcal{F}^{(1)} \to 0$ etc. and pushing forward by π yields a diagram of a form (dual to) Diagram 4.6. An argument similar to the one given there produces a quasi-isomorphism (of quasi-coherent, algebraic sheaves)

$$[\pi_*\mathcal{G} \to \pi_*\mathcal{H}] \longrightarrow [\pi_*^{(0)}\mathcal{F} \to \pi_*^{(1)}\mathcal{F}]. \tag{5.17}$$

For any point $s \in S$ we now choose a Stein refinement $\{U_i\}$ of the affine covering $\{V_i\}$ of the form given in Section 2.3 (we assume that the point $0, \infty \in \mathbb{P}^1$ ahev been chosen suitably). Possibly by enlarging the index set for $\{V_i\}$ we may assume that $\{U_i\}$ is indeed a shrinking of $\{V_i\}$ (in the Hausdorff-topology). Now the analytic sheaves $\mathcal{F}^{\mathrm{an}}$ etc. associated to $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ and the covering $\{U_i\}$ give rise to a similar quasi-isomorphism (of $\mathcal{O}_{S^{\mathrm{an}}}$-modules)

$$[\pi_*\mathcal{G}^{\mathrm{an}} \to \pi_*\mathcal{H}^{\mathrm{an}}] \longrightarrow [\pi_*^{(0)}\mathcal{F}^{\mathrm{an}} \to \pi_*^{(1)}\mathcal{F}^{\mathrm{an}}]. \tag{5.18}$$

We then obtain a restriction map from $\pi_*^{(i)}\mathcal{F} \otimes \mathcal{O}_{S^{\mathrm{an}}}$ to the corresponding $\mathcal{O}_{S^{\mathrm{an}}}$-module of analytic Čech cochains associated to $\{U_i\}$. Since π is a projective morphism, by Chows lemma sections of an analytic sheaf $\mathcal{L}$ over $\pi^{-1}(U)$ are fiberwise algebraic. In other words, $\pi_*\mathcal{G} \otimes \mathcal{O}_{S^{\mathrm{an}}} \simeq \pi_*\mathcal{G}^{\mathrm{an}}$ and similarly for $\mathcal{H}$. Compatibility of (5.18) with (5.17) tensored by $\mathcal{O}_{S^{\mathrm{an}}}$ and the restriction map now shows the required GAGA statement (just in this lemma we use for clarity the notation $\pi_*^{(i),\mathrm{an}}$ to denote relative bounded Čech cochains with respect to $\{U_i\}$ as opposed to $\pi_*^{(i)}$ for fiberwise algebraic Čech cochains with respect to $\{V_i\}$):

LEMMA 5.2. — *The restriction morphism*

$$[\pi_*^{(0)}\mathcal{F} \otimes \mathcal{O}_{S^{\mathrm{an}}} \to \pi_*^{(1)}\mathcal{F} \otimes \mathcal{O}_{S^{\mathrm{an}}}] \longrightarrow [\pi_*^{(0),\mathrm{an}}\mathcal{F}^{\mathrm{an}} \to \pi_*^{(1),\mathrm{an}}\mathcal{F}^{\mathrm{an}}]$$

is a quasi-isomorphism. □

As last ingredient (3) we give an analytic trace isomorphism. Let ρ_i be a partition of unity subordinate to U_i.

LEMMA 5.3. — *The map*

$$\Phi : \pi_*^{(1)}\omega_{X/S} \longrightarrow \mathcal{O}_S, \quad (\alpha_{ij})_{ij} \longmapsto \sum_{i,j} \int_{X/S} \alpha_{ij} \wedge \bar{\partial}\rho_i$$

induces an isomorphism $R^1\pi_*\omega_{X/S} \simeq \mathcal{O}_S$.

Proof. — The map Φ vanishes on the image of $\pi_*^{(0)}\omega_{X/S}$ under the Čech differential and thus induces the claimed map on the first cohomology. By duality on the fibers X_s of π it holds $H^1(X_s, \omega_{X_s}) = H^0(X_s, \mathcal{O}_{X_s}) = \mathbb{C}$ since X_s (being prestable) is reduced and connected. Naturality of relative dualizing sheaves shows $\omega_{X_s} = \omega_{X/S}|_{X_s}$. The map π being flat, Grauert's base change theorem thus implies that $R^1\pi_*\omega_{X/S}$ is locally free of rank one with fibers

$$R^1\pi_*\omega_{X/S}/\mathfrak{m}_s R^1\pi_*\omega_{X/S} \;\simeq\; H^1(X_s, \omega_{X_s}).$$

It thus suffices to find, for any prestable curve C, a 1-cocycle with values in ω_C with non-vanishing value under the integral defining Φ. And indeed, any $U_{ij} \subset C$ being an annulus $\Delta \setminus \bar{\Delta}_a$ (where we assume U_i meet the inner boundary $\partial\Delta_a$) we may put $\alpha_{ij} = z^{-1}dz$ with z the linear coordinate on Δ. Then using integration by parts and $\rho_i|_{\partial\Delta} \equiv 0$, $\rho_i|_{\partial\Delta_a} \equiv 1$ we obtain

$$\int_C \alpha_{ij} \wedge \bar{\partial}\rho_i \;=\; \int_{\Delta\setminus\Delta_a} z^{-1}dz \wedge \bar{\partial}\rho_i \;=\; \int_{\partial\Delta_a} z^{-1}dz \;=\; 2\pi i \;\neq\; 0. \qquad \square$$

Notice that an isomorphism $R^1\pi_*\omega_{X/S} \simeq \mathcal{O}_S$ is unique up to multiplication by an invertible holomorphic function. Thus the isomorphism given here differs from the analytification of the algebraic trace isomorphism only by such multiplication.

To complete the explicit (analytic) description of algebraic relative duality we just have to compose the trace isomorphism with the Yoneda pairing.

PROPOSITION 5.4. — *The algebraic duality morphism (5.15) is locally analytically given by*

$$\begin{array}{ccc}
\pi_*^{(0)}(\mathcal{F}^\vee \otimes \omega_{X/S}) \longrightarrow (\pi_*^{(1)}\mathcal{F})^\vee & & (\alpha_i) \longmapsto \left((\beta_{ij}) \mapsto \chi \cdot \sum_{i,j} \int_{\Gamma/Z} \left(\frac{\alpha_i+\alpha_j}{2}\right)(\beta_{ij}) \wedge \bar{\partial}\rho_i\right) \\
\downarrow \qquad\qquad\qquad \downarrow & & \\
\pi_*^{(1)}(\mathcal{F}^\vee \otimes \omega_{X/S}) \longrightarrow (\pi_*^{(0)}\mathcal{F})^\vee & & (\alpha_{ij}) \longmapsto \left((\beta_i) \mapsto \chi \cdot \sum_{i,j} \int_{\Gamma/Z} \alpha_{ij}\left(\frac{\beta_i+\beta_j}{2}\right) \wedge \bar{\partial}\rho_i\right)
\end{array}$$

for some invertible holomorphic function χ on S. $\square$

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