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WEAKLY-EINSTEIN HERMITIAN SURFACES

by V. APOSTOLOV and O. MUŠKAROV

1. Introduction.

A *Hermitian surface* (M, J, h) is a complex surface (M, J) endowed with a J-invariant Riemannian metric h. If the Kähler form $F(.,.) = h(J.,.)$ of (M, J, h) is closed we obtain a Kähler surface. The Riemannian metric h is said to be *Einstein* if its Ricci tensor Ric is a constant multiple of the metric, *i.e.*, if $\mathrm{Ric} = \lambda h$, where the constant 4λ is the scalar curvature of h. Many efforts have been done to study compact Einstein Hermitian surfaces (which, in general, give examples of *non-homogeneous* Einstein 4-spaces [10], [8]). The compact *Kähler-Einstein* surfaces have been described by completely resolving the corresponding complex Monge-Ampère equations, see [33], [3], [25], [28], [26], while the only known example of a compact, *non-Kähler*, Einstein Hermitian surface is the Hirzebruch surface $F_1 \cong \mathbb{CP}^2 \sharp \bar{\mathbb{CP}}^2$ with the Page metric [21]. Recently C. LeBrun [19] has proved that the only other compact complex surfaces that could admit non-Kähler Einstein Hermitian metrics are $\mathbb{CP}^2 \sharp 2\bar{\mathbb{CP}}^2$ and $\mathbb{CP}^2 \sharp 3\bar{\mathbb{CP}}^2$. It is thus natural to consider some weaker conditions on the Ricci tensor than the Einstein one in order to provide more examples of (non-Kähler) Hermitian metrics still having interesting geometric properties. A possible way to do this is to consider the two irreducible components of the traceless part of the Ricci tensor under the action of the unitary group $U(2)$ and to impose as a condition the vanishing of one of them.This leads to consider Hermitian surfaces with J-invariant Ricci tensor or with J-anti-invariant traceless

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Ricci tensor. If h is a Kähler metric, the first condition is automatically satisfied, whereas the second one means that h is Einstein. More generally, if h is conformal to a Kähler metric g (*i.e.* $h = f^{-2}g$ for a positive smooth function f on M), then the above two conditions are equivalent respectively to the following properties of the Kähler metric g and the conformal factor f (see [2] and [15]):

(i) $J\mathrm{grad}_g f$ is a Killing vector field of g;

(ii) $\gamma - 2i\dfrac{\partial\bar{\partial}f}{f}$ is a self-dual two form, where γ is the Ricci form of (g, J).

For any compact Einstein Hermitian surface (M, J, h) the Riemannian version of the Goldberg-Sachs theorem [24], [20] combined with the results of Derdziński [10] and Boyer [9] imply that h is conformal to a Kähler metric g, and the conformal factor f (which in this case satisfies simultaneously (i) and (ii)) is either a constant (*i.e.*, h is a Kähler-Einstein metric), or else g has a non-constant, everywhere positive scalar curvature s, and f is a constant multiple of s [10]. In the latter situation the condition (i) means that g is an *extremal* Kähler metric [10], while (ii) implies that the anti-canonical bundle of (M, J) is ample [15], [19].

Compact Hermitian surfaces with J-invariant Ricci tensor have been studied in [2] and it has been proved that if the first Betti number is even, then the Hermitian metric is still conformally Kähler. The existence of non-Kähler, Hermitian metrics with J-invariant Ricci tensor on compact complex surfaces with even first Betti number is then equivalent to the existence of Kähler metrics admitting non-trivial Killing vector fields with zeroes.

The purpose of this paper is to study the compact Hermitian surfaces (M, h, J) with J-anti-invariant traceless Ricci tensor which, in addition, are *locally conformally Kähler*. These surfaces can be characterized by the property that at any point the Kähler form is an eigenform of the curvature operator. The (almost) Hermitian manifolds satisfying the latter condition are known in the literature as (weakly) $*$-*Einstein* manifolds, cf. [29], and the corresponding eigenfunction (non-constant in general) is usually called $*$-scalar curvature. Since on a compact complex surface (M, J) with even first Betti number every locally conformally Kähler metric h is (globally) conformal to a Kähler metric $g = f^2 h$ [30], the $*$-Einstein condition on h is then equivalent to (ii).

As we have already mentioned for Kähler surfaces the Einstein and $*$-Einstein conditions coincide while, in general, the $*$-Einstein condition is weaker in view of the Riemannian Goldberg-Sachs theorem. In Section 2 we present large families of $*$-Einstein (but non-Einstein) Hermitian structures on $S^1 \times S^3$, $\mathbb{CP}^2 \sharp \bar{\mathbb{CP}}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, and $X \times Y$, where X and Y are compact oriented Riemann surfaces of genus $g(X) \geqslant 2,\ g(Y) \geqslant 1$. This shows that the problem for existence of $*$-Einstein (non-Kähler) Hermitian metrics is much more tractable than those for Einstein metrics.

On a compact complex surface (M, J) the Einstein and $*$-Einstein conditions can be also compared by considering the Hilbert functional

$$\mathbf{S}(g) = \frac{\int_M \mathrm{Scal}(g) dV_g}{[\int_M dV_g]^{\frac{1}{2}}},$$

where $\mathrm{Scal}(g)$ denotes the scalar curvature of a Riemannian metric g. It is well known that the Einstein metrics are the critical points of $\mathbf{S}$ acting on the space of all Riemannian metrics on M, while if $\mathbf{S}$ is restricted on the Hermitian metrics with respect to J, then its critical points are the metrics with J-anti-invariant traceless Ricci tensor and constant scalar curvature (cf. [8, ch. 4]). We ask *whether there are compact $*$-Einstein Hermitian surfaces of constant scalar curvature which are not Einstein*? One of our objectives here is to show that the answer to this question is *yes*. We will accomplish this by explicitly constructing such metrics on the conformally flat Hopf surfaces which, as it is well-known, do not admit Einstein metrics at all. According to [9], [23], [13], any such a surface (M, J) admits a unique (up to homothety) conformally flat Hermitian metric g with parallel Lee form θ, which is usually called *Vaisman metric*. Starting from g we construct a new Hermitian metric

$$h = g + \frac{1}{3|\theta|^2}(\theta \otimes \theta + J\theta \otimes J\theta), \tag{$*$}$$

which is $*$-Einstein, and has constant scalar and $*$-scalar curvatures (see Section 4). Moreover, we prove that the metrics h constructed as above can be in fact characterized by the latter property, *i.e.*, we have the following

THEOREM 1. — *Let (M, h, J) be a compact $*$-Einstein Hermitian surface of constant scalar and $*$-scalar curvatures. Then either (M, h, J) is a Kähler-Einstein surface, or (M, J) is a conformally flat Hopf surface and h is obtained from the Vaisman metric of (M, J) via $(*)$.*

A well known result of Jensen [16] says that any locally homogeneous, Einstein 4-manifold is locally symmetric. Concerning our weakly Einstein condition, it follows from Theorem 1 that the only locally homogeneous

$*$-Einstein, non-Einstein Hermitian metrics on compact complex surfaces are those obtained from the Vaisman metrics of the conformally flat Hopf surfaces via $(*)$.

The main point in the proof of Theorem 1 is to show that either the metric h is Kähler-Einstein, or (up to homothety) the eigenvalues of its Ricci tensor are equal to $0, 2, 1, 1$. To do this we use suitable estimates of the L^2-norm of the Ricci tensor involving the scalar curvature and the L^2-norm of the Lee form, obtained as a consequence of the second Bianchi identity. In the second case it follows from Gauduchon's Plurigenera theorem [11] that the Kodaira dimension of (M, J) is $-\infty$, *i.e.*, (M, J) belongs to class VII of the Kodaira-Enriques classification. The first Betti number of M is then equal to 1, cf. [5], and a Bochner type argument shows that the (Riemannian) universal cover of (M, h) is $\mathbb{R} \times N$, where N is a compact Sasakian 3-manifold. Therefore the Hermitian surface (M, h, J) is a generalized Hopf surface [31], *i.e.*, the Lee form of (h, J) is parallel. Now Theorem 1 follows by the observation that any $*$-Einstein generalized Hopf metric is obtained from a conformally flat one via $(*)$ (Section 4, Theorem 2), which amounts to a deformation of the induced Sasakian structure on N into an Einstein one (see [22], [14]).

As a by-product of the proof of Theorem 1 we show that the classification of the locally conformally Kähler metrics with parallel Lee form and constant, non-negative scalar curvature is equivalent to that of the conformally flat ones given in [9], [23], (Section 4, Remark 3).

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2. Examples of compact $*$-Einstein Hermitian surfaces.

In this section we present a detailed description of a number of examples of $*$-Einstein Hermitian metrics which are not Einstein.

2.1. $*$-Einstein metrics on $S^1 \times S^3$ and $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$.

Let h be a Riemannian metric in four dimensions with an isometric action of the group $U(2)$ and suppose that the generic orbit is 3-dimensional. It is known [27] that if h is Einstein, then (locally) it has the following *diagonal* form:

$$h = dt^2 + g^2(t)(\sigma_1^2 + \sigma_2^2) + f^2(t)\sigma_3^2, \tag{1}$$

where t is a coordinate transverse to the $U(2)$-orbits, $g(t)$ and $f(t)$ are positive smooth functions, and σ_i, $i = 1, 2, 3$ are the invariant 1-forms on the sphere S^3 satisfying $d\sigma_1 = \sigma_2 \wedge \sigma_3$ and cyclically. It is also known (cf. [8, 9.127]) that the only compact example of a non-homogeneous Einstein metric of the above form is the Page metric on $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$ (or its $\mathbb{Z}_2$-quotient). In this subsection we will look more carefully at the Page construction in order to provide a natural 1-parameter family of $U(2)$-invariant, $*$-Einstein Hermitian metrics on $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$; the only metric of constant scalar curvature in this family is the Page metric. We also construct a homogeneous, $*$-Einstein Hermitian metric on $\mathbb{C}^2 - \{0\} \cong \mathbb{R} \times S^3$, which can be performed on any conformally flat primary Hopf surface (diffeomorphic to $S^1 \times S^3$), cf. [13, sect. III].

Denote by $\frac{\partial}{\partial t}$, X_1, X_2, X_3 the dual vector fields of $dt, \sigma_1, \sigma_2, \sigma_3$, respectively. They satisfy the relations $[\frac{\partial}{\partial t}, X_i] = 0$ and $[X_1, X_2] = -X_3$, etc. It is known (cf. [7, exposé 15]) that the Ricci tensor Ric of the metric h is diagonal and it is given by

$$\begin{aligned} \mathrm{Ric}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) &= -2\frac{g''}{g} - \frac{f''}{f} \\ \mathrm{Ric}\Big(\frac{X_1}{g}, \frac{X_1}{g}\Big) = \mathrm{Ric}\Big(\frac{X_2}{g}, \frac{X_2}{g}\Big) &= -\frac{g''}{g} - \frac{f'g'}{fg} - \Big(\frac{g'}{g}\Big)^2 + \frac{4}{g^2} - 2\frac{f^2}{g^4} \\ \mathrm{Ric}\Big(\frac{X_3}{f}, \frac{X_3}{f}\Big) &= -\frac{f''}{f} + 2\Big(\frac{f^2}{g^4} - \frac{f'g'}{fg}\Big). \end{aligned} \tag{2}$$

Let J be the almost complex structure defined by $J\frac{\partial}{\partial t} = -\frac{X_3}{f}$, $JX_1 = X_2$. It is easily checked that J is integrable and compatible with h. The Lee form θ of the Hermitian structure (h, J) is then equal to $2\frac{(f + gg')}{g^2}dt$, hence $d\theta = 0$, *i.e.*, (h, J) is locally conformally Kähler. The Hermitian structure (h, J) is then $*$-Einstein iff its traceless Ricci tensor is J-anti-invariant (see Lemma 1 below). It follows from (2) that the $*$-Einstein condition is equivalent to the following ordinary differential equation for f and g:

$$\frac{f''}{f} = \left(\frac{g'}{g}\right)^2 + 3\frac{f^2}{g^4} - \frac{4}{g^2}. \tag{3}$$

An obvious solution of (3) is $g = \frac{\sqrt{3}}{2}f = const$. Therefore the product of the standard metric on $\mathbb{R}$ with the Berger metric $\sigma_1^2 + \sigma_2^2 + \frac{4}{3}\sigma_3^2$ on S^3 is a $*$-Einstein, non-Einstein Hermitian metric on $\mathbb{C}^2 - \{0\} \cong \mathbb{R} \times S^3$, which is clearly of constant scalar and $*$-scalar curvatures (it is, in fact, a locally homogeneous Hermitian structure). It can be shown that this construction actually provides $*$-Einstein Hermitian metrics on any conformally flat primary Hopf surface, the compact quotient of $\mathbb{C}^2 - \{0\}$ by a cyclic group $< \gamma >$, where $\gamma : (z_1, z_2) \mapsto (\alpha_1 z_1, \alpha_2 z_2)$; $|\alpha_1| = |\alpha_2| > 1$ (cf. [13]). Our main result, Theorem 1, states that these examples exhaust all compact, non-Kähler, $*$-Einstein Hermitian surfaces of constant scalar and $*$-scalar curvatures.

We will resolve (3) also on the compact manifold $\mathbb{CP}^2 \sharp \bar{\mathbb{CP}}^2$. Following [7, exposé 15], we first note that a metric h defined on $(-a, a) \times S^3$ by (1) induces a smooth metric on $\mathbb{CP}^2 \sharp \bar{\mathbb{CP}}^2$ iff the corresponding functions f and g satisfy the following boundary conditions:

$$\begin{aligned} &f'(-a) = -f'(a) = 1, f^{(2k)}(a) = f^{(2k)}(-a) = 0, \; k \geqslant 0, \\ &g(a) \neq 0 \neq g(-a), g^{(2k+1)}(a) = g^{(2k+1)}(-a) = 0, \; k \geqslant 0. \end{aligned} \tag{4}$$

We will look for solutions f and g of (3), such that $f = A(\sqrt{1-g^2})'$ where A is a constant. Let us introduce a new variable $x = \sqrt{1-g^2}$ and a new function $z(x)$ such that $x' = \sqrt{-z(x)}$. Then the equation (3) reduces to

$$\frac{d^2}{dx^2}(z(x)) - 2\frac{(x^2+3A^2)}{(1-x^2)^2}z(x) - \frac{8}{1-x^2} = 0. \tag{5}$$

The function $z(x) = \frac{4}{3A^2+1}(x^2-1)$ is a solution of (5) and the corresponding homogeneous equation reduces to a hyper-geometric equation of Gauss, which can be solved explicitly; the even solutions of (5) are given by

$$z(x) = \frac{2}{k(k+1)}(x^2-1) + C\frac{P(x)+P(-x)}{(1-x^2)^k},$$

where C is a constant, $k = \dfrac{\sqrt{6A^2+3}-1}{2}$, and $P(x) = (1+x)^{2k+1}(2k+1-x)$. To ensure the boundary conditions (4) we need to show that there exist constants C_0 and $x_0 \in (0,1)$ such that $z(x_0) = 0$ and $z'(x_0) = \dfrac{2}{A}$. This follows easily provided $A > \dfrac{1}{5}$. Denote by $t(x)$ the increasing function

$$t(x) = \int_0^x \frac{dx}{\sqrt{-z(x)}}, \quad x \in (-x_0, x_0),$$

and let $x(t)$ be its inverse function defined on the open interval $(-a,a)$, where $a = \lim_{x\to x_0} t(x)$. Put $f(t) = A\sqrt{-z((x(t))}$, $g(t) = \sqrt{1-x^2(t)}$, $t \in (-a,a)$. Then f and g are solutions of (3) satisfying the boundary conditions (4). Observe that the Page metric is obtained when $A = 1$. Moreover, computing the scalar curvature from (2), we see that it is constant iff $A = 1$.

Remark 1.— The metrics defined by (1) belong to the larger class of the so-called *diagonal Bianchi IX metrics* which have $SU(2)$-symmetries instead of $U(2)$ ones. These metrics can be written in the form

$$h = w_1w_2w_3dt^2 + \frac{w_2w_3}{w_1}\sigma_1^2 + \frac{w_1w_3}{w_2}\sigma_2^2 + \frac{w_1w_2}{w_3}\sigma_3^2,$$

where w_i, $i = 1,2,3$ are positive functions of t. Suppose that such a metric h is not Einstein, has no $U(2)$-symmetries, and admits a compatible complex structure J such that the traceless Ricci tensor of h is J-anti-invariant. Then it is easily seen that J itself must be $SU(2)$-invariant, and the existence of such a complex structure is then equivalent to the existence of a solution to a system of three ODE of second order for the functions w_i, $i = 1,2,3$. Taking appropriate local solutions of this system we see that there are germs of Hermitian metrics with J-anti-invariant traceless Ricci tensor whose self-dual Weyl tensor is non-degenerate. In contrast, the self-dual Weyl tensor of any Hermitian surface of J-invariant Ricci tensor is degenerate according to the Riemannian version of the Robinson-Shild theorem [2]. □

2.2. $*$-Einstein metrics on products of Riemann surfaces.

Suppose that (M,g,J) is a Kähler surface which is the product of two compact oriented Riemann surfaces (X,g_X) and (Y,g_Y). If the corresponding Gauss curvatures s_X and s_Y are both constant, an argument from [1, pp. 3059] shows that the conformal class of g contains a $*$-Einstein metric iff g itself is Einstein (*i.e.* iff $s_X = s_Y$).

In this subsection we consider the general case when g_X and g_Y are arbitrary metrics on X and Y, respectively. Let $g_X = e^{\psi}\bar{g}_X$, where $\bar{g}_X$ is a metric of constant Gauss curvature, and ψ is a (non-constant) smooth function on X. We will show that for $M = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $M = X \times Y$, where X and Y are Riemann surfaces of genus $g(X) \geqslant 2$, $g(Y) \geqslant 1$, there always exists a conformal metric $h = e^{-2\phi}g$ on (M, J), which is $*$-Einstein, but non-Einstein, provided that g_Y is taken to be of constant Gauss curvature. Indeed, if ϕ depends on X only, then the $*$-Einstein equation for the metric h reads as (see (ii))

$$\Delta\psi + \mu = s_Y e^{\psi}, \tag{6}$$

where Δ is the Laplacian of $\bar{g}_X$, μ is the function

$$\mu = -2\Delta\phi + 2\bar{g}_X(d\phi, d\phi) + \bar{s}_X,$$

and $\bar{s}_X$ denotes the constant Gauss curvature of $\bar{g}_X$. It follows from (6) that s_Y must be a constant, say α. Set

$$\mathrm{Vol}(\bar{g}_X)\mu_0 = \int_X \mu dV_{\bar{g}_X} = 4\pi\chi(X) + 2\int_X \bar{g}_X(d\phi, d\phi) dV_{\bar{g}_X},$$

where $\chi(X)$ is the Euler characteristic of X. Substituting $\eta = \psi + \gamma$, with γ determined by the conditions $\Delta\gamma = \mu - \mu_0$ and $\int_X \gamma dV_{\bar{g}_X} = 0$, the equation (6) takes the form

$$e^{-\eta}(\Delta\eta + \mu_0) = \alpha e^{-\gamma}. \tag{7}$$

So, any function ϕ on X such that (7) has a smooth solution η determines a Kähler metric g on $X \times Y$ with $h = e^{-2\phi}g$ being $*$-Einstein. The equation (7) presents fairly in the literature in connection with the problem of existence of Riemannian metrics with prescribed Gauss curvature, see for example [4, ch. 5] and the references included there. In particular, it is known that in the cases $\mu_0 < 0, \alpha < 0$ and $\mu_0 = \alpha = 0$ the equation (7) has a smooth solution on any compact Riemann surface (see [4, 5.9]), while if $\mu_0 \geqslant 0, \alpha > 0$ it may not admit any solution [17]. Notice that if $\chi(X) < 0$, then the condition $\mu_0 \leqslant 0$ can be always satisfied by rescaling ϕ if necessary. Thus, for any such a function ϕ (in the case $\chi(X) < 0$ and $\chi(Y) \leqslant 0$) we find a $*$-Einstein (non-Einstein) metric on $M = X \times Y$. The equation (7) can be also solved for $X = S^2$, $Y = S^2$ by taking ϕ to be invariant under the antipodal map of S^2 and then considering (7) on $\mathbb{RP}^2$ (see [4, Theorem 5.10]).

3. Second Bianchi identity for $*$-Einstein Hermitian surfaces.

Let (M,h) be a 4-dimensional oriented Riemannian manifold. We denote by $\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$ the bundle of 2-forms on M, where $\Lambda^+ M$, resp. $\Lambda^- M$, is the bundle of self-dual, resp. anti-self-dual, 2-forms, *i.e.*, the eigen-sub-bundle with respect to the eigenvalue $+1$, resp. -1, of the Hodge operator $*$ acting as an involution on $\Lambda^2 M$. We will freely identify vectors and covectors via the metric h and, accordingly, a 2-form ϕ with the corresponding skew-symmetric endomorphism of the tangent bundle TM by putting $h(\phi(X), Y) = \phi(X,Y)$ for any vector fields X, Y.

Considering the Riemannian curvature tensor R as a symmetric endomorphism of $\Lambda^2 M$ we have the following $SO(4)$-splitting:

$$R = \frac{s}{12}\mathrm{Id} + \widetilde{\mathrm{Ric}}_0 + W^+ + W^-,$$

where s is the scalar curvature, $\widetilde{\mathrm{Ric}}_0$ is the the Kulkarni-Nomizu extension of the traceless Ricci tensor Ric_0 to an endomorphism of $\Lambda^2 M$ anti-commuting with $*$, and $W^{\pm} = \frac{1}{2}(W \pm * \circ W)$ are respectively the self-dual and anti-self-dual parts of the Weyl tensor W. The self-dual Weyl tensor W^+ is viewed as a section of the bundle $\mathrm{Sym}_0(\Lambda^+ M)$ of symmetric, traceless endomorphisms of $\Lambda^+ M$ (also considered as a sub-bundle of the tensor product $\Lambda^+ M \otimes \Lambda^+ M$). Then, the codifferential δW^+ of W^+ is a section of the rank 8 real vector bundle $\mathcal{V} = \mathrm{Ker}\,(\mathrm{tr} : \Lambda^1 M \otimes \Lambda^+ M \mapsto \Lambda^1 M)$, where tr is defined by $\mathrm{tr}(\alpha \otimes \phi) = \phi(\alpha)$ on decomposed elements.

Let C be the *Cotton-York tensor* of (M,h), defined by

$$C_{X,Y,Z} = \frac{1}{2}[D_Z\Big(\frac{s}{12}h + \mathrm{Ric}_0\Big)(Y,X) - D_Y\Big(\frac{s}{12}h + \mathrm{Ric}_0\Big)(Z,X)].$$

(Here and henceforth D denotes the Levi-Civita connection of h). Then the *second Bianchi identity* reads as $C = \delta W$, where δW is the codifferential of W. In particular, we have

$$C^+ = \delta W^+, \tag{8}$$

where C^+ denotes the self-dual part of C.

Let (M, h, J) be a Hermitian surface, which means a 4-dimensional, oriented Riemannian manifold (M, h) endowed with a compatible complex structure J — *i.e.*, J is *h-orthogonal* $(h(JX, JY) = h(X, Y))$ and *positive* (the orientation induced by J coincides with the chosen orientation of M). We denote by F the corresponding *Kähler form* defined by $F(X, Y) = h(JX, Y)$.

The action of J extends to the cotangent bundle T^*M by putting $(J\alpha)(X) = -\alpha(JX)$, so as to be compatible with the Riemannian duality between TM and T^*M. This action further extends to an *involution* on $\Lambda^2 M$ by putting $(J\phi)(X, Y) = \phi(JX, JY)$, which in turn gives rise to the following orthogonal splitting of $\Lambda^+ M$:

$$\Lambda^+ M = \mathbb{R}.F \oplus \Lambda_0^+ M,$$

where $\Lambda_0^+ M$ denotes the bundle of J-anti-invariant real 2-forms.

We denote by

- θ, the *Lee form* of (h, J) defined by $dF = \theta \wedge F$ or, equivalently, $\theta = J\delta F$;
- $\Phi = (d\theta)_+$, the self-dual part of $d\theta$; it is easily checked that the inner product of $d\theta$ and F vanishes identically, so that Φ is actually a section of $\Lambda_0^+ M$;
- $\Psi = -J \circ (d\theta)_+$, again a section of $\Lambda_0^+ M$;
- κ, the *conformal scalar curvature*, defined by $\kappa = 3h(W^+(F), F)$; it is well known that κ is the scalar curvature with respect to h of the canonical *Weyl structure* associated to the Hermitian structure (h, J), see [32], [12]. The conformal scalar curvature is conformally covariant of weight -2, and it is related to the Riemannian scalar curvature s by (see *e.g.* [12])

$$\kappa = s - \frac{3}{2}(2\delta\theta + |\theta|^2); \tag{9}$$

- s^*, the $*$-*scalar curvature*, defined by $s^* = 2h(R(F), F)$; it is easily seen that $s^* = \frac{1}{3}(2\kappa + s)$ and hence the equality (9) can be rewritten as

$$s - s^* = 2\delta\theta + |\theta|^2. \tag{9$'$}$$

The self-dual Weyl tensor W^+ splits under the action of unitary group $U(2)$ (induced by J) into two pieces W_1^+ and W_2^+, defined as follows [29], [2, Lemma 1]:

$$
\begin{aligned}
W_1^+ &= \frac{\kappa}{8} F \otimes F - \frac{\kappa}{12} \mathrm{Id}; \\
W_2^+ &= -\frac{1}{4}(\Psi \otimes F + F \otimes \Psi).
\end{aligned}
\tag{10}
$$

Furthermore, we have that the following conditions are equivalent [2, Lemma 2]:

(i) $W_2^+ \equiv 0$;

(ii) $\Phi = (d\theta)_+ = 0$;

(iii) the spectrum of W^+ is degenerate;

(iv) F is an eigenform of W^+.

If M is compact, each of the above conditions is equivalent to $d\theta = 0$, *i.e.*, h being locally conformally Kähler metric, globally conformally Kähler if, in addition, the first Betti number $b_1(M)$ is even [30].

The vector bundle $\mathcal{V}$ splits as $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ [2], where:

$\mathcal{V}^+$ is identified with the (real) cotangent bundle T^*M by

$$
\begin{aligned}
&\alpha \in T^*M \mapsto A = J\alpha \otimes F - \frac{1}{2}\sum_{i=1}^{4} e_i \otimes (\alpha \wedge e_i - J\alpha \wedge Je_i), \\
&A \in \mathcal{V}^+ \mapsto \alpha = -\frac{1}{2} J < A, F >,
\end{aligned}
\tag{11}
$$

where $< A, F >$ denotes the 1-form defined by $X \mapsto (A_X, F)$;

$\mathcal{V}^-$ is identified with the real rank 4 vector bundle underlying the complex rank 2 vector bundle $\Lambda^{1,0}M \otimes K_M$.

We denote by $(\delta W^+)^+$, resp. $(\delta W^+)^-$, the component of δW^+ on $\mathcal{V}^+$, resp. $\mathcal{V}^-$. Then the corresponding 1-form α of $(\delta W^+)^+$ via (11) is given by [2]

$$
\alpha = -\frac{1}{8} J\Psi(\theta) + \frac{1}{4} J\delta\Psi - \frac{1}{8}\kappa\theta - \frac{1}{12} d\kappa. \tag{12}
$$

Moreover, according to [2, Theorem 1] we have that $(\delta W^+)^- \equiv 0$ iff $W_2^+ \equiv 0$.

The traceless Ricci tensor Ric_0 decomposes into the sum of two $U(2)$-irreducible components, its J-invariant and J-anti-invariant part. It is known (cf. [32]) that the J-anti-invariant part $\mathrm{Ric}_0^{\mathrm{anti}}$ of the traceless Ricci tensor of a Hermitian surface (M, h, J) is given by

$$\begin{aligned} \mathrm{Ric}_0^{\mathrm{anti}}(X,Y) = &-\frac{1}{4}\Big((D_X\theta)(Y) + (D_Y\theta)(X) - (D_{JX}\theta)(JY) \\ &- (D_{JY}\theta)(JX) + \theta(X)\theta(Y) - \theta(JX)\theta(JY)\Big). \end{aligned} \tag{13}$$

A Hermitian surface (M, h, J) is called $*$-Einstein if its Kähler form is an eigenform of the curvature operator. One can easily see (cf. [15, Lemma 3.2] or [1, Lemma 4.2]) that the $*$-Einstein condition can be expressed in terms of the $U(2)$-decomposition of the curvature operator as follows:

LEMMA 1. — *A Hermitian surface is $*$-Einstein if and only if its traceless Ricci tensor is J-anti-invariant and the spectrum of the self-dual Weyl tensor is degenerate. In particular, any* compact *$*$-Einstein Hermitian surface is locally conformally Kähler, globally conformally Kähler iff the first Betti number is even.*

Suppose from now on that (M, h, J) is a $*$-Einstein Hermitian surface. As a consequence of Lemma 1 we have $(\delta W^+)^- = 0$ (cf. [2]), and then the "half" second Bianchi identity (8) simply reduces to

$$\alpha = -\frac{1}{2}J < C^+, F > . \tag{14}$$

Using the fact that Ric_0 is J-anti-invariant, the Ricci identity $\delta(\mathrm{Ric}_0) = -\frac{ds}{4}$, and the integrability condition $D_{JX}J = J(D_XJ)$ for J, we compute

$$\begin{aligned} < C, F >_X &= -\frac{1}{2}\sum_{i=1}^{4} D_{E_i}(\frac{s}{12}h + \mathrm{Ric}_0)(JE_i, X) \\ &= \frac{1}{24}ds(JX) + \frac{1}{2}(\delta\mathrm{Ric}_0)(JX) \\ &+ \frac{1}{2}\sum_{i=1}^{4}\{\mathrm{Ric}_0((D_{E_i}J)(E_i), X) - \mathrm{Ric}_0(E_i, (D_{E_i}J)(X))\} \\ &= \frac{1}{24}ds(JX) - \frac{1}{8}ds(JX) + \frac{1}{2}\mathrm{Ric}_0(\theta, JX) \\ &= -\frac{1}{12}ds(JX) + \frac{1}{2}\mathrm{Ric}_0(\theta, JX), \end{aligned}$$

which, together with (12) and (14), eventually gives the following

LEMMA 2. — *For any $*$-Einstein Hermitian surface the following identity holds:*

$$ds^* + \kappa\theta = 2\mathrm{Ric}_0(\theta). \tag{15}$$

Furthermore, we compute

$$\begin{aligned}\delta(\mathrm{Ric}_0(\theta)) &= -\sum_{i=1}^{4}((D_{E_i}\mathrm{Ric}_0)(E_i,\theta) - \mathrm{Ric}_0(D_{E_i}\theta, E_i))\\ &= (\delta\mathrm{Ric}_0)(\theta) - h(\mathrm{Ric}_0, D\theta).\end{aligned}$$

Now, by the Ricci identity $\delta(\mathrm{Ric}_0) = -\frac{ds}{4}$, the equality (13), and Lemma 2, we infer

$$\begin{aligned}\delta(\mathrm{Ric}_0(\theta)) &= -\frac{ds(\theta)}{4} + \frac{1}{2}\mathrm{Ric}_0(\theta,\theta) + |\mathrm{Ric}_0|^2\\ &= \frac{1}{4}d(s^* - s)(\theta) + \frac{\kappa}{4}|\theta|^2 + |\mathrm{Ric}_0|^2.\end{aligned}$$

Taking the codifferential to both sides in (15) we finally reach the following expression for the square-norm of Ric_0:

LEMMA 3. — *For any $*$-Einstein Hermitian surface the square-norm of* Ric_0 *is given by*

$$4|\mathrm{Ric}_0|^2 = 2\Delta s^* + d(2s - 4s^*)(\theta) + \kappa(2\delta\theta - |\theta|^2). \tag{16}$$

Remark 2. — One can obtain similar formulas under the only hypothesis that the traceless Ricci tensor of (M,h,J) is J-anti-invariant. In this case we get

$$ds^* + \kappa\theta + \Phi(\theta) - 2\delta\Phi = 2\mathrm{Ric}_0(\theta); \tag{15$'$}$$

and

$$4|\mathrm{Ric}_0|^2 = 2|\Phi|^2 + 2\Delta s^* + d(2s + 4s^*)(\theta) + \kappa(2\delta\theta - |\theta|^2). \tag{16$'$}$$

□

If M is compact, we denote by c_1^2 the first Chern number of (M,J). Then we have

COROLLARY 1. — *Let (M, h, J) be a compact $*$-Einstein Hermitian surface. Then the following integral formulas hold*:

$$4\int_M |\mathrm{Ric}_0|^2 = \int_M \frac{(s-s^*)^2}{2} - s^*|\theta|^2 dV, \tag{17}$$

$$c_1^2 = \frac{1}{32\pi^2}\int_M (s^*)^2 + s^*|\theta|^2 dV. \tag{18}$$

Proof. — The first formula is obtained by integrating (16) and using (9)′. To obtain (18) we apply the well known Chern-Weil formula

$$c_1^2 = 2\chi + 3\sigma = \frac{1}{4\pi^2}\int_M 2|W^+|^2 + \frac{s^2}{24} - \frac{1}{2}|\mathrm{Ric}_0|^2 dV, \tag{19}$$

where χ and σ are the Euler characteristic and the signature of M. The square-norm of W^+ of a Hermitian surface can be easily computed by (10) (see also [9]): $16|W^+|^2 = \dfrac{2\kappa^2}{3} + 4|\Phi|^2$. Substituting (17) and the latter expression for $|W^+|^2$ in (19), we get (18). □

4. $*$-Einstein generalized Hopf surfaces.

In this section we consider compact (non-Kähler) Hermitian surfaces with odd first Betti number whose Lee form θ is parallel with respect to the Levi-Civita connection D. These are usually called *Generalized Hopf surfaces*; examples are the conformally flat Hopf surfaces with the Vaisman metric [9], [23], [31]. More generally, any principle flat S^1-bundle over a Sasakian 3-manifold admits a canonical structure of generalized Hopf surface [31]. The complete classification of the compact Generalized Hopf surfaces has been recently obtained in [6].

Our aim here is to show that the classification of $*$-Einstein generalized Hopf surfaces is in fact equivalent to that of the conformally flat ones. We begin with the following

LEMMA 4. — *A generalized Hopf surface (M, g, J) is $*$-Einstein if and only if its $*$-scalar curvature identically vanishes.*

Proof. — It follows from (13) that for every generalized Hopf surface the J-anti-invariant part of the Ricci tensor Ric is given by

$$\mathrm{Ric}(X,Y) - \mathrm{Ric}(JX,JY) = -\frac{1}{2}(\theta(X)\theta(Y) - \theta(JX)\theta(JY)). \tag{20}$$

Since θ is parallel, $\mathrm{Ric}(\theta) = 0$, and it follows from (20) that $\mathrm{Ric}(J\theta) = \frac{|\theta|^2}{2} J\theta$ — *i.e.*, two of the eigenvalues of the Ricci tensor are equal to 0 and $\frac{|\theta|^2}{2} = \frac{s - s^*}{2}$ (see $(9)'$). The identity (20) also shows that for any vector field X which is orthogonal to $\{\mathrm{span}(\theta, J\theta)\}$ we have $\mathrm{Ric}(JX) = J \circ \mathrm{Ric}(X)$; it thus follows that the other two eigenvalues of Ric coincide being equal to $\frac{s + s^*}{4}$. As the metric g is locally conformally Kähler it is $*$-Einstein iff the traceless Ricci tensor Ric_0 is J-anti-invariant (Lemma 1), *i.e.*, iff the spectrum of Ric_0 at any point is of the form $(a, -a, b, -b)$. The latter is clearly equivalent to $s^* \equiv 0$. □

Given a generalized Hopf surface (M, g, J) we associate a natural 1-parameter family of generalized Hopf metrics g_t as follows: For any real number $t < \frac{1}{2}$ we put

$$g_t = g - \frac{2t}{|\theta|^2}(\theta \otimes \theta + J\theta \otimes J\theta). \tag{21}$$

It is easily checked that the Levi-Civita connection D^t of g_t is given by

$$\begin{aligned} D^t_X Y = D_X Y + \frac{t}{|\theta|^2}[2\theta(JX)\theta(JY)\theta + (\theta(X)\theta(JY) \\ + \theta(JX)\theta(Y))J\theta] - t[\theta(JX)JY + \theta(JY)JX]. \end{aligned} \tag{22}$$

Then the Lee form θ^t of (g_t, J) is equal to $(1 - 2t)\theta$. Using (22) we obtain that θ^t is parallel with respect to D^t. By a direct computation we get for the Ricci tensor of g_t:

$$\mathrm{Ric}^t = \mathrm{Ric} + t(|\theta|^2 g - \theta \otimes \theta - J\theta \otimes J\theta) + 2t(t - 1)J\theta \otimes J\theta,$$

hence, the scalar and $*$-scalar curvatures s_t and s_t^* are given by

$$s_t = s + t|\theta|^2, \quad s_t^* = s^* + 3t|\theta|^2. \tag{23}$$

Now we are ready to prove the following

THEOREM 2. — *Let (M, g, J) be a generalized Hopf surface. Then g is $*$-Einstein metric if and only if the metric $g_{\frac{1}{8}}$ defined by (21) is conformally flat.*

Proof. — According to Lemma 4, g is $*$-Einstein iff $s^* = 0$, or equivalently, iff $s = |\theta|^2$ (see $(9)'$). By (23) we have that the conformal scalar curvature $\kappa_t = \frac{3s_t^* - s_t}{2}$ of (g_t, J) is equal to $\frac{1}{2}(8t - 1)|\theta|^2$; it vanishes for $t = \frac{1}{8}$. Since the metric $g_{\frac{1}{8}}$ is locally conformally Kähler, it follows from (10) that the self-dual Weyl tensor vanishes. Consider the (almost) complex structure $\bar{J}$ on M, defined on $\mathrm{span}(\theta, J\theta)$ to be equal to J and on the orthogonal complement of $\mathrm{span}(\theta, J\theta)$ to be equal to $-J$. It is easily seen that $(M, g, \bar{J})$ is a generalized Hopf surface and $\bar{J}$ is compatible with the inverse orientation of M. Moreover, the corresponding Lee form $\bar{\theta}$ is equal to θ, hence the $*$-scalar curvature of $(g, \bar{J})$ vanishes. Since the corresponding 1-parameter family of inverse oriented generalized Hopf structures is $(g_t, \bar{J})$, we conclude as above that the anti-self-dual Weyl tensor of $g_{\frac{1}{8}}$ vanishes as well, *i.e.*, $g_{\frac{1}{8}}$ is conformally flat.

Conversely, starting from a conformally flat generalized Hopf surface (M, g, J), we have by (10) that the conformal curvature κ vanishes, *i.e.*, $s = \frac{3}{2}|\theta|^2$ according to (9). Then it follows from (9)$'$, (23) and Lemma 4 that the metric $g^* = g_{-\frac{1}{6}}$ is $*$-Einstein. □

Remark 3. — The *Hermitian scalar curvature* u of a Hermitian surface (M, h, J) is by definition the trace of the Ricci form of the Hermitian connection of (h, J). It is known (see [12]) that u is related to the scalar and $*$-scalar curvatures of (h, J) by

$$u = \frac{s + s^*}{2} + |\theta|^2.$$

By (23) we see that the metrics g_t have the same Hermitian scalar curvature. Hence, the same reasoning as in the proof of Theorem 2 shows that a generalized Hopf surface (M, g, J) has constant positive Hermitian scalar curvature iff the family $g_t, t < \frac{1}{2}$, contains a conformally flat metric. In particular, any generalized Hopf metric with constant *non-negative* scalar curvature has the form $g_t, -\frac{3}{2} \leqslant t < \frac{1}{2}$, where g is a conformally flat generalized Hopf metric. □

5. Proof of Theorem 1.

Let (M, h, J) be a compact $*$-Einstein Hermitian surface of constant scalar and $*$-scalar curvatures s and s^*. By Lemma 2 we obtain $\mathrm{Ric}_0(\theta) = \frac{\kappa}{2}\theta$. Since Ric_0 is J-anti-invariant, we get $\mathrm{Ric}_0(J\theta) = -\frac{\kappa}{2}J\theta$. Thus, at any point where θ does not vanish, we have

$$
\begin{aligned}
4|\mathrm{Ric}_0|^2 &\geqslant \frac{4}{|\theta|^4}(\mathrm{Ric}_0(\theta,\theta)^2+\mathrm{Ric}_0(J\theta,J\theta)^2)\\
&= \frac{1}{2}(s-3s^*)^2, \qquad (24)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
4|\mathrm{Ric}_0|^2 &\geqslant \frac{1}{2}(s-s^*)(s-3s^*)+s^*(s^*-s)+2(s^*)^2\\
&\geqslant \frac{1}{2}(s-s^*)(s-3s^*)+s^*(s^*-s).
\end{aligned}
$$

On the other hand, if θ vanishes in a neighbourhood of a point $x \in M$, then according to (9)′ and (16), we have at x:

$$4|\mathrm{Ric}_0|^2 = \frac{1}{2}(s-s^*)(s-3s^*)+s^*(s^*-s)=0.$$

It thus follows that (25) holds everywhere on M. Integrating (25) over M we get

$$
\begin{aligned}
\int_M 4|\mathrm{Ric}_0|^2 dV &\geqslant \int_M \frac{1}{2}(s-s^*)(s-3s^*)+s^*(s^*-s)dV\\
&= \int_M \frac{1}{2}(s-s^*)(s-3s^*)-s^*|\theta|^2 dV \qquad (26)
\end{aligned}
$$

(here we made use of (9)′ and the fact that s^* is constant). Notice that equality in (26) (resp. in (24)) is achieved iff $s^*=0$, and (at any point where θ is non-zero) the spectrum of Ric_0 is equal to $\left(-\frac{s}{4},\frac{s}{4},0,0\right)$. Now, it follows from Corollary 1,(17) that

$$\int_M s^*|\theta|^2 dV \geqslant 0,$$

which shows that either $\theta=0$, — *i.e.*, (h,J) is Kähler —, or else $s^* \geqslant 0$.

Suppose (h,J) is non-Kähler. If $s^*>0$, then by Corollary 1, (18) we get $c_1^2>0$. According to the classification of compact complex surfaces (cf. [5]) the latter situation occurs only for complex surfaces of Kähler type, *i.e.*, $b_1(M)$ is even. Thus h is conformal to a Kähler metric $g=f^2h$ (see Lemma 1). Since for a Kähler surface the scalar and the $*$-scalar curvatures coincide, we obtain that the conformal factor f satisfies

$$4\Delta f+(s-s^*)f=0,$$

where Δ denotes the Laplacian of g (cf. [8, 1.161]). It follows from the maximum principle that $s - s^* = 0$, which contradicts $(9)'$.

Consider now the case $s^* = 0$. It follows from $(9)'$ that s is a positive constant. In particular, θ does not vanish on an open dense subset of M. As we have already mentioned, the equality in (25) implies that the Ricci tensor Ric has eigenvalues $\left(0, \frac{s}{2}, \frac{s}{4}, \frac{s}{4}\right)$ at any point where θ does not vanish, hence the Ricci tensor is non-negatively defined everywhere on M. Since the scalar curvature of (M, h, J) is positive, the Hermitian scalar curvature is also positive (cf. [1, Lemma 3.3]), hence all plurigenera of (M, J) vanish by [11, Plurigenera Theorem]. The same reasoning as in the case $s^* > 0$ shows that M is not of Kähler type. Thus, (M, J) is in class VII of the Kodaira-Enriques classification; in particular $b_1(M) = 1$ (cf. [5]). Denote by ξ a non-zero harmonic 1-form with respect to h. It follows from the Bochner formula that ξ is parallel and $\mathrm{Ric}(\xi) = 0$ at any point of M. The latter condition implies that $\theta = f\xi$, where $f = h\left(\theta, \frac{\xi}{|\xi|^2}\right)$. It also follows that $\mathrm{Ric}(J\xi) = \frac{s}{2}J\xi$, and $\mathrm{Ric}(X) = \frac{s}{4}X$, $\forall X \in \{\mathrm{span}(\xi, J\xi)\}^{\perp}$ holds on the dense subset of M where $\theta \neq 0$, hence, everywhere on M. Moreover, we have that $J\xi$ is a Killing vector field of constant length. Indeed, since ξ is parallel and $\theta = f\xi$, we get

$$f(D_X(J\xi)(Y) + D_Y(J\xi)(X)) = \theta((D_XJ)(Y) + (D_YJ)(X)) = 0$$

(for the latter equality we made use of the well known formula $D_XF = \frac{1}{2}(X \wedge J\theta + JX \wedge \theta)$, cf. [18].) The universal cover $\tilde{M}$ of (M, h) then splits as $\tilde{M} = \mathbb{R} \times N$, where N is a compact Riemannian 3-manifold whose Ricci tensor has eigenvalues $\left(\frac{s}{2}, \frac{s}{4}, \frac{s}{4}\right)$. Rescaling the metric h and the vector field ξ if necessary, we may assume that $s = 4$ and $|J\xi| = 1$, *i.e.*, $J\xi$ is a unit Killing vector field on N such that $R^N(J\xi, X) = J\xi \wedge X$, for any vector field X on N. Hence $J\xi$ determines a *Sasakian structure* on N, since N is 3-dimensional. It follows from [31] that the Lee form of (h, J) is parallel, *i.e.*, (M, h, J) is a generalized Hopf surface. Now applying Theorem 2 and the classification of compact conformally flat Hermitian surfaces [9], [23] we complete the proof of Theorem 1.

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