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# EIGENVALUE ASYMPTOTICS FOR THE PAULI OPERATOR IN STRONG NONCONSTANT MAGNETIC FIELDS

by Georgi D. RAIKOV

## 1. Introduction.

Let $\Pi(\mu) = (\Pi_1(\mu), \ldots, \Pi_m(\mu)) := -i\nabla - \mu A$, $m = 2, 3$, be the magnetic momentum operator, $A \in C^2_{\rm loc}(\mathbb{R}^m; \mathbb{R}^m)$ being the magnetic potential, and $\mu > 0$ – the magnetic-field coupling constant. The operators $\Pi_j(\mu)$, $j = 1, \ldots, m$, are defined originally on $C_0^\infty(\mathbb{R}^m)$ and then are closed in $L^2(\mathbb{R}^m)$. Introduce the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the unperturbed Pauli operator

$$\mathbf{H}_0(\mu) = \left( \sum_{j=1}^m \sigma_j \Pi_j(\mu) \right)^2$$

defined originally on $C_0^\infty(\mathbb{R}^m; \mathbb{C}^2)$ and then closed in $L^2(\mathbb{R}^m; \mathbb{C}^2)$, $m = 2, 3$. In what follows we shall denote the two-dimensional Pauli operator by $h_0(\mu)$, and the three-dimensional one – by $H_0(\mu)$.

Let at first $m = 2$. In this case the magnetic field $b$ is defined as

$$b(X) := \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}, \quad X := (x, y) \in \mathbb{R}^2.$$

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Throughout the paper we assume that the estimates

$$c_1 \leq b(X) \leq c_2, \quad |\nabla b(X)| \leq c_3, \quad X \in \mathbb{R}^2, \tag{1.1}$$

hold for some positive constants $c_1$, $c_2$, and $c_3$.

Introduce the operators

$$a(\mu) := \Pi_1(\mu) - i\Pi_2(\mu), \; a(\mu)^* := \Pi_1(\mu) + i\Pi_2(\mu).$$

Then we have

$$h_0(\mu) = \begin{pmatrix} h_0^-(\mu) & 0 \\ 0 & h_0^+(\mu) \end{pmatrix}$$

where

$$h_0^-(\mu) := a(\mu)a(\mu)^*, \; h_0^+(\mu) := a(\mu)^*a(\mu). \tag{1.2}$$

Note that $h_0^\pm \geq 0$. Moreover, $\operatorname{Ker} h_0^-(\mu) = \operatorname{Ker} a(\mu)^*$. On the other hand, it follows from $0 < c_1 \leq b(X)$ (see (1.1)), and a general result of I. Shigekawa (see [Sh], Lemma 3.3), that $\dim \operatorname{Ker} a(\mu)^* = \infty$. Hence,

$$0 \in \sigma_{\text{ess}}(h_0^-(\mu)) \subseteq \sigma_{\text{ess}}(h_0(\mu)). \tag{1.3}$$

Further, the commutation relation $[\Pi_1(\mu), \Pi_2(\mu)] = i\mu b$ implies

$$h_0^\pm(\mu) = \Pi_1(\mu)^2 + \Pi_2(\mu)^2 \pm \mu b. \tag{1.4}$$

Therefore, $h_0^+(\mu) = h_0^-(\mu) + 2\mu b \geq 2\mu c_1 \mathrm{Id}$, which entails

$$\sigma(h_0^+(\mu)) \subseteq [2\mu c_1, \infty). \tag{1.5}$$

An elementary supersymmetric argument yields

$$\sigma(h_0^+(\mu)) = \sigma(h_0^-(\mu)) \setminus \{0\} = \sigma(h_0(\mu)) \setminus \{0\}$$

which together with (1.5) implies

$$\sigma(h_0(\mu)) \setminus \{0\} \subseteq [2\mu c_1, \infty). \tag{1.6}$$

Let now $m = 3$. In this case we define the magnetic field as

$$B(\mathbf{X}) := \operatorname{curl} A(\mathbf{X}), \quad \mathbf{X} = (X, z) = (x, y, z) \in \mathbb{R}^3.$$

Throughout the paper we assume that $B$ has a constant direction, i.e.

$$B = (0, 0, b). \tag{1.7}$$

Since $\operatorname{div} B = 0$, $b$ is independent of $z$. Performing, if necessary, a gauge transform, we find that without any loss of generality we may assume that $A_j$, $j = 1, 2$, are independent of $z$, and $A_3 = 0$. Moreover, we again suppose that (1.1) is valid.

Introduce the operators

$$\tilde{H}_0^{\pm}(\mu) := \int_{\mathbb{R}} \oplus h_0^{\pm}(\mu)\, dz, \tag{1.8}$$

$$H_0^{\pm}(\mu) := \tilde{H}_0^{\pm}(\mu) + \Pi_3^2 = \sum_{j=1}^{3} \Pi_j^2 \pm \mu b,$$

acting in $L^2(\mathbb{R}^3)$ (see (1.2) and (1.4)). Then we have

$$H_0(\mu) = \begin{pmatrix} H_0^-(\mu) & 0 \\ 0 & H_0^+(\mu) \end{pmatrix}.$$

Note that $H_0^{\pm}(\mu) \geq 0$, the operators $\tilde{H}_0^-(\mu)$ and $\Pi_3 = -i\frac{\partial}{\partial z}$ commute, $\sigma(\Pi_3^2) = [0,\infty)$, and (1.3) entails $\inf \sigma(\tilde{H}_0^-(\mu)) = 0$. Thus we get

$$\sigma(H_0(\mu)) = \sigma_{\text{ess}}(H_0(\mu)) = [0, +\infty). \tag{1.9}$$

Further, let $V : \mathbb{R}^m \to \mathbb{R}$, $m = 2,3$, be the electric potential. We shall say that $V$ is in the class $\mathcal{L}_p$, $p \geq 1$, if and only if for each $\varepsilon > 0$ we can write $V = V_1 + V_2$ with $V_1 \in L^p(\mathbb{R}^m)$, and $\sup_{\mathbf{x}\in\mathbb{R}^m} |V_2(\mathbf{x})| \leq \varepsilon$. Introduce the Heaviside function

$$\theta(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{otherwise} \end{cases}$$

and set $\nu(s) := -\int_{\mathbb{R}^m} \theta(|V(\mathbf{x})| - s)\, d\mathbf{x}$. Then $V \in \mathcal{L}_p$ is equivalent to $\int_\varepsilon^\infty s^p\, d\nu(s) < \infty$, for each $\varepsilon > 0$.

Let $V \in \mathcal{L}_p$ with $p > 1$ if $m = 2$, and $p = 3/2$ if $m = 3$. Suppose that (1.1) and (1.7) (if $m = 3$) hold. Then the operator $|V|^{1/2}(-\Delta + 1)^{-1/2}$ and hence, by the diamagnetic inequality, the operator

$$|V|^{1/2}\Big(\sum_{j=1}^{m} \Pi_j(\mu)^2 + 1\Big)^{-1/2}$$

is compact. Since the magnetic field is bounded, we easily find that the operator $|V|^{1/2}(\mathbf{H}_0(\mu)+1)^{-1/2}$ is compact as well. Introduce the perturbed Pauli operator

$$\mathbf{H}(\mu) := \mathbf{H}_0(\mu) + VI_2 = \mathbf{H}_0(\mu) + V,$$

acting in $L^2(\mathbb{R}^m;\mathbb{C}^2)$, $m = 2,3$. Here $I_2$ is the unit $2 \times 2$ matrix, and the sum should be understood in the sense of the quadratic forms. We shall denote the two-dimensional perturbed Pauli operator by $h(\mu)$, and the three-dimensional one – by $H(\mu)$. Since the operator $|V|^{1/2}(\mathbf{H}_0(\mu)+1)^{-1/2}$ is compact, the essential spectra of $\mathbf{H}(\mu)$ and $\mathbf{H}_0(\mu)$ coincide. In particular, if $m = 2$, then (1.3) and (1.6) imply

$$0 \in \sigma_{\text{ess}}(h(\mu)),\ \sigma_{\text{ess}}(h(\mu)) \setminus \{0\} \subseteq [2\mu c_1, \infty),$$

while if $m = 3$, then (1.9) entails

$$\sigma_{\text{ess}}(H(\mu)) = [0, \infty).$$

However, the perturbation of $\mathbf{H}_0(\mu)$ by $V$ may generate some discrete spectrum in a vicinity of the origin. The aim of the present article is the analysis of the asymptotic behaviour as $\mu \to \infty$ of the discrete eigenvalues of $\mathbf{H}(\mu)$ adjoining the origin.

## 2. Statement of the main results.

**2.1.** Let $T$ be a selfadjoint operator in a Hilbert space. Denote by $P_{\mathcal{I}}(T)$ its spectral projection corresponding to the interval $\mathcal{I} \subset \mathbb{R}$. Set

$$\mathcal{N}(\lambda_1, \lambda_2; T) := \operatorname{rank} P_{(\lambda_1, \lambda_2)}(T), \ \lambda_1, \lambda_2 \in \mathbb{R}, \ \lambda_1 < \lambda_2,$$

$$N(\lambda; T) := \operatorname{rank} P_{(-\infty, \lambda)}(T), \ \lambda \in \mathbb{R},$$

$$n_{\pm}(s; T) := \operatorname{rank} P_{(s, +\infty)}(\pm T), s > 0.$$

If $T$ is a linear compact operator which is not necessarily selfadjoint, put

$$n_*(s; T) := \operatorname{rank} P_{(s^2, +\infty)}(T^*T), s > 0.$$

**2.2.** Let $m = 2$. Throughout the subsection we assume that (1.1) holds. Let $V \in \mathcal{L}_p$, $p > 1$. For $\lambda \neq 0$ set

$$\delta(\lambda) := \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \theta(\lambda - V(X)) b(X) dX & \text{if } \lambda < 0, \\ -\dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \theta(V(X) - \lambda) b(X) dX & \text{if } \lambda > 0. \end{cases} \tag{2.1}$$

Evidently, $\delta$ is a non-decreasing function on $(-\infty, 0)$ and $(0, \infty)$.

THEOREM 2.1. — *Let $m = 2$. Assume that (1.1) holds, and $V \in \mathcal{L}_p$, $p > 1$. Let $\lambda < 0$ be a continuity point of $\delta$. Then we have*

$$\lim_{\mu \to \infty} \mu^{-1} N(\lambda; h(\mu)) = \delta(\lambda). \tag{2.2}$$

THEOREM 2.2. — *Let $m = 2$. Assume that (1.1) holds, and $V \in \mathcal{L}_2$. Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $0 < \lambda_1 < \lambda_2$. Suppose that $\lambda_1$ and $\lambda_2$ are continuity points of $\delta$. Then we have*

$$\lim_{\mu \to \infty} \mu^{-1} \mathcal{N}(\lambda_1, \lambda_2; h(\mu)) = \delta(\lambda_2) - \delta(\lambda_1). \tag{2.3}$$

*Remark.* — Under the hypotheses of Theorems 2.1–2.2 the assumption that $\lambda \neq 0$ be a continuity point of $\delta$ is equivalent to $\text{vol}\left\{X \in \mathbb{R}^2 | V(X) = \lambda\right\} = 0$, where vol $\Omega$ denotes the Lebesgue measure of the set $\Omega \subset \mathbb{R}^2$.

We shall prove only Theorem 2.2 since the proof of Theorem 2.1 is quite similar and only simpler in comparison with that of Theorem 2.2 (see also the proof of Theorem 2.3 below).

**2.3.** Let $m = 3$. Throughout the subsection we assume that $V \in \mathcal{L}_{3/2}$, and (1.1) and (1.7) hold. Fix $X \in \mathbb{R}^2$ and set

$$\chi(X) = \chi_V(X) := -\frac{d^2}{dz^2} + V(X,.).$$

PROPOSITION 2.1. — *Let $V \in \mathcal{L}_{3/2}$. Then for almost every $X \in \mathbb{R}^2$ the operator $\chi(X)$ defined as a sum in sense of the quadratic forms is selfadjoint in $L^2(\mathbb{R})$. Moreover, for almost every $X \in \mathbb{R}^2$ we have*

$$\sigma_{\text{ess}}(\chi(X)) = [0, +\infty). \tag{2.4}$$

The proof of the proposition is contained in Section 5.

Let $\lambda < 0$. Introduce the magnetic integrated density of states

$$\mathcal{D}(\lambda) = \mathcal{D}_V(\lambda) := \frac{1}{2\pi} \int_{\mathbb{R}^2} N(\lambda; \chi_V(X)) b(X)\, dX.$$

PROPOSITION 2.2. — *Let $V \in \mathcal{L}_{3/2}$, and $\lambda < 0$. Then*

$$\mathcal{D}(\lambda) < \infty. \tag{2.5}$$

The proof of this proposition can also be found in Section 5.

THEOREM 2.3. — *Let $m = 3$. Assume that (1.1) and (1.7) hold, $V \in \mathcal{L}_{3/2}$. Let $\lambda < 0$ be a continuity point of $\mathcal{D}$. Then we have*

$$\lim_{\mu \to \infty} \mu^{-1} N(\lambda; H(\mu)) = \mathcal{D}(\lambda). \tag{2.6}$$

*Remark.* — The condition that $\lambda < 0$ be a continuity point of $\mathcal{D}$ is equivalent to

$$\text{vol}\left\{X \in \mathbb{R}^2 | \dim \text{Ker}\, (\chi(X) - \lambda) \geq 1\right\} = 0.$$

**2.4.** The present paper is closely related to the works [R1] and [R2] containing results on the asymptotic behaviour of the discrete spectrum for the Schrödinger, Pauli and Dirac operators in strong *constant* magnetic fields. In those articles the explicit spectral description of the unperturbed magnetic Hamiltonian played a crucial role at a certain stage of the proof. Since no such explicit description is known in the case of non-constant magnetic fields, significant modifications of the arguments of [R1]–[R2] were needed. Some of the main difficulties in this respect were overcome by using a result of L. Erdös on the strong-magnetic-field asymptotics of the diagonal values of the heat kernel associated with $h_0^-(\mu)$ (see below Lemma 3.1). Moreover, as in [R1]–[R2], we reduce the analysis of the eigenvalue asymptotics as $\mu \to \infty$ for $\mathbf{H}(\mu)$ to the study of the spectrum of certain Wiener-Hopf families of compact operators. However, since only constant magnetic were considered in [R1]–[R2], it sufficed to apply there relatively simple arguments close to the ones used in the pioneering work [KMSz] for the investigation of semiclassical spectral asymptotics for Wiener-Hopf operators. In the present paper the absence of an explicit spectral description of $\mathbf{H}_0(\mu)$ forced us to use somewhat different techniques similar to the commutator calculus developed in [W] for the study of the spectral asymptotics for operators of Toeplitz type (see also [Hö], Theorem 2.9.17, Lemma 2.9.18). It should be also noted that the present article is influenced by the recent papers [IT1]–[IT2]. In [IT1] the authors impose restrictions on the magnetic field quite similar to (1.1) and (1.7), and study the asymptotics as $\lambda \uparrow 0$ of $N(\lambda; \mathbf{H}(1))$ in both cases $m = 2, 3$, as well as the asymptotics as $\lambda \downarrow 0$ of $\mathcal{N}(\lambda, \lambda_0; h(1))$ with a fixed $\lambda_0 \in (0, 2c_1)$ in the case $m = 2$. The assumptions in [IT1] concerning $V$ are more restrictive than those of Theorems 2.1–2.3 which is natural and due to the different type of asymptotics considered. However, the results in [IT1] for the two-dimensional case are equivalent to

$$N(\lambda; h(1)) = \delta(\lambda)(1 + o(1)), \ \lambda \uparrow 0,$$

which resembles formally (2.2), and

$$\mathcal{N}(\lambda, \lambda_0; h(1)) = -\delta(\lambda)((1 + o(1)), \ \lambda \downarrow 0, \ \lambda_0 \in (0, 2c_1),$$

which is similar to (2.3), while the result in [IT1] concerning the three-dimensional case could be written as

$$N(\lambda; H(1)) = \mathcal{D}(\lambda)(1 + o(1)), \lambda \downarrow 0,$$

which recalls (2.6). In [IT2] the magnetic field is assumed to be locally strictly positive but decaying at infinity and the asymptotics as $\lambda \uparrow 0$ of $N(\lambda; \mathbf{H}(1))$ is investigated in the cases $m = 2, 3$. The results of the present paper will be possibly extended in a future work to the cases of magnetic fields which decay, or grow unboundedly at infinity.

The paper is organized as follows. Section 3 contains miscellaneous auxiliary results: in Subsection 3.1 we reveal some necessary facts concerning the heat kernel of the operator $h_0^-(\mu)$, in Subsection 3.2 we formulate a suitable version of the Kac-Murdock-Szegö theorem, and in Subsection 3.3 we recall the classical Birman-Schwinger principle and certain generalizations of its. In Section 4 we establish some preliminary estimates. In Section 5 we demonstrate Propositions 2.1–2.2. Section 6 is devoted to the asymptotics as $\mu \to \infty$ of the traces of the positive powers of certain operators of Toeplitz type which depend on the parameter $\mu$. The proof of Theorem 2.2 can be found in Section 7, while the proof of Theorem 2.3 is contained in Section 8.

## 3. Auxiliary results.

**3.1.** In this subsection we summarize several estimates of the kernel $\mathcal{K}_\mu(t; X, Y)$ of the operator $e^{-th_0^-(\mu)}$, $t > 0$.

LEMMA 3.1. — *Let $m = 2$. Assume that (1.1) holds. Then for every $t > 0$ the kernel $\mathcal{K}_\mu(t; X, Y)$ is locally uniformly continuous with respect to $(X, Y) \in \mathbb{R}^2 \times \mathbb{R}^2$.*

*Moreover, for every $t > 0$ and $(X, Y) \in \mathbb{R}^2 \times \mathbb{R}^2$ we have*

$$|\mathcal{K}_\mu(t; X, Y)| \le \frac{1}{4\pi t} e^{-\frac{|X-Y|^2}{4t} + c_2 \mu t}. \tag{3.1}$$

*Finally, for every compact $K \subset \mathbb{R}^2$ there exists a number $s_K > 0$ such that for each $s \ge s_K$ the limiting relation*

$$\lim_{\mu \to \infty} \mu^{-1} \mathcal{K}_\mu \left( \frac{\log \mu}{s\mu}; X, X \right) = \frac{1}{2\pi} b(X) \tag{3.2}$$

*holds uniformly with respect to $X \in K$.*

*Sketch of the proof.* — The continuity of $\mathcal{K}_\mu(t; X, Y)$ is proved in [E], Theorem 2.1. The estimate (3.1) follows immediately from the Feynman-Kac-Itô formula for the heat kernel of $h_0^-(\mu)$ (see e.g. [E], (65)–(66)). Finally, (3.2) is demonstrated in [E], Main Lemma 2.2. □

**3.2.** In this subsection we formulate a suitable version of the Kac-Murdock-Szegö theorem.

In the sequel we shall denote by $S_\infty$ the space of linear compact operators acting in a given Hilbert space, and by $S_p$, $p \in [1, \infty)$, – the

Schatten–von Neumann spaces of operators $T \in S_\infty$ for which the norm $\|T\|_p := (\mathrm{Tr}\ |T|^p)^{1/p}$ is finite.

LEMMA 3.2. — *Let $\{T(\mu)\}_{\mu>0}$ be a family of selfadjoint compact operators satisfying the estimate $\|T(\mu)\| \le t_0$ with $t_0 > 0$ independent of $\mu$. Assume that the function $\nu : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is non-decreasing on $(-\infty, 0)$ and $(0,\infty)$, non-negative on $(-\infty, 0)$, and non-positive on $(0,\infty)$. Let $\nu(t) = 0$ for $|t| > t_0$. Suppose that there exists a real number $p \ge 1$ such that the following three conditions are fulfilled:*

(i) *$T(\mu) \in S_p$ for each $\mu > 0$;*

(ii) *the quantity $\int_{\mathbb{R}\setminus\{0\}} |t|^p \, d\nu(t)$ is finite;*

(iii) *the limiting relations*

$$\lim_{\mu\to\infty} \mu^{-1}\ \mathrm{Tr}\ T(\mu)^l = \int_{\mathbb{R}\setminus\{0\}} t^l \, d\nu(t)$$

*hold for each integer $l \ge p$.*

*Let $t \ne 0$ be a continuity point of $\nu$. Then we have*

$$\lim_{\mu\to\infty} \mu^{-1}\ n_+(t; T(\mu)) = -\nu(t), \quad \text{if} \quad t > 0,$$

$$\lim_{\mu\to\infty} \mu^{-1}\ n_-(-t; T(\mu)) = \nu(t), \quad \text{if} \quad t < 0.$$

The proof of the lemma can be found in [R1], Subsection 3.1. In this article we shall use it only with $t < 0$.

**3.3.** This subsection contains a formulation of the classical Birman-Schwinger principle concerning the number of the eigenvalues of a selfadjoint operator situated below the bottom of its essential spectrum (see below Lemma 3.3), as well a generalization of this principle (see below Lemma 3.4) suitable in the case where the discrete spectrum lying in a gap of the essential one is investigated.

LEMMA 3.3. — *Let $\mathcal{H}_0 \ge 0$ and $\mathcal{V}$ be two selfadjoint operators in Hilbert space, such that $|\mathcal{V}|^{1/2}(\mathcal{H}_0+1)^{-1/2} \in S_\infty$. For $\lambda < 0$ set*

$$\mathcal{R}(\lambda; \mathcal{H}_0) := (\mathcal{H}_0 - \lambda)^{-1/2}, \tag{3.3}$$

$$\mathcal{T}(\lambda; \mathcal{H}_0, \mathcal{V}) := \mathcal{R}(\lambda; \mathcal{H}_0)\mathcal{V}\mathcal{R}(\lambda; \mathcal{H}_0). \tag{3.4}$$

*Then we have*

$$N(\lambda; \mathcal{H}_0 + \mathcal{V}) = n_-(1; \mathcal{T}(\lambda; \mathcal{H}_0, \mathcal{V}))$$

*where the sum $\mathcal{H}_0 + \mathcal{V}$ should be understood in the sense of the quadratic forms. Moreover,*

$$\dim \operatorname{Ker} (\mathcal{H}_0 + \mathcal{V} - \lambda) = \dim \operatorname{Ker} (\mathcal{T}(\lambda; \mathcal{H}_0, \mathcal{V}) + 1).$$

LEMMA 3.4 [R2], Lemma 4.1. — *Let $\mathcal{H}_0$ be a linear selfadjoint operator in Hilbert space, and $\lambda_1, \lambda_2$ be real numbers such that $\lambda_1 < \lambda_2$ and $[\lambda_1, \lambda_2] \subset \rho(\mathcal{H}_0)$. Set*

$$\tilde{\mathcal{R}}(\lambda_1, \lambda_2; \mathcal{H}_0) := ((\mathcal{H}_0 - \lambda_1)(\mathcal{H}_0 - \lambda_2))^{-1/2}, \tag{3.5}$$

$$\mathcal{G}(\lambda_1, \lambda_2; \mathcal{H}_0) := \left(\mathcal{H}_0 - \frac{1}{2}(\lambda_1 + \lambda_2)\right) \tilde{\mathcal{R}}(\lambda_1, \lambda_2; \mathcal{H}_0). \tag{3.6}$$

*Further, let $\mathcal{V}$ be a symmetric operator on $D(\mathcal{H}_0)$ such that $\mathcal{V}(\mathcal{H}_0 + i)^{-1} \in S_\infty$. Put*

$$\begin{aligned} \tilde{\mathcal{T}}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V}) :=& \tilde{\mathcal{R}}(\lambda_1, \lambda_2; \mathcal{H}_0)\mathcal{V}^2\tilde{\mathcal{R}}(\lambda_1, \lambda_2; \mathcal{H}_0) \\ &+ 2\mathrm{Re}\, \mathcal{G}(\lambda_1, \lambda_2; \mathcal{H}_0)\mathcal{V}\tilde{\mathcal{R}}(\lambda_1, \lambda_2; \mathcal{H}_0). \end{aligned} \tag{3.7}$$

*Then we have*

$$\mathcal{N}(\lambda_1, \lambda_2; \mathcal{H}_0 + \mathcal{V}) = n_-(1; \tilde{\mathcal{T}}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V}))$$

*where the sum $\mathcal{H}_0 + \mathcal{V}$ should be understood in the operator sense.*

## 4. Preliminary estimates.

**4.1.** Let $m = 2$. Denote by $p(\mu)$ the orthogonal projection on $\operatorname{Ker} h_0^-(\mu) = \operatorname{Ker} a^*$.

LEMMA 4.1. — *Let $W \in L^2(\mathbb{R}^2)$. Then $Wp(\mu) \in S_2$, and the estimate*

$$\|Wp(\mu)\|_2^2 \leq c_4\mu \int_{\mathbb{R}^2} |W(X)|^2\, dX \tag{4.1}$$

*holds with $c_4$ independent of $\mu$ and $W$.*

*Proof.* — For every $t > 0$ we have

$$\begin{aligned} \|Wp(\mu)\|_2^2 &= \|We^{-th_0^-(\mu)}p(\mu)\|_2^2 \leq \|We^{-th_0^-(\mu)}\|_2^2 \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |W(X)\mathcal{K}_\mu(t; X, Y)|^2 dXdY. \end{aligned} \tag{4.2}$$

Employing (3.1), we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |W(X)|^2 |\mathcal{K}_\mu(t;X,Y)|^2\, dXdY \leq \frac{e^{2c_2\mu t}}{16\pi^2 t^2}$$

$$\int_{\mathbb{R}^2} |W(X)|^2\, dX \int_{\mathbb{R}^2} e^{-\frac{|Y|^2}{2t}}\, dY = \frac{e^{2c_2\mu t}}{8\pi t}\int_{\mathbb{R}^2} |W(X)|^2\, dX. \tag{4.3}$$

Minimizing the function $f(t) := t^{-1}e^{2c_2\mu t}$ by choosing $t = 1/2c_2\mu$, and combining (4.2) with (4.3), we get

$$\|Wp(\mu)\|_2^2 \leq \frac{ec_2}{4\pi}\mu \int_{\mathbb{R}^2} |W(X)|^2\, dX, \tag{4.4}$$

which is equivalent to (4.1) with $c_4 = ec_2/4\pi$. □

Fix the real numbers $\lambda_1$ and $\lambda_2$ such that $\lambda_1 \leq \lambda_2$ and $\lambda_1\lambda_2 > 0$. Assume that $\mu$ is large enough. If $\lambda_1 < \lambda_2$ set $r^{\pm}_{\lambda_1,\lambda_2}(\mu) := \tilde{\mathcal{R}}(\lambda_1, \lambda_2; h_0^{\pm}(\mu))$ (see (3.5)); if $\lambda_1 = \lambda_2$, extend the definition by continuity.

COROLLARY 4.1. — *Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \leq \lambda_2$, $\lambda_1\lambda_2 > 0$, and $W \in L^2(\mathbb{R}^2)$. Then the estimates*

$$\|Wr^-_{\lambda_1,\lambda_2}(\mu)p(\mu)\|_2^2 \leq c_5\mu \int_{\mathbb{R}^2} |W|^2\, dX, \tag{4.5}$$

$$n_*(\varepsilon; Wr^-_{\lambda_1,\lambda_2}(\mu)p(\mu)) \leq c_5\varepsilon^{-2}\mu \int_{\mathbb{R}^2} |W|^2\, dX,\ \varepsilon > 0, \tag{4.6}$$

*hold for $\mu$ large enough with $c_5 = c_4/\sqrt{\lambda_1\lambda_2}$.*

*Proof.* — In order to check (4.5), it suffices to note that $r^-_{\lambda_1,\lambda_2}(\mu)p(\mu) = \frac{1}{\sqrt{\lambda_1\lambda_2}}p(\mu)$, and to apply Lemma 4.1. Estimate (4.6) follows from (4.5) and the general inequality

$$n_*(\varepsilon; T) \leq \varepsilon^{-p}\|T\|_p^p,\ T \in S_p,\ p \geq 1,\ \varepsilon > 0. \tag{4.7}$$

□

Set $q(\mu) := \mathrm{Id} - p(\mu)$.

LEMMA 4.2. — *Under the hypotheses of Corollary 4.1 the estimate*

$$n_*(\varepsilon; Wr^-_{\lambda_1,\lambda_2}(\mu)q(\mu)) \leq c_6\varepsilon^{-2}\mu^{-1} \int_{\mathbb{R}^2} |W|^2\, dX,\ \varepsilon > 0, \tag{4.8}$$

*holds for $\mu$ large enough with $c_6$ independent of $\varepsilon$, $\mu$, and $W$. In particular, if $\mu > c_6\varepsilon^{-2}\int_{\mathbb{R}^2} |W|^2\, dX$, we have*

$$n_*(\varepsilon; Wr^-_{\lambda_1,\lambda_2}(\mu)q(\mu)) = 0. \tag{4.9}$$

*Proof.* — Making use of the resolvent identity

$$(h_0^-(\mu)+1)^{-1}-(\Pi_1^2+\Pi_2^2+\mu)^{-1} = (\Pi_1^2+\Pi_2^2+\mu)^{-1}(\mu+\mu b-1)(h_0^-(\mu)+1)^{-1},$$

we deduce

$$\begin{aligned} r^-_{\lambda_1,\lambda_2}(\mu)q(\mu) =&(\Pi_1^2 + \Pi_2^2 + \mu)^{-1} \left(1 + (\mu + \mu b - 1)(h_0^-(\mu) + 1)^{-1}\right) \\ &(h_0^-(\mu) + 1)r^-_{\lambda_1,\lambda_2}(\mu)q(\mu). \end{aligned} \tag{4.10}$$

It is easy to see that the estimate

$$\| \left(1 + (\mu + \mu b - 1)(h_0^-(\mu) + 1)^{-1}\right) (h_0^-(\mu) + 1)r^-_{\lambda_1,\lambda_2}(\mu)q(\mu)\| \leq c_7 \tag{4.11}$$

holds for $\mu$ large enough with $c_7$ independent of $\mu$. Therefore,

$$n_*(\varepsilon; Wr^-_{\lambda_1,\lambda_2}(\mu)q(\mu)) \leq n_*(\varepsilon c_7^{-1}; W(\Pi_1^2 + \Pi_2^2 + \mu)^{-1}). \tag{4.12}$$

By (4.7) with $p = 2$ we have

$$n_*(\eta; W(\Pi_1^2 + \Pi_2^2 + \mu)^{-1}) \leq \eta^{-2}\|W(\Pi_1^2 + \Pi_2^2 + \mu)^{-1}\|_2^2,\ \eta > 0. \tag{4.13}$$

The diamagnetic inequality (see [A.H.S]) and the Parseval identity yield

$$\begin{aligned} \|W(\Pi_1^2 + \Pi_2^2 + \mu)^{-1}\|_2^2 \leq&\|W(-\Delta + \mu)^{-1}\|_2^2 = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} |W(X)|^2\, dX \\ &\int_{\mathbb{R}^2} \frac{d\xi}{(|\xi|^2 + \mu)^2} = \frac{1}{4\pi\mu}\int_{\mathbb{R}^2} |W(X)|^2\, dX,\ \mu \geq 1. \end{aligned} \tag{4.14}$$

Now, (4.12)–(4.14) entail (4.8) with $c_6 = c_7^2/4\pi$. In order to see that (4.8) implies (4.9), it suffices to note that $n_*(\varepsilon; Wr^-_{\lambda_1,\lambda_2}(\mu)q(\mu))$ is integer-valued. □

Arguing in a completely analogous manner, we can prove the following lemma.

LEMMA 4.3. — *Under the hypotheses of Lemma 4.1 we have*

$$n_*(\varepsilon; Wr^+_{\lambda_1,\lambda_2}(\mu)) \leq c_6\varepsilon^{-2}\mu^{-1}\int_{\mathbb{R}^2} |W|^2\, dX,\ \varepsilon > 0,$$

*for $\mu$ large enough. In particular, if $\mu > c_6\varepsilon^{-2}\int_{\mathbb{R}^2} |W|^2\, dX$, then $n_*(\varepsilon; Wr^+_{\lambda_1,\lambda_2}(\mu)) = 0$.*

Estimates (4.6) and (4.8), and the Weyl inequalities for the singular numbers of compact operators imply the following corollary.

COROLLARY 4.2. — *Under the hypotheses of Corollary 4.1 we have*

$$n_*(\varepsilon; Wr^-_{\lambda_1,\lambda_2}(\mu)) \leq 4(c_5\mu + c_6\mu^{-1})\varepsilon^{-2}\int_{\mathbb{R}^2} |W|^2\, dX,\ \varepsilon > 0,$$

*for $\mu$ large enough.*

COROLLARY 4.3. — *Let $W \in L^2(\mathbb{R}^2)$, $\omega \in \mathbb{S}^1 := \{\zeta \in \mathbb{C} | |\zeta| = 1\}$. Then the estimate*

$$n_*(\varepsilon; W(h_0^-(\mu) - \omega)^{-1}) \leq (c_5'\mu + c_6'\mu^{-1})\varepsilon^{-2} \int_{\mathbb{R}^2} |W|^2 \, dX, \; \varepsilon > 0, \tag{4.15}$$

*holds for each $\varepsilon > 0$ and $\mu$ large enough, with $c_5'$ and $c_6'$ independent of $\varepsilon$, $\mu$, $\omega$, and $W$.*

*Proof.* — If $\mu$ is sufficiently large, we have

$$\|(h_0^-(\mu) + 1)(h_0^-(\mu) - \omega)^{-1}\| \leq c_8$$

with $c_8$ independent of $\mu$ and $\omega$. It remains to note that the operator $(h_0^-(\mu) + 1)^{-1}$ coincides with $r^-_{\lambda_1,\lambda_2}(\mu)$ with $\lambda_1 = \lambda_2 = -1$, and apply Corollary 4.2. □

**4.2.** Let $m = 3$. Fix $\lambda < 0$, and define the operator

$$R_\lambda^\pm(\mu) := \mathcal{R}(\lambda; H^\pm(\mu)) \tag{4.16}$$

(see (3.3)). Moreover, introduce the operator

$$\mathbf{R}_\lambda := \mathcal{R}(\lambda; \Pi_3^2) = (\Pi_3^2 - \lambda)^{-1/2}, \; \lambda < 0, \tag{4.17}$$

acting in $L^2(\mathbb{R}^3)$. Note that for each $u \in L^2(\mathbb{R}^3)$ we have

$$(\mathbf{R}_\lambda u)(x,y,z) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{i(z-z')\zeta}}{(\zeta^2 - \lambda)^{1/2}} u(x,y,z') \, d\zeta dz', \tag{4.18}$$

$$\begin{aligned}(\mathbf{R}_\lambda^2 u)(x,y,z) &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{i(z-z')\zeta}}{\zeta^2 - \lambda} u(x,y,z') d\zeta dz' \\ &= \frac{1}{2\sqrt{|\lambda|}} \int_{\mathbb{R}} e^{-\sqrt{|\lambda|}|z-z'|} u(x,y,z') dz'.\end{aligned} \tag{4.19}$$

Denote by $P(\mu)$ the orthogonal projection on $\operatorname{Ker} \tilde{H}_0^-(\mu)$ (see (1.8)). In other words,

$$P(\mu) = \int_{\mathbb{R}} \oplus p(\mu) \, dz.$$

LEMMA 4.4. — *Let $W \in L^p(\mathbb{R}^3)$, $p \geq 2$, and $\lambda < 0$. Then the estimate*

$$\|W R_\lambda^-(\mu) P(\mu)\|_p^p \equiv \|W \mathbf{R}_\lambda P(\mu)\|_p^p \leq c_9 \mu \int_{\mathbb{R}^3} |W(\mathbf{X})|^p \, d\mathbf{X}, \; \mu > 0, \tag{4.20}$$

*holds with $c_9 = c_9(p)$ which depends on $p$ and $\lambda$ but is independent of $\mu$ and $W$.*

*Proof.* — Assume at first $W \in L^\infty(\mathbb{R}^3)$. Evidently,

$$\|WR_\lambda^-(\mu)P(\mu)\| = \|W\mathbf{R}_\lambda P(\mu)\|$$
$$\leq \|W\|_{L^\infty(\mathbb{R}^3)} \sup_{\zeta\in\mathbb{R}} (\zeta^2-\lambda)^{-1/2} = |\lambda|^{-1/2}\|W\|_{L^\infty(\mathbb{R}^3)}. \tag{4.21}$$

Now assume $W \in L^2(\mathbb{R}^3)$. We have

$$\|WR_\lambda^-(\mu)P(\mu)\|_2^2 = \|W\mathbf{R}_\lambda e^{-t\tilde{P}_0^-(\mu)}P(\mu)\|_2^2 \leq \|W\mathbf{R}_\lambda e^{-t\tilde{P}_0^-(\mu)}\|_2^2. \tag{4.22}$$

Taking into account (4.18), the identity

$$\left(e^{-t\tilde{P}_0^-(\mu)}u\right)(X,z) = \int_{\mathbb{R}^2} \mathcal{K}_\mu(t;X,Y)u(Y,z)\,dY,\ u\in L^2(\mathbb{R}^3), X\in\mathbb{R}^2, z\in\mathbb{R},$$

(3.1), and the Parseval identity, we obtain

$$\|W\mathbf{R}_\lambda e^{-t\tilde{P}_0^-(\mu)}P(\mu)\|_2^2 \leq \frac{e^{2c_2\mu t}}{32\pi^3t^2}\int_{\mathbb{R}^3}|W(\mathbf{X})|^2\,d\mathbf{X}\int_{\mathbb{R}}\frac{d\zeta}{\zeta^2-\lambda}\,d\zeta\int_{\mathbb{R}^2}e^{-\frac{|Y|^2}{2t}}\,dY$$
$$= \frac{e^{2c_2\mu t}}{16\pi t|\lambda|^{1/2}}\int_{\mathbb{R}^3}|W(\mathbf{X})|^2\,d\mathbf{X}. \tag{4.23}$$

As in the derivation of (4.4), we find that estimates (4.22)–(4.23) entail

$$\|WR_\lambda^-(\mu)P(\mu)\|_2^2 \leq \frac{ec_2\mu}{8\pi|\lambda|^{1/2}}\int_{\mathbb{R}^3}|W(\mathbf{X})|^2\,d\mathbf{X}. \tag{4.24}$$

Interpolating between (4.21) and (4.24), we conclude that (4.20) holds with $c_9 = ec_2/8\pi|\lambda|^{(p-1)/2}$. □

COROLLARY 4.4. — *Let $W \in L^3(\mathbb{R}^3)$, and $\lambda < 0$. Then for each $\varepsilon > 0$ we have*

$$n_*(\varepsilon; WR_\lambda^-(\mu)P(\mu)) \leq c_9(3)\,\mu\varepsilon^{-3}\int_{\mathbb{R}^3}|W(\mathbf{X})|^3\,d\mathbf{X}. \tag{4.25}$$

Set $Q(\mu) := \mathrm{Id} - P(\mu)$.

LEMMA 4.5. — *Under the assumptions of Corollary 4.4 the estimate*

$$n_*(\varepsilon; WR_\lambda^-(\mu)Q(\mu)) \leq c_{10}\varepsilon^{-3}\int_{\mathbb{R}^3}|W(\mathbf{X})|^3\,d\mathbf{X} \tag{4.26}$$

*holds for each $\varepsilon > 0$ with $c_{10}$ independent of $\varepsilon$, $\mu$, and $W$. Moreover, for each $\varepsilon > 0$ there exists a $\mu_0 = \mu_0(\varepsilon)$ such that $\mu \geq \mu_0$ entails*

$$n_*(\varepsilon; WR_\lambda^-(\mu)Q(\mu)) = 0. \tag{4.27}$$

*Proof.* — The operator inequality

$$Q(\mu)(H_0^-(\mu)-\lambda)Q(\mu) \geq c_{11}Q(\mu)(\Pi(\mu)^2+\mu)Q(\mu)$$

with $\Pi(\mu)^2 := \sum_{j=1,2,3} \Pi_j(\mu)^2$ and $c_{11} > 0$ independent of $\mu$, implies

$$\|(\Pi(\mu)^2+\mu)^{1/2}R_\lambda^-(\mu)Q(\mu)\| \leq 1/\sqrt{c_{11}}.$$

Therefore, we have

$$n_*(\varepsilon; WR_\lambda^-(\mu)Q(\mu)) \leq n_*(\varepsilon c_{11}^{1/2}; W(\Pi(\mu)^2+\mu)^{-1/2}). \tag{4.28}$$

The Birman-Schwinger principle (see Lemma 3.3) implies

$$\begin{aligned} & n_*(\varepsilon c_{11}^{1/2}; W(\Pi(\mu)^2+\mu)^{-1/2}) \\ & = n_+(\varepsilon^2 c_{11}; (\Pi(\mu)^2+\mu)^{-1/2}|W|^2(\Pi(\mu)^2+\mu)^{-1/2}) \\ & = N(0; \Pi(\mu)^2 - \varepsilon^{-2}c_{11}^{-1}|W|^2+\mu). \end{aligned} \tag{4.29}$$

The "magnetic" Cwickel-Lieb-Rozenblioum estimate (see e.g. [A.H.S]) yields

$$N(0; \Pi(\mu)^2 - \varepsilon^{-2}c_{11}^{-1}|W|^2+\mu) \leq c_{12}\int_{\mathbb{R}^3} \left(\varepsilon^{-2}c_{11}^{-1}|W(\mathbf{X})|^2-\mu\right)_+^{3/2} d\mathbf{X}, \tag{4.30}$$

with $c_{12}$ independent of $\varepsilon$, $\mu$, and $W$. Putting together (4.28)–(4.30), we get

$$n_*(\varepsilon; WR_\lambda^-(\mu)Q(\mu)) \leq c_{12}\int_{\mathbb{R}^3} (\varepsilon^{-2}c_{11}^{-1}|W(\mathbf{X})|^2-\mu)_+^{3/2} d\mathbf{X},\ \mu > 0. \tag{4.31}$$

Note that the right-hand-side of (4.31) decreases monotonously as $\mu$ grows from 0 to $\infty$, and tends to zero as $\mu \to \infty$. Setting $\mu = 0$, we find that (4.31) entails (4.26) with $c_{10} = c_{11}^{-3/2}c_{12}$. Moreover, we may choose any $\mu_0 = \mu_0(\varepsilon)$ for which the right-hand-side of (4.31) is smaller then one; then (4.31) implies (4.27). □

Using a completely analogous argument, we can demonstrate the following lemma.

LEMMA 4.6. — *Under the assumptions of Corollary 4.4 we have*

$$n_*(\varepsilon; WR_\lambda^+(\mu)) \leq c_{10}\varepsilon^{-3}\int_{\mathbb{R}^3} |W(\mathbf{X})|^3 d\mathbf{X},\ \varepsilon > 0,\ \mu > 0.$$

*Moreover, the inequality* $\mu \geq \mu_0(\varepsilon)$ *entails* $n_*(\varepsilon; WR_\lambda^+(\mu)) = 0$.

Combining (4.25) with (4.26), we get the following corollary.

COROLLARY 4.5. — *Under the hypotheses of Corollary 4.4 we have*

$$n_*(\varepsilon; WR_\lambda^-(\mu)) \leq 8(c_9\mu + c_{10})\varepsilon^{-3} \int_{\mathbb{R}^3} |W(\mathbf{X})|^3 \, d\mathbf{X}, \ \varepsilon > 0. \tag{4.32}$$

COROLLARY 4.6. — *Let $W \in L^2(\mathbb{R}^3)$, $\omega \in \mathbb{S}^1$, $\lambda < 0$. Then the estimate*

$$\|W(\tilde{H}_0^-(\mu) - \omega)^{-1}\mathbf{R}_\lambda\|_2^2 \leq (c_9'\mu + c_{10}'\mu^{-1}) \int_{\mathbb{R}^3} |W(\mathbf{X})|^2 \, d\mathbf{X} \tag{4.33}$$

*holds for $\mu$ large enough with $c_9'$ and $c_{10}'$ independent of $W$, $\mu$, and $\omega$.*

*Proof.* — Obviously

$$\begin{aligned}\|W(\tilde{H}_0^-(\mu) - \omega)^{-1}\mathbf{R}_\lambda\|_2^2 &\leq 2\|W(\tilde{H}_0^-(\mu) - \omega)^{-1}\mathbf{R}_\lambda P(\mu)\|_2^2 \\ &\quad + 2\|W(\tilde{H}_0^-(\mu) - \omega)^{-1}\mathbf{R}_\lambda Q(\mu)\|_2^2.\end{aligned} \tag{4.34}$$

Applying (4.20) with $p = 2$, we get

$$\|W(\tilde{H}_0^-(\mu) - \omega)^{-1}\mathbf{R}_\lambda P(\mu)\|_2^2 = \|W\mathbf{R}_\lambda P(\mu)\|_2^2 \leq c_9(2)\mu \int_{\mathbb{R}^3} |W|^2 d\mathbf{X}. \tag{4.35}$$

Next we write the identity

$$\begin{aligned}&(\tilde{H}_0^-(\mu) - \omega)^{-1}Q(\mu) \\ &= (\Pi_1(\mu)^2 + \Pi_2(\mu)^2 + \mu)^{-1}\left(1 + (\mu + \mu b + \omega)(\tilde{H}_0^-(\mu) - \omega)^{-1}\right)Q(\mu),\end{aligned}$$

note that the estimate

$$\left\|\left(1 + (\mu + \mu b + \omega)(\tilde{H}_0^-(\mu) - \omega)^{-1}\right)Q(\mu)\right\|_2^2 \leq c_{13}$$

holds with $c_{13}$ independent of $\mu$ and $\omega$, take into account the fact that $\mathbf{R}_\lambda$ commutes with $\left(1 + (\mu + \mu b + \omega)(\tilde{H}_0^-(\mu) - \omega)^{-1}\right)$ and $Q(\mu)$, and obtain

$$\|W(\tilde{H}_0^-(\mu) - \omega)^{-1}\mathbf{R}_\lambda Q(\mu)\|_2^2 \leq c_{13}^2\|W(\Pi_1(\mu)^2 + \Pi_2(\mu)^2 + \mu)^{-1}\mathbf{R}_\lambda\|_2^2. \tag{4.36}$$

Using the diamagnetic inequality and the Parseval identity, we get

$$\begin{aligned}&\|W(\Pi_1(\mu)^2 + \Pi_2(\mu)^2 + \mu)^{-1}\mathbf{R}_\lambda\|_2^2 \\ &\leq \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} |W(\mathbf{X})|^2 \, d\mathbf{X} \int_{\mathbb{R}^2} \frac{d\xi}{(|\xi|^2 + \mu)^2} \int_{\mathbb{R}} \frac{d\zeta}{\zeta^2 - \lambda} d\zeta \\ &= \frac{1}{8\pi\mu|\lambda|^{1/2}} \int_{\mathbb{R}^3} |W(\mathbf{X})|^2 \, d\mathbf{X}.\end{aligned} \tag{4.37}$$

Combining (4.34)–(4.37), we derive (4.33) with $c_9' = 2c_9(2)$ and $c_{10}' = c_{13}^2/4\pi|\lambda|^{1/2}$. □

## 5. Proof of Propositions 2.1 – 2.2.

For $\lambda < 0$ introduce the operator $\varrho_\lambda := \left(-\frac{d^2}{dz^2} - \lambda\right)^{-1/2}$ acting in $L^2(\mathbb{R})$.

LEMMA 5.1. — *Let $v \in L^p(\mathbb{R})$, $p \geq 2$, $\lambda < 0$. Then $v\varrho_\lambda \in S_p$, and we have*

$$\|v\varrho_\lambda\|_p^p \leq c_{14} \int_{\mathbb{R}} |v(z)|^p \, dz \tag{5.1}$$

*where $c_{14} = c_{14}(\lambda)$ is independent of $v$.*

*Proof.* — Our argument will be very close to the proof of Lemma 4.4.

Assume at first $v \in L^\infty(\mathbb{R})$. Obviously

$$\|v\varrho_\lambda\| \leq \|v\|_{L^\infty(\mathbb{R})} \|\varrho_\lambda\| = |\lambda|^{-1/2} \|v\|_{L^\infty(\mathbb{R})}. \tag{5.2}$$

Assume now $v \in L^2(\mathbb{R})$. We have

$$\|v\varrho_\lambda\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |v(z)|^2 \, dz \int_{\mathbb{R}} \frac{d\zeta}{\zeta^2 - \lambda} = \frac{1}{2\sqrt{|\lambda|}} \int_{\mathbb{R}} |v(z)|^2 \, dz. \tag{5.3}$$

Interpolating between (5.2) and (5.3), we get (5.1) with $c_{14} = 1/2|\lambda|^{(p-1)/2}$.

□

Set

$$\tau_\lambda(X) := \varrho_\lambda V(X, .)\varrho_\lambda, \; X \in \mathbb{R}^2, \; \lambda < 0. \tag{5.4}$$

COROLLARY 5.1. — *Let $m = 3$, $V \in \mathcal{L}_{3/2}$. Then for every $\lambda < 0$ and almost every $X \in \mathbb{R}^2$ the operator $\tau_\lambda(X)$ is compact.*

*Proof.* — Choose a sequence $\{\varepsilon_r\}_{r\geq 1}$ such that $\varepsilon_r > 0$, $r \geq 1$, and $\lim_{r\to\infty} \varepsilon_r = 0$. Fix $r \geq 1$ and write $V = V_1^{(r)} + V_2^{(r)}$ with $V_1^{(r)} \in L^{3/2}(\mathbb{R}^3)$ and $\sup_{X\in\mathbb{R}^2} |V_2^{(r)}(X)| \leq \varepsilon_r$. Set

$$\Omega_r := \left\{ X \in \mathbb{R}^2 \middle| \int_{\mathbb{R}} \left|V_1^{(r)}(X, z)\right|^{3/2} dz < \infty \right\}, \quad \Omega := \bigcap_{r\geq 1} \Omega_r.$$

Evidently, $\mathrm{vol}\{\mathbb{R}^2 \setminus \Omega\} = 0$. Put

$$\tau_{j,\lambda}^{(r)}(X) := \varrho_\lambda V_j^{(r)}(X, .)\varrho_\lambda, \; j = 1, 2.$$

Lemma 5.1 implies that for each $X \in \Omega$ and $r \geq 1$ we have $|V_1^{(r)}|^{1/2}\varrho_\lambda \in S_3$, and therefore $\tau_{j,\lambda}^{(r)}(X) \in S_{3/2} \subset S_\infty$. Moreover,

$$\|\tau_\lambda(X) - \tau_{1,\lambda}^{(r)}(X)\| = \|\tau_{2,\lambda}^{(r)}(X)\| \leq |\lambda|^{-1}\varepsilon_r, \ X \in \Omega, \ r \geq 1.$$

Hence, for every $X \in \Omega$ (i.e. almost every $X \in \mathbb{R}^2$) the operator $\tau_\lambda(X)$ can be approximated in norm by compact operators. Therefore, $\tau_\lambda(X)$ is compact itself. □

Proposition 2.1 follows immediately from Corollary 5.1 and the Weyl theorem on the invariance of the essential spectrum under relatively compact perturbations.

For $\lambda < 0$ set

$$(5.5) \quad \tilde{\mathcal{D}}_\lambda(s) := \tilde{\mathcal{D}}_\lambda(s;V) = \begin{cases} \frac{1}{2\pi}\int_{\mathbb{R}^2} n_-(-s;\tau_\lambda(X))b(X)\,dX \text{ if } s<0, \\ -\frac{1}{2\pi}\int_{\mathbb{R}^2} n_+(s;\tau_\lambda(X))b(X)\,dX \text{ if } s>0. \end{cases}$$

The Birman-Schwinger principle (see Lemma 3.3) implies the following assertion.

LEMMA 5.2. — *Let* $m=3$, $V \in \mathcal{L}_{3/2}$.

(i) *We have*

$$(5.6) \qquad \mathcal{D}(\lambda) = \tilde{\mathcal{D}}_\lambda(-1).$$

(ii) *The function* $\mathcal{D}(.)$ *is continuous at* $\lambda < 0$ *if and only if the function* $\tilde{\mathcal{D}}_\lambda(.)$ *is continuous at* $-1$.

Proposition 2.2 follows almost immediately from Lemma 5.1 and Lemma 5.2 (i). In order to see this, fix $\lambda < 0$, $\varepsilon \in (0,|\lambda|)$ and write $V = V_1 + V_2$ where $V_1 \in L^{3/2}(\mathbb{R}^3)$ and $\sup_{X\in\mathbb{R}^2} |V_2(X)| \leq \varepsilon$. Set

$$(5.7) \qquad \tau_{j,\lambda}(X) := \varrho_\lambda V_j(X,.)\varrho_\lambda, \ j=1,2.$$

By (5.6) we have

$$(5.8) \qquad \mathcal{D}_V(\lambda) \leq \mathcal{D}_{V_1}(\lambda-\varepsilon) = \tilde{\mathcal{D}}_{\lambda-\varepsilon}(-1;V_1).$$

On the other hand, by (5.1) the estimate

(5.9)

$$n_-(1;\tau_{\lambda-\varepsilon}(X)) \leq n_*(1;|V_1(X,.)|^{1/2}\varrho_{\lambda-\varepsilon}) \leq c_{14}(\lambda-\varepsilon)\int_{\mathbb{R}} |V_1(X,z)|^{3/2}\,dz$$

holds for almost every $X \in \mathbb{R}^2$. Multiplying (5.9) by $b(X)$, and integrating with respect to $X \in \mathbb{R}^2$, we find that (5.8) implies

$$\mathcal{D}_V(\lambda) \leq \frac{c_{14}(\lambda - \varepsilon)}{2\pi} c_2 \int_{\mathbb{R}^3} |V_1(\mathbf{X})|^{3/2} \, d\mathbf{X} < \infty$$

which entails (2.5).

## 6. Trace asymptotics.

**6.1.** The main goal of this subsection is to prove the following proposition.

PROPOSITION 6.1. — *Let $m = 2$, $W \in C_0^\infty(\mathbb{R}^2)$, and (1.1) hold. Then for each integer $l \geq 1$ we have*

$$\lim_{\mu \to \infty} \mu^{-1} \mathrm{Tr}\, (p(\mu) W p(\mu))^l = \frac{1}{2\pi} \int_{\mathbb{R}^2} W(X)^l b(X) \, dX. \tag{6.1}$$

We shall divide the proof of Proposition 6.1 into several lemmas and corollaries.

For $\mu > 1$ and $s > 0$ set $\epsilon_{\mu,s} := \exp\left(-\frac{\log \mu}{\mu s} h_0^-(\mu)\right)$.

LEMMA 6.1. — *Let $m = 2$, $U \in C_0^\infty(\mathbb{R}^2)$, and (1.1) hold. Then for each $s \geq 2s_{\mathrm{supp}\, U}$ (see Lemma 3.1) we have*

$$\lim_{\mu \to \infty} \mu^{-1} \, \mathrm{Tr}\, \epsilon_{\mu,s} U \epsilon_{\mu,s} = \frac{1}{2\pi} \int_{\mathbb{R}^2} U(X) b(X) \, dX. \tag{6.2}$$

*Proof.* — Let $\tilde{U} \in C_0^\infty(\mathbb{R}^2)$ such that $\tilde{U} = 1$ on $\mathrm{supp}\, U$. Using (3.1), we get $e^{-th_0^-(\mu)} U \in S_2$, $\tilde{U} e^{-th_0^-(\mu)} \in S_2$, $t > 0$. Therefore,

$$\mathrm{Tr}\, e^{-th_0^-(\mu)} U e^{-th_0^-(\mu)} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{K}_\mu(t; X, Y) U(Y) \mathcal{K}_\mu(t; Y, X) \, dX dY, \; t > 0. \tag{6.3}$$

Utilizing the continuity of $\mathcal{K}_\mu(t; X, Y)$ (see Lemma 3.1), and the semigroup properties of $e^{-th_0^-(\mu)}$, we obtain

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{K}_\mu(t; X, Y) U(Y) \mathcal{K}_\mu(t; Y, X) \, dX dY = \int_{\mathbb{R}^2} U(Y) \mathcal{K}_\mu(2t; Y, Y) \, dY. \tag{6.4}$$

Putting together (6.3)–(6.4), we obtain

$$\operatorname{Tr} \epsilon_{\mu,s} U \epsilon_{\mu,s} = \int_{\mathbb{R}^2} U(X) \mathcal{K}_\mu \left( \frac{2 \log \mu}{\mu s}; X, X \right) dX, \ \mu > 1, \tag{6.5}$$

Multiplying (6.5) by $\mu^{-1}$, letting $\mu \to \infty$, and recalling (3.2), we deduce (6.2). □

LEMMA 6.2. — *Let $m = 2$, $\mathcal{U} \in L^2(\mathbb{R}^2)$, and (1.1) hold. Then the estimate*

$$\|U e^{-th_0^-(\mu)} Q(\mu)\|_2^2 \leq c_{15} t^{-1} \int_{\mathbb{R}^2} |U|^2 \, dX \tag{6.6}$$

*holds for every $t > 0$ with $c_{15}$ independent of $t$, $\mu$, and $U$.*

*Proof.* — Set $\alpha = 2c_1/(2c_1 + c_2)$, $\beta = c_2/(2c_1 + c_2)$, so that we have $\alpha + \beta = 1$, $c_2\alpha - 2c_1\beta = 0$. Write the inequality

$$\|U e^{-th_0^-(\mu)} Q(\mu)\|_2^2 \leq \|U e^{-\alpha t h_0^-(\mu)}\|_2^2 \, \|e^{-\beta t h_0^-(\mu)} Q(\mu)\|^2,$$

apply (3.1) together with

$$\|e^{-\beta t h_0^-(\mu)} Q(\mu)\| \leq e^{-2\beta c_1 \mu t},$$

in order to deduce the estimate

$$\begin{aligned} &\|U e^{-th_0^-(\mu)} Q(\mu)\|_2^2 \\ &\leq (4\pi\alpha t)^{-2} e^{2(c_2\alpha - 2\beta c_1)\mu t} \int_{\mathbb{R}^2} e^{-\frac{|Y|^2}{2\alpha t}} \, dY \int_{\mathbb{R}^2} |U(X)|^2 \, dX \\ &= \frac{1}{8\pi\alpha t} \int_{\mathbb{R}^2} |U(X)|^2 \, dX, \end{aligned}$$

which is equivalent to (6.6) with $c_{15} = 1/8\pi\alpha$. □

COROLLARY 6.1. — *Let $m = 2$, $U \in L^1(\mathbb{R}^2)$, and (1.1) hold. Then there exists $s_0 > 0$ such that the estimate*

$$\operatorname{Tr} p(\mu) U p(\mu) - \operatorname{Tr} \epsilon_{\mu,s} U \epsilon_{\mu,s} = O(\mu (\log \mu)^{-1/2}), \ \mu \to \infty, \tag{6.7}$$

*holds uniformly with respect to $s \geq s_0$.*

*Proof.* — Set $U_1 := |U|^{1/2}$, $U_2 := U|U|^{-1/2}$. Evidently,

$$\begin{aligned} \operatorname{Tr} \epsilon_{\mu,s} U \epsilon_{\mu,s} - \operatorname{Tr} p(\mu) U p(\mu) &= \operatorname{Tr} q(\mu) \epsilon_{\mu,s} U_1 U_2 \epsilon_{\mu,s} q(\mu) \\ &\quad + 2 \operatorname{Re} \operatorname{Tr} p(\mu) U_1 U_2 q(\mu) \epsilon_{\mu,s}. \end{aligned}$$

Therefore,

$$|\operatorname{Tr} p(\mu) U p(\mu) - \operatorname{Tr} \epsilon_{\mu,s} U \epsilon_{\mu,s}| \leq \|q(\mu) \epsilon_{\mu,s} U_1\|_2^2 + 2 \|p(\mu) U_1\|_2 \|U_1 q(\mu) \epsilon_{\mu,s}\|_2. \tag{6.8}$$

Using Lemma 4.1 and Lemma 6.2 with $t = \frac{\log \mu}{\mu s}$, we find that (6.8) implies (6.7). $\square$

Combining (6.2) and (6.7), we get the following corollary.

COROLLARY 6.2. — *Let $m = 2$, $U \in C_0^\infty(\mathbb{R}^2)$, and (1.1) hold. Then we have*

$$\lim_{\mu\to\infty} \mu^{-1} \operatorname{Tr} p(\mu) U p(\mu) = \frac{1}{2\pi}\int_{\mathbb{R}^2} U(X) b(X)\, dX.$$

*In particular, if $W \in C_0^\infty(\mathbb{R}^2)$,*

$$(6.9)\quad \lim_{\mu\to\infty} \mu^{-1} \operatorname{Tr} p(\mu) W^l p(\mu) = \frac{1}{2\pi}\int_{\mathbb{R}^2} W(X)^l b(X)\, dX, \quad l \in \mathbb{Z}, \quad l \geq 1.$$

It is clear that if $l \geq 2$, it is necessary to have some control on the $S_2$-norm of the commutator $[W, p(\mu)]$ in order to pass from (6.9) to (6.1).

LEMMA 6.3. — *Let $m = 2$, $W$ be in the Sobolev space $\mathrm{H}^1(\mathbb{R}^2)$, and (1.1) hold. Then we have*

$$(6.10)\qquad \|[W, p(\mu)]\|_2 = O(1), \ \mu \to \infty.$$

*Proof.* — Set $\partial W := i\frac{\partial W}{\partial x} + \frac{\partial W}{\partial y}$, $\overline{\partial} W := -i\frac{\partial W}{\partial x} + \frac{\partial W}{\partial y}$. Obviously,

$$[W, a(\mu)] = \partial W, \ [W, a(\mu)^*] = -\overline{\partial} W,$$

$$(6.11)\qquad [W, h_0^-(\mu)] = \partial W a(\mu)^* - a(\mu)\overline{\partial} W.$$

Further, if $\omega \in \mathbb{S}^1$ and $\mu$ is sufficiently large, then the operator $h_0^-(\mu) - \omega$ is invertible, and $\|(h_0^-(\mu) - \omega)^{-1}\| = 1$. Moreover,

$$[W, (h_0^-(\mu) - \omega)^{-1}] = (h_0^-(\mu) - \omega)^{-1}(a(\mu)\overline{\partial} W - \partial W a(\mu)^*)(h_0^-(\mu) - \omega)^{-1}.$$

On the other hand,

$$p(\mu) = -\frac{1}{2\pi i}\int_{\mathbb{S}^1} (h_0^-(\mu) - \omega)^{-1} d\omega.$$

Therefore

$$[W, p(\mu)] = \frac{1}{2\pi i}\int_{\mathbb{S}^1} (h_0^-(\mu) - \omega)^{-1} \partial W a(\mu)^* (h_0^-(\mu) - \omega)^{-1} d\omega$$
$$- \frac{1}{2\pi i}\int_{\mathbb{S}^1} (h_0^-(\mu) - \omega)^{-1} a(\mu)\overline{\partial} W (h_0^-(\mu) - \omega)^{-1} d\omega.$$

Hence, we obtain the estimate

(6.12)
$$\|[W,p(\mu)]\|_2 \leq 2 \sup_{\omega\in\mathbb{S}^1} \left(\|(h_0^-(\mu)-\omega)^{-1}\partial W\|_2 \|a(\mu)^*(h_0^-(\mu)-\omega)^{-1}\|\right).$$

Applying Corollary 4.3, we get

$$\|(h_0^-(\mu)-\omega)^{-1}\partial W\|_2^2 \leq (c_5'\mu + c_6'\mu^{-1}) \int_{\mathbb{R}^2} |\nabla W|^2 \, dX. \tag{6.13}$$

It is easy to check that the estimate

(6.14)
$$\|a(\mu)^*(h_0^-(\mu)-\omega)^{-1}\|^2 = \|(h_0^-(\mu)-\bar\omega)^{-1}h_0^-(\mu)(h_0^-(\mu)-\omega)^{-1}\| = O(\mu^{-1})$$

holds as $\mu\to\infty$ uniformly with respect to $\omega\in\mathbb{S}^1$.

Putting together (6.12)–(6.14), we deduce (6.10). □

COROLLARY 6.3. — *Let $m=2$, $W\in C_0^\infty(\mathbb{R}^2)$, and (1.1) hold. Then for every integer $l\geq 2$ we have*

$$\operatorname{Tr}(p(\mu)Wp(\mu))^l - \operatorname{Tr} p(\mu)W^l p(\mu) = O(\mu^{1/2}),\ \mu\to\infty. \tag{6.15}$$

*Proof.* — Evidently,

$$(p(\mu)Wp(\mu))^l - p(\mu)W^l p(\mu) = \sum_{k=1}^{l-1} p(\mu)W^k[W,p(\mu)](p(\mu)W)^{l-k-1}p(\mu).$$

Therefore,

$$\begin{aligned}
&|\operatorname{Tr}(p(\mu)Wp(\mu))^l - \operatorname{Tr} p(\mu)W^l p(\mu)| \\
&\qquad \leq \|(p(\mu)Wp(\mu))^l - p(\mu)W^l p(\mu)\|_1 \\
&\qquad \leq \sum_{k=1}^{l-1} \|p(\mu)W^k[W,p(\mu)]\|_1 \|(p(\mu)W)^{l-k-1}p(\mu)\| \\
&\qquad \leq \sum_{k=1}^{l-1} \|p(\mu)W^k\|_2 \|[W,p(\mu)]\|_2 \|W\|_{L^\infty(\mathbb{R}^2)}^{l-k-1}.
\end{aligned} \tag{6.16}$$

By Lemma 4.1

$$\|p(\mu)W^k\|_2 = O(\mu^{1/2}),\ \mu\to\infty,\ k\geq 1,$$

by Lemma 6.3

$$\|[W,p(\mu)]\|_2 = O(1),\ \mu\to\infty,$$

and $W$ is independent of $\mu$. Hence, (6.16) entails (6.15). □

Now, (6.1) follows from (6.9) and (6.15).

**6.2.** Let $m = 3$, $V \in \mathcal{L}_{3/2}$. Introduce the operator

$$T_\lambda^-(\mu) := R_\lambda^-(\mu) V R_\lambda^-(\mu), \ \lambda < 0, \tag{6.17}$$

(see (4.16)), which is compact in $L^2(\mathbb{R}^3)$.

Our main goal will be to demonstrate the following proposition which is the three-dimensional analogue of Proposition 6.1.

PROPOSITION 6.2. — *Let $m = 3$, $W \in C_0^\infty(\mathbb{R}^2)$, $\lambda < 0$, and (1.1), (1.7) hold. Then for each integer $l \geq 1$ we have*

$$\lim_{\mu \to \infty} \mu^{-1} \operatorname{Tr}(P(\mu) T_\lambda^-(\mu) P(\mu))^l = \frac{1}{2\pi} \int_{\mathbb{R}^2} \operatorname{Tr} \tau_\lambda(X)^l b(X) \, dX \tag{6.18}$$

*where the operator $\tau_\lambda(X)$ is defined in (5.4).*

*Proof.* — Our argument will follow the scheme of the proof of Proposition 6.1, and that is why we shall omit some details. Introduce the operator

$$\tilde{T}_\lambda := \mathbf{R}_\lambda V \mathbf{R}_\lambda, \ \lambda < 0,$$

(see (4.17)), acting in $L^2(\mathbb{R}^3)$. Note that the operator $\tilde{T}_\lambda$ is not compact but only bounded. Nevertheless, Lemma 4.4 implies that $|V|^{1/2} \mathbf{R}_\lambda P(\mu) \in S_2$, and hence $P(\mu) \tilde{T}_\lambda^l P(\mu) \in S_1$ for each integer $l \geq 1$. Set

$$\mathcal{E}_{\mu,s} := \int_{\mathbb{R}} \oplus \epsilon_{\mu,s} dz, \ \mu > 1, \ s > 0.$$

Obviously, $\mathcal{E}_{\mu,s} \tilde{T}_\lambda^l \mathcal{E}_{\mu,s} \in S_1$, $l \geq 1$. Our first step is to prove the asymptotic relation

$$\lim_{\mu \to \infty} \mu^{-1} \operatorname{Tr} \mathcal{E}_{\mu,s} \tilde{T}_\lambda^l \mathcal{E}_{\mu,s} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \operatorname{Tr} \tau_\lambda(X)^l b(X) \, dX, \ l \geq 1, \tag{6.19}$$

which is analogous with (6.2). To this end we utilize the identities

$$\operatorname{Tr} \tau_\lambda(X)^l = \frac{1}{(4|\lambda|)^{l/2}} \int_{\mathbb{R}^l} V(X, z_1) e^{-\sqrt{|\lambda|}|z_1 - z_2|} \ldots V(X, z_l) e^{-\sqrt{|\lambda|}|z_l - z_1|} \, dz_1 \ldots dz_l,$$

$$\begin{aligned} \operatorname{Tr} \mathcal{E}_{\mu,s} \tilde{T}_\lambda^l \mathcal{E}_{\mu,s} &= \frac{1}{(4|\lambda|)^{l/2}} \int_{\mathbb{R}^4} \int_{\mathbb{R}^l} \mathcal{K}_\mu \left( \frac{\log \mu}{\mu s}; X, Y \right) V(Y, z_1) e^{-\sqrt{|\lambda|}|z_1 - z_2|} \ldots \\ &\quad V(Y, z_l) e^{-\sqrt{|\lambda|}|z_l - z_1|} \, dz_1 \ldots dz_l \mathcal{K}_\mu \left( \frac{\log \mu}{\mu s}; Y, X \right) dX dY \\ &= \int_{\mathbb{R}^2} \mathcal{K}_\mu \left( \frac{\log \mu}{\mu s}; Y, Y \right) \operatorname{Tr} \tau_\lambda(Y)^l \, dY, \end{aligned}$$

(see (4.19)), take into account that $\operatorname{Tr} \tau_\lambda(.)^l \in C_0^\infty(\mathbb{R}^2)$, and apply Lemma 3.1. Next, by analogy with (6.6) we establish the estimate

$$\| |V|^{1/2} \mathbf{R}_\lambda e^{-tH_0^-(\mu)} Q(\mu) \|_2^2 \le c'_{15} t^{-1} \int_{\mathbb{R}^3} |V| d\mathbf{X}, \ t > 0,$$

with $c'_{15}$ independent of $t$, $\mu$ and $V$. Using this estimate together with (4.20) for $p = 2$, we get

$$\operatorname{Tr} P(\mu) \tilde{T}_\lambda^l P(\mu) - \operatorname{Tr} \mathcal{E}_{\mu,s} \tilde{T}_\lambda^l \mathcal{E}_{\mu,s} = O(\mu (\log \mu)^{-1/2})), \ \mu \to \infty, \tag{6.20}$$

by analogy with (6.7). Further, as in (6.10), we show that

$$\| [\tilde{T}_\lambda, P(\mu)] \|_2 = O(1), \ \mu \to \infty, \tag{6.21}$$

using Corollary 4.6. Finally, we notice that

$$\operatorname{Tr} (P(\mu) T_\lambda(\mu) P(\mu))^l = \operatorname{Tr} (P(\mu) \tilde{T}_\lambda P(\mu))^l, \ l \ge 1. \tag{6.22}$$

Employing (6.21), we deduce the estimate

$$\operatorname{Tr} (P(\mu) \tilde{T}_\lambda P(\mu))^l - \operatorname{Tr} P(\mu) \tilde{T}_\lambda^l P(\mu) = O(\mu^{1/2}), \ \mu \to \infty, \tag{6.23}$$

which is similar to (6.15). Putting together (6.19), (6.20), (6.22) and (6.23), we obtain (6.18). □

## 7. Proof of Theorem 2.2.

Throughout the section we assume that the hypotheses of Theorem 2.2 are fulfilled. In particular, $m = 2$, $V \in \mathcal{L}_2$, and $0 < \lambda_1 < \lambda_2$. Set

$$W = W_{V;\lambda_1,\lambda_2} := \frac{1}{\lambda_1 \lambda_2} V^2 - \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} V.$$

Further, for $s \neq 0$ put

$$\tilde{\delta}_{\lambda_1,\lambda_2}(s) = \tilde{\delta}_{\lambda_1,\lambda_2}(s) := \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \theta(s - W_{V;\lambda_1,\lambda_2}(X)) b(X)\, dX & \text{if } s < 0, \\ -\dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \theta(W_{V;\lambda_1,\lambda_2}(X) - s) b(X)\, dX & \text{if } s > 0. \end{cases} \tag{7.1}$$

Note that if $\lambda_1$ and $\lambda_2$ are continuity points of $\delta$, we have

$$\tilde{\delta}(-1; \lambda_1, \lambda_2) = \delta(\lambda_2) - \delta(\lambda_1)$$

(see (2.1)), since the inequality $W_{V;\lambda_1,\lambda_2}(X) < -1$ is equivalent to $\lambda_1 < V(X) < \lambda_2$. Set

$$t^-_{\lambda_1,\lambda_2}(\mu) := \tilde{T}(\lambda_1, \lambda_2; h_0^-(\mu), V)$$

(see (3.7)), and

$$g = g_{\lambda_1,\lambda_2}(\mu) := \mathcal{G}(\lambda_1, \lambda_2; h_0^-(\mu))$$

(see (3.6)). Obviously, $\|g_{\lambda_1,\lambda_2}(\mu)\| \leq g_0$ with $g_0 > 0$ independent of $\mu$.

LEMMA 7.1. — *Let $V \in C_0^\infty(\mathbb{R}^2)$, $0 < \lambda_1 < \lambda_2$. Suppose that (1.1) holds. Assume that $s < 0$ is a continuity point of $\tilde{\delta}$. Then we have*

$$\lim_{\mu\to\infty} \mu^{-1} n_-(-s; p(\mu)t^-_{\lambda_1,\lambda_2}(\mu)p(\mu))) = \tilde{\delta}_{\lambda_1,\lambda_2}(s). \tag{7.2}$$

*Proof.* — Note that $p(\mu)t^-_{\lambda_1,\lambda_2}(\mu)p(\mu) = p(\mu)W_{V;\lambda_1,\lambda_2}p(\mu)$. Apply Proposition 6.1 with $W = W_{V;\lambda_1,\lambda_2}$. Then (7.2) follows from Lemma 3.1 with $T(\mu) = p(\mu)W_{V;\lambda_1,\lambda_2}p(\mu)$, $t_0 = \sup_{X\in\mathbb{R}^2} |W_{V;\lambda_1,\lambda_2}(X)|$, $\nu = \tilde{\delta}_{\lambda_1,\lambda_2}$, and $p = 1$. □

PROPOSITION 7.1. — *Under the hypotheses of Lemma 7.1 we have*

$$\lim_{\mu\to\infty} \mu^{-1} n_-(-s; t^-_{\lambda_1,\lambda_2}(\mu)) = \tilde{\delta}_{\lambda_1,\lambda_2}(s). \tag{7.3}$$

*Proof.* — The minimax principle entails

$$n_-(-s; t^-_{\lambda_1,\lambda_2}(\mu)) \geq n_-(-s; p(\mu)t^-_{\lambda_1,\lambda_2}(\mu)p(\mu)), \; s < 0.$$

Hence, Lemma 7.1 implies

$$\liminf_{\mu\to\infty} \mu^{-1} n_-(-s; t^-_{\lambda_1,\lambda_2}(\mu)) \geq \tilde{\delta}_{\lambda_1,\lambda_2}(s). \tag{7.4}$$

Further, we have

$$\begin{aligned} t^-_{\lambda_1,\lambda_2}(\mu) &= p(\mu)t^-_{\lambda_1,\lambda_2}(\mu)p(\mu) + q(\mu)t^-_{\lambda_1,\lambda_2}(\mu)q(\mu) \\ &\quad + 2\operatorname{Re} p(\mu)r^-_{\lambda_1,\lambda_2}(\mu)V^2 r^-_{\lambda_1,\lambda_2}(\mu)q(\mu) \\ &\quad + 2\operatorname{Re} p(\mu)gVr^-_{\lambda_1,\lambda_2}(\mu)q(\mu) + 2\operatorname{Re} p(\mu)r^-_{\lambda_1,\lambda_2}(\mu)Vgq(\mu) \\ &= p(\mu)Wp(\mu) + q(\mu)t^-_{\lambda_1,\lambda_2}(\mu)q(\mu) \\ &\quad + 2\operatorname{Re} p(\mu)r^-_{\lambda_1,\lambda_2}(\mu)V^2 r^-_{\lambda_1,\lambda_2}(\mu)q(\mu) \\ &\quad + 4\operatorname{Re} p(\mu)gVr^-_{\lambda_1,\lambda_2}(\mu)q(\mu) \\ &\quad + 2\operatorname{Re} p(\mu)r^-_{\lambda_1,\lambda_2}(\mu)[V, h_0^-(\mu)]r^-_{\lambda_1,\lambda_2}(\mu)q(\mu). \end{aligned}$$

Recalling (6.11), we find that $[V, h_0^-(\mu)] = \partial V a^* - a\bar{\partial}V$. However, since $\operatorname{Ran} a = (\operatorname{Ker} a^*)^\perp$ we have $p(\mu)a = 0$. Hence, $p(\mu)r^-_{\lambda_1,\lambda_2}(\mu)a = r^-_{\lambda_1,\lambda_2}(\mu)p(\mu)a = 0$, and

$$p(\mu)r^-_{\lambda_1,\lambda_2}(\mu)[V, h_0^-(\mu)]r^-_{\lambda_1,\lambda_2}(\mu)q(\mu) = p(\mu)r^-_{\lambda_1,\lambda_2}(\mu)\partial V a^* r^-_{\lambda_1,\lambda_2}(\mu)q(\mu).$$

Therefore, for each $\eta \in (0,1)$ we have

$$\begin{aligned} t^-_{\lambda_1,\lambda_2}(\mu) &= p(\mu)Wp(\mu) + q(\mu)t^-_{\lambda_1,\lambda_2}(\mu)q(\mu) \\ &\quad + 2\operatorname{Re} p(\mu)r^-_{\lambda_1,\lambda_2}(\mu)V^2 r^-_{\lambda_1,\lambda_2}(\mu)q(\mu) \\ &\quad + 4\operatorname{Re} p(\mu)gVr^-_{\lambda_1,\lambda_2}(\mu)q(\mu) \\ &\quad + 2\operatorname{Re} p(\mu)r^-_{\lambda_1,\lambda_2}(\mu)\partial V a^* r^-_{\lambda_1,\lambda_2}(\mu)q(\mu) \\ &\geq p(\mu)(W - \eta r^-_{\lambda_1,\lambda_2}(\mu)(|V|^2 + |\nabla V|^2)r^-_{\lambda_1,\lambda_2}(\mu) - 2\eta g^2)p(\mu) \\ &\quad + q(\mu)(t^-_{\lambda_1,\lambda_2}(\mu) - 3\eta^{-1}r^-_{\lambda_1,\lambda_2}(\mu)V^2 r^-_{\lambda_1,\lambda_2}(\mu) \\ &\quad - \eta^{-1}r^-_{\lambda_1,\lambda_2}(\mu)aa^* r^-_{\lambda_1,\lambda_2}(\mu))q(\mu) \end{aligned}$$

and, hence,

$$\begin{aligned} n_-(-s; t^-_{\lambda_1,\lambda_2}(\mu)) &\leq n_-(-s; p(\mu)(W - \eta r^-_{\lambda_1,\lambda_2}(\mu) \\ &\qquad (|V|^2 + |\nabla V|^2)r^-_{\lambda_1,\lambda_2}(\mu) - 2\eta g^2)p(\mu)) \\ &\quad + n_-(-s; q(\mu)(t^-_{\lambda_1,\lambda_2}(\mu) - 3\eta^{-1}r^-_{\lambda_1,\lambda_2}(\mu)V^2 r^-_{\lambda_1,\lambda_2}(\mu) \\ &\quad - \eta^{-1}r^-_{\lambda_1,\lambda_2}(\mu)aa^* r^-_{\lambda_1,\lambda_2}(\mu))q(\mu)). \end{aligned} \tag{7.5}$$

At first we estimate the second term at the right-hand side of (7.5). Since we have $\lim_{\mu\to\infty} \|a^*(\mu)r^-_{\lambda_1,\lambda_2}(\mu)q(\mu)\| = 0$, the estimate

$$\eta^{-1}\|q(\mu)r^-_{\lambda_1,\lambda_2}(\mu)a(\mu)a^*(\mu)r^-_{\lambda_1,\lambda_2}(\mu)q(\mu)\| = \eta^{-1}\|a^*(\mu)r^-_{\lambda_1,\lambda_2}(\mu)\|^2 < -s/2,$$

holds for every fixed $\eta > 0$ and $s < 0$, provided that $\mu$ is great enough. Therefore,

$$\begin{aligned} n_-(-s; &q(\mu)(t^-_{\lambda_1,\lambda_2}(\mu) - 3\eta^{-1}r^-_{\lambda_1,\lambda_2}(\mu) \\ &V^2 r^-_{\lambda_1,\lambda_2}(\mu) - \eta^{-1}r^-_{\lambda_1,\lambda_2}(\mu)aa^* r^-_{\lambda_1,\lambda_2}(\mu))q(\mu)) \\ &\leq n_-(-s/2; q(\mu)(t^-_{\lambda_1,\lambda_2}(\mu) - 3\eta^{-1}r^-_{\lambda_1,\lambda_2}(\mu)V^2 r^-_{\lambda_1,\lambda_2}(\mu))q(\mu)) \\ &\leq n_+(-s/6; (3\eta^{-1} - 1)q(\mu)r^-_{\lambda_1,\lambda_2}(\mu) \\ &\qquad V^2 r^-_{\lambda_1,\lambda_2}(\mu)q(\mu)) + 2n_*(-s/6; g_0 V r^-_{\lambda_1,\lambda_2}(\mu)q(\mu)). \end{aligned} \tag{7.6}$$

Applying Lemma 4.2, we easily find that if $\mu$ is large enough, both terms at the right-hand side of (7.6) vanish.

Next, we deal with the first term at the right-hand side of (7.5). Fix $\varepsilon \in (0, -s)$, and choose $\eta$ so small that we have $2\eta g_0^2 < \varepsilon/2$. Hence,

$$\begin{aligned} n_-(-s; &p(\mu)(W - \eta r^-_{\lambda_1,\lambda_2}(\mu)(|V|^2 + |\nabla V|^2)\, r^-_{\lambda_1,\lambda_2}(\mu) - 2\eta g^2)p(\mu)) \\ &\leq n_-(-s - \varepsilon/2; p(\mu)Wp(\mu) - \eta p(\mu)r^-_{\lambda_1,\lambda_2}(\mu) \\ &\qquad (|V|^2 + |\nabla V|^2)\, r^-_{\lambda_1,\lambda_2}(\mu)p(\mu)) \\ &\leq n_-(-s - \varepsilon; p(\mu)Wp(\mu)) + n_*((\varepsilon/2\eta)^{1/2}; \\ &\qquad \sqrt{|V|^2 + |\nabla V|^2}\, r^-_{\lambda_1,\lambda_2}(\mu)p(\mu)). \end{aligned} \tag{7.7}$$

Employing Lemma 4.1 and (4.7) with $p = 2$, we get

$$n_*((\varepsilon/2\eta)^{1/2}; \sqrt{|V|^2 + |\nabla V|^2}\, r^-_{\lambda_1,\lambda_2}(\mu)p(\mu)) \leq 2c_4\mu\eta\varepsilon^{-1} \int_{\mathbb{R}^2} (|V|^2 + |\nabla V|^2)dX. \tag{7.8}$$

The combination of (7.5)–(7.8) entails that for every fixed $s < 0$, $\varepsilon \in (0, -s)$, and $\eta \in (0, \varepsilon g_0/4)$, we have

$$\limsup_{\mu\to\infty} \mu^{-1} n_-(-s; t^-_{\lambda_1,\lambda_2}(\mu)) \leq \limsup_{\mu\to\infty} \mu^{-1} n_-(-s-\varepsilon; p(\mu)t^-_{\lambda_1,\lambda_2}(\mu)p(\mu)) + 2c_4\mu\eta\varepsilon^{-1} \int_{\mathbb{R}^2} (|V|^2 + |\nabla V|^2)dX. \tag{7.9}$$

Letting $\eta \downarrow 0$, we get

$$\limsup_{\mu\to\infty} \mu^{-1} n_-(-s; t^-_{\lambda_1,\lambda_2}(\mu)) \leq \limsup_{\mu\to\infty} \mu^{-1} n_-(-s-\varepsilon; p(\mu)t^-_{\lambda_1,\lambda_2}(\mu)p(\mu)),\ \forall \varepsilon \in (0, -s). \tag{7.10}$$

Choose a sequence $\{\varepsilon_r\}_{r\geq 1}$ such that $\varepsilon_r \in (0, -s)$, $r \geq 1$, $\lim_{r\to\infty} \varepsilon_r = 0$, and $s + \varepsilon_r$, $r \geq 1$, are continuity points of $\tilde{\delta}_{\lambda_1,\lambda_2}$. Then Lemma 7.1 implies

$$\limsup_{\mu\to\infty} \mu^{-1} n_-(-s-\varepsilon_r; p(\mu)t^-_{\lambda_1,\lambda_2}(\mu)p(\mu)) = \lim_{\mu\to\infty} \mu^{-1} n_-(-s-\varepsilon_r; p(\mu)t^-_{\lambda_1,\lambda_2}(\mu)p(\mu)) = \tilde{\delta}_{\lambda_1,\lambda_2}(s+\varepsilon_r),\ \forall r \geq 1. \tag{7.11}$$

Putting together (7.10) and (7.11), we deduce the estimate

$$\limsup_{\mu\to\infty} \mu^{-1} n_-(-s; t^-_{\lambda_1,\lambda_2}(\mu)) \leq \tilde{\delta}_{\lambda_1,\lambda_2}(s+\varepsilon_r),\ \forall s < 0, \forall r \geq 1. \tag{7.12}$$

Letting $r \to \infty$ (hence, $\varepsilon_r \downarrow 0$) in (7.12), and taking into account that by assumption $s$ is a continuity point of $\tilde{\delta}_{\lambda_1,\lambda_2}$, we get

$$\limsup_{\mu\to\infty} \mu^{-1} n_-(-s; t^-_{\lambda_1,\lambda_2}(\mu)) \leq \tilde{\delta}_{\lambda_1,\lambda_2}(s). \tag{7.13}$$

The combination of (7.4) and (7.13) yields (7.3). □

PROPOSITION 7.2. — *Let $m = 2$, $V \in L^2(\mathbb{R}^2)$. Suppose that (1.1) holds. Let $0 < \lambda_1 < \lambda_2$. Assume that $\lambda_1$ and $\lambda_2$ are continuity points of $\delta$ (see (2.1)). Then (2.3) is valid.*

*Proof.* — By Lemma 3.4 we have

$$\mathcal{N}(\lambda_1, \lambda_2; h(\mu)) = n_-(1; t_{\lambda_1,\lambda_2}(\mu)) \tag{7.14}$$

where $t_{\lambda_1,\lambda_2}(\mu) := \tilde{\mathcal{T}}(\lambda_1, \lambda_2; h_0(\mu), V)$ (see (3.7)). Evidently,

$$n_-(1; t_{\lambda_1,\lambda_2}(\mu)) = n_-(1; t^+_{\lambda_1,\lambda_2}(\mu)) + n_-(1; t^-_{\lambda_1,\lambda_2}(\mu)), \tag{7.15}$$

where $t^+_{\lambda_1,\lambda_2}(\mu) := \tilde{\mathcal{T}}(\lambda_1, \lambda_2; h_0^+(\mu), V)$. Let us estimate the first term at the right-hand side of (7.15). Obviously,

$$n_-(1; t^+_{\lambda_1,\lambda_2}(\mu)) \leq n_*(1/\sqrt{3}; Vr^+_{\lambda_1,\lambda_2}(\mu)) + 2n_*(1/3; g_0|V|r^+_{\lambda_1,\lambda_2}(\mu)). \tag{7.16}$$

By Lemma 4.3, we find that both terms at the right-hand side of (7.16) vanish for sufficiently large $\mu$.

Next, we pass to the estimation of the second term at the right-hand side of (7.15). Choose a sequence $\{\eta_l\}_{l\geq 1}$, $\eta_l > 0$, $l \geq 1$, $\lim_{l\to\infty} \eta_l = 0$, and write $V = V_0 + V_1$ where $V_0 = V_{0,l} \in C_0^\infty(\mathbb{R}^2)$, $V_1 = V_{1,l} \in L^2(\mathbb{R}^2)$, and $\|V_{1,l}\|_{L^2(\mathbb{R}^2)} \leq \eta_l$. Introduce the operator $t^-_{0;\lambda_1,\lambda_2} := \tilde{\mathcal{T}}(\lambda_1, \lambda_2; h_0^-(\mu), V_0)$. Similarly, define the functions $\tilde{\delta}_{j;\lambda_1,\lambda_2}(s)$, $j = 0, 1$, $s \neq 0$, replacing $V$ by $V_{j,l}$ in (7.1). Finally, define the function $\delta_0(\lambda)$, $\lambda \neq 0$, substituting in (2.1) $V$ for $V_{0,l}$, $l \geq 1$.

Choose a sequence $\{\varepsilon_r\}_{r\geq 1}$, $\varepsilon_r \in (0, 1/2)$, $\lim_{r\to\infty} \varepsilon_r = 0$, so that $-1\pm\varepsilon_r$, $r \geq 1$, are continuity points of all the functions $\tilde{\delta}_{0;\lambda_1,\lambda_2}(s)$, $l \geq 1$. Evidently, we have

$$\begin{aligned} t^-_{\lambda_1,\lambda_2}(\mu) \geq (1-\varepsilon_r^2)t^-_{0;\lambda_1,\lambda_2} &+ (1-\varepsilon_r^{-2})r^-_{\lambda_1,\lambda_2}(\mu)V_1^2 r^-_{\lambda_1,\lambda_2}(\mu) \\ &+ 2\operatorname{Re} gV_1 r^-_{\lambda_1,\lambda_2}(\mu) + 2\varepsilon_r^2 \operatorname{Re} gV_0 r^-_{\lambda_1,\lambda_2}(\mu), \end{aligned}$$

$$\begin{aligned} t^-_{\lambda_1,\lambda_2}(\mu) \leq (1+\varepsilon_r^2)t^-_{0;\lambda_1,\lambda_2} &+ (1+\varepsilon_r^{-2})r^-_{\lambda_1,\lambda_2}(\mu)V_1^2 r^-_{\lambda_1,\lambda_2}(\mu) \\ &+ 2\operatorname{Re} gV_1 r^-_{\lambda_1,\lambda_2}(\mu) - 2\varepsilon_r^2 \operatorname{Re} gV_0 r^-_{\lambda_1,\lambda_2}(\mu). \end{aligned}$$

Therefore,

$$\begin{aligned} n_-(1; t^-_{\lambda_1,\lambda_2}(\mu)) &\leq n_-(1-\varepsilon_r; (1-\varepsilon_r^2)t^-_{0;\lambda_1,\lambda_2}) \\ &\quad + n_+(\varepsilon_r/3; (\varepsilon_r^{-2}-1)r^-_{\lambda_1,\lambda_2}(\mu)V_1^2 r^-_{\lambda_1,\lambda_2}(\mu)) \\ &\quad + n_-(\varepsilon_r/3; 2\operatorname{Re} gV_1 r^-_{\lambda_1,\lambda_2}(\mu)) \\ &\quad + n_-(\varepsilon_r/3; 2\varepsilon_r^2 \operatorname{Re} gV_0 r^-_{\lambda_1,\lambda_2}(\mu)) \\ &\leq n_-(1-\varepsilon_r; t^-_{0;\lambda_1,\lambda_2}) + 3n_*(1; c^+_{16}\varepsilon_r^{-3/2} V_1 r^-_{\lambda_1,\lambda_2}(\mu)) \\ &\quad + 2n_*(1; c^+_{17}\varepsilon_r V_0 r^-_{\lambda_1,\lambda_2}(\mu)), \end{aligned}$$

and

$$\begin{aligned} n_-(1; t^-_{\lambda_1,\lambda_2}(\mu)) &\geq n_-(1+\varepsilon_r; (1+\varepsilon_r^2)t^-_{0;\lambda_1,\lambda_2}) \\ &\quad - n_+(\varepsilon_r/3; (\varepsilon_r^{-2}+1)r^-_{\lambda_1,\lambda_2}(\mu)V_1^2 r^-_{\lambda_1,\lambda_2}(\mu)) \\ &\quad - n_+(\varepsilon_r/3; 2\mathrm{Re}\, gV_1 r^-_{\lambda_1,\lambda_2}(\mu)) \\ &\quad - n_-(\varepsilon_r/3; 2\varepsilon_r^2 \mathrm{Re}\, gV_0 r^-_{\lambda_1,\lambda_2}(\mu)) \\ &\geq n_-(1+\varepsilon_r; t^-_{0;\lambda_1,\lambda_2}) - 3n_*(1; c^-_{16}\varepsilon_r^{-3/2}V_1 r^-_{\lambda_1,\lambda_2}(\mu)) \\ &\quad - 2n_*(1; c^-_{17}\varepsilon_r V_0 r^-_{\lambda_1,\lambda_2}(\mu)), \end{aligned}$$

with $c^\pm_{16}$ and $c^\pm_{17}$ independent of $\mu$, $\delta_l$, and $\varepsilon_r$. Utilizing Proposition 7.1 and Lemma 4.1, we get

$$\limsup_{\mu\to\infty} \mu^{-1} n_-(1; t^-_{\lambda_1,\lambda_2}(\mu)) \leq \tilde{\delta}_{0;\lambda_1,\lambda_2}(-1+\varepsilon_r) + c^+_{18}\left(\varepsilon_r^{-3}\eta_l^2 + \varepsilon_r^2 \int_{\mathbb{R}^2} |V_0|^2 dX\right), \tag{7.17}$$

$$\liminf_{\mu\to\infty} \mu^{-1} n_-(1; t^-_{\lambda_1,\lambda_2}(\mu)) \geq \tilde{\delta}_{0;\lambda_1,\lambda_2}(-1-\varepsilon_r) - c^-_{18}\left(\varepsilon_r^{-3}\eta_l^2 + \varepsilon_r^2 \int_{\mathbb{R}^2} |V_0|^2 dX\right), \tag{7.18}$$

with $c^\pm_{18}$ independent of $\delta_l$ and $\varepsilon_r$.

Straightforward but tedious calculations show that the estimates

$$\begin{aligned} \tilde{\delta}_{0;\lambda_1,\lambda_2}(-1+\varepsilon_r) &\leq \tilde{\delta}_{\lambda_1,\lambda_2}(-1+2\varepsilon_r) + \tilde{\delta}_{\lambda_1,\lambda_2}(-c^+_{19}\varepsilon_r^{-1}) \\ &\quad + \tilde{\delta}_{1;\lambda_1,\lambda_2}(-c^+_{19}\varepsilon_r^{3/2}) - \tilde{\delta}_{1;\lambda_1,\lambda_2}(c^+_{19}\varepsilon_r^{3/2}), \end{aligned} \tag{7.19}$$

$$\begin{aligned} \tilde{\delta}_{0;\lambda_1,\lambda_2}(-1-\varepsilon_r) &\geq \tilde{\delta}_{\lambda_1,\lambda_2}(-1-2\varepsilon_r) + \tilde{\delta}_{\lambda_1,\lambda_2}(c^-_{19}\varepsilon_r^{-1}) \\ &\quad - \tilde{\delta}_{1;\lambda_1,\lambda_2}(-c^-_{19}\varepsilon_r^{3/2}) + \tilde{\delta}_{1;\lambda_1,\lambda_2}(c^-_{19}\varepsilon_r^{3/2}), \end{aligned} \tag{7.20}$$

hold with $c^\pm_{19}$ independent of $\delta_l$ and $\varepsilon_r$. Using the elementary estimate

$$\int_{\mathbb{R}^2} \theta(|U(X)|-\lambda)b(X)dX \leq \lambda^{-2}\int_{\mathbb{R}^2} |U(X)|^2 b(X)dX,\ U \in L^2(\mathbb{R}^2),\ \lambda > 0,$$

we conclude that (7.19)–(7.20) yield

$$\tilde{\delta}_{0;\lambda_1,\lambda_2}(-1+\varepsilon_r) \leq \tilde{\delta}_{\lambda_1,\lambda_2}(-1+2\varepsilon_r) + c^+_{20}\left(\varepsilon_r^{-3}\eta_l^2 + \varepsilon_r^2 \int_{\mathbb{R}^2} |V|^2 dX\right), \tag{7.21}$$

$$\tilde{\delta}_{0;\lambda_1,\lambda_2}(-1-\varepsilon_r) \geq \tilde{\delta}_{\lambda_1,\lambda_2}(-1-2\varepsilon_r) - c^-_{20}\left(\varepsilon_r^{-3}\eta_l^2 + \varepsilon_r^2 \int_{\mathbb{R}^2} |V|^2 dX\right), \tag{7.22}$$

with $c^\pm_{20}$ independent of $\delta_l$ and $\varepsilon_r$.

Letting at first $l \to \infty$ (hence, $\eta_l \downarrow 0$), and then $r \to \infty$ (hence, $\varepsilon_r \downarrow 0$), in (7.17), (7.18), (7.21), and (7.22), and taking into account the fact that the continuity of $\delta$ at $\lambda_1$ and $\lambda_2$ is equivalent to the continuity of $\tilde{\delta}_{\lambda_1,\lambda_2}$ at $-1$, we get

$$\lim_{\mu\to\infty} \mu^{-1} n_-(1; t^-_{\lambda_1,\lambda_2}(\mu)) = \tilde{\delta}_{\lambda_1,\lambda_2}(-1) = \delta(\lambda_2) - \delta(\lambda_1). \tag{7.23}$$

Now, (7.14)–(7.16), and (7.23) entail (2.3). □

In order to complete the proof of Theorem 2.2, we choose $\varepsilon > 0$, such that $\lambda_1 - 3\varepsilon > 0$ and $\lambda_1 + 3\varepsilon < \lambda_2 - 3\varepsilon$, and write $V = V_1 + V_2$ with $V_1 \in L^2(\mathbb{R}^2)$, $\sup_{X\in\mathbb{R}^2} |V_2(X)| \leq \varepsilon$. Define the function $\delta_1(\lambda)$, $\lambda \neq 0$, substituting $V$ for $V_1$ in (2.1). Choose $\eta > 0$ such that $\eta \in (0,\varepsilon)$, and $\lambda_j \pm \varepsilon \pm \eta$, $j = 1,2$, are continuity points of $\delta_1$. Evidently,

$$\begin{aligned}\mathcal{N}(\lambda_1,\lambda_2; h(\mu)) &\leq \mathcal{N}(\lambda_1 - \varepsilon, \lambda_2 + \varepsilon; h_0(\mu) + V_1)\\ &\leq \mathcal{N}(\lambda_1 - \varepsilon - \eta, \lambda_2 + \varepsilon + \eta; h_0(\mu) + V_1),\end{aligned}$$

$$\begin{aligned}\mathcal{N}(\lambda_1,\lambda_2; h(\mu)) &\geq \mathcal{N}(\lambda_1 + \varepsilon, \lambda_2 - \varepsilon; h_0(\mu) + V_1)\\ &\geq \mathcal{N}(\lambda_1 + \varepsilon + \eta, \lambda_2 - \varepsilon - \eta; h_0(\mu) + V_1).\end{aligned}$$

Utilizing Proposition 7.2, we get

$$\begin{aligned}\limsup_{\mu\to\infty} \mu^{-1}\mathcal{N}(\lambda_1,\lambda_2; h(\mu)) &\leq \delta_1(\lambda_2 + \varepsilon + \eta) - \delta_1(\lambda_1 - \varepsilon - \eta)\\ &\leq \delta_1(\lambda_2 + 2\varepsilon) - \delta_1(\lambda_1 - 2\varepsilon) \leq \delta(\lambda_2 + 3\varepsilon) - \delta(\lambda_1 - 3\varepsilon),\end{aligned} \tag{7.24}$$

$$\begin{aligned}\liminf_{\mu\to\infty} \mu^{-1}\mathcal{N}(\lambda_1,\lambda_2; h(\mu)) &\geq \delta_1(\lambda_2 - \varepsilon - \eta) - \delta_1(\lambda_1 + \varepsilon + \eta)\\ &\geq \delta_1(\lambda_2 - 2\varepsilon) - \delta_1(\lambda_1 + 2\varepsilon) \geq \delta(\lambda_2 - 3\varepsilon) - \delta(\lambda_1 + 3\varepsilon).\end{aligned} \tag{7.25}$$

Recalling that by assumption $\lambda_1$ and $\lambda_2$ are continuity points of $\delta$, we let $\varepsilon \downarrow 0$ in (7.24) and (7.25), and conclude that (2.3) holds for general $V \in \mathcal{L}_2$.

## 8. Proof of Theorem 2.3.

PROPOSITION 8.1. — *Let $m = 3$, $V \in C_0^\infty(\mathbb{R}^3)$, $\lambda < 0$. Suppose that (1.1) and (1.7) hold. Assume that $s < 0$ is a continuity point of $\tilde{\mathcal{D}}$ (see (5.5)). Then we have*

$$\lim_{\mu\to\infty} \mu^{-1} n_-(-s; P(\mu)T^-_\lambda(\mu)P(\mu)) = \tilde{\mathcal{D}}_\lambda(s) \tag{8.1}$$

*where the operator $T^-_\lambda(\mu)$ is defined in (6.17).*

*Proof.* — Asymptotics (8.1) follow immediately from Proposition 6.2 and Lemma 3.2 with $T(\mu) = P(\mu)T_\lambda^-(\mu)P(\mu)$, $t_0 = |\lambda|^{-1}\|V\|_{L^\infty(\mathbb{R}^3)}$, $p = 1$, and $\nu = \tilde{\mathcal{D}}_\lambda$. □

PROPOSITION 8.2. — *Let $m = 3$, $V \in L^{3/2}(\mathbb{R}^3)$. Suppose that (1.1) and (1.7) hold. Assume that $\lambda < 0$ is a continuity point of $\mathcal{D}$. Then (2.6) is valid.*

*Proof.* — By Lemma 3.3 we have

$$N(\lambda; H(\mu)) = n_-(1; T_\lambda(\mu)) \tag{8.2}$$

where $T_\lambda(\mu) := \mathcal{T}(\lambda; H_0(\mu), V)$ (see (3.4)). Obviously,

$$n_-(1; T_\lambda(\mu)) = n_-(1; T_\lambda^-(\mu)) + n_-(1; T_\lambda^+(\mu)) \tag{8.3}$$

where $T_\lambda^+(\mu) = \mathcal{T}(\lambda; H_0^+(\mu), V)$. Since $n_-(1; T_\lambda^+(\mu)) \leq n_*(1; |V|^{1/2}R_\lambda^+(\mu))$, Lemma 4.6 implies that for sufficiently large $\mu$ we have

$$n_-(1; T_\lambda^+(\mu)) = 0. \tag{8.4}$$

Further, we estimate the second term at the right-hand side of (8.3).

Choose a sequence $\{\eta_l\}_{l\geq 1}$ $\eta_l > 0$, $l \geq 1$, $\lim_{l\to\infty} \eta_l = 0$, and write $V = V_0 + V_1$ with $V_0 = V_{0,l} \in C_0^\infty(\mathbb{R}^2)$, $V_1 = V_{1,l} \in L^{3/2}(\mathbb{R}^2)$, $\|V_1\|_{L^{3/2}(\mathbb{R}^2)} \leq \eta_l$. Introduce the operators $T_{j,\lambda}^-(\mu)$, $j = 0, 1$, substituting $V$ for $V_j$ in (6.17). Similarly, define the function $\tilde{\mathcal{D}}_{0,\lambda}$, $\lambda \neq 0$, replacing $\tau_\lambda(X)$ by $\tau_{0;\lambda}(X)$ (see (5.4) and (5.7)) in (5.5).

Choose a sequence $\{\varepsilon_r\}_{r\geq 1}$, $\varepsilon_r \in (0, 1/3)$, $\lim_{r\to\infty} \varepsilon_r = 0$, such that $-1 \pm \varepsilon_r$, $r \geq 1$, are continuity points of all functions $\tilde{\mathcal{D}}_{1,\lambda}$. Evidently,

$$\begin{aligned} n_-(1; T_\lambda^-(\mu)) &\geq n_-(1; P(\mu)T_\lambda^-(\mu)P(\mu)) \\ &= n_-(1; P(\mu)T_{0,\lambda}^-(\mu)P(\mu) + P(\mu)T_{1,\lambda}^-(\mu)P(\mu)) \\ &\geq n_-(1+\varepsilon_r; P(\mu)T_{0,\lambda}^-(\mu)P(\mu)) - n_+(\varepsilon_r; P(\mu)T_{1,\lambda}^-(\mu)P(\mu)). \end{aligned} \tag{8.5}$$

Utilizing Corollary 4.4, we get

$$\begin{aligned} n_+(\varepsilon_r; P(\mu)T_{1,\lambda}^-(\mu)P(\mu)) &\leq n_*(\varepsilon_r^{1/2}; |V_1|^{1/2}R_\lambda^-(\mu)P(\mu))) \\ &\leq c_9\mu\varepsilon_r^{-3/2}\int_{\mathbb{R}^3} |V_1(\mathbf{X})|^{3/2}d\mathbf{X}. \end{aligned} \tag{8.6}$$

Recalling Proposition 8.1, we find that (8.5)–(8.6) yield

$$\liminf_{\mu\to\infty} \mu^{-1}n_-(1; T_\lambda^-(\mu)) \geq \tilde{\mathcal{D}}_{0;\lambda}(-1-\varepsilon_r) - c_9\varepsilon_r^{-3/2}\eta_l^{3/2}. \tag{8.7}$$

On the other hand we have

$$\begin{aligned}T_\lambda^-(\mu) &= T_{0,\lambda}^-(\mu) + T_{1,\lambda}^-(\mu) = P(\mu)T_{0,\lambda}^-(\mu)P(\mu)\\ &\quad + Q(\mu)T_{0,\lambda}^-(\mu)Q(\mu) + 2\,\mathrm{Re}\,P(\mu)T_{0,\lambda}^-(\mu)Q(\mu) + T_{1,\lambda}^-(\mu)\\ &\geq P(\mu)\left(T_{0,\lambda}^-(\mu) - \varepsilon_r^2 R_\lambda^-(\mu)|V_0|R_\lambda^-(\mu)\right)P(\mu)\\ &\quad + Q(\mu)\left(T_{0,\lambda}^-(\mu) - \varepsilon_r^{-2} R_\lambda^-(\mu)|V_0|R_\lambda^-(\mu)\right)Q(\mu) + T_{1,\lambda}^-(\mu).\end{aligned}$$

Therefore,

$$\begin{aligned}n_-(1;T_\lambda^-(\mu)) \leq & n_-\left(1-\varepsilon_r/2; P(\mu)\left(T_{0,\lambda}^-(\mu)-\varepsilon_r^2 R_\lambda^-(\mu)|V_0|R_\lambda^-(\mu)\right)P(\mu)\right)\\ & + n_-\left(1-\varepsilon_r/2; Q(\mu)\left(T_{0,\lambda}^-(\mu)-\varepsilon_r^{-2} R_\lambda^-(\mu)|V_0|R_\lambda^-(\mu)\right)Q(\mu)\right)\\ & + n_-\left(\varepsilon_r/2; T_{1,\lambda}^-(\mu)\right) \leq n_-\left(1-\varepsilon_r; P(\mu)T_{0,\lambda}^-(\mu)P(\mu)\right)\\ & + n_+\left(1; 2\varepsilon_r P(\mu)R_\lambda^-(\mu)|V_0|R_\lambda^-(\mu)P(\mu)\right)\\ & + n_-\left(\frac{1}{2}-\frac{\varepsilon_r}{4}; Q(\mu)T_{0,\lambda}^-(\mu)Q(\mu)\right)\\ & + n_+\left(\frac{1}{2}-\frac{\varepsilon_r}{4}; \varepsilon_r^{-2}Q(\mu)R_\lambda^-(\mu)|V_0|R_\lambda^-(\mu)Q(\mu)\right)\\ & + n_-\left(\varepsilon_r/2; T_{1,\lambda}^-(\mu)\right).\end{aligned} \tag{8.8}$$

Proposition 8.1 entails

$$\lim_{\mu\to\infty} \mu^{-1} n_-\left(1-\varepsilon_r; P(\mu)T_{0,\lambda}^-(\mu)P(\mu))\right) = \tilde{\mathcal{D}}_{0,\lambda}(-1+\varepsilon_r). \tag{8.9}$$

It follows easily from Lemma 4.5 that

$$\begin{aligned}& n_-\Big(\frac{1}{2}-\frac{\varepsilon_r}{4}; Q(\mu)T_{0,\lambda}^-(\mu)Q(\mu)\Big)\\ &\qquad = n_+\Big(\frac{1}{2}-\frac{\varepsilon_r}{4}; \varepsilon_r^{-2}Q(\mu)R_\lambda^-(\mu)|V_0|R_\lambda^-(\mu)Q(\mu)\Big) = 0,\end{aligned} \tag{8.10}$$

provided that $\mu$ is large enough.

Further, employing Corollary 4.4, we get

$$\begin{aligned}&\limsup_{\mu\to\infty} \mu^{-1} n_+\left(1; 2\varepsilon_r P(\mu)R_\lambda^-(\mu)|V_0|R_\lambda^-(\mu)P(\mu)\right)\\ &\qquad \leq 2^{3/2} c_9(3)\varepsilon_r^{3/2}\int_{\mathbb{R}^3}|V_0(\mathbf{X})|^{3/2}d\mathbf{X}.\end{aligned} \tag{8.11}$$

Finally, Corollary 4.5 entails

$$\limsup_{\mu\to\infty} \mu^{-1} n_-(\varepsilon_r/2; T_{1,\lambda}(\mu)) \leq 8^{3/2} c_9(3)\varepsilon_r^{-3/2}\int_{\mathbb{R}^3}|V_1(\mathbf{X})|^{3/2}d\mathbf{X}. \tag{8.12}$$

The combination of (8.8)–(8.12) yields

$$\limsup_{\mu\to\infty} \mu^{-1} n_-(1; T_\lambda^-(\mu)) \leq \tilde{\mathcal{D}}_{0,\lambda}(-1+\varepsilon_r) + c_9'' \left( \varepsilon_r^{-3/2}\eta_l^{3/2} + \varepsilon_r^{3/2} \int_{\mathbb{R}^3} |V_0(\mathbf{X})|^{3/2} d\mathbf{X} \right) \tag{8.13}$$

with $c_9''$ independent of $\varepsilon_r$ and $\eta_l$. Obviously,

$$\tilde{\mathcal{D}}_{0,\lambda}(-1+\varepsilon_r) \leq \tilde{\mathcal{D}}_\lambda(-1+2\varepsilon_r) + \frac{1}{2\pi}\int_{\mathbb{R}^2} n_-(\varepsilon_r; \tau_{1,\lambda}(X))\, dX, \tag{8.14}$$

$$\tilde{\mathcal{D}}_{0,\lambda}(-1-\varepsilon_r) \geq \tilde{\mathcal{D}}_\lambda(-1-2\varepsilon_r) - \frac{1}{2\pi}\int_{\mathbb{R}^2} n_+(\varepsilon_r; \tau_{1,\lambda}(X))\, dX. \tag{8.15}$$

By analogy with (5.9) we get

$$\int_{\mathbb{R}^2} n_\pm(\varepsilon_r; \tau_{1,\lambda}(X))\, dX \leq c_{14}(\lambda)\varepsilon_r^{-3/2}\eta_l^{3/2}. \tag{8.16}$$

The combination of (8.7), (8.15), and (8.16) implies

$$\liminf_{\mu\to\infty} \mu^{-1} n_-(1; T_\lambda^-(\mu)) \geq \mathcal{D}_\lambda(-1-2\varepsilon_r) - c_{21}^- \varepsilon_r^{-3/2}\eta_l^{3/2}, \tag{8.17}$$

while the combination of (8.13), (8.14) and (8.16) implies

$$\limsup_{\mu\to\infty} \mu^{-1} n_-(1; T_\lambda^-(\mu)) \leq \tilde{\mathcal{D}}_\lambda(-1+2\varepsilon_r) + c_{21}^+ \left( \varepsilon_r^{-3/2}\eta_l^{3/2} + \varepsilon_r^{3/2} \int_{\mathbb{R}^3} |V_0(\mathbf{X})|^{3/2} d\mathbf{X} \right), \tag{8.18}$$

where the quantities $c_{21}^-$ and $c_{21}^+$ are independent of $\varepsilon_r$ and $\eta_l$. Letting at first $l \to \infty$, and then $r \to \infty$ in (8.17) and (8.18), taking into account that by assumption $\lambda$ is a continuity point of $\mathcal{D}$, and recalling Lemma 5.2 (ii), we get

$$\lim_{\mu\to\infty} \mu^{-1} n_-(1; T_\lambda^-(\mu)) = \tilde{\mathcal{D}}_\lambda(-1) = \mathcal{D}(\lambda),$$

which together with (8.2)–(8.4) yields (2.6). □

Finally, the deduction of Theorem 2.3 where we consider general $V \in \mathcal{L}_{3/2}$ from Proposition 8.2 where potentials $V \in L^{3/2}(\mathbb{R}^3)$ are treated, is quite similar (and simpler) to the deduction of Theorem 2.2 from Proposition 7.2 (see the end of Section 7), and therefore we omit the details.

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