ANNALES DE L'INSTITUT FOURIER

GEORGES DLOUSSKY
KARL OELJEKLAUS

# Vector fields and foliations on compact surfaces of class $\mathrm{VII}_0$

*Annales de l'institut Fourier*, tome 49, nº 5 (1999), p. 1503-1545
<http://www.numdam.org/item?id=AIF_1999__49_5_1503_0>

Ann. Inst. Fourier, Grenoble
**49**, 5 (1999), 1503–1545

# VECTOR FIELDS AND FOLIATIONS ON COMPACT SURFACES OF CLASS $\mathrm{VII}_0$

by G. DLOUSSKY and K. OELJEKLAUS



A *surface* is a complex manifold of dimension 2. We denote by $b_i(S)$ the i-th Betti number of $S$. All surfaces with a GSS are supposed to be minimal.

## 0. Introduction.

A compact complex surface belongs to the class $\mathrm{VII}_0$ of Kodaira if it is minimal and the first Betti number satisfies $b_1 = 1$. The classification of this class is incomplete when $b_2 > 0$. All known examples contain global spherical shells (GSS), that is to say there exists in $S$ an open set $V \subset S$ such that $S \setminus V$ is connected and $V$ is biholomorphic to an open neighbourhood of the sphere $S^3$ in $\mathbb{C}^2 \setminus \{0\}$. The case $b_2(S) \geqslant 1$ has been investigated in several papers ([18], [5], [6], [7], [19], [24], [10], [25] and

*Keywords:* Compact complex surface – Class $\mathrm{VII}_0$ – Holomorphic vector field – Singular holomorphic foliation.
*Math. classification:* 32J15 – 32L30 – 57R30.

others). A surface with a GSS contains exactly $n := b_2(S)$ rational curves $D_0, \dots, D_{n-1}$, each of them being regular or with a double point. We define $\sigma(S) := -\sum_{i=0}^{n-1} D_i^2 + 2\,\mathrm{Card}\{\text{double points}\}$. By [5], $2n \leqslant \sigma(S) \leqslant 3n$.

A surface $S$ with GSS is a primary Hopf surface if and only if $b_2 = 0$. In the case $b_2(S) \geqslant 1$ the construction is, like for Hopf surfaces, quite simple, the description of the geometric properties is deeply related to the study of normal forms for singular germs of mappings $F = \Pi\sigma : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ which factorize through a finite number of blowing-ups.

If $\sigma(S) = 2n$, then $S$ is an "exceptional compactification " of an affine line bundle over an elliptic curve. These surfaces are well known (see [10], [8]).

If $\sigma(S) = 3n$, then the surface $S$ is called an Inoue-Hirzebruch surface (see [7], [17] and [24]).

It is well known that primary Hopf surfaces admit holomorphic vector fields and therefore singular holomorphic foliations.

In [8] the situation of generic and Inoue surfaces $S$ has been investigated: They all admit a unique global singular foliation. It is induced by a global vector field (in fact by a holomorphic $\mathbb{C}^\star$-action) if and only if $S$ is an Inoue surface. The crucial point is the construction of normal forms for the associated germs $F = \Pi\sigma$. These germs are exactly those for which the trace $\mathrm{tr}(S) = \mathrm{tr}\, DF(0)$ of the tangent mapping satisfies $0 < |\mathrm{tr}(S)| < 1$.

This article is devoted to the more complicated situation of surfaces with $\mathrm{tr}(S) = 0$, i.e. $2n < \sigma(S) \leqslant 3n$, or to the case of germs $F = \Pi\sigma$ where the sequence of blowing-ups is not generic. Our main result is the following

THEOREM. — *Let $S$ be a minimal compact complex surface with a GSS. Then there is always a global singular holomorphic foliation on $S$. Furthermore we have:*

1) *If $b_2(S) \geqslant 1$, then $S$ admits at most two foliations. There are two foliations if and only if $S$ is an Inoue-Hirzebruch surface.*

2) *If $2n < \sigma_n(S) < 3n$ and there exists a numerically anticanonical divisor (see Section 4), there exists a logarithmic deformation of $S$ into a surface admitting a global non-trivial vector field.*

We remark that this result contributes to the problem of classifying surfaces with non-trivial global holomorphic vector fields (see also [3], [11], [15] and references in these papers).

In [16] J. Hubbard and W. Oberste-Vorth study the dynamical system associated to a Henon automorphism $H$ of $\mathbb{C}^2$. The attraction bassin $U_+$ of $H$ may be completed with an infinite family of rational curves to a manifold $M$. The quotient of $M$ by the infinite cyclic group generated by $H$ is a compact surface $S$ with GSS, $b_2(S) = 3$ and $\mathrm{tr}(S) = 0$ (see [9]). This article may be considered as a generalization of [16], since we obtain similar results for every second Betti number $b_2 > 0$ and every germ $F = \Pi\sigma$.

The paper is organized as follows:

Section 1 introduces the basic notions which will be repetedly used.

In Section 2 a precise description of the quadratic transformations associated to singular and regular sequences of self-intersections allows us to define an invariant $k(S) \in \mathbb{N}$ having the following property: For curves $C$ such that $\hat{O}_C \in \hat{S}_C$ is not the intersection of two rational curves (see Section 1), there is a holomorphic function $f_C$ in a neighbourhood of $\hat{O}_C$ which satisfies the functional equation

$$f_C(F_C) = f_C^{k(S)}.$$

This function yields readily:

- a global singular foliation $\mathcal{F}$ on $S$,
- a twisted closed logarithmic 1-form $\omega \in H^0(S, \Omega(\operatorname{Log} D) \otimes L^{k(S)})$,
- a plurisubharmonic function $\tilde{G}$ (called Green function) on the universal covering space $\tilde{S}$ of $S$, which is pluriharmonic outside the rational curves and unique up to a multiplicative positive contant,
- a first step towards the classification of (super attractive) singular germs of mappings $F = \Pi\sigma : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with two zero eigenvalues.

Finally we describe the leaves of the foliation in the complement of the rational curves. They are isomorphic to $\mathbb{C}$ and dense in the level sets of the Green function. Here a solenoid phenomenon similar to that in [16] occurs.

Section 3 is intended to adapt the Baum-Bott, Camacho-Sad and Brunella-Suwa formulas for singular foliations to the case of surfaces containing GSS. These formulas provide in Section 4 two linear systems with coefficients in $\mathbb{Z}$. The first system gives an equivalent condition for the existence of a (positive) numerically anticanonical divisor. The second gives a necessary and sufficient condition for the existence of a positive divisor $D_\theta$ of zeros of a twisted holomorphic vector field $\theta \in H^0(S, \Theta \otimes \mathcal{O}(-D_\theta) \otimes L), L \in \mathrm{Pic}^0(S)$.

Hence we obtain numerical obstructions for the existence of sections $H^0(S, \Theta \otimes \mathcal{O}(-D_\theta) \otimes L)$ and $H^0(S, -K \otimes \mathcal{O}(-D_{-K}) \otimes L)$. The relation $D_{-K}^{\mathbb{Q}} = D_\theta^{\mathbb{Q}} + D$, where $D$ is the sum of all rational curves shows that $D_{-K}^{\mathbb{Q}}$ is a divisor if and only if $D_\theta^{\mathbb{Q}}$ is a divisor. If there is no numerical obstruction, an explicit parametrization of the flat line bundles $\mathrm{Pic}^0(S)$ by $\mathbb{C}^\star$ allows to find a unique complex number $\kappa$ such that $H^0(S, K^{-1} \otimes L^\kappa) \neq 0$. Considering logarithmic deformations $\mathcal{S} \to U$, we obtain a holomorphic function $\kappa$ on $U$.

In Section 5 we prove the existence of twisted vector fields if there is no numerical obstruction. A twisted vector field $\theta \in H^0(S, \Theta \otimes \mathcal{O}(-D_\theta) \otimes L^{\lambda(S)})$ is a vector field if and only if the flat line bundle $L^{\lambda(S)}$ is trivial. Given a surface $S$, we embed $S$ in a logarithmic family $\mathcal{S} \to \mathbb{C}^\star$ such that there is a non-constant holomorphic function $\lambda : \mathbb{C}^\star \to \mathbb{C}^\star$ with $\lambda(u) = \lambda(S_u)$. This function being surjective, the flat line bundle is trivial over the (non-empty) hypersurface $\{\lambda = 1\}$. Consequently, for surfaces over this hypersurface there are global holomorphic vector fields. We finish the section by computing the fundamental groups. In fact $\pi_1(\tilde{S} \setminus \tilde{D}) \simeq \mathbb{Z}\left[\frac{1}{k(S)}\right]$ and $\pi_1(S \setminus D) \simeq \mathbb{Z}\left[\frac{1}{k(S)}\right] \rtimes \mathbb{Z}$. We endow the universal covering space $Y$ of $S \setminus D$ with a non vanishing vector field tangent to our foliation with leaves isomorphic to $\mathbb{C}$. Using the Green function we show that $Y$ is a Riemann domain over $\Delta \times \mathbb{C}$.

## 1. Basic constructions.

### 1.1. Surfaces with global spherical shells.

In this section we recall notations and results from [5]. Details and proofs may be found there.

Let $F = \Pi\sigma = \Pi_0 \cdots \Pi_{n-1}\sigma : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be the contracting germ given by the following data. The map $\Pi_i$ is the quadratic transformation of the point $O_{i-1}$ where $O_i \in C_i = \Pi_i^{-1}(O_{i-1})$ for $0 \leqslant i \leqslant n-1$ and $O_{-1} = 0 \in \mathbb{C}^2$. The map $\sigma$ is a germ of an isomorphism with $\sigma(0) = O_{n-1}$. We associate to the germ $F$ a compact complex surface in the following way. We have a sequence of blowing-ups over the ball $B$

$$B_{n-1} \xrightarrow{\Pi_{n-1}} \cdots \longrightarrow B_i \xrightarrow{\Pi_i} B_{i-1} \longrightarrow \cdots \longrightarrow B_0 \xrightarrow{\Pi_0} B.$$

One can suppose that $\sigma$ is defined in a neighbourhood of the closed unit ball $\bar{B} \subset \mathbb{C}^2$ and that $\sigma(B)$ is relatively compact in $B_{n-1}$. Let $B' \subset\subset B$ be a slightely smaller ball. We remove the closed set $\overline{\sigma(B')}$ from $B_{n-1}$. Then one identifies isomorphic neighbourhoods of $\Sigma = \Pi^{-1}(\partial B)$ and $\sigma(\partial B)$ by $\sigma\Pi$ and obtains a minimal compact complex surface with a GSS, denoted by $S = S(\Pi, \sigma)$. The Betti numbers are $b_1(S) = 1$ and $b_2(S) = n$. The exceptional curve of the first kind in $B_{n-1}$ becomes a non-singular rational curve with self-intersection $\leqslant -2$ or a singular rational curve in $S(\Pi, \sigma)$, since $\sigma^{-1}(O_{n-1}) = 0$.

Let $F$ and $F'$ be two germs like above. If there is a germ of an isomorphism $\varphi$ under which $F$ and $F'$ are conjugate, then $\varphi$ induces an isomorphism between $S(\Pi, \sigma)$ and $S(\Pi', \sigma')$. Conversely, given a surface $S$ containing a GSS, Ma. Kato [18] has proved that $S$ is obtained by the above construction.

The **universal covering space** $(\tilde{S}, \tilde{\omega})$ of $S$ is obtained by glueing a sequence of copies $(A_i)_{i\in\mathbb{Z}}$ of $A_i = A := B_{n-1} \setminus \overline{\sigma(B)}$. The pseudoconcave boundary of $A_i$ is identified with the pseudoconvex boundary of $A_{i+1}$. The covering automorphism $\tilde{g} : \tilde{S} \to \tilde{S}$ sends $A_i$ onto $A_{i+1}$. In $\tilde{S}$ there is a countable family of rational curves with a canonical order induced by "the order of creation". In the case $\mathrm{tr}(S) = 0$, this order is not obviously understandable from the graph of the curves. Sometimes we shall denote by $C + 1$ the curve created after $C$. Given a curve $C$ in $\tilde{S}$ we construct a new surface $\hat{S}_C$ with a canonical morphism $p_C : \tilde{S} \to \hat{S}_C$ in the following way: Suppose $C \subset \bigcup_{i\leqslant p} A_i$. We fill in the hole of $A_p$ with a ball and obtain a surface with an exceptional curve of the first kind. If this curve is $C$, we have obtained $\hat{S}_C$. If not, we blow down successively the exceptional curves until we end up with $C$ . Finally the map $p_C$ is defined by blowing down the "half-infinite" number of curves created after $C$ and $\hat{O}_C$ is the image under $p_C$ of all curves $C' > C$. Now we have the following commutative diagram for every curve $C$:

$$\begin{array}{ccc}
\tilde{S} & \xrightarrow{\tilde{g}} & \tilde{S} \\
{\scriptstyle p_{C+1}}\downarrow & & \downarrow{\scriptstyle p_{C+1}} \\
\hat{S}_{C+1} & \xrightarrow{F_{C+1}} & \hat{S}_{C+1} \\
{\scriptstyle \Pi_C^{C+1}}\downarrow & & \downarrow{\scriptstyle \Pi_C^{C+1}} \\
\hat{S}_C & \xrightarrow{F_C} & \hat{S}_C
\end{array}$$

where $\hat{O}_C$ is the fix point and $\Pi_C^{C+1}$ is the blow-up in $\hat{O}_C$.

The automorphism $\tilde{g}$ induces for every $C$ the diagram:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{g}} & \tilde{S} \\ {\scriptstyle p_C}\downarrow & & \downarrow{\scriptstyle p_{C+n}} \\ \hat{S}_C & \xrightarrow{\sigma_C^{C+n}} & \hat{S}_{C+n} \end{array}$$

where $\sigma_C^{C+n}$ is an isomorphism such that $\sigma_C^{C+n}(\hat{O}_C) = \hat{O}_{C+n}$.

By diagram chasing one gets

$$F_C = \Pi_C^{C+n}\sigma_C^{C+n}$$

and

$$F_{C+n}\sigma_C^{C+n} = \sigma_C^{C+n}F_C,$$

i.e. the germs $F_C : (\hat{S}_C, \hat{O}_C) \to (\hat{S}_C, \hat{O}_C)$ and $F_{C+n} : (\hat{S}_{C+n}, \hat{O}_{C+n}) \to (\hat{S}_{C+n}, \hat{O}_{C+n})$ are conjugated. In general $F_C, ..., F_{C+(n-1)}$ are *not* conjugated and define the $n$ conjugacy classes associated to the surface $S$. It is easy to check that there are $n$ homotopy classes of GSS.

Now fix a curve $C = C_0$ in the universal covering space. We denote by

$$a(S) = (a_i)_{i\in\mathbb{Z}};\ a_i = -C_i^2$$

the family of opposite self-intersections of the curves in the universal covering space. The sequence $a(S)$ is periodic of period $n = b_2(S)$, i.e. $a_i = a_{i+n}$. It may be divided into **regular sequences** $r_m = (2, \dots, 2)$ of length $m$ and **singular sequences** $s_p = (p+2, 2, \dots, 2)$ of length $p$. The invariant $a(S)$ determines completely the intersection matrix $M(S)$ of the curves in $S$. The integer $\sigma_n(S) = \sum\limits_{i}^{i+n-1} a_i$ is independent of $i$ and satisfies the condition $2n \leqslant \sigma_n(S) \leqslant 3n$.

The **trace of a germ** $F = \Pi\sigma$ (resp. **of a surface** $S(\Pi, \sigma)$) is by definition the trace $\mathrm{tr}\, DF(O)$ of the tangent mapping $DF$ at the fixed point of $F$. The trace is independent of the choice of the GSS and depends only on the isomorphism class of $S$. So it is denoted by $\mathrm{tr}(S)$. The inequalities $0 \leqslant |\mathrm{tr}(S)| < 1$ always hold.

One has $\mathrm{tr}(S) \neq 0$ if and only if one of the following equivalent conditions are satisfied:

i) For every $0 \leqslant i \leqslant n-1$, the point $O_i$ is not in the intersection of $C_i$ with the strict transform of $C_k$ $(k < i)$ or of $\sigma^{-1}(C_{n-1})$;

ii) S contains a cycle $\Gamma$ of rational curves such that $\Gamma^2 = 0$;

iii) every rational curve of the universal covering space has self-intersection $-2$;

iv) the sum $\sigma_n(S) = 2n$.

On the other hand $tr(S) = 0$ if and only if

a) There exists an index $i$ such that $O_i$ is contained in the intersection of $C_i$ with another curve, or $\sigma(0)$ is in the intersection of $C_{n-1}$ with another curve, or the strict transform of $\sigma^{-1}(C_{n-1})$ by $\Pi_0$ contains $O_0$;

b) $\sigma_n(S) > 2n$;

c) $a(S)$ contains at least a singular sequence;

d) $M(S)$ is negative definite.

A germ of mapping (resp. a minimal surface) will be called **generic** if its trace is non vanishing.

A germ of mapping (resp. a minimal surface) will be called an **Inoue-Hirzebruch germ** (resp. **Inoue-Hirzebruch surface**), if $\sigma_n(S) = 3n$ or equivalently if $a(S)$ contains only singular sequences.

In order to calculate explicitely sequences of quadratic transformations we use througout the paper the following local coordinates on

$$\hat{\mathbb{C}}^2 = \{((z_1, z_2), [w_1 : w_2]) \in \mathbb{C}^2 \times \mathbb{P}_1(\mathbb{C}) \mid z_1 w_2 = z_2 w_1\},$$

i.e. the manifold obtained by blowing up at the origin of $\mathbb{C}^2$:

$$\phi : \mathbb{C}^2 \to \hat{\mathbb{C}}^2, \quad \phi(u, v) := ((uv, v), [u : 1])$$

and

$$\psi : \mathbb{C}^2 \to \hat{\mathbb{C}}^2, \quad \psi(u', v') := ((v', u'v'), [1 : u']).$$

The transition functions of these coordinates are

$$u' = \frac{1}{u}; \; v' = uv.$$

### 1.2. Flat line bundles.

We describe explicitely the subgroup of topologically trivial (flat) line bundles for a surface $S$ containing a GSS. First we have $\mathrm{Pic}^0(S) \simeq$

$H^1(S,\mathbb{C}^\star) \simeq H^1(S,\mathcal{O})/H^1(S,\mathbb{Z}) \simeq \mathbb{C}^\star$ for a surface of class $\mathrm{VII}_0$ with no non-constant meromorphic funtions (see [20], formulas (14), (102) and the commutative diagram which follows noticing that the argument works also for $b_2 > 0$). Following the notations of Subsection 1, let $B' \subset B \subset B''$ be three balls of radius $1-\varepsilon$ , 1 and $1+\varepsilon$ for small $\varepsilon$. Let $A = \Pi^{-1}(B)\setminus\sigma(\overline{B})$ , $A' = \Pi^{-1}(B)\setminus\sigma(\overline{B'})$ and $A'' = \Pi^{-1}(B'')\setminus\sigma(\overline{B'})$. The surface $S$ is obtained by gluing holomorphically the two connected components of the boundary of $A$ with $\sigma\Pi$ in $A''$. Let $i : A \hookrightarrow S$ be the natural inclusion and $V$ the canonical image of $B''\setminus\overline{B'}$ in $S$. We denote by $\mathcal{U} = \{A, V\}$ the so obtained open covering of $S$. It is clear that $A\cap V$ has two connected components and $i^{-1}(A\cap V) = U\cup W$ where $U = \Pi^{-1}(B\setminus\overline{B'})$ is taken to be the component on the "pseudoconvex side" of $A$.

Definition 1.1. — *For $f \in \mathcal{O}^\star(U)$ we define $L^f$ to be the holomorphic line bundle given by the cocycle $(f \in \mathcal{O}^\star(U), g = 1 \in \mathcal{O}^\star(W))$ in $H^1(\mathcal{U},\mathcal{O}^\star)$. For $f = \mathrm{const.} = \lambda \in \mathbb{C}^\star$ we call $L^\lambda$ the flat line bundle with parameter $\lambda$.*

*Remark* 1.2. — Clearly the function $f$ extends to a non-vanishing holomorphic function on $\Pi^{-1}(B)$, which we still denote by $f$.

Lemma 1.3. — *Let $L^f \in H^1(\mathcal{U},\mathcal{O}^\star)$ defined by $f \in \mathcal{O}^\star(U)$. Then $L^f$ is flat, $L^f = L^{f(\sigma(0))}$ and we have isomorphisms*

$$\begin{array}{ccccc} \mathbb{C}^\star & \xrightarrow{\cong} & H^1(\mathcal{U},\mathbb{C}^\star) & \xrightarrow{\cong} & H^1(S,\mathbb{C}^\star) \\ & & \iota\big\downarrow\cong & & \\ & & H^1(\mathcal{U},\mathcal{O}^\star) & & \end{array}$$

*Moreover, $f(\sigma(0))$ is independent of the choice of the GSS and therefore, the above identification between $\mathbb{C}^\star$ and $H^1(S,\mathbb{C}^\star)$ is canonical.*

*Proof.* — Since $\mathcal{U}$ is a Leray covering for the constant sheaf $\mathbb{C}^\star$, it is clear that the horizontal maps are isomorphisms.

First we prove that $\iota$ is injective. Suppose that $L^\lambda$ is trivial. Then there exists a non-vanishing section $s : S \to L^\lambda$, i.e. a holomorphic function $s : A' \to \mathbb{C}$ which satisfies $s(\sigma\Pi(z)) = \lambda s(z)$. The function $s$ extends to $B^\Pi := \Pi^{-1}(B)$ and since it is non-zero on $A'$, it does not vanish on $B^\Pi$. Let $O$ be the fixed point of $\sigma\Pi$. We have $s(O) = s(\sigma\Pi(O)) = \lambda s(O)$, therefore $\lambda = 1$. Now let $L^f \in H^1(\mathcal{U},\mathcal{O}^\star)$. We define a linear map $r : H^1(\mathcal{U},\mathcal{O}^\star) \to H^1(\mathcal{U},\mathbb{C}^\star)$ by $r(L^f) := L^{f(\sigma(0))}$. We shall prove that $L^f$

and $L^{f(\sigma(0))}$ are equal. We write $f(z) = \lambda_0(1 + a(z))$ with $a \in \mathcal{O}(\Pi^{-1}(B))$ and $a(\sigma(0)) = 0$. Let

$$g(z) := \prod_{i=0}^{\infty} \big(1 + a((\sigma\Pi)^i(z))\big).$$

It is easy to check that the infinite product converges and that

$$\frac{g(z)}{g(\sigma\Pi(z))} = (1 + a(z)) = \frac{f(z)}{\lambda_0}.$$

Now it follows that $\iota$ and $r$ are the inverse to each other.

Now, let $\Sigma$ be a GSS in $S$ and $\tilde{\Sigma}$ be a connected component of $\tilde{\omega}^{-1}(\Sigma)$ in $\tilde{S}$. We note by $\tilde{S}_+$ the strictly pseudoconvex component of $\tilde{S} \setminus \tilde{\Sigma}$. We remark that $f$ extends holomorphically to $\tilde{S}_+$ and that $L^f = L^{f(\sigma(0))}$. Hence $L^f$ is uniquely determined by the constant value of $f$ on the compact rational curves of $\tilde{S}_+$. This proves the last assertion of the theorem. □

### 1.3. Logarithmic deformations.

We recall that for an effective divisor $E$ on a compact complex surface $S$ the locally free sheaf $\Omega^1_S(\log E)$ defined by

$$\Omega^1_S(\log E)(U) := \{\omega \in \Omega^1_S(E_{\rm red})(U) | d\omega \in \Omega^2_S(E_{\rm red})(U)\}$$

is called the *sheaf of meromorphic forms with logarithmic poles* in $E$. If $z \in E$ is a regular point and $E = \{z_1 = 0\}$, then $\Omega^1_S(\log E)$ is generated by $\dfrac{dz_1}{z_1}$ and $dz_2$; if $z \in E$ is a singular point and $E = \{z_1 z_2 = 0\}$, then $\Omega^1_S(\log E)$ is generated by $\dfrac{dz_1}{z_1}$ and $\dfrac{dz_2}{z_2}$. A logarithmic deformation [23] is defined by cocycles in the dual sheaf $\Theta_S(-\log E) := \mathcal{H}om_{\mathcal{O}_S}(\Omega^1_S(\log E), \mathcal{O}_S)$. Therefore the configuration of curves is maintained by such a deformation.

## 2. The Green function on the universal covering.

### 2.1. The case $2n \leqslant \sigma_n(S) < 3n$.

We recall that a surface $S$ with $\sigma_n(S) = 2n$ is a compactification of an affine line bundle over an elliptic curve by a cycle $D$ of $n$ rational curves with $D^2 = 0$. These surfaces admit exactly one singular holomorphic foliation $\mathcal{F}$

which extends the affine fibration. This foliation is stable under deformation (see [8]), and is defined by a logarithmic 1-form $\tau \in H^0(S, \Omega(\log D))$. By [8] we have the following normal form for a germ defining $S$:

$$F(z) = \left( z_1 z_2^n t^n + \sum_{i=0}^{n-1} \alpha_i t^{i+1} z_2^{i+1}, t z_2 \right).$$

The leaves of $\mathcal{F}$ are the sets $\{z_2 = \text{constant}\}$ and $\tau = \dfrac{dz_2}{z_2}$. Furthermore there exists on the universal covering $(\tilde{S}, \tilde{\omega}, S)$ a holomorphic function $f$ such that $\dfrac{df}{f} = \tilde{\omega}^\star(\tau)$.

The aim of this section is devoted to the analog problem in the case $2n < \sigma_n(S) < 3n$. Given a defining germ for $S$, we shall prove that it is conjugate to a germ of the form

$$F(z) = (F_1(z), z_2^k)$$

and that there is a local foliation $\mathcal{F}$ induced by $\{z_2 = \text{constant}\}$. Contrarily to the above case there is no holomorphic function on the universal covering $(\tilde{S}, \tilde{\pi}, S)$ which globalizes $\mathcal{F}$. Nevertheless it is possible to define a plurisubharmonic function G (Green function) on $\tilde{S}$ which is pluriharmonic on the complement of the union of the rational curves $\tilde{D} := \bigcup_{i \in \mathbb{Z}} C_i$ in $\tilde{S}$. In order to globalize $\mathcal{F}$, we shall observe that the fibers of the function $G$ in $\tilde{S} \setminus \tilde{D}$ are foliated holomorphically and that all leaves are isomorphic to $\mathbb{C}$.

We recall that in $a(S)$ two successive singular sequences

$$s_p = (p+2, 2, \ldots, 2)$$

are separated by at most one regular sequence

$$r_m = (2, \ldots, 2).$$

The indices $p$ and $m$ indicate the lengths of the sequences. In this section we use frequently the local coordinate notations for sequences of quadratic transformations given in Section 1.1. The key result for the sequel is the following

LEMMA 2.4. — *Let $S$ be a surface with $2n \leqslant \sigma_n(S) < 3n$. Let $C$ be a rational curve in the universal covering $\tilde{S}$ and $p_C : \tilde{S} \to \hat{S}_C$ the canonical collapsing morphism onto the point $\hat{O}_C \in C$ (see Section 1). We suppose that the curve $C$ satisfies the following property:*

($\diamondsuit$) *$\hat{O}_C$ is not an intersection point of two rational curves.*

*Then we have*

*1) There is a local coordinate system in a neighbourhood of $\hat{O}_C$, say $z = (z_1, z_2)$, in which*

$$F_C(z) = F(z) = \left(az_2^l(1 + A(z)), bz_2^k(1 + B(z))\right) \quad \text{where} \quad a, b \in \mathbb{C}^\star.$$

*Moreover*

*i) the integer $k = k(S)$ is independant of the choice of the curve $C$ satisfying ($\diamondsuit$) and depends only on $a(S)$. Furthermore, $k(S) = 1$ if $\sigma_n(S) = 2n$ and $k(S) \geqslant 2$ if $\sigma_n(S) > 2n$.*

*ii) $A$ and $B$ are convergent series of order at least one and $z_2$ divides $B$.*

*iii) $l \geqslant 1$ and $l \geqslant 2$ if there are at least two singular sequences in a period of $a(S)$.*

*2) We denote by $\hat{O}_i = \hat{O}_{C+i} = (\alpha_i, 0)$, $0 \leqslant i \leqslant n-1$ the sequence of successively blown up points. Let $i_0$ be the smallest integer such that $\alpha_{i_0} \neq 0$. Let $\mathcal{S} \to \mathbb{C}^\star$ be the logarithmic deformation given by moving the point $\hat{O}_{i_0} = (\alpha_{i_0}, 0)$ along $C_{i_0}$ such that $\hat{O}_{i_0}$ does not meet an intersection point of two curves. Then there is a holomorphic family of germs*

$$F_{C,\alpha_{i_0}}(z) = F(\alpha_{i_0}, z) = \left(a(\alpha_{i_0})z_2^l(1 + A(z)), b(\alpha_{i_0})z_2^k(1 + B(z))\right),$$

*where $a, b : \mathbb{C}^\star \to \mathbb{C}^\star$ are holomorphic functions, such that the associated logarithmic deformation is isomorphic to $\mathcal{S} \to \mathbb{C}^\star$. Moreover*

*i) $A$ et $B$ do not depend on $\alpha_{i_0}$.*

*ii) There exists integers $U' < U$ such that $a = \alpha_{i_0}^{U'} a'$ and $b = \alpha_{i_0}^{U} b'$, where $a'$ and $b'$ do not depend on $\alpha_{i_0}$.*

*iii) For all $p \geqslant 1$*

$$\begin{array}{rccc} a^{-1}b^p : & \mathbb{C}^\star & \to & \mathbb{C}^\star \\ & \alpha_{i_0} & \mapsto & a(\alpha_{i_0})^{-1}b(\alpha_{i_0})^p \end{array}$$

*are non-constant holomorphic functions.*

*Proof.* — Suppose that $2n < \sigma_n(S)$.

Let $a(S) = (a_i)_{i \in \mathbb{Z}}$ be the sequence of opposite self-intersections of the curves in the universal covering of $S$. We choose the numbering of the curves in $\tilde{S}$ such that $C = C_{-1}$. The condition $\sigma_n(S) > 2n$ implies that $a(S)$ contains at least one singular sequence. On the other hand we have that $\sigma_n(S) < 3n$ which assures the existence of at least one regular sequence in $a(S)$. The hypothesis ($\diamondsuit$) on $C = C_{-1}$ shows that $a_{-2}$ and hence also $a_{n-2}$ belong to a regular sequence. Therefore there exists integers $p \geqslant 1$ and $m \geqslant 1$ such that

$$(\ldots, a_{n-p-m-1}, \ldots, a_{n-m-2}, a_{n-m-1}, \ldots, a_{n-2}, a_{n-1}, \ldots) = (\ldots, s_p, r_m, a_{n-1}, \ldots).$$

We first consider the case $m \geqslant 2$. For a ball $B$ centered at $0 = \hat{O}_C$ the sequence of blowing-ups of $B$ may be written in the following way (we denote by the same symbol a curve and its strict transform):

$$\begin{array}{ccccccccc} B_{n-1} & \xrightarrow{\Pi_{n-1}} & B_{n-2} & \xrightarrow{\Pi_{n-2}} & \cdots & \xrightarrow{\Pi_{n-m+2}} & B_{n-m+1} & \xrightarrow{\Pi_{n-m+1}} & B_{n-m} \\ (u,v) & \mapsto & (uv+\alpha_{n-2}, v) & \mapsto & \cdots & \mapsto & (u,v) & \mapsto & (uv+\alpha_{n-m}, v) \end{array}$$

$$\begin{array}{ccccccccc} B_{n-m} & \xrightarrow{\Pi_{n-m}} & B_{n-m-1} & \xrightarrow{\Pi_{n-m-1}} & \cdots & \xrightarrow{\Pi_{n-p-m+2}} & B_{n-p-m+1} & \xrightarrow{\Pi_{n-p-m+1}} & B_{n-p-m} \\ (u,v) & \mapsto & (uv, v) & \mapsto & \cdots & \mapsto & (u,v) & \mapsto & (uv, v) \end{array}$$

$$\begin{array}{ccccc} B_{n-p-m} & \xrightarrow{\Pi_{n-p-m}} & B_{n-p-m-1} & \xrightarrow{\Pi_{n-p-m-1}} & \cdots \\ (u',v') & \mapsto & (v'+\alpha_{n-p-m-1}, u'v') & \mapsto & \cdots \end{array}$$

In fact

- We have $C_{n-p-m-1}^2 = -(p+2)$ in $B_{n-1}$. Hence the curve $C_{n-p-m-1}$ with self-intersection $-1$ in $B_{n-p-m-1}$ has to be blown-up $p+1$ times. Therefore $O_{n-p-m} = C_{n-p-m-1} \cap C_{n-p-m} = \{v' = 0\} \cap \{u' = 0\}$ is the point at infinity of $C_{n-p-m}$ in $B_{n-p-m}$ with $\Pi_{n-p-m}(u', v') = (v' + \alpha_{n-p-m-1}, u'v')$,
- In $B_{n-p-m+1}$ the point $O_{n-p-m+1} = C_{n-p-m+1} \cap C_{n-p-m-1} = \{v = 0\} \cap \{u = 0\}$ is the origin of the chart in which $\Pi_{n-p-m+1}(u,v) = (uv, v)$ and $C_{n-p-m-1}^2 = -3$,
- Analogously $\Pi_i(u,v) = (uv, v)$ for $n - p - m + 1 \leqslant i \leqslant n - m$,
- Since $C_{n-p-m-1}^2 = -(p+2)$ and $O_{n-m} \neq C_{n-m} \cap C_{n-p-m-1} = \{v = 0\} \cap \{u = 0\}$ in $B_{n-m}$ and since there is a regular sequence after $s_p$, we have $O_{n-m} \neq C_{n-m} \cap C_{n-m-1} = \{v' = 0\} \cap \{u' = 0\}$. Thus $O_{n-m} = (\alpha_{n-m}, 0)$ with $\alpha_{n-m} \neq 0$ and $\Pi_{n-m+1}(u,v) = (uv + \alpha_{n-m}, v)$,
- For $n - m + 1 \leqslant i \leqslant n - 1$, we have $\Pi_i(u,v) = (uv + \alpha_{i-1}, v)$.

Finally we may suppose that $\sigma^{-1}(C_{n-1}) = \{z_2 = 0\}$ and thus

$$\sigma(z) = \big(\sigma_1(z) + \alpha_{n-1}, z_2(1 + \theta_2(z))\big).$$

We remark that the above conditions give no information about the self-intersection of $C_{n-1}$.

We stress the importance of the condition $\alpha_{n-m} \neq 0$ in the proof. We now calculate directly the compositions of quadratic transformations:

(1)

$$\begin{aligned}\Pi_{n-p-m} \cdots \Pi_{n-1}(u, v) &= \Pi_{n-p-m}\Big(uv^{p+m-1} + \sum_{i=n-m}^{n-2} \alpha_i v^{i-n+m+p}, v\Big)\\ &= \Big(v + \alpha_{n-p-m-1}, uv^{p+m} + \sum_{i=n-m}^{n-2} \alpha_i v^{i-n+m+p+1}\Big)\\ &= \Big(v + \alpha_{n-p-m-1}, \alpha_{n-m} v^{p+1}(1 + B(u, v))\Big)\end{aligned}$$

and

(2)

$$\begin{aligned}\Pi_{n-p-m} \cdots \Pi_{n-1}\sigma(z) &= \Big(z_2(1+\theta_2) + \alpha_{n-p-m-1},\\ &\qquad \sigma_1(z)[z_2(1+\theta_2)]^{m+p} + \sum_{i=n-m}^{n-1} \alpha_i [z_2(1+\theta_2)]^{i-n+m+p+1}\Big)\\ &= \Big(z_2(1+\theta_2) + \alpha_{n-p-m-1},\ \alpha_{n-m} z_2^{p+1}(1 + \beta(z))\Big).\end{aligned}$$

The orders of $\beta$ and $B$ are at least one, $z_2$ divides $\beta$ and $p + 1 \geqslant 2$.

Now let $m = 1$. The sequence of blowing-ups is now

$$\begin{array}{ccccccccc} B_{n-1} & \overset{\Pi_{n-1}}{\rightarrow} & B_{n-2} \rightarrow \cdots \rightarrow B_{n-p} & \overset{\Pi_{n-p}}{\rightarrow} & B_{n-p-1} & \rightarrow & B_{n-p-2} \\ (u, v) & \mapsto & (uv, v) \qquad\quad (u, v) & \mapsto & (uv, v) = (u', v') & \mapsto & (v' + \alpha_{n-p-2}, u'v'). \end{array}$$

We get

$$\Pi_{n-p-1} \cdots \Pi_{n-1}(u, v) = (v + \alpha_{n-p-2}, uv^{p+1}).$$

With $\sigma(z) = (\sigma_1(z) + \alpha_{n-1}, z_2(1 + \theta_2(z)))$ this yields

$$\Pi_{n-p-1} \cdots \Pi_{n-1}\sigma(z) = \big(z_2(1 + \theta_2(z)) + \alpha_{n-p-2}, \alpha_{n-1} z_2^{p+1}(1 + \beta(z))\big) \quad \text{where} \quad \alpha_{n-1} \neq 0$$

and we obtain the same formula as before.

We now calculate $F = \Pi\sigma$. We shall distinguish the following cases:

*First Case*: Suppose $a(S) = (\overline{s_p, r_m})$. Here $n = p + m$ and

$$F(z) = (z_2(1 + \theta_2(z)), \alpha_{n-m} z_2^{p+1}(1 + \beta(z))).$$

*Second Case*:

**a)** There is exactly one singular sequence before $r_m$. We write $a(S) = (\dots, s_{p'}, r_{m'}, s_p, r_m, a_{n-1}, \dots)$ with $m' \geqslant 1$. Thus $C^2_{n-p-m-2} = -2$ and therefore $\hat{O}_{C_{n-p-m-1}} \neq C_{n-p-m-1} \cap C_{n-p-m-2}$ and

$$\Pi_{n-p-m-1}(u, v) = (uv + \alpha_{n-p-m-2}, v).$$

The composition of the mappings corresponding to $s_{p'}, r_{m'}$ is of type **(1)** with $\alpha_{n-p-m-m'} \neq 0$ and we obtain

$$\begin{aligned}\Pi_{n-p-m-p'-m'} \cdots \Pi_{n-1}\sigma(z) = \big(&\alpha_{n-m} z_2^{p+1}(1 + \beta(z)) + \alpha_{n-p-m-p'-m'-1},\\ &\alpha_{n-p-m-m'}\alpha_{n-m}^{p'+1} z_2^{(p+1)(p'+1)}(1 + B')\big).\end{aligned}$$

**b)** There are several singular sequences before $r_m$, i.e.

$$a(S) = (\dots, s_{p_k}, \dots, s_{p_1}, r_m, a_{n-1}, \dots).$$

Each singular sequence $s_{p_i}$ corresponds to

$$(u, v) \mapsto (v, uv^{p_i}). \tag{3}$$

We define by induction sequences of polynomials $T_k = T_k(p_1, \dots, p_k) \in \mathbb{Z}[p_1, \dots, p_k]$ and $U_k = U_k(p_2, \dots, p_k) \in \mathbb{Z}[p_2, \dots, p_k]$ by $T_0 = 1$, $T_1 = p_1 + 1$ and for $k \geqslant 2$,

$$\begin{pmatrix} T_{k-1} \\ T_k \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & p_k \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & p_2 \end{pmatrix} \begin{pmatrix} 1 \\ p_1 + 1 \end{pmatrix}$$

and $U_0 = 0$, $U_1 = 1$, $U_2 = p_2$ and for $k \geqslant 3$,

$$\begin{pmatrix} U_{k-1} \\ U_k \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & p_k \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & p_3 \end{pmatrix} \begin{pmatrix} 1 \\ p_2 \end{pmatrix}.$$

For fixed values of $p_1, \dots, p_k$, one has that the sequences $(T_i)$, $(U_i)$ are increasing. It is easy to show by induction on $k \geqslant 1$ that the sequence $s_{p_k} \cdots s_{p_1} r_m$ corresponds to the composition of mappings of type **(1)** (with $\alpha_{n-p-m-1} = 0$) or **(3)** and hence

$$\begin{aligned}&\Pi_{n-m-p_1-\dots-p_k} \cdots \Pi_{n-1}(u, v)\\ &\quad= \Big(\alpha_{n-m}^{U_{k-1}} v^{T_{k-1}}(1 + A) + \alpha_{n-m-p_1-\dots-p_k-1}, \alpha_{n-m}^{U_k} v^{T_k}(1 + B)\Big),\end{aligned} \tag{4}$$

where $v$ divides $B$ and $A$, $B$ are independent of $\alpha_{n-m}$. So

$$\Pi_{n-m-p_1-\cdots-p_k}\cdots\Pi_{n-1}\sigma(z_1,z_2) \tag{5}$$

$$=\Big(\alpha_{n-m}^{U_{k-1}}z_2^{T_{k-1}}(1+A'(z))+\alpha_{n-m-p_1-\cdots-p_k-1},\alpha_{n-m}^{U_k}z_2^{T_k}(1+B'(z))\Big),$$

where $z_2$ divides $B'(z)$, and $A'$, $B'$ are independent of $\alpha_{n-m}$.

In the first case the mapping $F$ has the desired properties and we are done. In general $a(S)$ may be written as $a(S) = (\overline{\sigma_N,\ldots,\sigma_1})$, where $\sigma_i = s_{p^i_{k_i}}\cdots s_{p^i_1}r_{m_i}$. The cases 2a and 2b describe the compositions of quadratic transformations corresponding to $\sigma_i$ for all $i=1,\cdots,N$. Now one proves by induction on $N\geqslant 1$ that $F$ is a composition of a mapping of type (**5**) and mappings of type (**4**), with $\alpha_{-1}=0$.

One shows again by induction on $N\geqslant 1$ that $k(S)=\prod_{i=1}^{N}T^i$. This integer is independent of the choice of the curve $C$: If we choose another one, the factors of $k(S)$ are changed by a circular permutation.

The terms $a$ and $b$ are products of powers of coordinates of all base points of blow-ups which are not intersections of two curves. The number $\alpha_{i_0}\in\mathbb{C}^\star$ and the integers $U,U'$ appear when composing the mappings corresponding to the last sequence $\sigma_N$. One gets $a=\alpha_{i_0}^{U'}a'=\alpha_{i_0}^{U^{k-1}}a'$ and $b=\alpha_{i_0}^{U}b'=\alpha_{i_0}^{U^k}b'$, where $a'$ and $b'$ are independent of $\alpha_{i_0}$. Finally

$$\begin{pmatrix}U'\\U\end{pmatrix}=\begin{pmatrix}U_{k-1}\\U_k\end{pmatrix}=\begin{pmatrix}0&1\\1&p_k\end{pmatrix}\begin{pmatrix}U''\\U'\end{pmatrix},$$

with $U'>0, U''>0$.

Hence $a(\alpha_{i_0})^{-1}b(\alpha_{i_0})^p=\alpha_{i_0}^{-U'+pU}a'^{-1}b'^p$, with $pU-U'>0$. This achieves the proof of 2) and hence of 1) which is a particular case of 2).

The case $2n=\sigma_n(S)$ is easy and left to the reader. □

*Example* 2.5. — If $a(S) = (\overline{s_{p_1},r_{m_1},s_{p_2},r_{m_2},\ldots s_{p_l},r_{m_l}})$, i.e. every couple of singular sequences is separated by a regular sequence, then

$$k(S)=\prod_{i=1}^{l}(p_i+1).$$

LEMMA 2.6. — *For every $p \geqslant 1$,*

$$\begin{aligned} F^p(z) &= (F_1^p(z), F_2^p(z)) \\ &= \Big( ab^{l(1+k+\cdots+k^{p-2})} z_2^{lk^{p-1}} \big(1+B(z)\big)^{lk^{p-2}} \\ &\quad \cdots \big(1+B(F^{p-2}(z))\big)^l \big(1+A(F^{p-1}(z))\big), \\ &\quad b^{1+k+\cdots+k^{p-1}} z_2^{k^p} \big(1+B(z)\big)^{k^{p-1}} \\ &\quad \cdots \big(1+B(F^{p-2}(z))\big)^k \big(1+B(F^{p-1}(z))\big) \Big). \end{aligned}$$

*Moreover, with the notation of Lemma 2.4, $F^p$ depends holomorphically on $\alpha = \alpha_{i_0} \in \mathbb{C}^\star$.*

*Proof.* — By induction. □

LEMMA 2.7. — *For every curve $C$ such that $\hat{O}_C \in \hat{S}_C$ is not an intersection point of $C$ with another rational curve, there exists a holomorphic function $f = f_C$ defined on a neighbourhood $U_C$ of $\hat{O}_C$, such that*

$$df(\hat{O}_C) \neq 0 \quad \text{and} \quad f_C(F_C(z)) = b f_C^{k(S)}(z).$$

*Let $\bar{f} := \bar{f}_C := \gamma f_C$ with $b = \gamma^{k-1}$ and $k = k(S)$. We have $\bar{f}_C(F_C) = \bar{f}_C^k$ and for any domain $V_C$ on which $\bar{f}_C$ extends, one gets $\bar{f}_C(V_C) \subset \Delta$.*

*Moreover $f$ depends holomorphically on $\alpha = \alpha_{i_0} \in \mathbb{C}^\star$ but its domain of existence is independent of $\alpha$.*

*Proof.* — 1) We show the existence of the function $f_C$ by proving the convergence of an infinite product.

Let $0 < \varepsilon < 1$ such that $A$ and $B$ are defined on a neighbourhood of $\{||z|| \leqslant \varepsilon\}$. By Lemma 2.4, there exists a constant $K > 0$ independant of $\alpha \in \mathbb{C}^\star$ satisfying the conditions

$$|A(z)| \leqslant K||z||, \quad |B(z)| \leqslant K|z_2|, \quad \varepsilon(1+K\varepsilon) < \frac{1}{k}, \quad K\varepsilon < \frac{\pi}{k}$$

with $k = k(S)$. We show by induction on $i \geqslant 0$ that for $||z|| < \varepsilon$,

$$|B(F^i(z))| \leqslant \frac{K|z_2|}{k^i}.$$

If $i = 0$ the inequality is trivial. The induction hypothesis gives

$$\begin{aligned} |B(F^{i+1}(z))| &\leqslant \frac{K|F_2(z)|}{k^i} \leqslant \frac{K}{k^i}|z_2|^k(1+K||z||) \\ &\leqslant \frac{K|z_2|}{k^i}\varepsilon^{k-1}(1+K\varepsilon) \\ &\leqslant \frac{K|z_2|}{k^{i+1}}. \end{aligned}$$

For $||z|| < \varepsilon$ we obtain

$$|B(F^i(z))| \leqslant \frac{K\varepsilon}{k^i} \leqslant \frac{\pi}{k^{i+1}}.$$

For every $i \geqslant 0$ the $k^{i+1}$-th root

$$\big(1 + B(F^i(z))\big)^{\frac{1}{k^{i+1}}}$$

may be defined. Then the infinite product

$$p(z) = \prod_{i \geqslant 0} \big(1 + B(F^i(z))\big)^{\frac{1}{k^{i+1}}}$$

is clearly convergent.

2) We now define $f_C(z) = f(z) = z_2 p(z)$ and get

$$\begin{aligned} f(z) = \lim_{p\to\infty} z_2 \big(1+B(z)\big)^{\frac{1}{k}} & \frac{\big(1+B(z)\big)^{\frac{1}{k}}\big(1+B(F(z))\big)^{\frac{1}{k^2}}}{\big(1+B(z)\big)^{\frac{1}{k}}} \\ \cdots & \frac{\big(1+B(z)\big)^{\frac{1}{k}}\big(1+B(F(z))\big)^{\frac{1}{k^2}}\cdots\big(1+B(F^p(z))\big)^{\frac{1}{k^{p+1}}}}{\big(1+B(z)\big)^{\frac{1}{k}}\cdots\big(1+B(F^{p-1}(z))\big)^{\frac{1}{k^p}}} \\ = \lim_{p\to\infty} z_2 & \frac{(F_2(z))^{\frac{1}{k}}}{b^{\frac{1}{k}} z_2} \frac{(F_2^2(z))^{\frac{1}{k^2}}}{b^{\frac{1}{k^2}} (F_2(z))^{\frac{1}{k}}} \cdots \frac{(F_2^{p+1}(z))^{\frac{1}{k^{p+1}}}}{b^{\frac{1}{k^{p+1}}} (F_2^p(z))^{\frac{1}{k^p}}} \\ = b^{-\frac{1}{k-1}} \lim_{p\to\infty} z_2 & \frac{(F_2(z))^{\frac{1}{k}}}{z_2} \frac{(F_2^2(z))^{\frac{1}{k^2}}}{(F_2(z))^{\frac{1}{k}}} \cdots \frac{(F_2^{p+1}(z))^{\frac{1}{k^{p+1}}}}{(F_2^p(z))^{\frac{1}{k^p}}}. \end{aligned}$$

It follows that $f(F(z)) = bf(z)^k$ and $\bar{f}(F^k) = \bar{f}^{k^p}$. Let $V_C$ be a connected neighbourhood of $\hat{O}_C$ such that $\bar{f}_C$ extends holomorphically to a neighbourhood of $\overline{V_C}$. There exists $p \geqslant 1$ such that $F^p(\overline{V_C}) \subset\subset V_C$, since $F_C$ is contractant. This gives

$$\sup_{\partial V_C} |\bar{f}| = \Big(\sup_{\partial V_C} |\bar{f}|^{k^p}\Big)^{\frac{1}{k^p}} = \Big(\sup_{\partial V_C} |\bar{f}(F^p)|\Big)^{\frac{1}{k^p}} < \Big(\sup_{\partial V_C} |\bar{f}|\Big)^{\frac{1}{k^p}}$$

and the lemma is proved. $\square$

THEOREM 2.8.— *Let $F = \Pi\sigma : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a contracting germ composed of $n$ blowing-ups $\Pi_i$ and a germ of an isomorphism $\sigma$ such that the singular set of $F$ contains only one component. Let $S$ be the surface such that $2n < \sigma_n(S) < 3n$, associated to $F$. Then for every $\alpha \in \mathbb{C}^\star$, $F$ is conjugate to*

$$F'(z) = \varphi \circ F \circ \varphi^{-1}(z) = (az_2^l(1 + A'(z)), bz_2^k)$$

*where $\varphi(z) = \varphi(\alpha, z) = (z_1, f(\alpha, z))$ is an isomorphism which depends holomorphically on $\alpha$ defined on a fixed neighbourhood, $k = k(S)$ with $k \geqslant 2$, $l \geqslant 1$, $A'(0) = 0$ and $\dfrac{\partial A'}{\partial z_1}(0)$ does not depend on $\alpha$.*

*Proof.* — By Lemma 2.4, we can suppose that $F(z) = \Big(az_2^l(1 + A(z)), bz_2^k(1 + B(z))\Big)$. Since $f(z) = z_2p(z)$ with $p(0) = 1$, one has $\varphi^{-1}(z) = (z_1, z_2q(z))$ with $q(0) = 1$ and

$$\begin{aligned}\varphi F\varphi^{-1}(z) &= \Big(az_2^l q(z)^l(1 + A(\varphi^{-1}(z))), f(F(\varphi^{-1}(z)))\Big)\\ &= \Big(\star, bf^k(\varphi^{-1}(z))\Big) = \Big(\star, bz_2^k\Big).\end{aligned}$$

We set $1 + A'(z) = q(z)^l(1 + A(\varphi^{-1}(z)))$. Since $z_2$ divides $B$, we have $\dfrac{\partial q}{\partial z_1}(0) = \dfrac{\partial p}{\partial z_1}(0) = 0$. Therefore

$$\begin{aligned}\frac{\partial A'}{\partial z_1}(0) &= lq(z)^{l-1}\frac{\partial q}{\partial z_1}(z).(1 + A(\varphi^{-1}(z)))_{|z=0} + q(z)^l\frac{\partial}{\partial z_1}(A(\varphi^{-1}(z)))_{|z=0}\\ &= \frac{\partial}{\partial z_1}(A(\varphi^{-1}(z)))_{|z=0}\\ &= \frac{\partial A}{\partial z_1}(\varphi^{-1}(0)) + \frac{\partial A}{\partial z_2}(\varphi^{-1}(z))z_2\frac{\partial q}{\partial z_1}(z)_{|z=0}\\ &= \frac{\partial A}{\partial z_1}(0)\end{aligned}$$

does not depend on $\alpha$. □

*Remark* 2.9. — It is easy to see that $F(z) = \Big(az_2^l(1 + A(z)), bz_2^k(1 + B(z))\Big)$ is conjugate by a diagonal linear map to a similar germ with $a = b = 1$. Combining this with the preceding theorem we get that $F$ is conjugate to $F'(z) = (z_2^l(1 + A'(z)), z_2^k)$. However we maintain the constants, since $a$ and $b$ depend holomorphically on the "generic" points in the sequence of blow-ups. This dependence will be important for the construction of surfaces with global vector fields.

PROPOSITION 2.10. — *Let $C$ be a curve in the universal covering $\tilde S$ such that $\hat O_C \in \hat S_C$ is not an intersection point of two compact curves. Then there exists a plurisubharmonic function $G_C : \hat S_C \to [-\infty, 0[$, which satisfies the following properties:*

i) *The polar set of $G_C$ is the union $D_C$ of the rational curves on $\hat S_C$ and $G_C$ is pluriharmonic in the complement of $D_C$;*

ii) *$G_C$ satisfies the functional equation $G_C \circ F_C = k(S)G_C$;*

iii) *the function $G_C : \hat{S}_C \setminus D_C \to ]-\infty, 0[$ is surjective and submersive;*

iv) *for every $x < 0$, the level set $\hat{S}_C(x) := \{z \in \hat{S}_C | G_C(z) = x\}$ is a real 3-fold homeomorphic to the complement $S^3 \setminus \Sigma_k$ of a closed set with empty interior $\Sigma_k$ in the unit sphere $S^3$.*

*Remark* 2.11. — The closed set $\Sigma_k$ is an analogue of the solenoid in [16].

*Proof.* — i) and ii) For a connected neighbourhood $V_C$ on which $f_C$ is defined, we consider $G_C := \log|f_C|$. One has $G_C(F_C(z)) = \log|f_C^k| = kG_C(z)$. For an arbitrary point $z$ in $\hat{S}_C$ there exists $p$ such that $F_C^p(z) \in V_C$. We define

$$G_C(z) = \frac{G_C(F^p(z))}{k^p}.$$

This definition is independent of the choice of $p$. The function $G_C$ satisfies condition i) and the functional equation ii), since $f_C \circ F_C = f_C^k$.

iii) $G_C(\hat{S}_C \setminus D_C)$ is an intervall $]-\infty, \alpha[$ where $\alpha \leqslant 0$. Since $F_C$ is an automorphism of $\hat{S}_C \setminus D_C$, by ii) we see that $]-\infty, \alpha[$ is invariant under multiplication by $k$ and $1/k$. Hence $\alpha = 0$ and the surjectivity follows. There is a neighbourhood $U$ of $\hat{O}_C$ such that $df_C(z) \neq 0$ for $z \in U$ by Lemma 2.7. For $z \in U \setminus D_C$, we have $dG_C(z) = \partial f_C / f_C(z) + \overline{\partial f_C} / \bar{f}_C(z) \neq 0$. The functional equation of $G_C$ implies that $G_C$ is submersive.

iv) We choose a neighbourhood of $\hat{O}_C$ isomorphic to the unit ball $B$ on which $f_C$ is defined. We write $F_C = \Pi_C^{C+n} \sigma_C^{C+n}$ (see Section 1.1). The manifold $\hat{S}_C$ is isomorphic to a union of annuli and of the ball $B$: $\hat{S}_C = \bigcup_{i \leqslant 0} A_i \cup B$. For every $i < 0$, one has $F_C(A_i) = A_{i+1}$ and $F_C(A_0) = B$. In view of Theorem 2.8, there is a system of coordinates $(z_1, z_2)$ on $B$ such that $f_C(z) = z_2$. Therefore $G_C(z) = \log|z_2|$. Let $x \in ]-\infty, 0[$. By replacing if necessary the level set $\hat{S}_C(x) := {G_C}^{-1}(x)$ by an image $F_C^q(\hat{S}_C(x))$, we suppose that $\hat{S}_C(x)$ meets $B$. For $i \leqslant 0$, we set $A'_i = \bigcup_{i \leqslant j \leqslant 0} A_j \cup B$, $T = \hat{S}_C(x) \cap B$ and $T_i = \hat{S}_C(x) \cap A'_i$. With these notations $(T_i)_{i \leqslant 0}$ is an increasing sequence and

$$\hat{S}_C(x) = T \cup \bigcup_{i \leqslant 0} T_i.$$

For $x := \log|\lambda|$, we have that $T = \{z \in B \mid \log|z_2| = x\} = \{z \in B \mid |z_2| = |\lambda|\}$ is a solid torus. For every $i \leqslant 0$, $A'_i$ is isomorphic to the $n(i+1)$-times blown-up ball. Since $F_C(A'_i) = A'_{i+1}$, we have the following commutative diagram:

$$\begin{array}{ccccccccccccc}
T & \xrightarrow{I} & T_0 & \xrightarrow{I_0} & T_{-1} & \xrightarrow{I_{-1}} \cdots \xrightarrow{I_{-(p-2)}} & T_{-(p-1)} & \xrightarrow{I_{-(p-1)}} & T_{-p} & \xrightarrow{I_{-p}} \cdots \\
{\scriptstyle F_C}\downarrow\cong & & {\scriptstyle F_C}\downarrow\cong & & {\scriptstyle F_C^2}\downarrow\cong & & {\scriptstyle F_C^p}\downarrow\cong & & {\scriptstyle F_C^{p+1}}\downarrow\cong & \\
F_C(T) & \xrightarrow{\mathbf{I}} & \mathbf{T}(|\lambda|^k) & \xrightarrow{F_C} & \mathbf{T}(|\lambda|^{k^2}) & \xrightarrow{F_C} \cdots \xrightarrow{F_C} & \mathbf{T}(|\lambda|^{k^p}) & \xrightarrow{F_C} & \mathbf{T}(|\lambda|^{k^{p+1}}) & \xrightarrow{F_C} \cdots
\end{array}$$

Here $I$ and $I_{-p}$ are inclusions and $\mathbf{T}(r) := \{z \in B \mid |z_2| = r\}$ for $r > 0$. Therefore $\hat{S}_C(x)$ is an increasing union of solid tori. The embedding $\mathbf{I} : F_C(T) \to \mathbf{T}(|\lambda|^k)$ has degree 1. For $p \geqslant 1$, $F_C : \mathbf{T}(|\lambda|^{k^p}) \to \mathbf{T}(|\lambda|^{k^{p+1}})$ has degree $k$. This means that the image $F_C(\mathbf{T}(|\lambda|^{k^p}))$ is winding around $k$ times in the interior of $\mathbf{T}(|\lambda|^{k^{p+1}})$. For every $r > 0$, the continuous mapping

$$\begin{array}{rccc}
h: & \bar{B} \setminus \{z_2 = 0\} & \to & S^3 \setminus \{z_2 = 0\} \\
 & (z_1, z_2) & \mapsto & \left(z_1, \dfrac{z_2}{|z_2|}\sqrt{1 - |z_1|^2}\right)
\end{array}$$

induces by restriction the homeomorphism

$$\begin{array}{rccc}
h: & \mathbf{T}(r) := \{z \in \bar{B} \mid |z_2| = r\} & \to & \mathbf{U}(r) := \{z \in S^3 \mid |z_2| \geqslant r\} \\
 & (z_1, z_2) & \mapsto & \left(z_1, \dfrac{z_2}{|z_2|}\sqrt{1 - |z_1|^2}\right).
\end{array}$$

This shows that $T_{-p}$ is homeomorphic to $\mathbf{U}(|\lambda|^{k^{p+1}})$ and that the complement

$$S^3 \setminus \mathbf{U}(|\lambda|^{k^{p+1}})$$

of the image of $T_{-p}$ in $S^3$ is a solid torus without boundary . Similarly, let

$$\mathbf{T}'(r) = \{z \in \bar{B} \mid |z_1| = \sqrt{1 - r^2}\}, \quad \mathbf{U}'(r) = \{z \in S^3 \mid |z_1| \geqslant \sqrt{1 - r^2}\},$$

$$\begin{array}{rccc}
h': & \mathbf{T}'(r) & \to & \mathbf{U}'(r) \\
 & (z_1, z_2) & \mapsto & \left(\dfrac{z_1}{|z_1|}\sqrt{1 - |z_2|^2}, z_2\right).
\end{array}$$

In order to simplify the notations we set

$$\mathbf{T}_j = \mathbf{T}(|\lambda|^{k^{-j+1}}), \quad \mathbf{T}'_j = \mathbf{T}'(|\lambda|^{k^{-j+1}}),$$
$$\mathbf{U}_j = \mathbf{U}(|\lambda|^{k^{-j+1}}), \quad \mathbf{U}'_j = \mathbf{U}'(|\lambda|^{k^{-j+1}}).$$

Of course $S^3 = \mathbf{U}_0 \cup \mathbf{U}'_0$. We want to prove that $\hat{S}_C$ is homeomorphic to $\mathbf{U}_0 \cup \mathbf{U}'_0 \backslash \Sigma_k$ (the set $\Sigma_k$ will be defined below). For $r > 0$, let

$$\begin{array}{rccc} R: & \mathbf{T}(r) & \rightarrow & \mathbf{T}'(r^{\frac{1}{k}}) \\ & (z_1, z_2) & \mapsto & \left( \dfrac{\sqrt{1 - r^{\frac{2}{k}}}}{r} \, z_2, \ \dfrac{r^{\frac{1}{k}}}{\sqrt{1 - r^2}} \, z_1 \right). \end{array}$$

Finally we define for $j \leqslant 0$,

$$H_j : \mathbf{T}_j \overset{R_j}{\rightarrow} \mathbf{T}'_{j+1} \overset{h'_{j+1}}{\rightarrow} \mathbf{U}'_{j+1}, \text{ and } \tau'_j : \mathbf{U}'_j \overset{H_{j-1}^{-1}}{\rightarrow} \mathbf{T}_{j-1} \overset{F_C}{\rightarrow} \mathbf{T}_{j-2} \overset{H_{j-2}}{\rightarrow} \mathbf{U}'_{j-1},$$

where $R_j$ and $H_j$ are induced by $R$ and $H$.

This allows to complete the previous commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "I_{-p+1}"] & T_{-p} \arrow[r, "I_{-p}"] \arrow[d, "F_C^{p+1}"', "\cong"] & T_{-p-1} \arrow[r, "I_{-p-1}"] \arrow[d, "F_C^{p+2}"', "\cong"] & \cdots \arrow[r, "I_{j+1}"] & T_j \arrow[r, "I_j"] \arrow[d, "F_C^{-j+1}"', "\cong"] & T_{j-1} \arrow[r, "I_{j-1}"] \arrow[d, "F_C^{-j+2}"', "\cong"] & \cdots \\
\cdots \arrow[r, "F_C"] & \mathbf{T}_{-p} \arrow[r, "F_C"] \arrow[d, "H_{-p}"', "\cong"] & \mathbf{T}_{-p-1} \arrow[r, "F_C"] \arrow[d, "H_{-p-1}"', "\cong"] & \cdots \arrow[r, "F_C"] & \mathbf{T}_j \arrow[r, "F_C"] \arrow[d, "H_j"', "\cong"] & \mathbf{T}_{j-1} \arrow[r, "F_C"] \arrow[d, "H_{j-1}"', "\cong"] & \cdots \\
\cdots \arrow[r, "\tau'_{-p+2}"] & \mathbf{U}'_{-p+1} \arrow[r, "\tau'_{-p+1}"] & \mathbf{U}'_{-p} \arrow[r, "\tau'_{-p}"] & \cdots \arrow[r, "\tau'_j"] & \mathbf{U}'_{j+1} \arrow[r, "\tau'_{j+1}"] & \mathbf{U}'_j \arrow[r, "\tau'_j"] & \cdots
\end{tikzcd}$$

in which $\tau'_j$ is also an embedding of degree $k$. Let $\tau'_1 := h'_0 \circ R_1 \circ F_C \circ h_0^{-1} : \mathbf{U}'_1 := \mathbf{U}_0 \rightarrow \mathbf{U}'_0$. We notice that $\left(\mathbf{U}'_j, \tau'_j, j \leqslant 1\right)$ defines a direct limit system in the category of topological spaces and that

$$\hat{S}_C(x) = \varinjlim \left(\mathbf{U}'_j, \tau'_j, j \leqslant 1\right).$$

This means that $\hat{S}_C$ is obtained in the following way: We start with the solid torus $\mathbf{U}_0$ and we add $\mathbf{U}'_0$ minus $\tau'_1 \mathbf{U}_0$ winding around $k$ times in $\mathbf{U}'_0$. We fill the hole $\tau'_1 \mathbf{U}_0$ by $\mathbf{U}'_{-1} \setminus \tau'_0 \mathbf{U}'_0$. The new hole is therefore a torus winding around $k^2$ times in $\mathbf{U}'_0$. Repeating this procedure we finally get

$$\Sigma_k := \varprojlim \left(\mathbf{U}'_j, \tau'_j, j \leqslant 1\right)$$

i.e. $\Sigma_k$ is the intersection of the removed tori. □

DEFINITION 2.12. — *The psh function $G_C$ on $\hat{S}_C$ is called the Green function associated to the curve $C$.*

COROLLARY 2.13. — *On the universal covering $\tilde{S}$ of $S$ there exists a plurisubharmonic function $\tilde{G}_C := G_C \circ p_C : \tilde{S} \rightarrow [-\infty, 0[$ with connected fibers, the polar set of which is the union of all rational curves $\tilde{D}$. On*

*$\tilde{S} \setminus \tilde{D}$ the function $\tilde{G}_C$ is pluriharmonic, submersive and is surjective onto* $]-\infty, 0[$.

THEOREM 2.14. — *Let $S$ be a compact surface with GSS such that $n = b_2(S) > 0$ and $2n < \sigma_n(S) < 3n$. Then there exists a singular foliation $\mathcal{F}$ on $S$ with the following properties:*

i) *The singular set* $\mathrm{Sing}(\mathcal{F})$ *is the union of the $n$ intersection points of the rational curves;*

ii) *the complement of* $\mathrm{Sing}(\mathcal{F})$ *in each rational curve is a leaf;*

iii) *all other leaves are isomorphic to $\mathbb{C}$ and dense in the real threefolds*

$$\tilde{\omega}(\{\tilde{G}_C = \text{constant}\})$$

*which are homeomorphic to the complement $S^3 \setminus \Sigma_k \subset S \setminus D$. Moreover the closure in $S$ of a such a leaf contains the maximal divisor $D$ of $S$.*

*Proof.* — Let $C$ be a curve such that $\hat{O}_C$ is not the intersection point of two curves and $G_C$ the associated Green function on $\hat{S}_C$. By Theorem 2.8 and Remark 2.9, there exists a ball $B$ centered at $\hat{O}_C$ on which $F(z) = F_C(z) = (z_2^l(1 + A(z)), z_2^k)$. In $B$ the curves $\{z_2 = \lambda\} \cap B$ are plaques of leaves of a foliation $\hat{\mathcal{F}}$ on $\hat{S}_C$ which induces a foliation $\mathcal{F}$ on $S$. The properties i) and ii) are already known (see [19]).

Let $\hat{L}_0 \subset \hat{S}_C$ be a leaf in the complement of the rational curves and $L_0$ its image in $S$. We shall prove that $\hat{L}_0$ and $L_0$ are both isomorphic to $\mathbb{C}$. Since $F_C : \hat{S}_C \to \hat{S}_C$ is an isomorphism outside the rational curves of $\hat{S}_C$, we have that $\hat{L}_p := F_C^p(\hat{L}_0)$ and $\hat{L}_0$ are isomorphic. We consider $\hat{S}_C$ as the union $\bigcup\limits_{i\leqslant 0} A_i \cup B$ of annuli and the ball $B$. We have $\hat{L}_0 = \cup_i(\hat{L}_0 \cap A_i) \cup (\hat{L}_0 \cap B)$. Let $-q$ be the greatest value for $i$ such that $\hat{L}_0 \cap A_i \neq \varnothing$. Replacing, if necessary $\hat{L}_0$ by $\hat{L}_{q+1}$, we may suppose that $\hat{L}_0 \cap B \neq \varnothing$. Notice that each connected component of $\hat{L}_0 \cap B$ is a disc. Let $\Delta_0 = \{z_2 = \lambda\}$ be one of them and $\Delta_{p,q} = \{z_2 = e^{\frac{2i\pi q}{k^p}}\lambda\}$. For every $p \geqslant 0$ and every $0 \leqslant q \leqslant k^p - 1$, $F^p(\Delta_{p,q}) \subset \{z_2 = \lambda^{k^p}\} \subset \hat{L}_p$. Therefore $\hat{L}_p$ contains the disc $\{z_2 = \lambda^{k^p}\}$ and $\hat{L}_0 = F^{-p}(\hat{L}_p)$ contains all the discs $\Delta_{p,q}$. Since the $k^p$-th roots of unity are dense in $S^1$, it follows that $\hat{L}_0 \cap B$ is dense in the solid torus $T_0 = \{|z_2| = |\lambda|\} = \{G_C = \log|\lambda|\} \cap B$.

For every $i \geqslant 0$ define now the disc $\delta_i = \{z \in B \mid z_2 = \lambda^{k^i}\}$. So $\delta_0 = \Delta_0$ and $F_C^i(\Delta_0) \subset \delta_i$. For every $i \geqslant 0$, we have that $\Delta_i := F_C^{-i}(\delta_i)$ is a disc contained in $\hat{L}_0$ and $Y_i = \Delta_i \setminus \bar{\Delta}_{i-1}$ is a 1-dimensional annulus.

Since $Y_i = F_C^{-i}\left(\delta_i \setminus \overline{F_C(\delta_{i-1})}\right)$, the sequence of moduli of $(Y_i)$ is increasing. This shows that $\cup\Delta_i$ is isomorphic to $\mathbb{C}$. Furthermore $\hat{L}_0$ is clearly simply connected and contains an open set isomorphic to $\mathbb{C}$, hence $\hat{L}_0 = \underset{i\geqslant 0}{\cup} \Delta_i \simeq \mathbb{C}$. As a consequence one has that $\hat{L}_0 \cap B = \underset{p,q}{\cup} \Delta_{p,q}$ and that $\hat{L}_0$ is dense in $\{G_C = \log|\lambda|\}$. Finally, since $F_C^{i+1}(\hat{L}_0 \cap A_i) \subset \{z \in B \mid |z_2| = |\lambda|^{k^{i+1}}\} \subset \hat{L}_{i+1}$, $\hat{L}_0$ is mapped isomorphically by $\tilde{\omega} \circ p_C^{-1}$ to $L_0$ in $S$. Hence $L_0$ is also isomorphic to $\mathbb{C}$. All the leaves $\hat{L}_i$ have the same image $L_0$ in $S$. Therefore the closure of $L_0$ in $S$ contains $D$. □

PROPOSITION 2.15. — *Let $C$ and $C'$ be two curves such that $\hat{O}_C$ and $\hat{O}_{C'}$ are are not intersection points of two compact curves in $\hat{S}_C$ and $\hat{S}_{C'}$ respectively. Then the two psh funtions $\tilde{G}_C$ and $\tilde{G}_{C'}$ differ by a positive multiplicative constant.*

*Proof.* — By Theorem 2.14 the fibers of $\tilde{G}_C$ and $\tilde{G}_{C'}$ in $\tilde{S} \setminus \tilde{D}$ are densely foliated by copies of $\mathbb{C}$. Furthermore these functions are pluriharmonic on $\tilde{S} \setminus \tilde{D}$ and bounded from above by 0. Let $L$ be a complex leaf of a fiber of $\tilde{G}_C$. Then the restriction of $\tilde{G}_{C'}$ to $L$ is constant. Since the fibers of $\tilde{G}_C$ are 3-dimensinal, $L$ is a complex leaf of a fiber of $\tilde{G}_{C'}$. This shows that the two functions have the same level sets and differ therefore by a multiplicative positive real-analytic function $a$. The fact that both are pluriharmonic implies that $a$ is constant. □

### 2.2. The case of Inoue-Hirzebruch surfaces: $\sigma_n = 3n$.

We recall (cf. [6] and [7]) that any Inoue-Hirzebruch surface (first constructed in [17]) can be defined by mappings

$$\begin{array}{rccc} F = F_C : & (\mathbb{C}^2, 0) & \to & (\mathbb{C}^2, 0) \\ & (z_1, z_2) & \mapsto & (z_1^p z_2^q, z_1^r z_2^s) \end{array}$$

where $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is the product of $n = b_2(S)$ matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, with at least one matrix of the second type. The matrix $A$ has two distinct eigenvalues $\lambda_i$, $i = 1, 2$, which are real quadratic and $\det A = \pm 1$.

Let $(a_i, b_i)^t$ be the eigenvectors, i.e. $(a_i, b_i)A^t = \lambda_i(a_i, b_i)$, $i = 1, 2$. We do not have the analogue of the functions $f_C$ because $f(z) = z_1^{a_i} z_2^{b_i}$ is not defined in a neighbourhood of 0.

Nevertheless, for every curve $C$ in the universal covering space $\tilde{S}$ of $S$, we have two Green functions $G_{C,i}$ on $\hat{S}_C$, defined by

$$G_{C,i}(z) = a_i \log|z_1| + b_i \log|z_2|, \quad i = 1, 2,$$

in a neighbourhood of 0 and extended on $\hat{S}_C$ in the same way as for the case $2n < \sigma_n < 3n$, since $G_{C,i}(F_C(z)) = \lambda_i G_{C,i}(z)$. The function $G_{C,i}$ is plurisubharmonic and pluriharmonic outside the rational curves. We recover two foliations by holomorphic curves in the level sets of $G_{C,i}$, $i = 1, 2$.

The two foliations which exist on $S$ by [19] are given by twisted vector fields. In fact, since a twisted vector field on $S$ becomes a vector field on $\tilde{S}$, a twisted vector field has to be tangent to the rational curves. Choosing any curve $C$, $F_C$ can be written in a neighbourhood of $\hat{O}_C$ as $F_C(z) = F(z) = (z_1^p z_2^q, z_1^r z_2^s)$ (see [6], [7]), where the invertible matrix $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ has two distinct quadratic eigenvalues, $\lambda_1$ and $\lambda_2$. Since the two rational curves passing through $\hat{O}_C$ have local equations $\{z_i = 0\}$, $i = 1, 2$, the local defining vector fields of the two foliations are of the form $\theta(z) = z_1 a(z) \dfrac{\partial}{\partial z_1} + z_2 b(z) \dfrac{\partial}{\partial z_2}$. So we have to find $\lambda \in \mathbb{C}^\star$ such that $DF(z)\theta(z) = \lambda\theta(F(z))$. The holomorphic functions $a$ and $b$ have to satisfy the equations

$$(\diamondsuit) \qquad \begin{cases} pa(z) + qb(z) &= \lambda a(F(z)) \\ ra(z) + sb(z) &= \lambda b(F(z)). \end{cases}$$

If $a(0) = 0$ or $b(0) = 0$, then $a(0) = b(0) = 0$. By induction on the degree of the homogeneous parts of $a$ and $b$, it is easy to check that $a = b = 0$. The only values of $\lambda$ for which $(\diamondsuit)$ has a non-trivial solution are precisely the two eigenvalues $\lambda_1$ and $\lambda_2$ of $A$. This gives the solutions $\theta_i(z) = z_1 a_i \dfrac{\partial}{\partial z_1} + z_2 b_i \dfrac{\partial}{\partial z_2}$.

We furthermore define twisted meromorphic 1-forms with logarithmic poles

$$(\star) \qquad \omega_i(z) = \frac{a_i'}{z_1} dz_1 + \frac{b_i'}{z_2} dz_2 \in H^0(S, \Omega^1(Log\, D) \otimes L^{\lambda_i}),$$

where $(a_i', b_i')^t$, $i = 1, 2$ are the eigenvectors of $A^t$ corresponding to $\lambda_i$, $i = 1, 2$.

The results of the following theorem are partially proved in [17]:

THEOREM 2.16. — *Let $S$ be an Inoue-Hirzebruch surface. Then*

1) *There are exactly two holomorphic foliations on $S$. In the complement of the rational curves, the leaves are all isomorphic to $\mathbb{C}$ or all isomorphic to the disc $\Delta$.*

2) *The universal covering of $S \setminus D$ is isomorphic to $\Delta \times \mathbb{C}$ and $\pi_1(\tilde{S} \setminus \tilde{D}) = \mathbb{Z}^2$.*

3) *We have the following exact sequence:*

$$0 \to \mathbb{Z}^2 \to \pi_1(S \setminus D) \to \mathbb{Z} \to 0$$

*and, more precisely, $\pi_1(S \setminus D) = \mathbb{Z}^2 \rtimes \mathbb{Z}$.*

*Proof.* — The characteristic polynomial of $A$ is $P_A(X) = X^2 - (p + s)X + \det A$. The eigenvalues are

$$\lambda_1, \lambda_2 = \frac{(p+s) \pm \sqrt{(p+s)^2 - 4\det A}}{2},$$

where $\lambda_1$ is chosen as being the greatest eigenvalue. We have $\lambda_1 > 1$ and since $\lambda_1\lambda_2 = \det A = \pm 1$, $0 < |\lambda_2| < 1$. We denote by $x_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ (resp. $x_2 = \begin{pmatrix} c \\ d \end{pmatrix}$) an eigenvector associated to $\lambda_1$ (resp. $\lambda_2$).

The equality $pa + qb = \lambda_1 a$ shows that $a$ and $b$ have the same sign, say $a > 0$ and $b > 0$. From $pc + qd = \lambda_2 c$, we deduce similarly that $c$ and $d$ have opposite signs, say $c > 0$, $d < 0$ (see [6], lemme 2.5).

The following diagram:

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C} & \xrightarrow{A} & \mathbb{C} \times \mathbb{C} \\ {\scriptstyle \exp}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\ \mathbb{C}^\star \times \mathbb{C}^\star & \xrightarrow{F} & \mathbb{C}^\star \times \mathbb{C}^\star \end{array}$$

where $\exp(\zeta_1, \zeta_2) := (\exp\zeta_1, \exp\zeta_2)$, is commutative. The real plane $(\mathrm{Re}\,\zeta_1, \mathrm{Re}\,\zeta_2)$ is divided into two parts by the line $\mathbb{R}x_2$. Set $H^+ = \mathbb{R}_+^\star x_1 \times \mathbb{R}x_2$ and $H^- = \mathbb{R}_-^\star x_1 \times \mathbb{R}x_2$. We have $A^m(\alpha x_1 + \beta x_2) = \alpha\lambda_1^m x_1 + \beta\lambda_2^m x_2$, $\lim_{m\to\infty} \beta\lambda_2^m = 0$ and $\lim_{m\to\infty} \alpha\lambda_1^m = \pm\infty$ for $\alpha x_1 + \beta x_2 \in H^\pm$. Therefore the attraction basin of 0 for $F$ is

$$\{z_1 z_2 = 0\} \cup \exp\big(\{\zeta \in \mathbb{C}^2 \mid (\mathrm{Re}\,\zeta_1, \mathrm{Re}\,\zeta_2) \in H^-\}\big).$$

Obviously $\{\zeta \in \mathbb{C}^2 \mid (\mathrm{Re}\,\zeta_1, \mathrm{Re}\,\zeta_2) \in H^-\} \simeq \Delta \times \mathbb{C}$. One can check that this attraction basin of 0 for $F$ is isomorphic to

$$\hat{S}_C \setminus \bigcup_{\hat{O}_C \notin C'} C'.$$

Hence, the universal covering of $S \setminus D$ is isomorphic to $\Delta \times \mathbb{C}$. Furthermore $\exp\big(\{\zeta \in \mathbb{C}^2 \mid (\mathrm{Re}\,\zeta_1, \mathrm{Re}\,\zeta_2) \in H^-\}\big) \simeq \tilde{S} \setminus \tilde{D}$. Therefore $\pi_1(\tilde{S} \setminus \tilde{D}) = \mathbb{Z}^2$. The two complex directions $x_1$ and $x_2$ induce two linear foliations on $H^- \oplus i\mathbb{R}^2$: The leaves of the first one are isomorphic to $\Delta$ and the leaves of the second one are isomorphic to $\mathbb{C}$. These foliations induce via exp and $F$ two foliations on $S$ having leaves isomorphic to $\Delta$ (resp. $\mathbb{C}$) in the complement of the rational curves. These foliations on $S$ are furthermore transversal in $S \setminus D$.

3) is a straightforward consequence of the exact homotopy sequence for a fibration. □

## 3. Baum-Bott formulas for foliations on surfaces with a GSS.

In this section we apply known formulas for foliations of compact complex surfaces to surfaces containing a GSS.

First we recall basic properties of these surfaces.

Let $p$ be a singularity of $\mathcal{F}$. Then the foliation is locally defined by a holomorphic vector field

$$\theta(z,w) = A(z,w)\frac{\partial}{\partial z} + B(z,w)\frac{\partial}{\partial w}$$

with $p = (0,0)$. We call *order* of the singularity at $p$ the order of the first non trivial jet of $\theta$. Let $J(z,w)$ be the Jacobian matrix of the mapping $(A,B)$ and let $\lambda$, $\mu$ be the eigenvalues of $J(p)$. We shall say that the singularity is *simple*, if $\lambda\mu \neq 0$ and $\frac{\lambda}{\mu} \in \mathbb{Q}^\star_-$.

THEOREM 3.1 [19]. — *Let $S$ be a minimal surface with a GSS and $b_2(S) > 0$ with a (reduced) foliation $\mathcal{F}$. The following statements hold:*

1) *The rational curves are invariant.*

2) *The singularities of $\mathcal{F}$ are exactly the $n$ intersection points of the curves and their order is one.*

3) *If $S$ is not an Inoue-Hirzebruch surface, the singularities are simple.*

*Proof.* — We use the notations of [5] and [19].

*Case A:* All points $\hat{O}_C$ are singular points for the foliation $\mathcal{F}_C$: this is the case studied by F. Kohler [19], p.171.

*Case B:* There is a point $\hat{O}_C$ which is not a singularity for $\mathcal{F}_C$, then we apply Remarks 1 and 2, A of [19], p.164. □

Following [1] and [2], we define the two indices

$$\mathrm{Det}(p,\mathcal{F}) = \mathrm{Res}_{(0,0)} \frac{\det J(z,w)}{A(z,w)B(z,w)} dz \wedge dw$$

$$\mathrm{Tr}(p,\mathcal{F}) = \mathrm{Res}_{(0,0)} \frac{(\mathrm{tr}\ J(z,w))^2}{A(z,w)B(z,w)} dz \wedge dw$$

where $\mathrm{Res}_{(0,0)}$ is the residue at $(0,0)$ (see [13]). If $J(0,0)$ has two eigenvalues $\lambda$ and $\mu$ different from 0, then

$$\mathrm{Det}(p,\mathcal{F}) = 1, \quad \text{and} \quad \mathrm{Tr}(p,\mathcal{F}) = 2 + \frac{\lambda}{\mu} + \frac{\mu}{\lambda}.$$

Furthermore we set

$$\mathrm{Det}(\mathcal{F}) = \sum_{p\in \mathrm{Sing}(\mathcal{F})} \mathrm{Det}(p,\mathcal{F}) \quad \text{and} \quad \mathrm{Tr}(\mathcal{F}) = \sum_{p\in \mathrm{Sing}(\mathcal{F})} \mathrm{Tr}(p,\mathcal{F}).$$

Let $\lambda_i$, $\mu_i$ be the eigenvalues of the singularity $p_i$, $i = 0,\ldots,n-1$. Then we have

COROLLARY 3.2. — *If $S$ is a minimal surface containing a GSS such that $n = b_2(S) > 0$ and $\mathcal{F}$ is a foliation on $S$, then*

$$\mathrm{Det}(\mathcal{F}) = n \quad \text{and} \quad \mathrm{Tr}(\mathcal{F}) = 2n + \sum_{i=0}^{n-1} \frac{\lambda_i}{\mu_i} + \frac{\mu_i}{\lambda_i}.$$

**Baum-Bott formulas** [1], [2]: Given a foliation $\mathcal{F}$ with isolated singularities on a compact complex surface $S$, one can associate a tangent line bundle $T_\mathcal{F} \subset TS$ on $S\backslash \mathrm{Sing}(\mathcal{F})$ which extends to $S$ (see [12]). However, on $S$ the bundle $T_\mathcal{F}$ is not a subbundle of $TS$. In the same way the conormal bundle $N^\star_\mathcal{F}$ extends to $S$. The line bundles $T^\star_\mathcal{F}$ and $N_\mathcal{F}$ are defined by duality. We have the following formulas:

$$\mathrm{Det}(\mathcal{F}) = c_2(S) - c_1(T_\mathcal{F}).c_1(S) + c_1^2(T_\mathcal{F}) \tag{1}$$

$$\mathrm{Tr}(\mathcal{F}) = c_1^2(S) - 2c_1(T_\mathcal{F}).c_1(S) + c_1^2(T_\mathcal{F}). \tag{2}$$

**Camacho-Sad formula** [4]: Let $\mathcal{F}$ be a foliation on a minimal surface with GSS and $C$ be a regular invariant curve of $\mathcal{F}$. Let $\lambda_i$ and $\mu_i$ be the eigenvalues of the foliation at the singular points $p_i \in C$, where $\lambda_i$ is the eigenvalue corresponding to the eigenvector tangent to $C$. Then

$$\sum_i \frac{\mu_i}{\lambda_i} = C^2.$$

Suppose that all rational curves on the surface are regular. It follows that

$$\mathrm{Tr}(\mathcal{F}) = 2n + \sum_{i=0}^{n-1} D_i^2 = 2n - \sigma_n(S).$$

The equation

$$\mathrm{Tr}(\mathcal{F}) = 2n - \sigma_n(S) \tag{3}$$

still holds if there are singular rational curves on $S$. This can be checked case by case, using a $k$-to-one covering of $S$ ($k = 2, 3$) in order to reduce to the regular situation (see also [27]).

For minimal surfaces with $b_1(S) = 1$ and $n = b_2(S) > 0$ we have $c_2(S) = -c_1^2(S) = n$. Therefore (1), (2) and (3) yield

$$-n \leqslant c_1^2(T_{\mathcal{F}}) = c_1(T_{\mathcal{F}})c_1(S) = \sigma_n(S) - 3n \leqslant 0. \tag{4}$$

By [2], one has $K_S = T_{\mathcal{F}}^{\star} \otimes N_{\mathcal{F}}^{\star}$. Thus $c_1(S) = c_1(T_{\mathcal{F}}) + c_1(N_{\mathcal{F}})$. Using equation (2), we obtain

$$c_1^2(N_{\mathcal{F}}) = \mathrm{Tr}(\mathcal{F}) = 2n - \sigma_n(S) \tag{5}$$

$$c_1(T_{\mathcal{F}})c_1(N_{\mathcal{F}}) = 0. \tag{6}$$

**Brunella-Khanedani-Suwa formulas** [2], [22]: Let $C$ be an invariant curve of the foliation $\mathcal{F}$, $p \in C$ a singularity of $\mathcal{F}$ and $\theta$ a local defining vector field of $\mathcal{F}$ with isolated singularity $p \in C$. In [2], Brunella defines an index $Z(p, C, \mathcal{F})$. If $C$ is regular, this index coincides with the vanishing order at $p \in C$ of the restriction $\theta_{|C}$.

Let $Z(C, \mathcal{F}) = \sum_{p \in \mathrm{Sing}(\mathcal{F}) \cap C} Z(p, C, \mathcal{F})$ and $\chi(C) := -KC - C^2 = 2 - 2g(C)$ the virtual Euler characteristic. Then, by [2], we have

$$c_1(N_{\mathcal{F}}) \cdot C = C^2 + Z(C, \mathcal{F}) \quad \text{and} \quad c_1(T_{\mathcal{F}}) \cdot C = \chi(C) - Z(C, \mathcal{F}).$$

For a regular rational curve $C$ a local defining vector field for $\mathcal{F}$ in a neighbourhood of the singularity $p=0$ is of the form $\theta(x,y)=x\dfrac{\partial}{\partial x}+\lambda y\dfrac{\partial}{\partial y}$, with $\lambda\neq 0$. For $\sigma_n=3n$, this is a consequence of Theorem 3.1 in [19]; for $2n\leqslant\sigma_n<3n$, it follows from PARTIE A, p. 171-p. 177 (in particular étape 4, p. 174) and Remarques 1 & 2 p. 164 in [19].

Therefore $Z(p,C,\mathcal{F})=1$, and

$$c_1(N_{\mathcal{F}})\cdot C=C^2+\operatorname{Card}\{\operatorname{Sing}(\mathcal{F})\cap C\} \tag{7}$$

$$c_1(T_{\mathcal{F}})\cdot C=2-\operatorname{Card}\{\operatorname{Sing}(\mathcal{F})\cap C\}. \tag{8}$$

If $C$ is a rational curve with an ordinary double point $x$, we have

$$a(S)=(\overline{s_{n-1}r_1})=(\overline{n+1,2,\dots,2,2}),$$
$$\text{or}\quad a(S)=(\overline{s_{n-1}s_1})=(\overline{n+1,2,\dots,2,3}).$$

(See [5], Thm. II.2.25.) (The index "0", which appears for $n=1$ indicates the empty sequence.)

Let $\Pi:S'\to S$ be the blowing-up of $S$ at the point $x$, and $\mathcal{F}'$ be the induced foliation on $S'$. The multiplicity of $C$ at $x$ is $\mu=2$ and the vanishing order of a defining vector field of $\mathcal{F}$ is $m=1$, therefore if $C'$ is the strict transform of $C$, and $E$ is the exceptional curve, then

$$[C']=\Pi^\star[C]-\mu[E],\quad\text{which implies}\quad {C'}^2=C^2-\mu^2,$$

and

$$\begin{aligned}Z(C',\mathcal{F}')&=c_1(N_{\mathcal{F}'}).C'-{C'}^2=(\Pi^\star c_1(N_{\mathcal{F}})-m[E]).(\Pi^\star[C]-\mu[E])-C^2+\mu^2\\&=Z(C,\mathcal{F})-\mu(m-\mu).\end{aligned}$$

Here, it yields $Z(C,\mathcal{F})=Z(C',\mathcal{F}')+\mu(m-\mu)=Z(C',\mathcal{F}')-2$. Furthermore we have $Z(C',\mathcal{F}')=\operatorname{Card}\{\operatorname{Sing}(\mathcal{F}')\cap C'\}$.

*Case 1*: $n>1$ and $a(S)=(\overline{s_{n-1}r_1})$ , then $\operatorname{Card}\{\operatorname{Sing}(\mathcal{F}')\cap C'\}=3$ and $Z(C,\mathcal{F})=1$.

*Case 2*: $n=1$ or $a(S)=(\overline{s_{n-1}s_1})=(\overline{n+1,2,\dots,2,3})$, then $\operatorname{Card}\{\operatorname{Sing}(\mathcal{F}')\cap C'\}=2$ and $Z(C,\mathcal{F})=0$.

Finally, the Brunella formulas yield

$$c_1(N_{\mathcal{F}}).C=\begin{cases}C^2+1 & \text{in Case 1}\\ C^2 & \text{in Case 2}\end{cases} \tag{9}$$

$$c_1(T_{\mathcal{F}}).C=\begin{cases}-1 & \text{in Case 1}\\ 0 & \text{in Case 2.}\end{cases} \tag{10}$$

## 4. Numerically anti-canonical and numerically tangent divisors.

Let $S$ be a surface with a GSS and $\mathrm{tr}(S)=0$. Under this assumption the only curves on $S$ are the $n$ rational curves given by the construction and these curves generate $H^2(S,\mathbb{Q})$. We denote by $D=\sum_{i=0}^{n-1} D_i$ the maximal divisor.

Let $L\in H^1(S,\mathbb{C}^\star)$ be a flat line bundle and let $\theta\in H^0(S,\Theta\otimes L)$ be a global twisted vector field. By [19] Theorem 2.1.1. p. 169, $\theta$ may vanish only on the rational curves.

DEFINITION 4.1. — *Let $S$ be a minimal surface with a global spherical shell satisfying $b_2(S)=n\geqslant 1$ and $\mathrm{tr}(S)=0$.*

*1) A divisor $D_{-K}=\sum_{i=0}^{n-1} k_iD_i$, $k_i\in\mathbb{Z}$, is called numerically anticanonical if there exists a flat line bundle $L$ such that $-K\otimes L=[D_{-K}]$. (Here $[D]$ denotes the line bundle associated to a divisor $D$.)*

*2) A divisor $D_\theta=\sum_{i=0}^{n-1} t_iD_i$, $t_i\in\mathbb{Z}$, is called a numerically tangent divisor if there exists a flat line bundle $L$ such that $\Theta\otimes L$ admits a global meromorphic section $\theta$ which satisfies $(\theta)=D_\theta$, in particular $h^0(S,\Theta\otimes L\otimes\mathcal{O}(-D_\theta))\geqslant 1$. We denote by $\mathcal{F}$ the foliation given by $\theta$.*

By [7], p. 671, $D_{-K}\geqslant D$ is a strictly positive divisor. More precisely, we have

LEMMA 4.2. — *For a minimal surface $S$ with GSS and $\mathrm{tr}(S)=0$ let $M(S)=\Big(D_iD_j\Big)_{ij}$ be the intersection matrix of the curves on $S$. Then*

*1) If it exists, a numerically anticanonical divisor $D_{-K}=\sum_{i=0}^{n-1} k_iD_i$ satisfies the linear system*

$$M(S)\begin{pmatrix} k_0\\ \vdots\\ k_j\\ \vdots\\ k_{n-1}\end{pmatrix}=\begin{pmatrix} D_0^2+2-2g(D_0)\\ \vdots\\ D_i^2+2-2g(D_i)\\ \vdots\\ D_{n-1}^2+2-2g(D_{n-1})\end{pmatrix} \tag{1}$$

*where $g(D_i)$ is the genus of the curve $D_i$. Conversely, if the Cramer system* **(1)** *admits a solution $(k_0, ..., k_{n-1})$ in $\mathbb{Z}^n$, then $\sum_{i=0}^{n-1} k_i D_i$ is a numerically anticanonical divisor. In both cases the divisor $D_{-K}$ is unique.*

2) *If it exists, a numerically tangent divisor $D_\theta$ is unique and satisfies the linear system*

$$\textbf{(2)} \qquad M(S)\begin{pmatrix} t_0 \\ \vdots \\ t_j \\ \vdots \\ t_{n-1} \end{pmatrix} = \begin{pmatrix} 2-2g(D_0)-Z(D_0,\mathcal{F}) \\ \vdots \\ 2-2g(D_i)-Z(D_i,\mathcal{F}) \\ \vdots \\ 2-2g(D_{n-1})-Z(D_{n-1},\mathcal{F}) \end{pmatrix}$$

*where*

$$2-2g(D_i)-Z(D_i,\mathcal{F}) = \begin{cases} 2-\mathrm{Card}\{\mathrm{Sing}(\mathcal{F})\cap D_i\} & \text{if } D_i \text{ is regular} \\ -1 & \text{if } D_i \text{ is singular and meets another curve} \\ 0 & \text{if } D_i \text{ is singular and meets no other curve} \end{cases}$$

3) *The $\mathbb{Q}$-divisors $D^{\mathbb{Q}}_{-K}$ and $D^{\mathbb{Q}}_{\theta}$, which are defined by the linear systems (1) and (2) respectively, satisfy the relation*

$$D^{\mathbb{Q}}_{-K} = D^{\mathbb{Q}}_{\theta} + D.$$

*In particular $D^{\mathbb{Q}}_{-K}$ is a divisor if and only if $D^{\mathbb{Q}}_{\theta}$ is a divisor. In this case $D_\theta \geqslant 0$.*

*Proof.* — 1) is a consequence of the adjunction formula and the fact that the curves $D_0, ..., D_{n-1}$ are a $\mathbb{Q}$-basis of $H^2(S, \mathbb{Q})$. 2) a consequence of the formulas (8) and (10) in Section 2, noticing that $T_{\mathcal{F}} = L^{-1} \otimes [D_\theta]$.

3) The intersection matrix is negative definite and

$$D_i^2 + Z(D_i, \mathcal{F}) = D \cdot D_i.$$

The Cramer systems yield for all $i = 0, \cdots, n-1$

$$\det M(S)(k_i - t_i) = \det \begin{array}{c} \begin{pmatrix} D_0^2 & \dots & D_0\cdot D & \dots & D_0 D_{n-1} \\ D_0 D_1 & \dots & D_1 \cdot D & \dots & D_1 D_{n-1} \\ \vdots & & \vdots & & \vdots \\ D_0 D_{n-1} & \dots & D_{n-1}\cdot D & \dots & D_{n-1}^2 \end{pmatrix}. \\ i\text{th column} \end{array}$$

We see that the i-th column is precisely the sum of all the columns of $M(S)$. Therefore $\det M(S)(k_i - t_i) = \det M(S)$. □

*Remark* 4.3. — We shall prove the converse of 2) in Theorem 5.2.

*Example* 4.4. — Let $S$ be an Inoue-Hirzebruch surface. By [5], we have $\sigma_n(S) = 3n$, and $D$ is the sum of one or two cycles. Therefore we get $D_\theta = 0$ and $D_{-K} = D$. Conversely, if $D_\theta = 0$, then $S$ is an Inoue-Hirzebruch surface. □

Suppose that there exists a global twisted vector field $\theta \in H^0(S, \Theta \otimes L)$, where $L$ is flat. Since $T_{\mathcal{F}} = [D_\theta] \otimes L^{-1}$, we have by formula (4), Section 3,

$$-n \leqslant D_\theta^2 = -3n + \sigma_n(S) \leqslant 0. \tag{3}$$

Moreover by [5], p. 107, $D^2 = 2n - \sigma_n(S)$. Using (3) and Lemma 4.2, 3) we obtain

$$-n = c_1(S)^2 = D_{-K}^2 = (D_\theta + D)^2 = D_\theta^2 + 2D_\theta D + D^2 = -n + 2D_\theta D,$$

and finally

$$D_\theta D = 0. \tag{4}$$

THEOREM 4.5. — *Let $S$ be a compact surface with a GSS such that $n = b_2(S)$ and $2n < \sigma_n(S) < 3n$.*

*1) We suppose that for the intersection matrix $M = M(S)$ of $S$ the linear system (1) of Lemma 3.2 has a solution in $\mathbb{Z}^n$. Then there exists a numerically anticanonical divisor $D_{-K}$, i.e. there is a unique complex number $\kappa = \kappa(S)$ such that $K^{-1} \otimes L^\kappa = [D_{-K}]$. In particular*

$$H^0(S, K^{-1} \otimes L^\kappa) \neq 0.$$

*2) Let $\mathcal{S} \to U$ be a logarithmic family of surfaces with fixed intersection matrix $M(S)$. Then there exists a holomorphic function $\kappa$ on $U$ such that $\kappa(S_u) = \kappa(u)$. Surfaces in this deformation which admit a global 2-vector field are exactly those over the (possibly empty) hypersurface $\{\kappa = 1\}$.*

*Proof.* — 1) This is a direct consequence of Lemma 3.2.

2) We consider the family $\mathcal{K} \to U \times \mathbb{C}^\star$ of line bundles $-K_{S_u} \otimes L^\alpha$. By Grauert's semi-continuity theorem, there is an analytic subset $\Gamma \subset U \times \mathbb{C}^\star$

over which $-K_{S_u} \otimes L^{\alpha}$ has a non-zero section. Let $p : \Gamma \to U$ be the projection. The map $p$ is bijective by 1) and therefore there is a point $x \in \Gamma$ for which exists a neighbourhood $U(x) \subset \Gamma$ such that $p_{|U(x)}$ is biholomorphic on its image. The closure of $\Gamma$ in $U \times \mathbb{P}^1(\mathbb{C})$ cannot contain $U(x) \times \{0\}$ or $U(x) \times \{\infty\}$. Hence by the Remmert-Stein theorem, $\bar{\Gamma}$ is an analytic subset of $U \times \mathbb{P}^1(\mathbb{C})$. Now $p : \bar{\Gamma} \to U$ is proper and finite and thus a ramified covering. Since there is only one sheet, it is a graph and $\bar{\Gamma} = \Gamma$. This gives the function $\kappa : U \to \mathbb{C}^\star$. □

*Remark* 4.6.— If $S$ is a surface with $\sigma(S) = 3n$, i.e. $S$ is an Inoue-Hirzebruch surface, then $D_{-K} = D$, $h^0(-K \otimes \mathcal{O}(-D)) = 1$ and every logarithmic deformation of $S$ is trivial.

LEMMA 4.7.— *Let $S$ be a surface with GSS and $2n < \sigma_n(S) < 3n$. We suppose that there is $\lambda \in \mathbb{C}^\star$ and a non-trivial twisted holomorphic vector field $\theta \in H^0(S, \Theta \otimes L^{\lambda})$. For the zero-divisor $D_\theta = \sum_0^{n-1} t_j D_j$ the following statement holds: If there is an index $i \in \{0, \cdots, n-1\}$ for which $t_i = 0$ , then the curve $D_i$ is the top of a tree.*

*Proof.*— Under the hypothesis ($t_i = 0$), one has

$$D_i.D_\theta = \sum_0^{n-1} t_j D_i.D_j \geqslant 0$$

and, since $\sigma_n(S) < 3n$, $D_i$ necessarily meets another curve (see [5]). By Lemma 4.2 it follows that $D_i$ is regular and therefore

$$(*) \qquad 0 \leqslant \sum t_j D_i.D_j = 2 - \mathrm{Card}\{\mathrm{Sing}(\mathcal{F}) \cap D_i\}.$$

We have to show that the right hand side of (*) is equal to 1. Suppose that it is zero. Then there are the following two cases:

• The curve $D_i$ intersects two other curves $D_{j_1}$ and $D_{j_2}$ in one point respectively. By (*) it follows that the coefficents $t_{j_1}$ and $t_{j_2}$ vanish. Using the explicit form of the divisor $D$ (see [5]) and repeating if necessary this procedure, we find a curve $D_k$ which is the root of a tree in $D$ with $t_k = 0$. But $D_k$ has three intersection points with other curves. A contradiction.

• The curve $D_i$ intersects another curve $D_l$ in two points. By (*), $t_l = 0$. On the other hand $D_i + D_l$ is a cycle, hence at least one of these curves is the root of a tree. Contradiction. □

*Example* 4.8. — 1) If $a(S) = (\overline{3\underbrace{2\cdots 2}_{n-1}})$ , then $\det M(S) = 1$. It follows that numerically anticanonical and numerically tangent divisors exist, since the Cramer systems admit integer-valued solutions.

2) Suppose that for $S$ with $\mathrm{tr}(S) = 0$ there are regular sequences in $a(S)$ and each of these is of the form $r_m = r_1$. Furthermore suppose that all curves in $S$ are regular and that there is at least one curve $D_k$ which is not the top of a tree with $D_k^2 \leqslant -4$. Then numerically anticanonical and numerically tangent divisors do not exist.

*Proof.* — 1) is evident.

2) Suppose that a numerically tangent divisor exists. By (4) of Section 4, $0 = DD_\theta = \sum_i t_i\big(D_i^2 + (D - D_i)D_i\big)$. If $D_i$ is not the root of a tree, $(D - D_i)D_i \leqslant 2$. Thus $D_i^2 + (D - D_i)D_i \leqslant 0$. If $D_i$ is a root, then by [5], pp.113-114, $D_i^2 \leqslant -3$, hence $D_i^2 + (D - D_i)D_i \leqslant 0$ and for every $i$, $t_i\big(D_i^2 + (D - D_i)D_i\big) \leqslant 0$. The hypotheses on $D_k$ imply $D_k^2 + (D - D_k)D_k \leqslant -1$. This yields a contradiction. □

*Example* 4.9. — The case of surfaces with $a(S) = (\overline{3,2})$. We show that there are surfaces $S$ with $a(S) = (\overline{3,2})$ which admit a global meromorphic non-vanishing differential 2-form or equivalently, a holomorphic section of the anticanonical bundle. In Section 5 we shall give examples of surfaces with global vector fields.

Let $S$ be the surface defined by the germ $F = \Pi\sigma = \Pi_0\Pi_1\sigma$, where

$$\Pi_1(u', v') = (v'+\alpha, u'v') \text{ with } \alpha \neq 0, \quad \Pi_0(u, v) = (uv, v), \quad \text{and} \quad \sigma(z) = z.$$

One gets $F(z) = \big(z_1z_2(\alpha + z_2), z_1z_2\big)$. Remark that $\sigma(0) = O_1$ is the intersection of the two exceptional curves $D_0$ and $D_1$ (we denote by the same symbols the curves in the blown-up ball and the corresponding curves in the surface $S$). The surface $S$ contains a singular rational curve $D_0$ and a regular rational curve $D_1$. Their intersection numbers are $D_0^2 = -1$, $D_1^2 = -2$, $D_0D_1 = 1$. The Cramer systems have integer solutions. More precisely

$$D_{-K} = 2D_0 + D_1 \quad \text{and} \quad D_\theta = D_0.$$

The existence of $D_{-K}$ implies that there is $\delta \in \mathbb{C}^\star$ and a non-vanishing meromorphic 2-form $\eta$ satisfying the functional equation

$$(F^\star\eta)(z) = \eta(F(z)) \det DF(z) = \delta\eta(z)$$

where $\delta \in \mathbb{C}^\star$. Since $\sigma^{-1}(D_0) = \{z_1 = 0\}$ and $\sigma^{-1}(D_1) = \{z_2 = 0\}$, it follows that

$$\eta(z) = \frac{a(z)}{z_1^2 z_2} dz_1 \wedge dz_2.$$

The holomorphic function $a$ satisfies $a(0) \neq 0$. The invariance condition becomes

$$(\diamondsuit) \qquad -a(F(z)) = \delta(\alpha + z_2)^2 a(z), \quad \text{with} \quad \alpha \neq 0.$$

For $z = 0$ we get $-a(0) = \delta\alpha^2 a(0)$. Hence $\delta\alpha^2 = -1$. On the other hand, if $\delta\alpha^2 = -1$ and the j-th iteration of $F$ is denoted by $F^j(z) = (F_1^j(z), F_2^j(z))$, then

$$a(z) = \frac{a(0)}{\prod_{j=0}^{\infty}\left(1 + \frac{F_2^j(z)}{\alpha}\right)^2}$$

is an infinite convergent product which is the solution of $(\diamond)$. Since $\kappa(S) = \dfrac{1}{\delta}$ (see Theorem 4.5), it follows that

$$\kappa(S) = -\alpha^2.$$

The equivalent condition for the existence of a non-twisted meromorphic 2-forms is $\delta = 1$. Hence $\alpha = \pm i$ and in these cases $K = [-2D_0 - D_1]$. We want to stress that $\alpha$ is the parameter of the logarithmic versal deformation (we shall not prove this point here). In this deformation the surfaces with a non-zero anticanonical section are those over the hypersurface $\{\alpha = \pm i\}$.

## 5. Flat line bundles and global vector fields.

THEOREM 5.1. — *Let $S$ be a compact surface with a GSS and $2n < \sigma_n(S) < 3n$. Then*

$$\omega = \frac{df}{f} \in H^0(S, \Omega^1(\operatorname{Log} D) \otimes L^k)$$

*where $k$ is the integer defined in Lemma 2.4, $f$ is the holomorphic function of Lemma 2.7. Furthermore $\omega$ is closed and defines the foliation $\mathcal{F}$ of Theorem 2.14.*

*Proof.* — In a neighbourhood of 0, $f(F) = f^k$, therefore

$$F^\star\left(\frac{df}{f}\right)(z) = \frac{df(F(z))}{f(F(z))} DF(z) = \frac{d(f \circ F)}{f \circ F}(z) = \frac{df^k}{f^k}(z) = k\frac{df(z)}{f(z)}.$$

The foliation $\mathcal{F}$, given originally by the function $f$, is evidently also defined by $\ker \omega$. □

THEOREM 5.2. — 1) *Let $S$ be a compact surface with a GSS such that $n = b_2(S)$, $2n < \sigma_n(S) < 3n$. We suppose that for the intersection matrix $M = M(S)$ of $S$ the linear system (2) of Lemma 4.2 has solutions in $\mathbb{Z}$. Then there exists a unique numerically tangent divisor $D_\theta$ and a unique complex number $\lambda(S) = k(S)\kappa(S) \in \mathbb{C}^\star$ (See Theorem 4.5 for the definition of $\kappa$) such that*

$$H^0(S, \Theta(-D_\theta) \otimes L^{\lambda(S)}) \neq 0.$$

2) *Let $\mathcal{S} \to U$ be a logarithmic versal family of surfaces with fixed intersection matrix $M$. Then there exists a holomorphic function $\lambda$ on $U$ such that $\lambda(S_u) = \lambda(u)$ and surfaces which admit a global vector field are those over the hypersurface $\{u \mid \lambda(u) = 1\}$.*

3) *Every surface $S_0$ with intersection matrix $M(S_0) = M$ may be deformed by a logarithmic deformation $\mathcal{S} \to U$ into a surface $S_1$ admitting a global vector field. In this case $H_{vf} := \{u \in U \mid \lambda(u) = 1\}$ is a non-empty hypersurface.*

4) *Every surface $S_0$ with intersection matrix $M(S_0) = M$ may be deformed by a logarithmic deformation $\mathcal{S} \to U$ into a surface $S_2$ admitting a global anticanonical section. In this case $H_{ac} := \{u \in U \mid \kappa(u) = 1\}$ is a non-empty hypersurface. Moreover $H_{vf} \cap H_{ac} = \varnothing$.*

*Proof.* — By Lemma 4.2 and Theorem 4.5, there exists a unique anticanonical divisor $D_{-K} = D_\theta + D$ and a unique flat line bundle $L^{\kappa(S)}$ such that $-K \otimes L^{\kappa(S)} = [D_{-K}]$ has a non-trivial holomorphic section $s$. By Theorem 5.1, $\theta = \langle \omega, s \rangle \in H^0(S, \Theta(-D_\theta) \otimes L^{k\kappa(S)})$ is a non-trivial holomorphic section. Therefore $\lambda = k\kappa(S)$ satisfies the condition of 1). Remark that $\langle \omega, \theta \rangle = 0$. It remains to prove the unicity of $\lambda$. For this, let $a \in \mathbb{C}^\star$ and suppose the existence of a non-trivial section $\theta' \in H^0(S, \Theta \otimes L^a)$. Since $(\theta')$ is a numerically tangent divisor, we have that $\theta' \in H^0(S, \Theta(-D_\theta) \otimes L^a)$ and therefore $\tau = \theta \wedge \theta' \in H^0(S, -K(-2D_\theta) \otimes L^{a\lambda})$. If $\tau \neq 0$, then $0 \neq s' := \langle \omega, \theta' \rangle \in H^0(S, \mathcal{O}(D - D_\theta) \otimes L^{ak})$. Hence $s'$ is a meromorphic section of the flat line bundle $L^{ak}$. Since $\sigma_n(S) > 2n$ and $M(S)$ is negative-definite, there is no flat divisor. Therefore $L^{ak}$ is holomorphically trivial and $D = D_\theta$. This implies $D^2 = D.D_\theta = 0$, which gives a contradiction.

So $\theta$ and $\theta'$ are colinear twisted vector fields. But then $\frac{\theta}{\theta'}$ is again a non-trivial meromorphic section in the flat line bundle $L^{\frac{k\kappa(S)}{a}}$. This bundle is therefore trivial and $a = k\kappa(S) = \lambda$.

2) is clear, since by Theorem 4.5, $\kappa(u)$ is holomorphic and $\lambda(u) = k\kappa(u)$.

3) and 4) Let $\mathcal{S} \to \mathbb{C}^\star$ be the logarithmic deformation defined in Lemma 2.4, 2). We have

$$F(z) = F_{C,\alpha_{i_0}}(z) = \left(a(\alpha_{i_0})z_2^l(1 + A(z)), b(\alpha_{i_0})z_2^k\right),$$

$$DF(z) = \begin{pmatrix} az_2^l \frac{\partial A}{\partial z_1} & \star \\ 0 & bkz_2^{k-1} \end{pmatrix}.$$

We are looking for a vector field $\theta$ which is necessarily tangent to the foliation $\mathcal{F}$, by Theorem 3.1. Hence $\theta(z) = z_2^p H(z)\frac{\partial}{\partial z_1}$, with $p \geqslant 1$ by Lemma 4.7, because the curve $C$ is not the top of a tree. Moreover $H(0) \neq 0$, since the foliation $\mathcal{F}_C$ is non-singular at $\hat{O}_C$. The invariance condition for a global section of $\Theta \otimes L^{\lambda(S)}$ is

$$(F_\star\theta)(F(z)) = \lambda^{-1}\theta(F(z)).$$

We obtain

$$\begin{pmatrix} az_2^{p+l}H(z)\frac{\partial A}{\partial z_1}(z) \\ 0 \end{pmatrix} = \begin{pmatrix} az_2^l \frac{\partial A}{\partial z_1} & \star \\ 0 & bkz_2^{k-1} \end{pmatrix} \begin{pmatrix} z_2^p H(z) \\ 0 \end{pmatrix} = DF(z).\theta(z)$$
$$= \lambda^{-1}\begin{pmatrix} b^p z_2^{kp} H(F(z)) \\ 0 \end{pmatrix},$$

i.e.

$$(\triangle) \qquad az_2^{p+l}H(z)\frac{\partial A}{\partial z_1}(z) = \lambda^{-1}b^p z_2^{kp}H(F(z)).$$

We know by 1) and 2) that the holomorphic function $\lambda : \mathbb{C}^\star \to \mathbb{C}^\star$ exists and that $(\triangle)$ admits a solution. By Theorem 2.8, $\frac{\partial A}{\partial z_1}(0)$ is independent of $\alpha_{i_0}$, therefore considering the terms of lowest degree, we see that there exists a constant $C \in \mathbb{C}^\star$ such that

$$\lambda(\alpha_{i_0}) = Ca^{-1}(\alpha_{i_0})b^p(\alpha_{i_0}).$$

By Lemma 2.4 this function is not constant. Hence it is surjective and the hypersurface $H_{vf}$ is not empty. Since $\lambda = k\kappa$ and $k \geqslant 2$, $H_{ac} \neq \varnothing$ and $H_{vf} \cap H_{ac} = \varnothing$. $\square$

*Example* 5.3. — Surfaces with $a(S) = (\overline{3,2})$: We give examples of surfaces with non-trivial global vector fields:

We have $k(S) = 2$ and $\kappa(S) = -\alpha^2$ by (4.9). Hence $\lambda(S) = k(S)\kappa(S) = -2\alpha^2$. We have a non-trivial holomorphic vector field if and only if $\lambda(S) = 1$, i.e. $\alpha = \pm\frac{i\sqrt{2}}{2}$.

The following lemma recovers results of I. Nakamura ([24], §3).

LEMMA 5.4. — *Let $S$ be a surface containing a GSS with $2n < \sigma_n(S) \leqslant 3n$. Then*

$$l := \mathrm{Card}\{\lambda \in \mathbb{C}^\star \mid h^0(S, \Omega^1(\mathrm{Log}\, D) \otimes L^\lambda) > 0\} \leqslant 2$$

*and equality holds if and only if $S$ is an Inoue-Hirzebruch surface.*

*Proof.* — If $\lambda_1 \neq \lambda_2$, then $\omega_1 \wedge \omega_2 \neq 0$, because otherwise $\frac{\omega_1}{\omega_2}$ is a non-trivial meromorphic section of a non-trivial flat line bundle, which is impossible. Therefore $l \geqslant 3$ contradicts Theorem 4.5. For an Inoue-Hirzebruch surface $S$, we see with ($\star$) that $l = 2$. On the other hand, suppose that there are $\lambda_i \in \mathbb{C}^\star$, $i = 1,2$ with $\lambda_1 \neq \lambda_2$ and $0 \neq \omega_i \in H^0(S, \Omega^1(\mathrm{Log}\, D) \otimes L^{\lambda_i})$. Then $0 \neq \tau = \omega_1 \wedge \omega_2 \in H^0(S, K(D) \otimes L^{\lambda_1\lambda_2})$. Hence $D = D_{-K}$ and $D_\theta = 0$, i.e. $S$ is an Inoue-Hirzebruch surface (see Example 4.4). □

THEOREM 5.5. — *Let $\mathcal{G}$ be a foliation on a surface $S$ containing a GSS with $n = b_2(S) > 0$. Then*

i) *If $\sigma_n(S) < 3n$ there exists a non-trivial d-closed section $\omega \in H^0(S, \Omega^1(\mathrm{Log}\, D) \otimes L^{k(S)})$ which defines $\mathcal{G}$. Moreover $h^0(S, \Omega^1(\mathrm{Log}\, D) \otimes L^{k(S)}) = 1$ and the foliation is unique i.e. $\mathcal{G} = \mathcal{F}$, where $\mathcal{F}$ is the foliation given by Theorem 2.14.*

ii) *If $\sigma_n(S) = 3n$, i.e. if $S$ is an Inoue-Hirzebruch surface, $\mathcal{G}$ is one of the two foliations defined by*

$$\omega_i \in H^0(S, \Omega^1(\mathrm{Log}\, D) \otimes L^{\lambda_i}), \quad i = 1,2,$$

*where $\lambda_i$ are quadratic real numbers (see Section 2.2).*

*Proof.* — The case of Inoue-Hirzebruch surfaces has been studied in [19], §3, therefore we shall suppose that $\sigma_n(S) < 3n$. By [19] Partie A, there exists a curve $C$ of the universal covering, such that $\hat{O}_C$ is a regular point of the foliation $\hat{\mathcal{G}}_C$ induced by $\mathcal{G}$ on $\hat{S}_C$. By [19], Proposition 2.2.1, $\hat{\mathcal{G}}_C$ is defined by a reduced 1-form $\hat{\omega}_C$ such that

$$(*) \qquad F_C^\star \hat{\omega}_C = \lambda_C \hat{\omega}_C,$$

where $\lambda_C$ is a holomorphic function which vanishes only on $C$. We set $\lambda_C(z) = u(z)z_2^p$ with $p \geqslant 0$ and $u(0) \neq 0$. Since $C$ is invariant under the foliation, $\hat{\omega}_C(z) = a(z)z_2 dz_1 + b(z)dz_2$, where $b(0) \neq 0$.

We suppose that $\sigma_n(S) > 2n$. By virtue of Theorem 2.8, there is a coordinate system in a neighbourhood of $\hat{O}_C$, in which $C = \{z_2 = 0\}$ and $F_C(z) = (F_1(z), z_2^k)$, where $k = k(S)$. So

$$DF_C(z) = \begin{pmatrix} \dfrac{\partial F_1}{\partial z_1}(z) & \dfrac{\partial F_1}{\partial z_2}(z) \\ 0 & kz_2^{k-1} \end{pmatrix}.$$

We have to prove that $\mathcal{G}$ is equal to the foliation $\mathcal{F}$. Using $(*)$, one gets

$$\begin{cases} a(F)z_2^k \dfrac{\partial F_1}{\partial z_1}(z) & = \quad u(z)z_2^{p+1}a(z) \qquad (\mathbf{A}) \\ a(F)z_2^k \dfrac{\partial F_1}{\partial z_2}(z) + kz_2^{k-1}b(F) & = \quad u(z)z_2^p b(z) \qquad (\mathbf{B}) \end{cases}$$

Considering the order, equation (**B**) implies that $p = k-1$. Cancelling the extra factors we obtain

$$\begin{cases} a(F)\dfrac{\partial F_1}{\partial z_1}(z) & = \quad u(z)a(z) \qquad (\mathbf{C}) \\ a(F)z_2\dfrac{\partial F_1}{\partial z_2}(z) + kb(F) & = \quad u(z)b(z) \qquad (\mathbf{D}) \end{cases}$$

By (**D**), one has $u(0) = k$ and since $F_1(z)$ is a multiple of $z_2^l$ with $l \geqslant 1$, condition (**C**) implies that $a(0) = 0$. By induction we prove that the homogeneous part of degree $i \geqslant 0$ of $a$ vanishes, i.e. $a = 0$. Hence

$$\hat{\omega}_C(z) = b(z)dz_2 \quad , \quad \text{with} \quad b(F_C(z)) = \frac{u(z)}{u(0)}b(z).$$

The expression of $\hat{\omega}_C$ shows that $\mathcal{G}$ is the foliation defined by $\dfrac{df_C}{f_C}$ or by the Green function $G_C$. We shall describe $\hat{\omega}_C$ with more accuracy: The last condition on $b$ implies that it is a section of the flat line bundle $L^{\frac{u(z)}{u(0)}}$. By virtue of Lemma 1.3, $L^{\frac{u(z)}{u(0)}}$ is the trivial bundle. Hence $b$ is constant and $\lambda_C(z) = kz_2^{k-1}$. Finally $\mu(z) := b\dfrac{dz_2}{z_2}$, with $b \in \mathbb{C}^\star$, fulfills the condition

$F_C^\star \mu = k\mu$. Therefore $\mu \in H^0(S, \Omega^1(\operatorname{Log} D) \otimes L^k)$. Now apply Lemma 5.4 to finish the proof.

In the case $\sigma_n(S) = 2n$, we know by [8] Prop. 1.8 that $F_C$ is conjugate to $F(z) = (F_1(z), tz_2)$ with $0 < |t| < 1$. A similar computation yields the result with $k(S) = 1$. □

THEOREM 5.6. — *Let $S$ be a surface with GSS and $2n < \sigma_n(S) < 3n$. Let $D$ be the maximal divisor, $(\tilde{S}, \tilde{\omega}, S)$ the universal covering and $\tilde{D} = \tilde{\omega}^\star(D)$. Then*

*1) The fundamental group $\pi_1(\tilde{S} \setminus \tilde{D})$ is isomorphic to $\mathbb{Z}[\frac{1}{k}]$, i.e. the rational number having powers of $k$ as denominators.*

*2) We have the following exact sequence:*

$$0 \to \mathbb{Z}\left[\frac{1}{k}\right] \to \pi_1(S \setminus D) \to \mathbb{Z} \to 0.$$

*More precisely, $\pi_1(S \setminus D) = \mathbb{Z}[\frac{1}{k}] \rtimes \mathbb{Z}$.*

*3) The universal covering space $(Y, \tilde{p}, S \setminus D)$ of $S \setminus D$ (and therefore also of $(\tilde{S} \setminus \tilde{D})$) is a Riemann domain spread over $\Delta \times \mathbb{C}$.*

*Proof.* — 1) The complement $\tilde{S} \setminus \tilde{D}$ of $\tilde{D}$ in the universal covering space is isomorphic to the complement of the union of rational curves $\hat{C} = \bigcup_{C' \leqslant C} C'$ in

$$\hat{S}_C = B \cup \bigcup_{i \leqslant 0} A_i.$$

For every $i \geqslant 0$, we have $\pi_1(A_i \setminus \hat{C}) = \pi_1(B \setminus \{z_2 = 0\}) = \mathbb{Z}$. Let $\gamma_i$ be a positive generator of $\pi_1(A_i \setminus \hat{C})$, i.e. $\gamma_i$ is winding around one time positively in the plane of variable $z_2$. By Theorem 2.8, the patching of $A_i$ with $A_{i+1}$ sends $\gamma_i$ onto $k\gamma_{i+1}$. Since $A_i \cap A_{i+1}$ is simply connected, the group $\pi_1(\hat{S}_C \setminus \hat{C})$ is isomorphic to the quotient of the free abelian group generated by the $\{\gamma_i \mid i \leqslant 0\}$ by the subgroup generated by the relations $\gamma_i = k\gamma_{i+1}$. Therefore

$$\pi_1(\tilde{S} \setminus \tilde{D}) \simeq \mathbb{Z}\left[\frac{1}{k}\right].$$

2) Since the fibre $F$ is isomorphic to $\mathbb{Z}$, the homotopy exact sequence of a fibration (see [26], p. 377) applied to the covering $(\tilde{S} \setminus \tilde{D}, \tilde{\omega}, S \setminus D)$ yields

$$1 \to \pi_1(\tilde{S} \setminus \tilde{D}, \tilde{z}) \to \pi_1(S \setminus D, z) \to \pi_0(F, \tilde{z}) \to 1.$$

This gives immediately the desired exact sequence. Consider the subgroup of $\pi_1(S \setminus D)$ generated by a loop obtained from a path joining two identified points, one on the pseudoconvex boundary and the other on the pseudoconcave boundary of $A_0$. This group is sent onto $\pi_0(F, \tilde{z})$. Therefore $\pi_1(S \setminus D, z)$ is a semi-direct product.

3) There exists an integer $m \geqslant 1$, such that the linear system

$$M(S)\Big(k_i\Big)_i = m\Big(D_i^2 + 2 - g(D_i)\Big)_i,$$

has solutions in $\mathbb{Z}$. Therefore there exists a positive divisor $D_m$ and a flat line bundle $L$ such that $-mK \otimes \mathcal{O}(-D_m) \otimes L$ is trivial. For the universal covering $(\tilde{S}, \tilde{\omega}, S)$ of $S$, we set $\tilde{D}_m = \tilde{\omega}^\star D_m$ and $\tilde{K} = \tilde{\omega}^\star K$, which is the canonical bundle of $\tilde{S}$. Then we have $-m\tilde{K} \otimes \mathcal{O}(-\tilde{D}_m) = \tilde{\omega}^\star\big(-mK \otimes \mathcal{O}(-D_m) \otimes L\big)$ and $-m\tilde{K}$ is trivial on $\tilde{S} \setminus \tilde{D}$. By [21] Thm 33, p. 698, there exists a m-to-one covering $(X, q, \tilde{S} \setminus \tilde{D})$ of $\tilde{S} \setminus \tilde{D}$ such that $q^\star(\tilde{K}_{|\tilde{S}\setminus\tilde{D}})$ is trivial. Hence, there exists on $Y$ a holomorphic 2-field $\tilde{\tau}$ which does not vanish. Besides, $\omega \in H^0(S, \Omega^1(\operatorname{Log} D) \otimes L^k)$ induces on $Y$ a non-vanishing holomorphic form $\alpha$. Therefore $\tilde{\theta} = \langle \tilde{\tau}, \alpha \rangle$ is a non-vanishing vector field on $Y$, tangent to the foliation induced by $\alpha$, since $\langle \tilde{\theta}, \alpha \rangle = 0$.

On $\tilde{S} \setminus \tilde{D}$, the Green function $\tilde{G} : \tilde{S} \setminus \tilde{D} \to \mathbb{R}^{<0}$ is pluriharmonic, surjective and submersive with connected fibers (see Corollary 2.13). It is locally the real part of a holomorphic function. On the universal covering $Y$, this yields a holomorphic function $\tilde{f} : Y \to \mathbb{H}^g$, where $\mathbb{H}^g := \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$. The image of $\tilde{f}$ is connected, invariant under the group of translations $2\pi i\mathbb{Z}$ and under the group of homotheties $\{k^p \mid p \in \mathbb{Z}\}$. Obviously, this shows that $\tilde{f}$ is submersive and surjective onto $\mathbb{H}^g$ with connected fibers isomorphic to $\mathbb{C}$ by Theorem 2.14.

Denote by $\beta$ the section of $T^\star\mathcal{F}$ dual to $\tilde{\theta} \in H^0(Y, T\mathcal{F})$. Then $\beta$ is a vertical non-vanishing 1-form. Let $\{U_i\}$ be a covering of $\mathbb{H}^g$, such that on each $U_i$ there is a section $s_i : U_i \to Y$ of the fiber space $\tilde{f} : Y \to \mathbb{H}^g$. Then integration of $\beta$ along curves with starting point $s_i(x)$ in the fiber $\tilde{f}^{-1}(x)$, $x \in U_i$ yields a holomorphic function $g_i$ on $\tilde{f}^{-1}(U_i)$ with $\tilde{\theta}(g_i) = 1$. The difference of two such funtions $g_i - g_j =: g_{ij}$ is in fact a holomorphic function constant on the fibers of $\tilde{f}$ and therefore a function on $U_i \cap U_j$. The collection $\{g_{ij}\}$ defines a (trivial) cocycle of $H^1(\mathbb{H}^g, \mathcal{O})$, trivalized by, say, $h_i \in \mathcal{O}(U_i)$. Now $g := g_i - h_i$ is a global holomorphic function on $Y$ with $\tilde{\theta}(g) = 1$. Finally the map $(\tilde{f}, g) : Y \to \mathbb{H}^g \times \mathbb{C}$ realizes $Y$ as a Riemann domain spread over $\mathbb{H}^g \times \mathbb{C}$. □

## BIBLIOGRAPHY

[1] P. BAUM, R. BOTT, On the zeroes of meromorphic vector fields, Essais en l'honneur de G. De Rham, (1970), 29-47.

[2] M. BRUNELLA, Feuilletages holomorphes sur les surfaces complexes compactes, Annales Scient. Ec. Norm. Sup., $4^e$ série, tome 30 (1997), 569-594.

[3] J. CARRELL, A. HOWARD, C. KOSNIOWSKI, Holomorphic vector fields on complex surfaces, Math. Ann., 204 (1973), 73-81.

[4] C. CAMACHO, P. SAD, Invariant varieties through singularities of holomorphic vector fields, Annals of Math., 115 (1982), 579-595.

[5] G. DLOUSSKY, Structure des surfaces de Kato, Mémoire de la S.M.F 112, n° 14 (1984).

[6] G. DLOUSSKY, Sur la classification des germes d'applications holomorphes contractantes, Math. Ann., 280 (1988), 649-661.

[7] G. DLOUSSKY, Une construction élémentaire des surfaces d'Inoue-Hirzebruch, Math. Ann., 280 (1988), 663-682.

[8] G. DLOUSSKY, F. KOHLER, Classification of singular germs of mappings and deformations of compact surfaces of the $VII_0$ class, Ann. Math. Pol., LXX (1998), 49-83.

[9] G. DLOUSSKY, K. OELJEKLAUS, Surfaces de la classe $VII_0$ et automorphismes de Hénon, C.R.A.S., 328, Série I (1999), 609-612.

[10] I. ENOKI, Surfaces of class $VII_0$ with curves, Tôhoku Math. J., 33 (1981), 453-492.

[11] C. GELLHAUS, P. HEINZNER, Komplexe Flächen mit holomorphen Vektorfeldern, Abh. Math. Sem. Hamburg, 60 (1990), 37-46.

[12] X. GOMEZ-MONT, Singularités d'équations différentielles, Astérisque 150-151, SMF (1987).

[13] P. GRIFFITH, J. HARRIS, Principles of Algebraic Geometry, Wiley & Sons, 1978.

[14] H. HIRONAKA, Introduction to the theory of infinitesimaly near singular points, Memorias de Mathematica del Instituto "Jorge Juan" 28, Madrid 1974.

[15] J. HAUSEN, Zur Klassifikation glatter kompakter $\mathbb{C}^*$-Flaechen, Math. Ann., 301 (1995), 763-769.

[16] J.H. HUBBARD, R.W. OBERSTE-VORTH, Hénon mappings in the complex domain I, Pub. Math. IHES, 79 (1994), 5-46.

[17] M. INOUE, New surfaces with no meromorphic functions II, Complex Analysis and Alg. Geom., 91-106, Iwanami Shoten Pb. (1977).

[18] Ma. KATO, Compact complex manifolds containing "global spherical shells", Proceedings of the Int. Symp. Alg. Geometry, Kyoto 1977. Kinokuniya Book Store, Tokyo, 1978.

[19] F. KOHLER, Feuilletages holomorphes singuliers sur les surfaces contenant une coquille sphérique globale, Ann. Inst. Fourier, 45-1 (1995), 161-182. Erratum Ann. Inst. Fourier, 46-2 (1996).

[20] K. KODAIRA, On the structure of compact complex analytic surfaces I, Am. Jour. Math., 86 (1964), 651-698.

[21] K. KODAIRA, On the structure of compact complex analytic surfaces II, Am. Jour. Math., 88 (1966), 682-721.

[22] B. KHANEDANI, T. SUWA, First variation of holomorphic forms and some applications, Hokkaido Math. Jour., Vol. XXVI, No. 2 (1997), 323-335.

[23] Y. KAWAMATA, On deformations of compactifiable complex manifolds, Math. Ann., 235 (1978), 247-265.

[24] I. NAKAMURA, On surfaces of class $VII_0$ with curves, Invent. Math., 78 (1984), 393-443.

[25] I. NAKAMURA, On surfaces of class $VII_0$ with curves II, Tohoku Math. J., 42 (1990), 475-516.

[26] E.H. SPANIER, Algebraic Topology, Mc Graw-Hill, 1966.

[27] T. SUWA, Indices of holomorphic vector fields relative to invariant curves, Proc. AMS., 123, No.10 (1995), 2989-2997.

Manuscrit reçu le 4 décembre 1997,
révisé le 4 juin 1999,
accepté le 22 juin 1999.

G. DLOUSSKY and K. OELJEKLAUS,
CMI-Université d'Aix-Marseille I
LATP-UMR(CNRS) 6632
39, rue Joliot-Curie
F-13453 Marseille Cedex 13.
dloussky@gyptis.univ-mrs.fr
karloelj@gyptis.univ-mrs.fr