ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 49, nº 4 (1999), p. 1369-1385

<http://www.numdam.org/item?id=AIF_1999__49_4_1369_0>

Ann. Inst. Fourier, Grenoble
49, 4 (1999), 1369–1385

A NOTE ON PROJECTIVE LEVI FLATS AND MINIMAL SETS OF ALGEBRAIC FOLIATIONS

by Alcides LINS NETO

1. Introduction.

A non trivial minimal set of a singular foliation $\mathcal{F}$ on a complex compact manifold M, is a closed set $\mathcal{M} \subset M$ with the following properties:

a) $\mathcal{M}$ is invariant for $\mathcal{F}$.

b) $\mathcal{M} \neq \varnothing$.

c) $\mathcal{M}$ does not contain singular points of $\mathcal{F}$.

d) $\mathcal{M}$ is minimal with respect to properties a), b) and c).

We shall use the abbreviation n.t.m.s. to denote "non trivial minimal set".

In [CLS1] the problem of the existence or not of n.t.m.s. for codimension one foliations of $\mathbb{CP}^2$ was studied. In particular it was proved that for any $k \geqslant 2$, the space of foliations of degree k contains an open non empty set, say A_k, such that any foliation $\mathcal{F} \in A_k$ has no n.t.m.s. It was proved also that if a foliation has an algebraic leaf then it has no n.t.m.s. Since foliations of degree 0 or 1 have always algebraic leaves, they cannot have n.t.m.s. In general, the question of the existence of foliations in $\mathbb{CP}^2$ having n.t.m.s. remains open. Concerning this problem we will prove in §2.1 the following result:

Keywords: Holomorphic foliation – Non trivial minimal set – Levi flat.
Math. classification: 58A17 – 32F15 – 32F25.

THEOREM 1. — *Codimension 1 foliations on* $\mathbb{CP}^n$, $n \geqslant 3$, *have no n.t.m.s..*

In fact Theorem 1 will be a consequence of the following result:

THEOREM 1′. — *Any closed invariant set of a holomorphic codimension one foliation of* $\mathbb{CP}^n, n \geqslant 3$, *contains a singularity of the foliation.*

Another problem that we will consider is the existence of Levi flats in projective spaces. Let M be a complex manifold of complex dimension n and L be a C^1 submanifold of real codimension 1. Given $p \in L$, the tangent space $T_p(L)$ contains an unique complex subspace of complex dimension $n-1$ that we shall denote C_p. This defines a distribution $\mathcal{C}$ on L.

We say that L is a Levi flat if the distribution $\mathcal{C}$ is integrable. The integrability of $\mathcal{C}$ implies that L has a C^{k-1} foliation $\mathcal{F}$, whose leaves are tangent to the subspaces $C_p, p \in L$. Since the subspaces C_p are complex the leaves of $\mathcal{F}$ are holomorphic immersed submanifolds of complex dimension $n-1$.

In §2.2 we shall prove the following result:

THEOREM 2. — *For* $n \geqslant 3$ *there are no real analytic Levi flats on* $\mathbb{CP}^n$.

In §3 we will generalize Theorem 2. In order to state the main result that will be used, we consider the following situation:

Let M be a holomorphic manifold of complex dimension $n \geqslant 2$. Let V be an open set of M satisfying the following properties:

a) All connected components of V are Stein.

b) The closure $\overline{V}$ of V, is compact and connected.

We will denote by K the boundary $\overline{V} \setminus V$ of V. Given a neighborhood U of K, $0 \leqslant j \leqslant 2n$, and $p \in K$, let

$$(*) \qquad h_j : H_j(U,\mathbb{Z}) \longrightarrow H_j(\overline{V},\mathbb{Z}) \text{ and } i_j : \Pi_j(U,p) \longrightarrow \Pi_j(\overline{V},p)$$

be the homomorphisms induced by the inclusion $i : U \longrightarrow \overline{V}$ in the homology and homotopy groups respectively. The following result is a kind of generalization of Lefschetz Theorem on hyperplane sections (cf. [M]):

THEOREM 3. — *For any neighborhood A of K in $\overline{V}$ there is a neighborhood U of K such that $U \subset A$ and h_j and i_j as in (*) are isomorphisms for $j \leqslant n-2$ and are onto for $j = n-1$. In particular we have the following:*

(i) *K is connected.*

(ii) *If K is a C^1 real submanifold of M then the homomorphisms below are isomorphisms for $j \leqslant n-2$ and are onto for $j = n-1$:*

$$h_j : H_j(K, \mathbb{Z}) \longrightarrow H_j(\tilde{V}, \mathbb{Z}) \text{ and } i_j : \Pi_j(K, p) \longrightarrow \Pi_j(\tilde{V}, p).$$

(iii) *If K is a C^1 real submanifold of M and $n \geqslant 3$ then $\Pi_1(K,p)$ and $\Pi_1(\overline{V}, p)$ are isomorphic.*

It is not difficult to see that Lefschetz Theorem on hyperplane sections is a consequence of Theorem 3. Another consequence is the following:

COROLLARY — *Let M be a compact complex manifold of complex dimension $n \geqslant 3$, with finite fundamental group. Then M cannot contain a real analytic Levi flat L such that all connected components of $M \backslash L$ are Stein.*

The above corollary follows from (iii) of Theorem 3 and Haefliger's Theorem, which says that a real analytic manifold with finite fundamental group admits no real analytic foliations (cf. [Ha]).

I would like to thank A. Douady and M. Brunella for helpful conversations and ideas and the referee for some useful suggestions for the final version of this paper.

2. Non trivial minimal sets and Levi flats.

In this section we prove Theorems 1′ and 2. In the proof we will use the following:

THEOREM [T]. — *Let U be an open connected subset of $\mathbb{CP}^n, n \geqslant 2$, satisfying the following properties:*

(i) *The boundary ∂U of U is not empty.*

(ii) *For any $p \in \partial U$ there exists a holomorphic embedding $f_p : B^{n-1} \longrightarrow \mathbb{CP}^n$, where B^{n-1} is the unit ball in $\mathbb{C}^{n-1}$, such that $f_p(0) = p$ and $f_p(B^{n-1}) \cap U = \varnothing$.*

Then U is Stein.

Note that condition (ii) implies that U is locally pseudo-convex, that is, if $p \in \partial U$, then there exists a neighborhood V of p such that $V \cap U$ is

Stein. Therefore Theorem [T] is a consequence of a Theorem of Takeuchi (cf. [T] and [E]).

As a consequence we have the following:

COROLLARY — *Let $K \subset \mathbb{CP}^n, n \geqslant 2$, be either a Levi flat or a closed invariant set of a holomorphic singular foliation of codimension 1 of $\mathbb{CP}^n$, which does not contain singular points of the foliation. Then every connected component of $U = \mathbb{CP}^n \setminus K$ is Stein.*

2.1 Proof of Theorem 1′.

The idea is to prove that the singular set of a holomorphic foliation of codimension 1 on $\mathbb{CP}^n$ must have at least one irreducible component of complex codimension 2. This fact together with the corollary of Theorem [T] implies Theorem 1′. In fact, if a foliation $\mathcal{F}$ on $\mathbb{CP}^n, n \geqslant 3$, had a closed invariant set K such that $K \cap \operatorname{sing}(\mathcal{F}) = \varnothing$, then, by the corollary of Theorem [T], the connected components of the open set $U = \mathbb{CP}^n \setminus K$ would be Stein. On the other hand the singular set of $\mathcal{F}$ contains some irreducible component, say S, of dimension $\geqslant 1$. Since a Stein open set cannot contain a compact analytic subset of dimension greater than zero, this would imply that $S \cap K \neq \varnothing$, which is a contradiction.

We will consider the following situation:

Let $\mathcal{F}$ be a foliation of degree k on $\mathbb{CP}^2$, with finite singular set $\operatorname{sing}(\mathcal{F})$. Given $p \in \operatorname{sing}(\mathcal{F})$ let X be a holomorphic vector field in a neighborhood V of p which is tangent to $\mathcal{F}$. Suppose that the linear part $A = DX(p)$ of X at p is non singular. The Baum-Bott index of $\mathcal{F}$ at p is defined in this case by

$$BB(\mathcal{F},p) \;=\; (\operatorname{trace}(A))^2/\det(A).$$

The Baum-Bott index can be defined for any isolated singularity (cf. [BB] and [MB]), but we will use it only in the non degenerate case.

We will use the following result, wich is a consequence of a theorem of Baum and Bott (cf. [AL-N]):

PROPOSITION 1. — *Let $\mathcal{F}$ be a foliation of degree k on $\mathbb{CP}^2$, with singular set $\operatorname{sing}(\mathcal{F})$ of complex codimension 2. Then*

$$\sum_{p \in \operatorname{sing}(\mathcal{F})} BB(\mathcal{F},p) \;=\; (k+2)^2.$$

In particular $\sum_{p\in \mathrm{sing}(\mathcal{F})} BB(\mathcal{F},p)$ *is positive for any foliation* $\mathcal{F}$ *on* $\mathbb{CP}^2$.

Now let $\mathcal{G}$ be a codimension one holomorphic foliation on $\mathbb{CP}^n$, $n \geqslant 3$, and suppose by contradiction that all irreducible components of $\mathrm{sing}(\mathcal{G})$ have complex codimension $\geqslant 3$. Let $E \subset \mathbb{CP}^n$ be a 2-plane in general position with respect to $\mathcal{G}$. The 2-plane E is a linear embedding of $i : \mathbb{CP}^2 \longrightarrow \mathbb{CP}^n$, where $E = i(\mathbb{CP}^2)$, with the following properties:

a) E is not containned in any leaf of $\mathcal{G}$.

b) E intersects transversally all smooth strata of $\mathrm{sing}(\mathcal{G})$.

c) Outside $E \cap \mathrm{sing}(\mathcal{G})$, the set of tangencies of $\mathcal{G}$ with E has codimension at most 2 in E.

In Proposition 1 of [CLS2] it is proved that the set of all 2-planes satisfying (a), (b) and (c) is open and dense in the Grassmanian of 2-planes in $\mathbb{CP}^n$. It follows from (a) that the $i^*(\mathcal{G}) = \mathcal{F}$ is a codimension one foliation on $\mathbb{CP}^2$. Condition (b) and the fact that all irreducible components of $\mathrm{sing}(\mathcal{G})$ have codimension $\geqslant 3$, imply that $E \cap \mathrm{sing}(\mathcal{G}) = \varnothing$, so that the singularities of $\mathcal{F}$ correspond to the tangencies of E with some leaves of $\mathcal{G}$, and so (c) implies that $\mathrm{sing}(\mathcal{F})$ is finite.

Let $p \in \mathrm{sing}(\mathcal{F})$. Since $i(p) = p'$ is not a singular point of $\mathcal{G}$, this foliation has a holomorphic first integral, say g, defined in a neighborhood U of p', where $dg(p') \neq 0$. Let $f = g \circ i$. Observe that f is a local first integral of $\mathcal{F}$. Since p' is a tangency of $\mathcal{G}$ with E, we must have $df(p) = 0$, so that p is an isolated singularity of f and of $\mathcal{F}$. We say that p' is a tangency of Morse type if p is a Morse singularity for f, that is in some coordinate system (x, y) around p such that $x(p) = y(p) = 0$ we have $f(x, y) = f(p) + xy$. If this is the case, then the foliation $\mathcal{F}$ is defined in a neighborhood of $p = (0, 0)$ by the vector field $x\partial/\partial x - y\partial/\partial y$, so that $BB(\mathcal{F}, p) = 0$.

Now observe that the 2-plane E can be deformed a little bit to a 2-plane E' in such a way that all tangencies of E' with $\mathcal{G}$ are of Morse type. This assertion is an easy consequence of the following facts:

(i) Morse type singularities are stable by small perturbations.

(ii) Let $g : B^n(0, 2) \longrightarrow \mathbb{C}$ be a holomorphic function, where $g(0) = 0$ and $dg(0) \neq 0$, say $\partial g/\partial x_n(0) \neq 0$. Let $f(x, y) = g(x, y, 0, \ldots, 0)$. Assume that $df(0, 0) = 0$ and that $df(x, y) \neq 0$ for $(x, y) \in B^2(0, 2) \setminus \{0\}$. Then, given $\epsilon > 0$, there are $a, b, c \in \mathbb{C}$, with $|a|$, $|b|$, $|c| < \epsilon$ and such that all singularities of $h(x,y) = g(x,y,\ldots,ax+by+c)$ in $B^2(0,1)$ are of Morse type.

We leave the details of the proof for the reader.

Finally, observe that we have obtained a foliation $\mathcal{F}' = \mathcal{G}|_{E'}$ such that for any $p \in \operatorname{sing}(\mathcal{F}')$ we have $BB(\mathcal{F}',p) = 0$. This contradicts Proposition 1, so that $\operatorname{sing}(\mathcal{G})$ must have some component of codimension 2.

2.2. Proof of Theorem 2.

The idea is to use Theorem 1′, and the following result:

THEOREM 4 . — *Let L be a real analytic Levi flat in $\mathbb{CP}^n, n \geqslant 2$. Then there exists a holomorphic codimension one foliation $\mathcal{F}$ on $\mathbb{CP}^n$ such that L is $\mathcal{F}$-invariant.*

Clearly Theorem 4 follows from the corollary of Theorem [T] and of the following lemmas:

LEMMA 1. — *Let M be a complex manifold and $L \subset M$ be a real analytic Levi flat. Then, there exist a neighborhood U of L and a holomorphic codimension one foliation $\mathcal{G}$ on U such that L is $\mathcal{G}$-invariant.*

Observe that, if $\mathcal{F}$ is the foliation on L defined by the integrable distribution $\mathcal{C}$ of complex hyperplanes in L, then $\mathcal{G}|_L = \mathcal{F}$.

LEMMA 2 . — *Let V be a Stein manifold and $K \subset V$ be a compact set such that $U = V \setminus K$ is connected. Then any holomorphic codimension one foliation $\mathcal{G}$ on U, such that $\operatorname{cod}(\operatorname{sing}(\mathcal{G})) \geqslant 2$, can be extended to a holomorphic foliation on V.*

2.2.1 - Proof of Lemma 1.

We will use the following fact in the proof:

ASSERTION. — For any $p \in L$ there exists a holomorphic function H, defined in a neighborhood U of p, such that $dH(p) \neq 0$ and $L \cap U = v^{-1}(0)$, where $v = \Im(H)$.

The above assertion is well known and can be proved by using Frobenius Theorem and the results of [To]. Since its proof is not long we will give it at the end of §2.2.1. Let us prove Lemma 1 from the assertion.

Observe first that the assertion implies that for any $p \in L$ there exists a holomorphic coordinate system

$$\phi = (x, y) : U \longrightarrow \mathbb{C}^n, \text{ where } x : U \longrightarrow \mathbb{C}^{n-1} \text{ and } y : U \longrightarrow \mathbb{C}$$

such that $L \cap U = y_2^{-1}(0)$, where $y_2 = \Im(y)$.

It follows from the above observation that it is possible to find a holomorphic atlas of a neighborhood V of L

$$\mathcal{U} = \{U_j, \phi_j = (x_j, y_j)\}_{j \in J} \text{ where } (x_j, y_j) : U_j \longrightarrow \mathbb{C}^{n-1} \times \mathbb{C}$$

such that

(i) If $i, j \in J$, then $U_{i,j} = U_i \cap U_j$ is connected and homeomorphic to a ball in $\mathbb{C}^n$, if not empty.

(ii) For any $j \in J$ we have $U_j \cap L \neq \varnothing$ and is homeomorphic to a ball in L.

(iii) For any $i, j \in J$ such that $U_{i,j} \neq \varnothing$ then $L \cap U_{i,j} \neq \varnothing$ and is homeomorphic to a ball in L.

(iv) For any $j \in J$ we have $U_j \cap L = \{\Im(y_j) = 0\}$.

Now let $i, j \in J$ be such that $U_i \cap U_j \neq \varnothing$ and consider the change of chart

$$\phi = \phi_{i,j} = \phi_i \circ (\phi_j)^{-1} : \phi_j(U_{i,j}) \longrightarrow \phi_i(U_{i,j}).$$

In order to simplify the notations let us call $x_i = x$, $y_i = y$, $x_j = z$ and $y_j = w$, so that $\phi(x, y) = (z(x, y), w(x, y))$. It follows from (i), (ii) and (iii) above, that the domain of w, $\phi_j(U_{i,j})$, is homeomorphic to a ball and contains a ball $B \subset \mathbb{C}^{n-1} \times \mathbb{R} \subset \mathbb{C}^{n-1} \times \mathbb{C}$. Moreover, condition (iv) implies that if $(x, t) \in B$ ($t \in \mathbb{R}$), then $w(x, t) \in \mathbb{R}$, so that the holomorphic function $x \mapsto w(x, t)$ (t fixed) must be constant. This implies that the map $(x, t) \in B \mapsto w(x, t)$ does not depend on x. Since w is holomorphic it follows that $(x, y) \mapsto w(x, y)$ does not depends on x.

The above argument implies the coordinate changes $\phi_{i,j}$, are of the form

$$\phi_{i,j}(x_j, y_j) = (x_i(x_j, y_j), y_i(y_j)).$$

Therefore the atlas $\mathcal{U}$ defines a codimension one foliation on the neighborhood $V = \cup_j U_j$ of L.

Proof of the assertion. — Let $\mathcal{F}$ be the foliation on L defined by the distribution of complex hyperplanes $\mathcal{C}$. Since L is real analytic $\mathcal{F}$ is also real analytic. Fix a point $p \in L$ and a holomorphic coordinate system $(\phi = (x, y), U)$ with the following properties:

a) $x = (x_1, \dots, x_{n-1}) : U \longrightarrow \mathbb{C}^{n-1}$ and $y : U \longrightarrow \mathbb{C}$.

b) $p \in U$ and $\phi(p) = 0$.

c) The surface $\{x = 0\}$ is transversal to the leaves of $\mathcal{F}$.

Condition (c) implies that $L \cap \{x = 0\}$ is a real analytic curve $\gamma(t) = (0, y(t))$. After a holomorphic change of variables in the coordinate y we can suppose that $y(t) = t$, which means that $L \cap \{x = 0\}$ is the real axis in the y-plane. Let F_t be the leaf of $\mathcal{F}$ through $\gamma(t)$. Since F_t is holomorphic it can be written locally as the graph of a function, say $y = \varphi(x, t)$, where φ is real analytic, holomorphic with respect to x and $\varphi(0, t) = t$. If we set $\varphi = u + iv$, where $u = \mathbb{R}(\varphi)$ and $v = \Im(\varphi)$, then the hypersurface L can be defined locally around p by eliminating t in the equation $y_1 - u(x, t) = 0$ $(y = y_1 + iy_2)$, say $t = h(x, y_1)$, and substituting h in $y_2 - v(x, t)$, so that in a neighborhood of p, L is given by $y_2 = v(x, h(x, y_1))$.

Now, let $\varphi(x, t) = \sum_{n=1}^{\infty} a_n(x).t^n$ be the Taylor series in t of φ around (0,0). Extend φ to a neighborhood of (0,0) in $\mathbb{C}^{n-1} \times \mathbb{C}$ by setting $\varphi(x, z) = \sum_{n=1}^{\infty} a_n(x).z^n$, $z \in \mathbb{C}$. Let $F(x, y, z) = y - \varphi(x, z)$. Since $F(0, 0, 0) = 0$ and $\partial F/\partial z(0, 0, 0) \neq 0$, by the implicit function theorem, there exists a holomorphic function $H(x, y)$, defined in a neighborhood of (0,0), such that $\varphi(x, H(x, y)) = y$. Finally, observe that L can be defined in a neighborhood of (0,0) by $\Im(H(x, y)) = 0$. We leave the proof of this fact for the reader. This ends the proof of Lemma 1.

2.2.2. Proof of Lemma 2.

Let $f : V \longrightarrow \mathbb{R}$ be a strictly-pluri-subharmonic C^∞ exhaustion of V. Since $\lim_{p \to \infty} f(p) = +\infty$, the sets $M_t = \{p \in V;\ f(p) \leqslant t\}$ are compact and $K \subset M_t$ for $t \geqslant t_0$, so that the foliation $\mathcal{G}$ is defined on $V_t = V \setminus M_t$, for $t \geqslant t_0$. The idea is to prove the following:

(*) Suppose that $\mathcal{G}_t$ is a codimension one holomorphic foliation defined on V_t, such that $\mathrm{cod}(\mathrm{sing}(\mathcal{G}_t)) \geqslant 2$. Then there exists $\epsilon > 0$ such that $\mathcal{G}_t$ can be extended to a foliation on $V_{t-\epsilon}$.

Since $m = \inf\{f\} > -\infty$, it is clear that (*) implies Lemma 2. On the other hand, since $f^{-1}(t)$ is compact, it is not difficult to see that (*) is a consequence of the following:

(**) Let $\mathcal{G}_t$ be as in (*) and $p \in f^{-1}(t)$. Then $\mathcal{G}_t$ can be extended to $V_t \cup W$, where W is a neighborhood of p.

In order to prove (**) we use the following results:

LEMMA A (cf. [ST]). — *Given $p \in f^{-1}(t)$ there exists a biholomorphism $\phi : W \longrightarrow W' \subset \mathbb{C}^n$, where W is a neighborhood of p, and a Hartog's domain $H \subset W'$ such that $\phi^{-1}(H) \subset V_t$ and $p \in \phi^{-1}(\widehat{H})$.*

A Hartog's domain is an open set H of $\mathbb{C}^n, n \geqslant 2$ of the form $H = (U' \times \Delta(r)) \cup (U \times (\Delta(r) \setminus \overline{\Delta}(r')))$, where U and U' are open sets of $\mathbb{C}^{n-1}$, U connected, $U \supset U' \neq \varnothing$, $\Delta(r)$ is the disk of radius r in $\mathbb{C}$ and $0 < r' < r$. The set $\widehat{H}$ is, by definition $U \times \Delta(r)$. We observe that in Lemma A, we can suppose that U and U' are polydisks.

LEVI'S THEOREM (cf. [S]). — *Let H be a Hartog's domain and f be a meromorphic function on H. Then f can be extended to a meromorphic function on $\widehat{H}$.*

Let us finish the proof of Lemma 2. Consider the biholomorphism ϕ as in Lemma A and let $\mathcal{G}'$ be the restriction of $\phi_\star(\mathcal{G}_t)$ to $H \subset \mathbb{C}^n$. We will prove that there exists an integrable holomorphic 1-form ω on $\widehat{H}$ such that $\mathcal{G}'$ is defined by the differential equation $\omega \mid_H = 0$. This will prove the lemma.

The foliation $\mathcal{G}'$ is defined locally by integrable 1-forms, so that there exist a covering of H by open sets $\mathcal{U} = (U_j)_{j \in J}$ and collections $(\omega_j)_{j \in J}$, $(h_{i,j})_{U_{i,j} \neq \varnothing}$ $(U_{i,j} = U_i \cap U_j)$, such that

a) ω_j is an integrable 1-form on U_j such that $\mathrm{cod}(\mathrm{sing}(\omega_j)) \geqslant 2$ and $\mathcal{G}' \mid_{U_j}$ is defined by $\omega_j = 0$.

b) If $U_{i,j} \neq \varnothing$ then $h_{i,j} \in \mathcal{O}^\star(U_{i,j})$ and on $U_{i,j}$ we have $\omega_i = h_{i,j} \,.\, \omega_j$.

Since $\widehat{H} \subset \mathbb{C}^n$ we can write

$$\omega_j \;=\; \sum_{i=1}^{n} g_i^j.dx_i \ , \ \text{where} \ \ g_i^j \in \mathcal{O}(U_j).$$

Observe that condition (b) implies that if $U_{j,\ell} \neq \varnothing$ then

$$(*) \qquad g_i^j = h_{j,\ell}.g_i^\ell \ \text{ for all } \ i = 1, \ldots, n.$$

Since $\widehat{H}$ is connected, it follows that for some $i \in \{1, \ldots, n\}$ we must have $g_i^j \not\equiv 0$ for all $j \in J$. We suppose $i = n$, so that g_i^j / g_n^j defines a meromorphic function f_i^j on U_j for all $i = 1, \dots, n-1$. Now, $(*)$ implies that if $U_{j,\ell} \neq \varnothing$, then $f_i^j = f_i^\ell$ on $U_{j,\ell}$, so that, for all $i = 1, \dots, n-1$, there exists a meromorphic function f_i on H, such that $f_i\ |_{U_j} = f_i^j$. It follows from Levi's Theorem that f_i can be extended to a meromorphic function on $\widehat{H}$, which we call still f_i.

Consider the meromorphic 1-form η defined on $\widehat{H}$ by

$$\eta = dx_n + \sum_{i=1}^{n-1} f_i \, . \, dx_i$$

Since $\widehat{H}$ is a polydisk, it follows that there exists $h \in \mathcal{O}(\widehat{H})$ and a holomorphic 1-form ω on $\widehat{H}$ such that $\mathrm{cod}(\mathrm{sing}(\omega) \geqslant 2$ and $\eta = \dfrac{1}{h}.\omega$. It is not difficult to see that for all $j \in J$ we have $\omega\ |_{U_j} = g_j.\omega_j$ for some $g_j \in \mathcal{O}^\star(U_j)$, so that ω is integrable and the foliation defined by $\omega = 0$ on $\widehat{H}$ extends $\mathcal{G}'$. This ends the proof of Lemma 2.

3. Theorem 3 and its corollary.

In this section we prove Theorem 3 and its corollary.

Let V, $\overline{V}$ and K$=\overline{V} \setminus V$, be as in Theorem 3. Fix a neighborhood A of K in $\overline{V}$. We need a lemma.

LEMMA 3. — *There exist neighborhoods U_1 and U of K in $\overline{V}$ and a flow $\varphi : \mathbb{R} \times \overline{V} \longrightarrow \overline{V}$ such that*

(a) $U_1 \subset \overline{U}_1 \subset U \subset A$.

(b) $\varphi_t(p) = p, \ \forall\ p \in \overline{U_1}$, *where* $\varphi_t(p) = \varphi(t,p)$.

(c) $\varphi\ |_{\mathbb{R}\times V}$ *is* C^∞. *Let X be the C^∞ vector field on V which generates φ.*

(d) *All singularities of X in $V \setminus \overline{U}_1$ are hyperbolic.*

(e) *If $p \in V \setminus \overline{U_1}$ is a singularity of X, then its stable manifold, $W^S(p)$, has (real) dimension at most n.*

(f) *If $p \in V \setminus U$ is a singularity of X, then $W^S(p)$ is contained in $V \setminus U$.*

(g) *If $F \subset V$ is a compact subset of $\overline{V}$ such that $F \cap W^S(q) = \varnothing$ for any singularity q of X in $V \setminus U$, then there exists $s_0 > 0$ such that $\varphi_t(F) \subset U$ for $t \geqslant s_0$.*

3.1. Proof of Lemma 3.

Let N be a connected component of V. Since N is Stein, there exists on N a C^∞ strictly-pluri-subharmonic exhaustion, say f. We will use the following facts:

(1) The set of exhaustions of N, is open in the Whitney C^0 topology of $C^0(N,\mathbb{R})$.

(2) The set of C^2 Morse functions is open and dense in the Whitney C^2 topology of $C^2(N,\mathbb{R})$.

(3) The set of C^∞ functions is dense in the Whitney C^2 topology of $C^2(N,\mathbb{R})$.

(4) The set of strictly-pluri-subharmonic functions of class C^2 on N, say $\mathrm{SPSH}^2(N,\mathbb{R})$, is open in the Whitney C^2 topology of $C^2(N,\mathbb{R})$.

The definition of the Whitney topology and the proofs of (1), (2) and (3) can be found in [H]. Let us prove (4).

Let $h \in C^2(N,\mathbb{R})$ and let $\mathcal{L}_h$ be the Levi form of h, which is defined in a holomorphic coordinate system $(x = (x_1,\dots,x_n), U)$ of N by

$$\mathcal{L}_h(p).v \;=\; \sum_{i,j} \partial^2 h/\partial x_i \partial \overline{x_j}(x)\ .v_i.\overline{v_j}\ ,$$

$$\text{where } (p,v) \in TU \text{ and } (x,(v_1,\dots,v_n)) = Tx(p,v).$$

By definition, h is strictly-pluri-subharmonic if, and only if, $\mathcal{L}_h(p).v > 0$ for all $(p,v) \in TN$, with $v \neq 0$. Let us fix a riemannian metric g on N with norm $|\ .\ |$. Given $h \in \mathrm{SPSH}^2(N,\mathbb{R})$ define $k_h : N \longrightarrow \mathbb{R}$ by

$$k_h(p) = \inf\{\mathcal{L}_h(p).v\ ;\ v \in T_pN \text{ and } \mid v\mid_p = 1\}$$

so that $k_h > 0$ on N. Now, for a fixed $h_0 \in \mathrm{SPSH}^2(N,\mathbb{R})$ let

$$\mathcal{U} \;=\; \{h \in C^2(N,\mathbb{R});\ \mid \mathcal{L}_h(p).v - \mathcal{L}_{h_0}(p).v \mid < 1/2\ .k_{h_0}(p) \quad \forall\ (p,v) \in N \text{ with } \mid v\mid_p = 1\}.$$

Since $\mathcal{L}_h$ and k_h depend only on the second jet of h, it follows from the definition of the Whitney topology that $\mathcal{U}$ is a neighborhood of h_0 in $C^2(N,\mathbb{R})$. Moreover if $h \in \mathcal{U}$, then $k_h > 0$, so that $h \in \mathrm{SPSH}^2(N,\mathbb{R})$, which proves (4).

It follows from (1), (2),(3) and (4) that we can suppose that f is a Morse function. Let g be the riemannian metric on N fixed before. Let

$Y = \mathrm{grad}_{\mathrm{g}}(f)$, which is defined by $df_p.v = \mathrm{g}_p(v, Y(p))$, for any $(p, v) \in TN$. Let Y_t be the flow of Y.

The following facts are well known (cf. [M] and [Sm]):

(5) f is strictly increasing along non singular orbits of Y.

(6) The singularities of Y are the points p of N for which $df_p = 0$.

(7) Given p such that $df_p = 0$, there exists a C^∞ coordinate system $x = (x_1, \ldots, x_{2n})$ around p such that

$$(*) \qquad f(x) \;=\; f(\mathrm{p}) + \sum_{j=1}^{2n} b_j.(x_j)^2 \;+\; \text{h.o.t.}\,, \quad \text{where } \; b_j \in \mathbb{R} \setminus \{0\}.$$

The number $i(p)$ of negative $b_{j's}$ in $(*)$, is an invariant of f and p and is called the Morse index of f at p.

(8) If p and $i(p)$ are as in (7), then p is a hyperbolic singularity of Y and its stable manifold, $W^s(p)$, has (real) dimension $i(p)$.

We need the following:

ASSERTION. — For any singularity p of Y, we have $i(p) \leqslant n$.

Proof. — It follows from Theorem 1.4.15, pg. 29 of [HL], that there exists a holomorphic coordinate system $(z = (z_1, \ldots, z_n), W)$ around p, such that $z(p) = 0$ and the expression of f in W is of the form

$$(*) \qquad f(z) \;= f(p) + \sum_{j=1}^{n} [(1 + a_j).x_j^2 + (1 - a_j).y_j^2] \;+ \text{h.o.t.}\,,$$

where $z_j = x_j + i\; y_j$ and $1 \neq a_j \geqslant 0$ (because f is a Morse function). This implies the assertion.

Let us consider now the open set $A' = A \cap N$. Since $\overline{V}$ is compact, it follows that $N \setminus A'$ is compact, which implies that there exists $t_0 \in \mathbb{R}$ such that $f^{-1}((-\infty, t]) \supset N \setminus A'$, for $t \geqslant t_0$. Fix $t_2 > t_1 > t_0$ and let $N_j = f^{-1}(t_j, +\infty)$, $j = 1, 2$, so that $A' \supset \overline{N_1} \supset N_1 \supset \overline{N_2}$. We choose t_1 in such a way that $f^{-1}(t_1)$ does not contain singularities of Y, which implies that Y is transverse to $f^{-1}(t_1)$ and points inward N_1. Observe also that Y has finitely many singularities on $N \setminus \overline{N_2}$.

Now, let $p \in N \setminus \overline{N_j}$, $j = 1, 2$ be a singularity of Y. It follows from (5), from the definition of N_j and from the fact that

$$W^s(p) \;=\; \{q \;;\; \lim_{t \to +\infty} Y_t(q) = p\}$$

that

(9) $W^s(p) \subset N \setminus \overline{N_j}$, for $j = 1, 2$.

Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^∞ function such that $\phi(t) = 0$ for $t \geqslant t_2$, $\phi(t) = 1$ for $t \leqslant t_1$ and $\phi(t) > 0$ for $t \in (t_1, t_2)$.

Let X^N be the C^∞ vector field on N defined by $X^N(p) = \phi(f(p)).Y(p)$, and denote by X_t^N its flow. It is not difficult to verify the following facts:

(10) $X_t^N(p) = p$, $\forall \ p \in \overline{N_2}$.

(11) If $p \in N \setminus \overline{N_2}$ and $o_X(p)$, $o_Y(p)$ are the orbits of X^N and Y through p, respectively, then $o_X(p) = o_Y(p) \cap (N \setminus \overline{N_2})$.

(12) f is strictly increasing along non singular orbits of X^N.

(13) The singularities of X^N on $N \setminus \overline{N_2}$ are hyperbolic and are singularities of Y.

(14) If $p \in N \setminus \overline{N_j}$, $j = 1, 2$, is a singularity of X^N , then its stable manifold coincides with $W^s(p)$, the stable manifold of p with respect to Y. In particular (13) is true for X^N.

(15) If $p \in N \setminus \overline{N_1}$ does not belongs to the stable manifold of some singularity of X^N in $N \setminus \overline{N_1}$, then there exists $s_0 > 0$ and a neighborhood W_p of p such that if $t > s_0$, then $X_t^N(W_p) \subset N_1$.

This last assertion follows from the continuity of the flow and from (12).

Let us finish the proof of Lemma 3. Consider the decomposition of V in connected components, $V = \underset{j \in J}{\cup} N^j$. For each $j \in J$, let f^j be a Morse strictly-pluri-subharmonic exhaustion of N^j. Let $A^j = A \cap N^j$ and $N_2^j \subset N_1^j$ be like N_1 and N_2 considered before. Let $X^{N^j} = X^j$ be a vector field on N^j which satisfies properties (10), …, (15). Define the flow $\varphi : \mathbb{R} \times \overline{V} \longrightarrow \overline{V}$ by $\varphi(t, p) = p$ if $p \in K$ and $\varphi(t, p) = X_t^j(p)$ if $p \in N^j$. Observe that φ satisfies (c) of Lemma 3.

Since f^j is an exhaustion of N^j for any $j \in J$, it is not difficult to see that $U = \cup_j N_1^j \cup K$ and $U_1 = \underset{j}{\cup} N_2^j \cup K$ are neighborhoods of K and satisfy (a) of Lemma 3. Observe that this fact and (10) imply that φ is continuous and satisfies (b), (d) and (f) of Lemma 3. On the other hand property (g) of Lemma 3 can be easily checked from (15). We leave the details for the reader. This ends the proof of Lemma 3.

3.2. Proof of Theorem 3.

Let $V, \overline{V}, K$ and A be as in Theorem 3. Let U, U_1 and φ be as in Lemma 3. Fix $p \in K$ and consider the homomorphisms induced by the inclusion $i : U \longrightarrow \overline{V}$:

$$(*)\ h_q : H_q(U, \mathbb{Z}) \longrightarrow H_q(\overline{V}, \mathbb{Z}) \text{ and } i_q : \Pi_q(U, p) \longrightarrow \Pi_q(\overline{V}, p).$$

Let us consider first the homotopy case. Consider i_q as above and let us prove that it is onto for $1 \leqslant q \leqslant n-1$. We will consider a class $[g]$ in $\Pi_q(\overline{V}, p)$ represented by a continuous map $g : S^q \longrightarrow \overline{V}$, where S^q is the unit sphere in $\mathbb{R}^{q+1}$ and $g(e) = p$ for some fixed $e \in S^q$. Let $V_1 = V \setminus \overline{U_1}$ and consider the open set $A = g^{-1}(V_1) \subset S^q$. It follows from standard arguments of differential topology that it is possible to find a continuous map $h : S^q \longrightarrow \overline{V}$ such that

(1) h coincides with g in $S^q \setminus A$.

(2) h is homotopic to g.

(3) h is C^∞ in A.

Let $p_1, \ldots, p_m$ be the singularities of X in $V \setminus \overline{U}$, and $W = \bigcup_{j=1}^{m} W^s(p_j)$. Since for all $j = 1, \ldots, m$ we have $q + \dim_{\mathbb{R}}(W^s(p_j)) \leqslant 2n - 1 < 2n = \dim_{\mathbb{R}}(V)$, it follows from transversality theory that it is possible to find h in such a way that

(4) $h(A) \cap W = \varnothing$, so that $h(S^q) \cap W = \varnothing$ (by (f) of Lemma 3 and (1)).

It follows from (g) of Lemma 3 that there exists $t > 0$ such that $\varphi_t(h(S^q)) \subset U$. Since φ_t is homotopic to the identity of $\overline{V}$, this implies that i_q is onto.

Let us prove that i_q is injective if $1 \leqslant q \leqslant n-2$. Let $[g] \in \Pi_q(U, p)$ be such that $i_q[g] = 0$ and $g : S^q \longrightarrow U$ be a representative of $[g]$. Let $\overline{B^{q+1}}$ be the closed unit ball in $\mathbb{R}^{q+1}$ and $G : \overline{B^{q+1}} \longrightarrow \overline{V}$ be a continuous map such that $G\ |_{S^{q+1}} = g$. Let $A = G^{-1}(V_1)$. It follows from standard arguments of differential topology that it is possible to find a continuous map $H : \overline{B^{q+1}} \longrightarrow \overline{V}$ such that

(5) H coincides with G in $\overline{B^{q+1}} \setminus A$.

(6) H is homotopic to G.

(7) H is C^∞ in A.

Moreover, since for all $j = 1, \dots, m$ we have $q+1+\dim_{\mathbb{R}}(W^{S}(p_j)) \leqslant 2n - 1 < 2n = \dim_{\mathbb{R}}(V)$, it follows from transversality theory that it is possible to find H in such a way that

(8) $H(S^q) \cap W = \varnothing$.

It follows from (g) of Lemma 3 that there exists $t > 0$ such that $\varphi_t(H(\overline{B^{q+1}})) \subset U$. This implies that $[g] = 0$ in $\Pi_q(U,p)$, so that i_q is injective.

The proof in the homology case is similar. Let us sketch it.

We will work in the singular homology theory, with the notations of [G]. If $c = \sum_{j=1}^{k} \nu_j \sigma_j$ is a q-chain in $\overline{V}$ then each simplex σ_j is a continuous map from the standard simplex

$$\Delta_q = \left\{p \in \mathbb{R}^q;\ p = \sum_{i=0}^{q} t_i.E_i\ ,\ 0 \leqslant t_i \leqslant 1, \sum_i t_i \leqslant 1\right\}$$

into $\overline{V}$. We will use the notation $\operatorname{supp}(c) = \bigcup_j \sigma_j(\Delta_q)$. A sub-simplex σ_j^I, $I = (i_0 < i_1 < \dots < i_r)$, $r \leqslant q$, is, by definition, the restriction of σ_j to Δ_q^I, where

$$\Delta_q^I = \left\{p \in \mathbb{R}^q;\ p = \sum_{j=0}^{r} t_j.E_{i_j}\ ,\ 0 \leqslant t_j \leqslant 1, \sum_j t_j \leqslant 1\right\}.$$

We will say that c is C^∞ if for all j and all I the restriction of σ_j^I to the open subsets $(\sigma_j^I)^{-1}(V)$ of Δ_q^I is C^∞.

Let us prove that h_q is onto if $0 \leqslant q \leqslant n-1$. Let $[c] \in H_q(\overline{V}, \mathbb{Z})$ and $c = \sum_{j=1}^{k} \nu_j \sigma_j$ be a representative of $[c]$. It follows from standard arguments of homology theory that we can suppose that all σ_j^I are C^∞ and transversal to W. Since $0 \leqslant q \leqslant n-1$ this implies that $W \cap \operatorname{supp}(c) = \varnothing$. On the other hand, (g) of Lemma 3 implies that there exists $t > 0$ such that $\varphi_t(\operatorname{supp}(c)) \subset U$. Since φ_t is homotopic to the identity, it follows that h_q is onto.

Let us prove that h_q is injective if $0 \leqslant q \leqslant n-2$. Let $[c] \in H_q(U, \mathbb{Z})$ be such that $h_q[c] = 0$. Let c be a C^∞ representative of [c]. It follows from standard arguments of homology theory that there exists a C^∞ (q+1)-chain $c' = \sum_j \nu_j.\sigma_j$ on $\overline{V}$, such that $\partial c' = c$ and all σ_j^I are transversal to W. Since q+1$\leqslant n-1$ this implies that $W\cap \text{supp}(c')=\varnothing$. On the other hand, (g) of Lemma 3 implies that there exists $t>0$ such that $\varphi_t(\text{supp}(c'))\subset U$. Since φ_t is homotopic to the identity, it follows that h_q is injective.

It remains to prove (i) and (ii) of Theorem 3 (since (iii) follows from (ii)).

Proof of (i). — Since $n\geqslant 2$, it follows from the theorem that there exists a a collection $\{U_n\}_{n=1}^{\infty}$ of open neighborhoods of K such that $\cap_n U_n = K$ and for all n

$$h_0 : H_0(U_n,\mathbb{Z}) \longrightarrow H_0(\overline{V},\mathbb{Z})$$

is an isomorphism. On the other hand, this implies that $\overline{U_n}$ is compact and connected for all n. Therefore K is connected.

Proof of (ii). — Let us suppose now that K is a C^1 submanifold of M. Let B be a tubular neighborhood of K with projection $\pi : B\longrightarrow K$. We have two possibilities:

1st - $B\setminus K$ has one connected component.

2nd - $B\setminus K$ has two connected components, say B_1 and B_2 (if K has real codimension 1).

In the first case $\overline{V}\cap B=B$ and B is a neighborhood of K in $\overline{V}$. In the second case we have two possibilities: either $\overline{V}\cap B=B$, or $\overline{V}\cap B=B_j\cup K$ for $j=1$ or 2 and $B_j\cup K$ is a neighborhood of K in $\overline{V}$. In any case we will set $A=\overline{V}\cap B$, so that A is a neighborhood of K in $\overline{V}$. It is well known that the homomorphisms induced by the inclusion $K\longrightarrow A$

$$h_q^{'} : H_q(K,\mathbb{Z}) \longrightarrow H_q(A,\mathbb{Z}) \text{ and } i_q^{'} : \Pi_q(K,p)\longrightarrow \Pi_q(A,p)$$

are isomorphisms for all q.

On the other hand there exists a neighborhood U of K in $\overline{V}$ such that $U\subset A$ and the homomorphisms induced by the inclusion $U\longrightarrow\overline{V}$

$$h_q^{''} : H_q(U,\mathbb{Z}) \longrightarrow H_q(\overline{V},\mathbb{Z}) \text{ and } i_q^{''} : \Pi_q(U,p)\longrightarrow \Pi_q(\overline{V},p)$$

are onto if $q\leqslant n-1$ and injectives if $q\leqslant n-2$. It is not difficult to see that this implies (ii). This finishes the proof of Theorem 3.

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Manuscrit reçu le 15 décembre 1998,
accepté le 19 janvier 1999.

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