ANNALES DE L'INSTITUT FOURIER

BERNHARD KRÖTZ

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Annales de l'institut Fourier, tome 49, nº 4 (1999), p. 1241-1264

<http://www.numdam.org/item?id=AIF_1999__49_4_1241_0>

Ann. Inst. Fourier, Grenoble
49, 4 (1999), 1241–1264

NORM ESTIMATES FOR UNITARIZABLE HIGHEST WEIGHT MODULES

by Bernhard KRÖTZ

Introduction.

A simple non-compact Lie algebra $\mathfrak{g}$ over the real numbers admits a non-trivial unitarizable highest weight module if and only if it is *hermitian*, i.e., $\mathfrak{z}(\mathfrak{k}) \neq \{0\}$ for $\mathfrak{k} \subseteq \mathfrak{g}$ a maximal compactly embedded subalgebra.

Assume now that $\mathfrak{g}$ is hermitian. Then every Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$ and we have $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{z}(\mathfrak{k})$, where $\mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{k}'$. Let Δ be the root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$. Let $X_0 \in \mathfrak{z}(\mathfrak{k})$ be such that $\operatorname{Spec}(\operatorname{ad}(X_0)) = \{-i, 0, i\}$ and let Δ^+ be a positive system such that the positive non-compact roots are given by $\Delta_n^+ := \{\alpha \in \Delta : \alpha(iX_0) = 1\}$. Then for every $\lambda \in i\mathfrak{t}^*$ there exists a unique irreducible highest weight module $L(\lambda)$ with respect to Δ^+ and highest weight λ. If Δ_k^+ denotes the positive compact roots and λ is dominant integral with respect to Δ_k^+, then the module $L(\lambda)$ is the unique irreducible quotient of the *generalized Verma module*

$$N(\lambda) = \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \underset{\mathcal{U}(\mathfrak{q})}{\otimes} F(\lambda),$$

where $\mathfrak{q}$ is the parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ corresponding to $\Delta_k \cup \Delta_n^+$ and $F(\lambda)$ is the irreducible $\mathfrak{k}_{\mathbb{C}}$-module with highest weight λ. By the work of Jakobsen (cf. [Jak83]) and Enright, Howe and Wallach (cf. [EHW83]) we

Keywords: Highest weight module – Unitary representation – Semisimple Lie group – Semisimple Lie algebra.
Math. classification: 22E45 – 22E46.

nowadays know exactly which $L(\lambda)$ are unitarizable (see also [EJ90] for a different approach).

Let $\zeta \in i\mathfrak{z}(\mathfrak{k})^*$ be such that $\zeta(iX_0) = 1$. For $\lambda^0 \in i\mathfrak{t}_0^*$ dominant integral with respect to Δ_k^+ and $z \in \mathbb{R}$ we set $\lambda_z := \lambda^0 + z\zeta$, and

$$l(\lambda^0) := \{z \in \mathbb{R}: L(\lambda_z) \text{ is unitarizable}\}.$$

For each $z \in l(\lambda^0)$ we denote by $\langle\cdot,\cdot\rangle_z$ the *Shapovalov form* on $N(\lambda_z)$, which is a certain contravariant hermitian form on $N(\lambda_z)$.

If we identify $N(\lambda_z)$ with $N(\lambda^0)$ as $\mathfrak{k}'_{\mathbb{C}}$-modules, then our first result (cf. Theorem 2.7) says that for $z, z' \in l(\lambda^0)$ with $z \le z'$ we have $\langle v, v\rangle_{z'} \le \langle v, v\rangle_z$ for all $v \in N(\lambda^0)$. We also obtain estimates in the converse direction on the various irreducible $\mathfrak{k}_{\mathbb{C}}$-types (cf. Theorem 2.8).

In Section 3 we apply the obtained inequalities to representation theory. Let G denote a simply connected Lie group corresponding to $\mathfrak{g}$. If $L(\lambda)$ is unitarizable we denote by $\mathcal{H}_\lambda$ the globalization of $L(\lambda)$ in the $F(\lambda)$-valued holomorphic functions on G/K. Then, if we identify $F(\lambda_z)$ with $F(\lambda^0)$ and normalize the inner products on $(\mathcal{H}_{\lambda_z})_{z\in l(\lambda^0)}$ so that they coincide on $F(\lambda^0)$, we obtain contractive inclusions $\mathcal{H}_{\lambda_{z'}} \to \mathcal{H}_{\lambda_z}$ for $z \le z'$. Further we use the inequalities of Section 2 to characterize the hyperfunction vectors of $\mathcal{H}_\lambda$ (cf. Theorem 3.9).

Finally we apply our results to spherical highest weight representations. Let τ be an involutive automorphism of G and H the corresponding fixed point group. We assume that the symmetric space G/H is *compactly causal* (cf. [HiÓl96]). Let $\chi: H \to \mathbb{C}$ denote a continuous character of H. Then for $L(\lambda) = N(\lambda)$ it turns out that $\mathcal{H}_\lambda$ is (H, χ)-spherical if and only if $F(\lambda)$ is $(H \cap K, \chi|_{H\cap K})$-spherical (cf. Theorem 3.14). We also give an example that this becomes false if $L(\lambda) \ne N(\lambda)$ (cf. Remark 3.15).

1. Generalities on highest weight modules.

Hermitian Lie algebras.

Let $\mathfrak{g}$ be a simple real Lie algebra and $\mathfrak{k} \subseteq \mathfrak{g}$ a maximal compactly embedded subalgebra. We call $\mathfrak{g}$ *hermitian* if $\mathfrak{z}(\mathfrak{k}) \ne \{0\}$. We collect some basic facts concerning hermitian Lie algebras (cf. [Hel78], Ch. VIII).

The center of $\mathfrak{k}$ is one-dimensional, i.e., $\mathfrak{z}(\mathfrak{k}) = \mathbb{R}X_0$ for some $0 \ne X_0 \in \mathfrak{k}$. We can normalize X_0 such that $\mathrm{Spec}(X_0) = \{-i, 0, i\}$. Further, every Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$. Note that $\mathfrak{z}(\mathfrak{k}) \subseteq \mathfrak{t}$. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$ and Δ the root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$.

A root $\alpha \in \Delta$ is called *compact* if $\alpha(X_0) = 0$ and *non-compact* otherwise. We denote by Δ_k, resp. Δ_n, the collection of compact, resp. non-compact roots. We fix a positive system $\Delta^+ \subseteq \Delta$ such that

$$\Delta_n^+ := \Delta_n \cap \Delta^+ = \{\alpha \in \Delta_n : \alpha(iX_0) = 1\}.$$

We set $\mathfrak{p}^{\pm} := \{X \in \mathfrak{g}_{\mathbb{C}} : [iX_0, X] = \pm X\}$ and note that

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-.$$

As $\operatorname{Spec}(X_0) = \{-i, 0, i\}$ and $X_0 \in \mathfrak{z}(\mathfrak{k})$ it follows that $[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}^{\pm}] \subseteq \mathfrak{p}^{\pm}$, $[\mathfrak{p}^+, \mathfrak{p}^-] \subseteq \mathfrak{k}_{\mathbb{C}}$, $[\mathfrak{p}^+, \mathfrak{p}^+] = \{0\}$ and $[\mathfrak{p}^-, \mathfrak{p}^-] = \{0\}$.

Highest weight modules algebraically.

In this subsection we collect some basic facts concerning highest weight modules from an abstract algebraic point of view. As reference for the forthcoming facts may serve [EHW83] or [Ne99], Ch. IX.

Let $\mathfrak{n}^+$ denote the sum of all positive root spaces and $\mathfrak{n}^-$ defined accordingly. Then $\mathfrak{b} = \mathfrak{n}^+ \rtimes \mathfrak{t}_{\mathbb{C}}$ is a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$. For $\lambda \in i\mathfrak{t}^*$ let $\mathbb{C}_\lambda$ be the one-dimensional $\mathfrak{b}$-module, where $X \in \mathfrak{t}_{\mathbb{C}}$ acts by $\lambda(X)$ and the elements of the nilradical $\mathfrak{n}^+$ of $\mathfrak{b}$ act trivially. Associated to λ we define the *Verma module*

$$M(\lambda) := \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda$$

and note that $M(\lambda)$ is a highest weight module for $\mathfrak{g}_{\mathbb{C}}$ with highest weight λ with respect to Δ^+.

Let $X \mapsto \overline{X}$ denote the conjugation in $\mathfrak{g}_{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$. Then the map $X \mapsto X^* := -\overline{X}$ extends to an involutive antilinear antiautomorphism of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ which we denote by the same symbol.

A hermitian form $\langle \cdot, \cdot \rangle$ on a $\mathfrak{g}_{\mathbb{C}}$-module V is called *contravariant* if

$$(\forall X \in \mathfrak{g}_{\mathbb{C}})(\forall v, w \in V) \quad \langle X.v, w \rangle = \langle v, X^*.w \rangle.$$

Contravariant forms on $M(\lambda)$ are unique up to real scalar multiples whenever they exist. Their construction is described as follows. According to the Poincaré-Birkhoff-Witt-Theorem, the universal envoloping algebra of $\mathfrak{g}_{\mathbb{C}}$ decomposes as

$$\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\mathfrak{t}_{\mathbb{C}}) \oplus \left(\mathfrak{n}^{-}\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) + \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{n}^{+}\right).$$

We denote by $P: \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{U}(\mathfrak{t}_{\mathbb{C}})$ the projection onto the first component. We extend $\lambda: \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$ to an algebra homomorphism $\lambda: \mathcal{S}(\mathfrak{t}_{\mathbb{C}}) \to \mathbb{C}$. The *Shapovalov form* on $M(\lambda)$ is defined by

$$\langle X \otimes 1, Y \otimes 1 \rangle_{\lambda} := \lambda(P(Y^{*}X))$$

for all $X, Y \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. It is easy to see that $\langle \cdot, \cdot \rangle_{\lambda}$ is contravariant. Its radical is the unique maximal submodule of $M(\lambda)$ and the corresponding irreducible quotient is denoted by $L(\lambda)$. In particular, the Shapovalov form factors to a contravariant form on $L(\lambda)$ which we also denote by $\langle \cdot, \cdot \rangle_{\lambda}$. We call $L(\lambda)$ *unitarizable* if $\langle \cdot, \cdot \rangle_{\lambda}$ is positive definite.

For every root $\alpha \in \Delta$ we denote by $\check{\alpha} \in i\mathfrak{t}$ the corresponding *coroot*, i.e., $\check{\alpha} \in [\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{-\alpha}]$ such that $\alpha(\check{\alpha}) = 2$.

LEMMA 1.1. — *Let $\lambda \in i\mathfrak{t}^{*}$. If $L(\lambda)$ is unitarizable, then*

(i) *The highest weight λ is dominant integral with respect to Δ_k^{+}, i.e., $\lambda(\check{\alpha}) \in \mathbb{N}_0$ holds for all $\alpha \in \Delta_k^{+}$.*

(ii) *We have $\lambda(\check{\alpha}) \leq 0$ for all $\alpha \in \Delta_n^{+}$.* □

Remark 1.2. — (a) If a simple non-compact Lie algebra admits a non-trivial unitarizable highest weight module, then it has to be hermitian. Moreover, up to sign, the positive system in question has to be as above.

(b) The conditions under (ii) in Lemma 1.1 do not characterize unitarizable highest weight modules. One has to impose further conditions on λ to guarantee unitarizability which will be explained in the next subsection. □

Abstract classification of unitarizable highest weight modules.

Let $\lambda \in i\mathfrak{t}^{*}$ be dominant integral with respect to Δ_k^{+} and write $F(\lambda)$ for the corresponding irreducible $\mathfrak{k}_{\mathbb{C}}$-module. Let $\mathfrak{q} = \mathfrak{p}^{+} \rtimes \mathfrak{k}_{\mathbb{C}}$ and turn $F(\lambda)$ into a $\mathfrak{q}$-module by letting $\mathfrak{p}^{+}$ act trivially. We define the *generalized Verma module* associated to λ by

$$N(\lambda) = \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \underset{\mathcal{U}(\mathfrak{q})}{\otimes} F(\lambda).$$

Note that the generalized Verma module is a quotient of the Verma module $M(\lambda)$ and hence a highest weight module with respect to Δ^+ and highest weight λ. Thus $L(\lambda)$ is a quotient of $N(\lambda)$ and the Shapovalov form on $M(\lambda)$ defines a contravariant form on $N(\lambda)$ which is also denoted by $\langle \cdot, \cdot \rangle_\lambda$.

We set $\mathfrak{t}_0 := \operatorname{span}\{i\check{\alpha} : \alpha \in \Delta_k\}$ and note that $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{z}(\mathfrak{k})$. Let $\zeta \in i\mathfrak{z}(\mathfrak{k})^*$ be defined by $\zeta(iX_0) = 1$ and note that $\zeta(\check{\alpha}) > 0$ for all $\alpha \in \Delta_n^+$. For $\lambda^0 \in i\mathfrak{t}_0^*$ be dominant integral with respect to Δ_k^+ and $z \in \mathbb{C}$ we set

$$\lambda_z = \lambda^0 + z\zeta.$$

Further we define

$$l(\lambda^0) := \{z \in \mathbb{R} : L(\lambda_z) \text{ is unitarizable}\}.$$

THEOREM 1.3 (Enright-Howe-Wallach, Jakobsen). — *Let $\lambda^0 \in i\mathfrak{t}_0^*$ be dominant integral with respect to Δ_k^+. Then the following assertions hold:*

(i) *There exists real numbers $A(\lambda^0) \leq 0$ and $C(\lambda^0) > 0$ such that*

$$l(\lambda^0) = \{z \in \mathbb{R} : z < A(\lambda^0)\} \dot\cup \{z_0, \dots, z_n\},$$

where $z_0 = A(\lambda^0)$ and $z_{i+1} - z_i = C(\lambda^0)$ for all $0 \leq i \leq n-1$.

(ii) *For $z \in l(\lambda^0)$ we have $L(\lambda_z) \cong N(\lambda_z)$ if and only if $z < A(\lambda^0)$.*

(iii) *For $z \in l(\lambda^0)$ the parameter λ_z belongs to the relative holomorphic discrete series if and only if $z < \langle \lambda^0 + \rho, \beta \rangle < 0$ with β the unique simple non-compact root. Further we have $\langle \lambda^0 + \rho, \beta \rangle < A(\lambda^0)$.*

Proof. — [EHW83], Th. 2.4. □

Two lemmas of Parthasarathy.

The main tool in the classification of unitarizable highest weight modules are two lemmas of Parthasarathy which we cite now and use later on.

If V is a $\mathfrak{t}_\mathbb{C}$-module, then we denote by $\mathcal{P}(V)$ the set of $\mathfrak{t}_\mathbb{C}$-weights of V. For each $\mu \in \mathcal{P}(V)$ we write V^μ for the corresponding weight space. For every $\alpha \in \Delta_n^+$ let $X_\alpha \in \mathfrak{g}_\mathbb{C}^\alpha$ be such that $\kappa(X_\alpha, X_\alpha^*) = -1$, where κ denotes the Cartan-Killing form on $\mathfrak{g}_\mathbb{C}$.

LEMMA 1.4 (Parthasarathy). — *Let $\mu \in \mathcal{P}(N(\lambda))$ and $v \in N(\lambda)^\mu$ be a primitive element with respect to Δ_k^+. Then we have*

$$(\|\mu+\rho\|^2 - \|\lambda+\rho\|^2)\langle v, v\rangle_\lambda = 2\sum_{\alpha\in\Delta_n^+} \langle X_\alpha.v, X_\alpha.v\rangle_\lambda.$$

Proof. — [EHW83], Lemma 3.6 or [Ne99], Lemma IX.5.3. □

LEMMA 1.5 (Parthasarathy). — *If $L(\lambda)$ is unitarizable, then every $\mu \in \mathcal{P}(L(\lambda))\backslash\{\lambda\}$ which is primitive for Δ_k^+ satisfies the inequality*

$$\|\mu+\rho\| > \|\lambda+\rho\|.$$

Proof. — [EHW83], Prop. 3.9 or [Ne99], Th. IX.5.4. □

2. The inequality for unitary highest weight modules.

Let $z \in l(\lambda^0)$. The action map $\mathcal{U}(\mathfrak{p}^-)\otimes F(\lambda_z) \to N(\lambda_z)$ gives rise to an isomorphism of $\mathfrak{k}_\mathbb{C} \ltimes \mathfrak{p}^-$-modules

$$N(\lambda_z) \cong \mathcal{S}(\mathfrak{p}^-)\otimes F(\lambda_z),$$

where the action of $\mathfrak{k}_\mathbb{C} \ltimes \mathfrak{p}^-$ on $\mathcal{S}(\mathfrak{p}^-)\otimes F(\lambda)$ is given by

$$\begin{aligned} &(\forall X \in \mathfrak{k}_\mathbb{C}) \qquad X.(p\otimes v) := [X,p]\otimes v + p\otimes X.v \\ &(\forall Y \in \mathfrak{p}^-) \qquad Y.(p\otimes v) := Yp\otimes v \end{aligned}$$

for all $p \in \mathcal{S}(\mathfrak{p}^-)$ and $v \in F(\lambda)$.

LEMMA 2.1. — *For all $z \in \mathbb{C}$ the module $N(\lambda_z)$ is isomorphic to $N(\lambda^0)$ as $\mathfrak{k}'_\mathbb{C} \ltimes \mathfrak{p}^-$-module.*

Proof. — As $F(\lambda_z) = F(\lambda^0)\otimes \mathbb{C}_{z\zeta}$, the module $F(\lambda_z)$ is isomorphic to $F(\lambda^0)$ as a $\mathfrak{k}'_\mathbb{C}$-module. From that the assertion follows. □

Realization in holomorphic functions.

Let G denote a connected Lie group with Lie algebra $\mathfrak{g}$ sitting in its universal complexification $G_\mathbb{C}$. We write K for the analytic subgroup of G

corresponding to $\mathfrak{k}$ and set $P^\pm := \exp(\mathfrak{p}^\pm) \subseteq G_{\mathbb{C}}$. In the following we refer to [Sa80] or [Ne99], Ch. XII for a detailed discussion of the facts mentioned below.

The mapping

$$P^+ \times K_{\mathbb{C}} \times P^- \to G_{\mathbb{C}}, \quad (p_+, k, p_-) \mapsto p_+ k p_-$$

is biholomorphic onto its open image $P^+ K_{\mathbb{C}} P^-$. We write

$$\zeta: P^+ \times K_{\mathbb{C}} \times P^- \to P^+ \quad \text{and} \quad \kappa: P^+ \times K_{\mathbb{C}} \times P^- \to K_{\mathbb{C}}$$

for the associated holomorphic projections.

We have $G \subseteq P^+ K_{\mathbb{C}} P^-$. Further $\mathcal{D} := \log \zeta(G) \subseteq \mathfrak{p}^+$ is a bounded symmetric domain and the map

$$G/K \to \mathcal{D}, \quad gK \mapsto \log \zeta(g)$$

is an analytic isomorphism, called the *Harish Chandra realization* of G/K. For $g \in G$ and $z \in \mathcal{D}$ we set $g.z = \log \zeta(gz) \in \mathcal{D}$.

We define the cocycle

$$J: G \times \mathcal{D} \to K_{\mathbb{C}}, \quad (g, z) \mapsto \kappa(g \exp(z))$$

and set

$$K_{\mathcal{D}}: \mathcal{D} \times \mathcal{D} \to K_{\mathbb{C}}, \quad (z, w) \mapsto \kappa(\exp(-\overline{w}) \exp(z))^{-1}.$$

Let $\widetilde{G}$ and $\widetilde{K_{\mathbb{C}}}$ denote the universal coverings of G, resp. $K_{\mathbb{C}}$. Then the maps J, $K_{\mathcal{D}}$ lift uniquely to mappings

$$\tilde{J}: \widetilde{G} \times \mathcal{D} \to \widetilde{K_{\mathbb{C}}} \quad \text{and} \quad \widetilde{K}_{\mathcal{D}}: \mathcal{D} \times \mathcal{D} \to \widetilde{K_{\mathbb{C}}}$$

with $\tilde{J}(\mathbf{1}, 0) = \mathbf{1}$ and $\widetilde{K}_{\mathcal{D}}(0,0) = \mathbf{1}$ (cf. [Ne99], Lemma XII.1.7).

For $\lambda \in i\mathfrak{t}^*$ dominant integral with respect to Δ_k^+ we write $(\sigma_\lambda, F(\lambda))$ for the corresponding representation of $\widetilde{K_{\mathbb{C}}}$. We set $J_\lambda := \sigma_\lambda \circ \tilde{J}$ and $K_\lambda := \sigma_\lambda \circ \widetilde{K}_{\mathcal{D}}$.

Let $\operatorname{Hol}(\mathcal{D}, F(\lambda))$ denote the space of $F(\lambda)$-valued functions on $\mathcal{D}$. We equip this space with the topology of compact convergence turning it into a Fréchet space. In view of [Ne99], Prop. XII.1.8, the prescription $\widetilde{G} \times \operatorname{Hol}(\mathcal{D}, F(\lambda)) \to \operatorname{Hol}(\mathcal{D}, F(\lambda))$, $(g, f) \mapsto \pi_\lambda(g).f$, where

$$(\pi_\lambda(g).f)(z) := J_\lambda(g^{-1}, z)^{-1}.f(g^{-1}.z), \tag{2.1}$$

defines a smooth representation of $\widetilde{G}$. The corresponding derived representation of the subalgebra $\mathfrak{k}_{\mathbb{C}} \ltimes \mathfrak{p}^+$ is given by

$$\begin{aligned} &(\forall X \in \mathfrak{k}_{\mathbb{C}}) && (X.f)(z) = d\sigma_\lambda(X).f(z) + df(z)([z, X]) \\ &(\forall W \in \mathfrak{p}^+) && (W.f)(z) = df(z)(-W) \end{aligned}$$

for all $f \in \mathrm{Hol}(\mathcal{D}, F(\lambda))$.

We write $\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda)$ for the holomorphic $F(\lambda)$-valued polynomials on $\mathfrak{p}^+$. By taking restrictions, we also consider $\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda)$ as a subspace of $\mathrm{Hol}(\mathcal{D}, F(\lambda))$. Note that $L(\lambda)$ is a locally finite $\mathcal{U}(\mathfrak{k}_{\mathbb{C}} \ltimes \mathfrak{p}^+)$-module.

LEMMA 2.2. — *If $p_{F(\lambda)}: L(\lambda) \to F(\lambda)$ denotes the orthogonal projection along the sum of all other $\mathfrak{k}_{\mathbb{C}}$-types, then the mapping*

$$L(\lambda) \to \mathrm{Pol}(\mathfrak{p}^+, F(\lambda)), \quad v \mapsto \big(z \mapsto p_{F(\lambda)}(\exp(z)^{-1}.v)\big)$$

defines a $\mathfrak{g}_{\mathbb{C}}$-equivariant embedding.

Proof. — This follows from [Ne94a], Prop. V.13. □

To distinguish $L(\lambda)$ from its realization in $\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda)$ we denote the latter by $L(\lambda)_{\mathrm{hol}}$.

LEMMA 2.3. — *For each $z \in \mathbb{C}$, the module $L(\lambda_z)_{\mathrm{hol}}$ is $\mathfrak{k}'_{\mathbb{C}} \ltimes \mathfrak{p}^+$-isomorphic to a submodule of* $\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda^0)$.

Proof. — This follows from Lemma 2.2 and the fact that $\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda_z)$ is isomorphic to $\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda^0)$ as $\mathfrak{k}'_{\mathbb{C}} \ltimes \mathfrak{p}^+$-module.

Duality.

LEMMA 2.4. — *For each $\lambda \in it^*$ which is dominant integral with respect to Δ_k^+ the following assertions hold:*

(i) *The prescription*

$$N(\lambda) \times \big(\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda)\big) \to \mathbb{C}, \quad (p \otimes v, f) \mapsto \langle v, p^*.f(0)\rangle$$

defines a non-degenerate sesquilinear contravariant pairing.

(ii) *The $\mathfrak{g}_{\mathbb{C}}$-module $L(\lambda)_{\mathrm{hol}}$ is the unique minimal submodule of* $\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda)$ *and the pairing in* (i) *gives rise to a contravariant pairing*

$$\langle \cdot, \cdot \rangle : L(\lambda) \times L(\lambda)_{\mathrm{hol}} \to \mathbb{C}.$$

Proof. — This follows from [Ne94a], V.12-V.14. □

Remark 2.5. — The point is that the pairing $\langle \cdot, \cdot \rangle : N(\lambda_z) \times \big(\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda_z)\big) \to \mathbb{C}$ respects identifications, i.e., if we identify $N(\lambda_z)$ with $N(\lambda^0)$

as $\mathfrak{k}'_{\mathbb{C}} \ltimes \mathfrak{p}^-$-module and $\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda_z)$ with $\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda^0))$ as $\mathfrak{k}'_{\mathbb{C}} \rtimes \mathfrak{p}^+$-module, then $\langle \cdot, \cdot \rangle$ is independent of z. □

LEMMA 2.6. — *Let $z, z' \in l(\lambda^0)$ with $z \leq z'$.*

(i) *As $\mathfrak{k}'_{\mathbb{C}} \rtimes \mathfrak{p}^-$-modules $L(\lambda_{z'})$ is a quotient of $L(\lambda_z)$.*

(ii) *As $\mathfrak{k}'_{\mathbb{C}} \rtimes \mathfrak{p}^+$-modules $L(\lambda_{z'})_{\mathrm{hol}}$ is a submodule of $L(\lambda_z)_{\mathrm{hol}}$.*

Proof. — (i) Note that both $L(\lambda_z)$ and $L(\lambda_{z'})$ are, as $\mathfrak{k}'_{\mathbb{C}} \ltimes \mathfrak{p}^-$-modules, quotients of $N(\lambda^0)$. If I_z and $I_{z'}$ denote the corresponding submodules of $N(\lambda^0)$, then the arguments in the proof of [EHW83], Prop. 3.15, imply that $I_z \subseteq I_{z'}$. From that the assertion follows.

(ii) This follows from (i) by the use of duality (cf. Lemma 2.4(ii) and Remark 2.5). □

For every $n \in \mathbb{N}_0$ denote by $\mathrm{Pol}(\mathfrak{p}^+)^n$ the space of homogeneous polynomials of degree n. Then we have

$$\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda) = \bigoplus_{n=0}^{\infty} \mathrm{Pol}(\mathfrak{p}^+)^n \otimes F(\lambda)$$

inducing a grading of $\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda)$. Note that $L(\lambda)_{\mathrm{hol}}$ is a graded submodule of $\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda)$.

THEOREM 2.7 (Inequality for unitary highest weight modules). *Let $\lambda^0 \in i\mathfrak{t}_0^*$ be dominant integral with respect to Δ_k^+. Let $z, z' \in l(\lambda^0)$ be such that $z \leq z'$ and set $\lambda := \lambda_z$, $\lambda' := \lambda_{z'}$.*

(i) *If we identify $L(\lambda')_{\mathrm{hol}}$ as a $\mathfrak{k}'_{\mathbb{C}} \ltimes \mathfrak{p}^+$ submodule of $L(\lambda)_{\mathrm{hol}}$ (cf. Lemma 2.3, Lemma 2.6(ii)) and normalize the Shapovalov forms $\langle \cdot, \cdot \rangle_\lambda$ and $\langle \cdot, \cdot \rangle_{\lambda'}$ so that they coincide on $F(\lambda^0)$, then we have*

$$(\forall v \in L(\lambda')_{\mathrm{hol}}) \quad \langle v, v \rangle_\lambda \leq \langle v, v \rangle_{\lambda'}.$$

(ii) *If we identify $N(\lambda)$ and $N(\lambda')$ with $N(\lambda^0)$ as $\mathfrak{k}'_{\mathbb{C}} \ltimes \mathfrak{p}^-$ modules (cf. Lemma II.1) and normalize the Shapovalov forms $\langle \cdot, \cdot \rangle_\lambda$, $\langle \cdot, \cdot \rangle_{\lambda'}$ so that they coincide on $F(\lambda^0)$, then we have*

$$(\forall v \in N(\lambda^0)) \quad \langle v, v \rangle_{\lambda'} \leq \langle v, v \rangle_\lambda.$$

Proof. — (i) We may identify $L(\lambda')_{\mathrm{hol}}$ and $L(\lambda)_{\mathrm{hol}}$ with a $\mathfrak{k}'_{\mathbb{C}} \ltimes \mathfrak{p}^+$-submodule of $\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda)$.

We proceed by induction on the degree of the grading of $L(\lambda')_{\mathrm{hol}}$. For $n = 0$ we have $L(\lambda')^0 = F(\lambda)$ and the assertion follows from the normalization of the contravariant forms. Let $n \in \mathbb{N}_0$ and assume that the assertion is true for all elements in $\bigoplus_{j=0}^{n} L(\lambda')^j$. Let $v \in L(\lambda')^{n+1}$. In view of Schur's Lemma, we may assume that v is a $\mathfrak{t}_{\mathbb{C}}$-weight vector which is primitive for Δ_k^+. Let μ, resp. μ', be the corresponding weight of v in $L(\lambda)$, resp. $L(\lambda')$. Then Lemma 1.4 shows that

$$(\|\mu+\rho\|^2 - \|\lambda+\rho\|^2)\langle v, v\rangle_\lambda = 2 \sum_{\alpha\in\Delta_n^+} \langle X_\alpha.v, X_\alpha.v\rangle_\lambda \tag{2.2}$$

$$(\|\mu'+\rho\|^2 - \|\lambda'+\rho\|^2)\langle v, v\rangle_{\lambda'} = 2 \sum_{\alpha\in\Delta_n^+} \langle X_\alpha.v, X_\alpha.v\rangle_{\lambda'}. \tag{2.3}$$

By induction we know that the right hand side of (2.2) is smaller than the right hand side of (2.3). Thus

$$(\|\mu'+\rho\|^2 - \|\lambda'+\rho\|^2)\langle v, v\rangle_{\lambda'} \geq (\|\mu+\rho\|^2 - \|\lambda+\rho\|^2)\langle v, v\rangle_\lambda. \tag{2.4}$$

Now $\lambda' = \lambda + w\zeta$ and $\mu' = \mu + w\zeta$ for $w = z' - z \geq 0$, and so

$$\|\mu'+\rho\|^2 - \|\lambda'+\rho\|^2 = \|\mu+\rho\|^2 - \|\lambda+\rho\|^2 + 2w\langle \mu - \lambda, \zeta\rangle. \tag{2.5}$$

As $\lambda - \mu \in \mathbb{N}_0[\Delta_n^+]$ (cf. [Ne99], Ch. IX) and $w \geq 0$, it follows that $w\langle\mu - \lambda, \zeta\rangle \leq 0$ and so

$$\|\mu+\rho\|^2 - \|\lambda+\rho\|^2 \geq \|\mu'+\rho\|^2 - \|\lambda'+\rho\|^2. \tag{2.6}$$

In view of the Parthasarathy inequality (cf. Lemma 1.5), we have $\|\mu'+\rho\|^2 - \|\lambda'+\rho\|^2 > 0$. Therefore the induction step follows from (2.4) and (2.6).

(ii) We denote by $L(\lambda_w)_{\mathrm{hol}}^\sharp$, $w \in l(\lambda^0)$, the $\mathfrak{g}_{\mathbb{C}}$-module of all $\mathcal{U}(\mathfrak{k}_{\mathbb{C}})$-finite vectors in the algebraic antidual of $L(\lambda_w)_{\mathrm{hol}}$. Then the contravariant pairing $L(\lambda_w) \times L(\lambda_w)_{\mathrm{hol}} \to \mathbb{C}$ (cf. Lemma 2.4(ii)) defines an isomorphism of $\mathfrak{g}_{\mathbb{C}}$-modules

$$\varphi\colon L(\lambda_w) \to L(\lambda_w)_{\mathrm{hol}}^\sharp, \quad v \mapsto \langle v, \cdot\rangle.$$

The contravariant form $\langle\cdot,\cdot\rangle_{\lambda_w}$ induces via φ a contravariant form $\langle\langle\cdot,\cdot\rangle\rangle_{\lambda_w}$ on $L(\lambda_w)_{\mathrm{hol}}^\sharp$. Since φ respects identifications (cf. Remark 2.5), we only have to show that the forms $\langle\langle\cdot,\cdot\rangle\rangle_{\lambda_w}$ are decreasing. This is now seen as follows.

Denote by $(\cdot|\cdot)_{\lambda_w}$ the normalized contravariant form on $L(\lambda_w)_{\mathrm{hol}}$. Then the map

$$L(\lambda_w)_{\mathrm{hol}} \to L(\lambda_w)_{\mathrm{hol}}^\sharp, \quad v \mapsto (v|\cdot)_{\lambda_w}$$

is a $\mathfrak{g}_\mathbb{C}$-equivariant isomorphism. By the unicity of contravariant forms, the pullback of $\langle\langle\cdot,\cdot\rangle\rangle_{\lambda_w}$ under this map coincides with $(\cdot|\cdot)_{\lambda_w}$. In particular, we obtain for all $u \in L(\lambda)^\sharp_{\text{hol}}$ that

$$\sqrt{\langle\langle u,u\rangle\rangle_{\lambda_w}} = \sup_{\substack{v\in L(\lambda_w)_{\text{hol}} \\ (v|v)_{\lambda_w}=1}} |u(v)|. \tag{2.7}$$

Now $(\cdot|\cdot)_{\lambda_w}$ are increasing by (i), and so (2.7) implies that $\langle\langle\cdot,\cdot\rangle\rangle_{\lambda_w}$ satisfy the reverse monotonicity as $(\cdot|\cdot)_{\lambda_w}$, i.e., they are monotonically decreasing. This completes the proof of (ii). □

We conclude this section with an estimate in the other direction on each level of the grading of $L(\lambda)_{\text{hol}}$ which we need later on.

THEOREM 2.8. — *Let $\lambda^0 \in i\mathfrak{t}_0^*$ be dominant integral with respect to Δ_k^+. Let $z, z' \in l(\lambda^0)$ be such that $z' \geq z$ and set $\lambda := \lambda_z$, $\lambda' := \lambda_{z'}$. If we identify $L(\lambda')_{\text{hol}}$ with a $\mathfrak{k}'_\mathbb{C} \ltimes \mathfrak{p}^+$-submodule of $L(\lambda)_{\text{hol}}$ and normalize the Shapovalov forms $\langle\cdot,\cdot\rangle_\lambda$ and $\langle\cdot,\cdot\rangle_{\lambda'}$ so that they coincide on $F(\lambda^0)$, then there exists constants $C, N > 0$ such that*

$$(\forall n \in \mathbb{N}_0)(\forall v \in L(\lambda')^n_{\text{hol}}) \qquad \langle v,v\rangle_{\lambda'} \leq C(1+n)^N\langle v,v\rangle_\lambda.$$

Proof. — We will show that there exists constants $c_1, c_2 > 0$ such that

$$(\forall n \in \mathbb{N})(\forall v \in L(\lambda')^n_{\text{hol}}) \qquad \langle v,v\rangle_{\lambda'} \leq c_1\Big(\prod_{j=1}^n \big(1+\frac{c_2}{j}\big)\Big)\langle v,v\rangle_\lambda \tag{2.8}$$

holds. In view of Lemma 2.9(i) below, the theorem then follows.

First we claim that there exists an $N \in \mathbb{N}$ and positive constants c, c' such that

$$cn^2 \leq \|\mu'+\rho\|^2 - \|\lambda'+\rho\|^2 \leq c'n^2 \tag{2.9}$$

for all Δ_k^+-primitive weights μ' in $L(\lambda')^n_{\text{hol}}$ with $n \geq N$. In fact, by [Ne99], Ch. IX, we can write μ' as

$$\mu' = \lambda' - \sum_{\alpha\in\Delta_n^+} n_\alpha\alpha \quad \text{with} \quad n = \sum_{\alpha\in\Delta_n^+} n_\alpha \tag{2.10}$$

and $n_\alpha \in \mathbb{N}_0$. Thus we get for large n

$$\begin{aligned}\|\mu'+\rho\|^2 - \|\lambda'+\rho\|^2 &= \|\mu'\|^2 + 2\langle\mu',\rho\rangle + \|\rho\|^2 - \|\lambda'+\rho\|^2 \\ &\leq c''(\|\mu'\|^2 + 2\langle\mu',\rho\rangle) \\ &\leq c''\Big(\sum_{\alpha\in\Delta_n^+} n_\alpha n_\beta|\langle\alpha,\beta\rangle| + 2\sum_{\alpha\in\Delta_n^+} n_\alpha|\langle\alpha,\rho-\lambda'\rangle| + \|\lambda'\|^2\Big) \leq c'n^2\end{aligned}$$

for some constants $c', c'' > 0$. Similarly we obtain an ineqality in the converse direction, proving (2.9).

Let $w = z' - z$. If μ' is a Δ_k^+-primitive weight of $L(\lambda')_{\rm hol}^n$ and μ is the corresponding weight of $L(\lambda)_{\rm hol}^n$, then (2.5) and (2.10) imply that

$$\begin{aligned}\frac{\|\mu+\rho\|^2 - \|\lambda+\rho\|^2}{\|\mu'+\rho\|^2 - \|\lambda'+\rho\|^2} &= 1 - \frac{2w\langle \mu - \lambda, \zeta\rangle}{\|\mu'+\rho\|^2 - \|\lambda'+\rho\|^2}\\ &= 1 + \frac{2wn}{\|\mu'+\rho\|^2 - \|\lambda'+\rho\|^2}.\end{aligned}$$

Thus if $n \geq N$, then (2.9) implies that

$$\frac{\|\mu+\rho\|^2 - \|\lambda+\rho\|^2}{\|\mu'+\rho\|^2 - \|\lambda'+\rho\|^2} \leq 1 + \frac{2w}{cn}. \tag{2.11}$$

Let $c_2 := \frac{2w}{c}$ and choose $c_1 > 0$ such that $\langle v, v\rangle_{\lambda'} \leq c_1\Big(\prod_{j=1}^{n}\big(1 + \frac{c_2}{j}\big)\Big)\langle v, v\rangle_\lambda$ for all $v \in L(\lambda')^n$ and $1 \leq n \leq N$.

To prove (2.8) we proceed now by induction on $n - N$. If $n - N \leq 0$, then the assertion is clear by the choice of c_1. Assume now that the asertion is true for all elements in $\bigoplus_{j=0}^{n} L(\lambda')_{\rm hol}^j$ for some $n \in \mathbb{N}$ with $n - N \geq 0$. Let $v \in L(\lambda')^{n+1}$. W.l.o.g. we may assume that v is a $\mathfrak{t}_\mathbb{C}$-weight vector which is primitive for Δ_k^+. Let μ, resp. μ', denote the corresponding weight of v in $L(\lambda)$, resp. $L(\lambda')$. Then it follows from (2.2), (2.3) and induction that

$$\langle v, v\rangle_{\lambda'} \leq \frac{\|\mu+\rho\|^2 - \|\lambda+\rho\|^2}{\|\mu'+\rho\|^2 - \|\lambda'+\rho\|^2} c_1\Big(\prod_{j=1}^{n}\big(1 + \frac{c_2}{j}\big)\Big)\langle v, v\rangle_\lambda.$$

Now the induction step follows from (2.11), proving (2.8) and hence the theorem. □

LEMMA 2.9. — *Let $c > 0$.*

(i) *For all $n \in \mathbb{N}$ we have*

$$\prod_{j=1}^{n}\Big(1 + \frac{c}{j}\Big) \leq e^c(1+n)^c.$$

(ii) *For all $t > 0$ there exists a constant $C > 0$ such that*

$$e^{-tn}\prod_{j=1}^{n}\Big(1 + \frac{c}{j}\Big) \leq C$$

holds for all $n \in \mathbb{N}$.

Proof. — (i) In view of $1+y \leq e^y$ for all $y \geq 0$, we have

$$\prod_{j=1}^{n}\left(1+\frac{c}{j}\right) \leq e^{c\sum_{j=1}^{n}\frac{1}{j}}.$$

Now $\sum_{j=1}^{n}\frac{1}{j} \leq 1+\int_1^{n+1}\frac{1}{x}\,dx = 1+\log(n+1)$ implies that

$$\prod_{j=1}^{n}\left(1+\frac{c}{j}\right) \leq e^{c(1+\log(n+1))} = e^c(1+n)^c,$$

as was to be shown.

(ii) This is immediate from (i). □

3. The characterization of hyperfunction vectors.

Globalization as holomorphic functions.

We identify for all $z \in l(\lambda^0)$ the module $L(\lambda_z)_{\mathrm{hol}}$ with a $\mathfrak{k}'_{\mathbb{C}} \ltimes \mathfrak{p}^+$-submodule of $\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda^0)$. Let $\mathcal{H}_{\lambda_z}$ denote the Hilbert completion of $L(\lambda_z)_{\mathrm{hol}}$ in $\mathrm{Hol}(\mathcal{D}, F(\lambda^0))$. Recall that the representation of $\mathfrak{g}_{\mathbb{C}}$ on $L(\lambda_z)_{\mathrm{hol}}$ integrates to a unitary representation $(\pi_{\lambda_z}, \mathcal{H}_{\lambda_z})$ of the simply connected group $\widetilde{G}$, which is given by (2.1) (cf. [Ne94a], Lemma VI.14). Note that $\mathcal{H}_\lambda$ is a reproducing kernel Hilbert space with reproducing kernel K^λ defined in Section 2.

THEOREM 3.1. — *If one normalizes the* $\widetilde{G}$*-invariant inner products on* $(\mathcal{H}_{\lambda_z})_{z\in l(\lambda^0)}$ *such that they coincide on the constant functions, then the following assertions hold:*

(i) *For* $z \leq z'$, $z < A(\lambda^0)$ *and* $\lambda := \lambda_z$, $\lambda' := \lambda_{z'}$ *we have* $\mathcal{H}_{\lambda'} \subseteq \mathcal{H}_\lambda$ *and the inclusion mapping*

$$T_{\lambda,\lambda'}\colon \mathcal{H}_{\lambda'} \to \mathcal{H}_\lambda$$

is contractive.

(ii) *If* $f \in \big(\mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda^0)\big) \backslash L(\lambda_{A(\lambda^0)})_{\mathrm{hol}}$, *then we have*

$$\lim_{\substack{z \to A(\lambda^0) \\ z < A(\lambda^0)}} \langle f, f\rangle_{\lambda_z} = \infty.$$

Proof. — (i) This is immediate from Theorem 2.7(ii).

(ii) W.l.o.g we may assume that f belongs to an irreducible K-subspace which does not belong to $L(\lambda_{A(\lambda^0)})_{\text{hol}}$. Let $\lambda = \lambda_z$ for some $z < A(\lambda^0)$ and note that $L(\lambda)_{\text{hol}} = \text{Pol}(\mathfrak{p}^+) \otimes F(\lambda)$ (cf. Theorem 1.3(ii)). Then $N(\lambda)$ and $\text{Pol}(\mathfrak{p}^+) \otimes F(\lambda)$ are $\mathfrak{g}_{\mathbb{C}}$-isomorphic and we denote by $\tilde{f}$ the corresponding element in $N(\lambda)$. Then Theorem 2.7(i) implies that

$$\lim_{\substack{z \to A(\lambda^0) \\ z < A(\lambda^0)}} \langle \tilde{f}, \tilde{f} \rangle_{\lambda_z} = 0.$$

As by (2.7) in $\text{Pol}(\mathfrak{p}^+) \otimes F(\lambda^0)$ the reverse statement must hold, the assertion follows. □

Hyperfunction vectors.

DEFINITION 3.2. — *Let G be a Lie group, $\mathcal{H}$ a Hilbert space and $(\pi, \mathcal{H})$ a unitary representation of G.*

(a) *An element $v \in \mathcal{H}$ is called a smooth vector if the orbit map $G \to \mathcal{H}$, $g \mapsto \pi(g).v$ is smooth. We denote the space of all smooth vectors by $\mathcal{H}^\infty$ and equip it with the locally convex topology induced by the family of seminorms $\{p_U\}_{U \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})}$, where $p_U(v) := \|d\pi(U).v\|$. Note that this topology is complete, i.e., $\mathcal{H}^\infty$ is a Fréchet space. The strong antidual of $\mathcal{H}^\infty$ is denoted by $\mathcal{H}^{-\infty}$ and its elements are called distribution vectors.*

(b) *(cf. [KNÓ97], App.) We say that $v \in \mathcal{H}$ is an analytic vector if the corresponding orbit map is analytic and write $\mathcal{H}^\omega$ for the collection of all analytic vectors.*

If $v \in \mathcal{H}^\omega$, then there exists an open connected 0-neighborhood $U \subseteq \mathfrak{g}_{\mathbb{C}}$ and a holomorphic map $\gamma_{v,U} : U \to \mathcal{H}$ with $\gamma_{v,U}(0) = v$ and $\gamma_{v,U}(X) = \pi(\exp X).v$ for $X \in U \cap \mathfrak{g}$. Let $\mathcal{H}_U \subseteq \mathcal{H}$ denote the subspace of all elements v for which $\gamma_{v,U}$ exists. Then we have a natural linear embedding

$$\eta_U : \mathcal{H}_U \to \text{Hol}(U, \mathcal{H}), \quad v \mapsto \gamma_{v,U}.$$

Thus we may think of $\mathcal{H}_U$ as a subspace of $\text{Hol}(U, \mathcal{H})$ and thus $\mathcal{H}^\omega = \bigcup_U \mathcal{H}_U$ as a subspace of $E := \bigcup_U \text{Hol}(U, \mathcal{H})$. We endow the space $\text{Hol}(U, \mathcal{H})$ with the topology of uniform convergence on compact subsets of U and put on E the natural inductive limit topology. We equip $\mathcal{H}^\omega$ with the subspace topology of E. We write $\mathcal{H}^{-\omega}$ for the strong antidual of $\mathcal{H}^{\omega}$. The elements of $\mathcal{H}^{-\omega}$ are called *hyperfunction vectors.*

Note that there is a natural chain of continuous inclusions

$$\mathcal{H}^{\omega} \subseteq \mathcal{H}^{\infty} \subseteq \mathcal{H} \subseteq \mathcal{H}^{-\infty} \subseteq \mathcal{H}^{-\omega}. \tag{3.1}$$

The corresponding representation of G on these spaces is denoted by $(\pi^{\omega}, \mathcal{H}^{\omega})$, $(\pi^{\infty}, \mathcal{H}^{\infty})$ etc. $\square$

LEMMA 3.3. — *Let $X_0 \in \mathfrak{z}(\mathfrak{k})$ such that $\Delta_n^+ = \{\alpha \in \Delta : \alpha(iX_0) = 1\}$. Then the following assertions hold:*

(i) *For all $t > 0$ we have*

$$\overline{\exp_{G_{\mathbb{C}}}(-itX_0).\mathcal{D}} \subseteq \mathcal{D}.$$

(ii) *Let $\lambda = \lambda^0 + z\zeta$ for some $z \in \mathbb{C}$ and set $\Pi^+ := \{u \in \mathbb{C} : \operatorname{Re} u \geq 0\}$. Then the prescription*

$$(\gamma_\lambda(u).f)(w) = e^{uz} f(\exp_{G_{\mathbb{C}}}(-iuX_0)w)$$

defines a semigroup homomorphism $\gamma_\lambda : \Pi^+ \to B\big(\mathrm{Hol}(\mathcal{D}, F(\lambda))\big)$, $u \mapsto \gamma_\lambda(u)$ which satisfies $\pi_\lambda(\exp(xX_0)) = \gamma_\lambda(-ix)$ for all $x \in \mathbb{R}$.

(iii) *If $L(\lambda)_{\mathrm{hol}}$ is unitarizable and $(\pi_\lambda, \mathcal{H}_\lambda)$ is the corresponding globalization in the holomorphic functions on $\mathcal{D}$, then $\gamma_\lambda(\Pi^+)$ leaves $\mathcal{H}_\lambda$ invariant and all operators $\gamma_\lambda(u)$, $u \in \operatorname{int} \Pi^+$, are injective contractive trace class operators.*

Proof. — (i) This follows from [HiÓl96], Th. 5.4.20.

(ii) In view of (i), the map γ_λ is well defined. It remains to show that $\gamma_\lambda(-ix) = \pi_\lambda(\exp(xX_0))$. In fact we have for all $f \in \mathrm{Hol}(\mathcal{D}, F(\lambda))$, $w \in \mathcal{D}$ and $x \in \mathbb{R}$

$$\begin{aligned}
\big(\pi_\lambda(\exp(xX_0)).f\big)(w) &= J_\lambda(\exp(-xX_0), w)^{-1}.f(\exp(-xX_0).w) \\
&= \sigma_\lambda(\exp(xX_0)).f(\exp(-xX_0).w) \\
&= e^{-izx} f(\exp(-xX_0).w) = \gamma_\lambda(-ix),
\end{aligned}$$

concluding the proof of (ii).

(iii) This follows from [Ne94b], Th. 3.8. $\square$

Even though in general the topology on the spaces of analytic and hyperfunction vectors is hard to get a hand on, one has a quite good picture for unitary highest weight representations.

LEMMA 3.4. — *Suppose that $L(\lambda)$ is unitarizable and let $(\pi_\lambda, \mathcal{H}_\lambda)$ be the corresponding globalization as a space of holomorphic functions. For all $t > 0$ we set $\mathcal{H}_\lambda^t := \gamma_\lambda(t).\mathcal{H}_\lambda \subseteq \mathcal{H}_\lambda$* (cf. Lemma 3.3).

(i) *The space of analytic vectors of $(\pi_\lambda, \mathcal{H}_\lambda)$ is given by*

$$\mathcal{H}_\lambda^\omega = \bigcup_{t>0} \mathcal{H}_\lambda^t.$$

Further the topology on $\mathcal{H}_\lambda^\omega$ is the finest locally convex topology on $\mathcal{H}_\lambda^\omega$ making for all $t > 0$ the maps $\mathcal{H}_\lambda \to \mathcal{H}_\lambda^\omega$, $f \mapsto \gamma_\lambda(t).f$ continuous.

(ii) *We equip $\mathcal{H}_\lambda^t$ with a Hilbert space structure by $\langle \gamma_\lambda(t).v, \gamma_\lambda(t).v\rangle_{\lambda,t} := \langle v, v\rangle_\lambda$ and write $\|\cdot\|_{\lambda,t}$ for the corresponding norm on $\mathcal{H}_\lambda^t$. Then*

(a) *For $0 < s < t$ the inclusion mapping*

$$i_{s,t}: (\mathcal{H}_\lambda^t, \|\cdot\|_{\lambda,t}) \to (\mathcal{H}_\lambda^s, \|\cdot\|_{\lambda,s})$$

is contractive and of trace class.

(b) *The topology on $\mathcal{H}_\lambda^t$ induced from $\mathcal{H}_\lambda^\omega$ is coarser than the topology induced from $\|\cdot\|_{\lambda,t}$.*

(c) *The space $\mathcal{H}_\lambda^\omega$ is the inductive limit of the Hilbert spaces $(\mathcal{H}_\lambda^t, \langle\cdot,\cdot\rangle_{\lambda,t})$, $t > 0$, i.e.,*

$$\mathcal{H}^\omega = \varinjlim_{t\to 0} \mathcal{H}_\lambda^t.$$

Proof. — (i) [KNÓ97], Prop. A5.

(ii)(a) Note that for all $t > 0$ the mapping $i_t: \mathcal{H}_\lambda \to \mathcal{H}_\lambda^t$, $v \mapsto \gamma_\lambda(t).v$ is an isometric isomorphism. In particular we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_\lambda^t & \xrightarrow{\quad i_{s,t}\quad} & \mathcal{H}_\lambda^s \\ \uparrow{\scriptstyle i_t} & & \uparrow{\scriptstyle i_s,} \\ \mathcal{H}_\lambda & \xrightarrow{\quad \tilde{i}_{s,t}\quad} & \mathcal{H}_\lambda \end{array}$$

where $\tilde{i}_{s,t}(v) = \gamma_\lambda(t-s).v$. In view of Lemma 3.3(iii), this proves (a).

(b) The Hilbert space topology on $\mathcal{H}_\lambda^t$ is the finest locally convex topology on $\mathcal{H}_\lambda^t$ which makes the map $i_t: \mathcal{H}_\lambda \to \mathcal{H}_\lambda^t$ continuous. In view of (i), this proves (b).

(c) This follows from (i) and (a). □

LEMMA 3.5. — *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation of $\widetilde{G}$ and K^λ the corresponding reproducing kernel. For all $z \in \mathcal{D}$*

and $v \in F(\lambda)$ the prescription $K^{\lambda}_{z,v}(w) := K^{\lambda}(w,z).v$ defines an element of $\mathcal{H}^{\omega}_{\lambda}$ and the mapping

$$r_{\lambda}: \mathcal{H}^{-\omega}_{\lambda} \to \mathrm{Hol}(\mathcal{D}, F(\lambda)), \quad \langle r_{\lambda}(\nu)(z), v\rangle = \nu(K^{\lambda}_{z,v})$$

defines a $\widetilde{G}$-equivariant realization of the hyperfunction vectors as holomorphic functions.

Proof. — [KNÓ97], Sect. 6. □

In the sequel we will be concerned with range and continuity properties of the map r_{λ}. In view of Lemma 3.5, we may consider $\mathcal{H}^{-\omega}_{\lambda}$ from now on as a subspace of $\mathrm{Hol}(\mathcal{D}, F(\lambda))$.

PROPOSITION 3.6. — *For all $t > 0$ let*

$$\mathcal{H}^{-t}_{\lambda} = \gamma_{\lambda}(t)^{-1}(\mathcal{H}_{\lambda}) = \{f \in \mathrm{Hol}(\mathcal{D}, F(\lambda)) : \gamma_{\lambda}(t).f \in \mathcal{H}_{\lambda}\},$$

equip $\mathcal{H}^{-t}_{\lambda}$ with the Hilbert space structure $\langle v, v\rangle_{\lambda,t} := \langle \gamma_{\lambda}(t).v, \gamma_{\lambda}(t).v\rangle_{\lambda}$ and write $\|\cdot\|_{\lambda,-t}$ for the corresponding norm on $\mathcal{H}^{-t}_{\lambda}$.

(i) *Every element $f \in \mathcal{H}^{-t}_{\lambda}$ defines via $\tilde{f}(\gamma_{\lambda}(t).v) := \langle \gamma_{\lambda}(t).f,\, v\rangle_{\lambda}$ a continuous antilinear functional on $\mathcal{H}^{t}_{\lambda}$. Moreover, the prescription $f \mapsto \tilde{f}$ defines an isomorphism between $\mathcal{H}^{-t}_{\lambda}$ and the strong antidual of $\mathcal{H}^{t}_{\lambda}$.*

(ii) *For $0 < s < t$ the inclusion mapping*

$$i'_{s,t}: (\mathcal{H}^{-s}_{\lambda}, \|\cdot\|_{\lambda,-s}) \to (\mathcal{H}^{-t}_{\lambda}, \|\cdot\|_{\lambda,-t})$$

is contractive and of trace class.

(iii) *We have*

$$\mathcal{H}^{-\omega}_{\lambda} = \bigcap_{t>0} \mathcal{H}^{-t}_{\lambda}$$

and the topology on $\mathcal{H}^{-\omega}_{\lambda}$ is the one induced from the seminorms $(\|\cdot\|_{\lambda,-t})_{t>0}$. Moreover $\mathcal{H}^{-\omega}_{\lambda}$ is a nuclear Fréchet space.

(iv) *The space $\mathcal{H}^{-\omega}_{\lambda}$ is reflexive.*

Proof. — (i) Let $(\mathcal{H}^{t}_{\lambda})^{\sharp}$ denote the strong antidual of $\mathcal{H}^{t}_{\lambda}$. We claim that the mapping

$$\psi_t: \mathcal{H}^{-t}_{\lambda} \to (\mathcal{H}^{t}_{\lambda})^{\sharp}, \quad f \mapsto \tilde{f}$$

is isometric. In fact we have

$$\begin{aligned}\|\tilde{f}\| &= \sup_{\substack{v \in \mathcal{H}^{t}_{\lambda} \\ \|v\|_{\lambda,t}=1}} |\tilde{f}(v)| = \sup_{\substack{w \in \mathcal{H}_{\lambda} \\ \|w\|_{\lambda}=1}} |\tilde{f}(\gamma_{\lambda}(t).w)| \\ &= \sup_{\substack{w \in \mathcal{H}_{\lambda} \\ \|w\|_{\lambda}=1}} |\langle \gamma_{\lambda}(t).f, w\rangle_{\lambda}| = \|f\|_{\lambda,-t},\end{aligned}$$

proving our claim. It follows from the Hahn-Banach Theorem that im ψ_t is dense. Thus ψ_t is in fact an isomorphism.

(ii) In view of (i), we may identify $i'_{s,t}$ with the antiadjoint of $i_{s,t}$. Now the assertion follows from the fact that antiadjoints of trace class operators are of trace class.

(iii) Since the topological antidual of an inductive limit of locally convex spaces is the corresponding projective limit of the antiduals (cf. [Kö69], p. 290), the first assertion follows.

As a projective limit of Hilbert spaces is complete, it follows that $\mathcal{H}_\lambda^{-\omega}$ is complete. Further the countable family $(\|\cdot\|_{\lambda,-\frac{1}{n}})_{n\in\mathbb{N}}$ suffices to define the topology, and so $\mathcal{H}_\lambda^{-\omega}$ is a Fréchet space. Finally it follows from (ii) that $\mathcal{H}_\lambda^{-\omega}$ is nuclear.

(iv) By (iii) we know that $\mathcal{H}_\lambda^{-\omega}$ is a nuclear Fréchet space. Since nuclear Frechet spaces are Montel spaces (cf. [Tr67], p. 520, Cor. 3) and Montel spaces are reflexive (cf. [Tr67], p. 376, Cor.), the assertion follows. □

LEMMA 3.7. — *Let $\overline{L(\lambda)_{\rm hol}}$ denote the closure of $L(\lambda)_{\rm hol}$ in the Fréchet space* $\mathrm{Hol}(\mathcal{D}, F(\lambda))$ *and let $r_\lambda: \mathcal{H}_\lambda^{-\omega} \to \mathrm{Hol}(\mathcal{D}, F(\lambda))$ be the realization map from* Lemma 3.5. *Then*

(i) *The map r_λ is continuous.*

(ii) *We have* $\mathrm{im}\, r_\lambda \subseteq \overline{L(\lambda)_{\rm hol}}$.

Proof. — (i) We identify $\mathcal{H}_\lambda^{-\omega}$ with a subspace of $\mathrm{Hol}(\mathcal{D}, F(\lambda))$. Recall from Proposition 3.6(iii) that $\mathcal{H}_\lambda^{-\omega}$ is a Fréchet space and that $f_n \to f$ in $\mathcal{H}_\lambda^{-\omega}$ if and only if $\gamma_\lambda(t).f_n \to \gamma_\lambda(t).f$ in $\mathcal{H}_\lambda$ for all $t > 0$. Since the Hilbert space topology on $\mathcal{H}_\lambda$ is finer than the topology of compact convergence, we conclude in particular that $\gamma_\lambda(t).f_n \to \gamma_\lambda(t).f$ in $\mathrm{Hol}(\mathcal{D}, F(\lambda))$. But in view of the concrete formula for γ_λ (cf. Lemma 3.3), it follows that $f_n \to f$ in $\mathrm{Hol}(\mathcal{D}, F(\lambda))$, as was to be shown.

(ii) Since $L(\lambda)_{\rm hol}$ is dense in $\mathcal{H}_\lambda$, it follows from (3.1), the reflexivity of $\mathcal{H}_\lambda^{-\omega}$ (cf. Proposition 3.6(iv)) and the Hahn-Banach Theorem that $L(\lambda)_{\rm hol}$ is even dense in $\mathcal{H}_\lambda^{-\omega}$. This proves (ii). □

LEMMA 3.8. — *Let $z \in l(\lambda^0)$ and set $\lambda = \lambda_z$. If λ_z belongs to the relative holomorphic discrete series, then the mapping $r_\lambda: \mathcal{H}_\lambda^{-\omega} \to$* $\mathrm{Hol}(\mathcal{D}, F(\lambda))$ *is an isomorphism of Fréchet spaces.*

Proof. — Recall that the bracket on $\mathcal{H}_\lambda$ is up to scalar multiple given by

$$\langle f, f\rangle_\lambda = \int_{\mathcal{D}} \langle K^\lambda(z,z)^{-1}.f(z).f(z)\rangle \, d\mu_{\mathcal{D}}(z),$$

where $\mu_{\mathcal{D}}$ denotes a $\widetilde{G}$-invariant positive measure on $\mathcal{D}$ (cf. [Ne94a], Sect. VIII]). In view of Theorem 1.3(ii),(iii), we have $L(\lambda)_{\mathrm{hol}} = \mathrm{Pol}(\mathfrak{p}^+) \otimes F(\lambda)$ and so

$$\mathcal{H}_\lambda = \{f \in \mathrm{Hol}(\mathcal{D}, F(\lambda)) \colon \langle f, f\rangle_\lambda < \infty\}.$$

As all constant functions belong to $\mathcal{H}_\lambda$, we see in particular that the Banach space $\mathrm{Hol}(\mathcal{D}, F(\lambda))_b$ of all bounded holomorphic functions is contained in $\mathcal{H}_\lambda$. In fact, if $\|\cdot\|_\infty$ denotes the sup-norm on $\mathrm{Hol}(\mathcal{D}, F(\lambda))_b$, then there exists a constant $C > 0$ such that

$$(\forall f \in \mathrm{Hol}(\mathcal{D}, F(\lambda))_b) \quad \|f\|_\lambda \leq C\|f\|_\infty. \tag{3.2}$$

In view of Proposition 3.6(iii), a holomorphic function f belongs to $\mathcal{H}_\lambda$ if and only if $\gamma_\lambda(t).f \in \mathcal{H}_\lambda$ for all $t > 0$. Since $\mathcal{D} \subseteq \mathfrak{p}^+$ is bounded and $\exp(-itX_0).\overline{\mathcal{D}} \subseteq \mathcal{D}$ for all $t > 0$ (cf. Lemma 3.3(i)), we conclude that

$$\gamma_\lambda(t).\mathrm{Hol}(\mathcal{D}, F(\lambda)) \subseteq \mathrm{Hol}(\mathcal{D}, F(\lambda))_b \subseteq \mathcal{H}_\lambda.$$

In particular r_λ is a bijection.

In view of Lemma 3.7, the map r_λ is continuous. As both $\mathrm{Hol}(\mathcal{D}, F(\lambda))$ and $\mathcal{H}_\lambda^{-\omega}$ are Fréchet spaces (cf. Proposition 3.6(iii)), the Open Mapping Theorem implies that r_λ is an isomorphism. This proves the lemma. □

THEOREM 3.9 (Characterization of hyperfunction vectors). — *Suppose that $L(\lambda)$ is unitarizable and let $\mathcal{H}_\lambda$ be the associated globalization as a space of holomorphic functions with reproducing kernel K^λ. If $\overline{L(\lambda)_{\mathrm{hol}}}$ denotes the closure of $L(\lambda)_{\mathrm{hol}}$ in the Fréchet space* $\mathrm{Hol}(\mathcal{D}, F(\lambda))$, *then the mapping*

$$r_\lambda \colon \mathcal{H}_\lambda^{-\omega} \to \overline{L(\lambda)_{\mathrm{hol}}}, \quad \langle r_\lambda(\nu)(z), v\rangle = \nu(K^\lambda_{z,v})$$

is a $\widetilde{G}$-equivariant isomorphism of nuclear Fréchet spaces.

Proof. — In view of Lemma 3.7 and the Open Mapping Theorem, we only have to show that r_λ is onto.

Let $\lambda = \lambda_z$ for some $z \in l(\lambda^0)$ and let $\lambda' := \lambda_{z'}$ with $z' < z$ belong to the holomorphic discrete series. Now let $f \in \overline{L(\lambda)_{\mathrm{hol}}}$. In view of Proposition 3.6(iii), we have to show that $\gamma_\lambda(t).f \in \mathcal{H}_\lambda$ holds for all

$t > 0$. Let $f = \sum_{n=0}^{\infty} f_n$ be the expansion in homogeneous polynomials. Then $\gamma_\lambda(t).f_n = e^{tz}e^{-tn}f_n$ holds for all $n \in \mathbb{N}_0$ by Lemma 3.3(ii). Since $z > z'$, Theorem 2.8 implies that

$$\begin{aligned}\langle \gamma_\lambda(t).f, \gamma_\lambda(t).f\rangle_\lambda &= \sum_{n=0}^{\infty} \langle \gamma_\lambda(t).f_n, \gamma_\lambda(t).f_n\rangle_\lambda = \sum_{n=0}^{\infty} e^{2tz}e^{-2tn}\langle f_n, f_n\rangle_\lambda \\ &= e^{t(2z-z')} \sum_{n=0}^{\infty} e^{-tn}\langle \gamma_{\lambda'}(\tfrac{t}{2}).f_n, \gamma_{\lambda'}(\tfrac{t}{2}).f_n\rangle_\lambda \\ &\leq Ce^{t(2z-z')} \sum_{n=0}^{\infty} e^{-tn}(1+n)^N\langle \gamma_{\lambda'}(\tfrac{t}{2}).f_n, \gamma_{\lambda'}(\tfrac{t}{2}).f_n\rangle_{\lambda'}\end{aligned}$$

for some positive constants $C, N > 0$. Thus Lemma 2.9(ii) implies that there exists a positive constant $c > 0$ such that

$$\langle \gamma_\lambda(t).f, \gamma_\lambda(t).f\rangle_\lambda \leq c\langle \gamma_{\lambda'}(\tfrac{t}{2}).f, \gamma_{\lambda'}(\tfrac{t}{2}).f\rangle_{\lambda'}$$

holds for all $t > 0$. By Lemma 3.8, the right hand side is finite for all $t > 0$, proving the theorem. □

COROLLARY 3.10. — *If $z < A(\lambda_0)$ and $\lambda = \lambda_z$, then the mapping*

$$r_\lambda : \mathcal{H}_\lambda^{-\omega} \to \mathrm{Hol}(\mathcal{D}, F(\lambda)), \quad \langle r_\lambda(\nu)(z), v\rangle = \nu(K_{z,v}^\lambda)$$

is a $\widetilde{G}$-equivariant isomorphism of nuclear Fréchet spaces.

Proof. — This follows from Theorem 1.3(ii) and Theorem 3.9. □

Remark 3.11 (Characterization of distribution vectors). — The characterization of distribution vectors of a unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ has recently been obtained by J.-L. Clerc (cf. [Cl98]; see also [ChFa98] for the scalar case). In the realization of $(\pi_\lambda, \mathcal{H}_\lambda)$ in $\mathrm{Hol}(\mathcal{D}, F(\lambda))$ the distribution vectors are those functions in $\overline{\mathcal{H}_\lambda}$ of moderate growth on $\mathcal{D}$. Clerc has obtained his result with a similar strategy: First prove the statement for the relative discrete series and then use our Theorem 2.8 for making a shifting process to obtain the characterization for all parameters.

But also in other aspects the characterization of hyperfunction and distribution vectors are similar. To describe $\mathcal{H}_\lambda^\omega$ one needs only one operator, namely $\mathrm{id}\pi_\lambda(X_0)$ (cf. Lemma 3.4(i)). For the smooth vectors one has

$$\mathcal{H}_\lambda^\infty = \bigcap_{n\in\mathbb{N}} \mathcal{D}(\mathrm{id}\pi_\lambda(X_0)^n),$$

where $\mathcal{D}(\mathrm{i}d\pi_\lambda(X_0)^n)$ denotes the domain of definition of the unbounded selfadjoint operator $\mathrm{i}d\pi_\lambda(X_0)^n$ (cf. [Ne97]). $\square$

Applications to spherical representations.

DEFINITION 3.12. — *Let G be a Lie group and $H \subseteq G$ a closed subgroup. We write $\mathbb{X}(H)$ for the group of all continuous characters $\chi\colon H \to \mathbb{C}^*$ of H.*

For a unitary representation $(\pi, \mathcal{H})$ of G and $\chi \in \mathbb{X}(H)$, we write $(\mathcal{H}^{-\omega})^{(H,\chi)}$ for the set of all those elements $\nu \in \mathcal{H}^{-\omega}$ satisfying $\pi^{-\omega}(h).\nu = \chi(h).\nu$ for all $h \in H$. The unitary representation $(\pi, \mathcal{H})$ is called (H,χ)-*spherical* if there exists a cyclic vector $\nu \in (\mathcal{H}^{-\omega})^{(H,\chi)}$. Note that for G semisimple and $\chi = \mathbf{1}$ one has $(\mathcal{H}^{-\omega})^{(H,\chi)} = (\mathcal{H}^{-\infty})^{(H,\chi)}$ by [BrDe92], Théorème 1, so that our definition of spherical representation coincides with the usual one. $\square$

DEFINITION 3.13. — *Let $\mathfrak{g}$ be a hermitian Lie algebra and $\tau\colon \mathfrak{g} \to \mathfrak{g}$ an involution on it. The $+1$ eigenspace of τ is denoted by $\mathfrak{h}$, the -1 eigenspace by $\mathfrak{q}$. Note that $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$. Further let θ be a Cartan involution commuting with τ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. The hermitian symmetric Lie algebra $(\mathfrak{g}, \tau)$ is called compactly causal if $\mathfrak{z}(\mathfrak{k}) \subseteq \mathfrak{q}$ (cf. [HiÓl96]).*

If $\widetilde{G}$ denotes a simply connnected Lie group associated to $\mathfrak{g}$, then we set $H = \langle \exp_{\widetilde{G}}(\mathfrak{h})\rangle$, $K = \exp_{\widetilde{G}}(\mathfrak{k})$ and $H_K := H \cap K$. $\square$

THEOREM 3.14. — *Let $(\mathfrak{g}, \tau)$ be a compactly causal symmetric Lie algebra and $\chi \in \mathbb{X}(H)$ a continuous character of H. If $z < A(\lambda^0)$, then $(\pi_{\lambda_z}, \mathcal{H}_{\lambda_z})$ is (H,χ)-spherical if and only if $F(\lambda_z)$ is $(H_K, \chi|_{H_K})$-spherical.*

Proof. — It follows from [KNÓ97], Prop. VI.5, that $F(\lambda_z)$ is $(H_K, \chi|_{H_K})$-spherical whenever $\mathcal{H}_{\lambda_z}$ is (H,χ)-spherical.

Conversely, we have a linear bijection

$$F(\lambda_z)^{(H_K,\chi|_{H_K})} \to \mathrm{Hol}(\mathcal{D}, F(\lambda_z))^{(H,\chi)}$$

(cf. [KNÓ97], Th. 2.11). Thus if $F(\lambda_z)$ is $(H_K, \chi|_{H_K})$-spherical, we find a non-zero (H,χ)-fixed holomorphic function f on $\mathcal{D}$. Since $z < A(\lambda^0)$, we have $\mathcal{H}_{\lambda_z}^{-\omega} = \mathrm{Hol}(\mathcal{D}, F(\lambda_z))$ by Corollary 3.10. Thus $f \in \mathcal{H}_{\lambda_z}^{-\omega}$, proving the converse. □

Remark 3.15. — In general it is not true that $\mathcal{H}_\lambda$ is (H,χ)-spherical if the minimal K-type $F(\lambda)$ is $(H_K, \chi|_{H_K})$-spherical.

Let $\widetilde{G} := \widetilde{\mathrm{Sp}(n,\mathbb{R})}$ be the universal covering group of $\mathrm{Sp}(n,\mathbb{R})$ and $\mathfrak{g} = \mathfrak{sp}(\mathfrak{n},\mathbb{R})$ the corresponding Lie algebra. Let $I_{n,n} := \mathrm{diag}(I_n, -I_n) \in \mathfrak{gl}(2n,\mathbb{R})$. The prescription

$$\tau\colon \mathfrak{g} \to \mathfrak{g}, \quad X \mapsto I_{n,n} X I_{n,n}$$

defines an involution on $\mathfrak{g}$ turning $(\mathfrak{g},\tau)$ into a compactly causal symmetric Lie algebra. We have $H = \widetilde{G}^\tau \cong \mathrm{GL}(n,\mathbb{R})_+$ (cf. [KNÓ98], Lemma 2.1) and we identify from now on H with $\mathrm{GL}(n,\mathbb{R})_+$.

Let $\chi = \mathbf{1}$ and $(\pi_\lambda, \mathcal{H}_\lambda)$ be the even metapletic representation of $\widetilde{G}$ modelled on the $L^2(\mathbb{R}^n)$-completion of the span of the even Hermite polymomials, i.e., the space of even functions in $L^2(\mathbb{R}^n)$. The action of $H = \mathrm{GL}(n,\mathbb{R})_+$ is given by

$$(\pi_\lambda(h).f)(x) = (\det h)^{-\frac{1}{2}} f(h^{-1}.x) \tag{3.3}$$

for all $h \in H$, $f \in \mathcal{H}_\lambda$ and $x \in \mathbb{R}^n$. Note that $F(\lambda) = \mathbb{C}\{e^{-\pi\langle x,x\rangle}\}$ is one-dimensional. Since $H_K = \mathrm{SO}(n,\mathbb{R})$, it follows from (3.3) that $F(\lambda)$ is H_K-spherical. We assert that $(\pi_\lambda, \mathcal{H}_\lambda)$ is H-spherical if and only if $n = 1$.

By [BrDe92], Théorème 1, we know that $(\mathcal{H}_\lambda^{-\omega})^H = (\mathcal{H}_\lambda^{-\infty})^H$. In view of [Fo89], Ch. IV, we know that $\mathcal{H}_\lambda^{-\infty}$ is the closure of $\mathcal{H}_\lambda$ in the tempered distributions, i.e., $\mathcal{H}_\lambda^{-\infty} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Thus we are searching for H-invariant tempered distributions.

Let $\nu \in \mathcal{S}'(\mathbb{R}^n)^H$. Since ν is H-fixed, it is fixed under $H_K = \mathrm{SO}(n)$ which in view of (3.3) means that ν is rotation invariant. Further considering the action of $Z(H) \cong \mathbb{R}^+$, we deduce that ν is homogeneous of degree $-\dfrac{n}{2}$.

If $n = 1$, then $H = Z(H) = \mathbb{R}^+$ and $\nu(x) = |x|^{\frac{1}{2}}$ defines a non-zero H-fixed element of $\mathcal{H}_\lambda^{-\omega}$.

If $n \geq 2$, there exists up to normalization only one rotation invariant distribution which is homogeneous of degree $-\dfrac{n}{2}$, namely $\nu(x) = r^{-\frac{n}{2}}$, where $r = \sqrt{x_1^2 + \ldots + x_n^2}$ (cf. Lemma 3.16 below). But since $n \geq 2$, this distribution cannot be H-invariant. This proves our assertion. □

LEMMA 3.16. — *Let $n \in \mathbb{N}$. Then for each $\mu \in \mathbb{C}$ the space of μ-homogeneous $\mathrm{O}(n, \mathbb{R})$-invariant distributions on $\mathbb{R}^n$ is one-dimensional. Moreover, if $n \geq 2$ we may replace $\mathrm{O}(n, \mathbb{R})$ by $\mathrm{SO}(n, \mathbb{R})$.*

Proof. — The first assertion is [HoTa92], Ch. IV, Prop. 3.1.2. Further the arguments in [HoTa92], Sect. IV.3, show that this remains true for $\mathrm{O}(n, \mathbb{R})$ replaced by $\mathrm{SO}(n, \mathbb{R})$, provided $n \geq 2$. □

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Manuscrit reçu le 11 septembre 1998,
accepté le 1er février 1999.

Bernhard KRÖTZ,
Technische Universität Clausthal
Mathematisches Institut
Erzstraße 1
D-38678 Clausthal-Zellerfeld (Germany).
mabk@math.tu-clausthal.de