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CHRISTOPHE KAPOUDJIAN

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SIMPLICITY OF NERETIN'S GROUP OF SPHEROMORPHISMS

by Christophe KAPOUDJIAN

Introduction.

Answering a question of I.M. Gelfand on the existence of analogues of highest-weight representations of the diffeomorphism group of the circle in the case of p -adic transformation groups, Yu.A. Neretin constructed a group of transformations of the boundary $\partial\mathcal{T}_p$ of the regular tree \mathcal{T}_p (cf. [12] and [13]): the group N_p of spheromorphisms (§1). When p is a prime integer, the boundary $\partial\mathcal{T}_p$ is naturally homeomorphic to the projective line on the field of p -adic numbers, and in any case, to a Cantor set.

Roughly speaking, a spheromorphism is a transformation induced in the boundary by a “piecewise” tree automorphism. The spheromorphism group is generated by two groups: on the one hand a Higman-Thompson group (§2), which is countable and almost-acts on the tree, respecting a local orientation of the edges, and on the other hand, the tree automorphism group (§3).

Exploiting simplicity theorems known for the generating two groups, and adapting some arguments of a simplicity theorem of Epstein, we finally prove the simplicity of N_p (the analogue of M.R. Herman's theorem on the simplicity of the orientation-preserving diffeomorphism group of the circle, cf. [7]), and of some of its subgroups (§4):

Keywords: Cantor set – Higman-Thompson groups – p -adic numbers – Simple groups – Spheromorphism – Tree – Tree automorphism group.

Math. classification: 20E08 – 20E32 – 22E65 – 54H15.

THEOREM. — *For each integer $p \geq 2$, the spheromorphism group N_p is simple.*

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1. The Neretin group of spheromorphisms.

1.1. Let \mathcal{T}_n be the regular tree whose vertices have valence $n+1$, with $n \geq 2$, and $\partial\mathcal{T}_n$ its boundary, or set of “ends”, see e.g. [14] or [6].

We may describe the boundary $\partial\mathcal{T}_n$ as a compact ultrametric space: choose a vertex o of the tree \mathcal{T}_n . Each end is defined by a unique chain (i.e. a sequence of consecutive vertices ($o = x_0, x_1, \dots$) with $x_{i+2} \neq x_i$) starting from the origin o . The metric on $\partial\mathcal{T}_n$ is defined in the following way: Let $\omega, \omega' \in \partial\mathcal{T}_n$ be respectively represented by the chains ($o = x_0, x_1, \dots$) and ($o = x'_0, x'_1, \dots$).

- If the intersection of the supports of the chains is reduced to $\{o\}$, then declare the distance between ω and ω' to be equal to 1: $d(\omega, \omega') = 1$.

- If $x_i = x'_i$ for $i = 0, \dots, k$ and $x_{k+1} \neq x'_{k+1}$, then define $d(\omega, \omega') = \frac{n}{n+1}n^{-k}$.

It follows that a closed ball of radius $\frac{n}{n+1}n^{-k}$ is the set of all points of $\partial\mathcal{T}_n$ represented by chains containing a fixed finite chain ($o = x_0, x_1, \dots, x_k$), and that it is an open set. In fact, $\partial\mathcal{T}_n$ endowed with the metric d is a compact ultrametric space, homeomorphic to a Cantor set.

When p is prime, \mathcal{T}_p is the Bruhat-Tits building of the p -adic Lie group $SL_2(\mathbb{Q}_p)$, just as the Poincaré disk D is the symmetric space of the real group $SL_2(\mathbb{R})$. The boundary $\partial\mathcal{T}_p$, which can be identified with $\mathbb{Q}_p P^1$, the projective line on \mathbb{Q}_p , may thus be viewed as the p -adic analogue of the circle.

1.2. Let $\partial\mathcal{T}_n$ still denote the boundary of the tree \mathcal{T}_n , $n \geq 2$. The group of spheromorphisms N_n can be defined as the group of transformations of $\partial\mathcal{T}_n$ induced by “piecewise” tree automorphisms:

Take a finite subtree of \mathcal{T}_n . Its complementary has finitely many connected components L_1, \dots, L_k , called *branches*, all isomorphic to an infinite n -ary complete rooted tree. A subset ∂L of the boundary is

naturally associated to each branch L : it consists of all the ends represented by the chains running over this branch. The k disjoint sets ∂L_j , $j = 1, \dots, k$ cover the boundary. We call (L_1, \dots, L_k) a *broom*.

Remark. — Each ball for the metric d is of the form ∂L , and each ∂L is a finite union of balls. The family $\{\partial L : L \text{ branch}\}$ is a basis of closed-open sets for the topology defined by d .

Let (L_1, \dots, L_k) and (L'_1, \dots, L'_k) be two brooms of \mathcal{T}_n , σ a permutation of $\{1, \dots, k\}$. Let $\phi_j : L_j \rightarrow L'_{\sigma(j)}$ be a rooted tree isomorphism, $j = 1, \dots, k$. These k mappings induce a bijection $\phi = (\partial\phi_j : \partial L_j \rightarrow \partial L'_{\sigma(j)})_{j=1, \dots, k}$ of the boundary. Such a broom appearing in the definition of ϕ is called ϕ -*adapted*, and is obviously not uniquely associated to ϕ . It is clear that the set of all the ϕ 's defined by this procedure is a group of homeomorphisms of the boundary.

DEFINITION 1.1 (Spheromorphism group, [13]). — *For each $n \geq 2$, the set of all bijections $\phi = (\partial\phi_j : \partial L_j \rightarrow \partial L'_{\sigma(j)})_{j=1, \dots, k}$ of the boundary $\partial\mathcal{T}_n$ is the spheromorphism group of Neretin, and is denoted N_n .*

Remarks. — 1) In view of this description, the automorphism group $\text{Aut } \mathcal{T}_n$ of the tree embeds as a subgroup of N_n . The image of $\text{Aut } \mathcal{T}_n$ in N_n is the set of spheromorphisms which possess an adapted broom with two branches.

2) When p is a prime integer, $\partial\mathcal{T}_p$ is homeomorphic to $\mathbb{Q}_p P^1$, and N_p contains the group An_p of locally analytic bijections of $\mathbb{Q}_p P^1$ (see [13]).

2. Higman-Thompson groups.

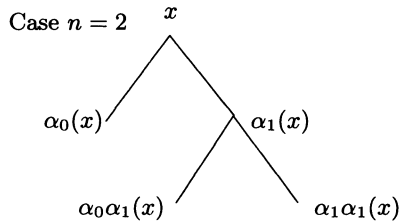
2.1. Definition of Higman-Thompson groups. In 1965, R.J. Thompson, interested in finitely presented groups with non-solvable word problem, introduced a group (denoted $G_{2,1}$ in the following) which happened to be the first known example of finitely generated infinite simple group [11]. Thompson's group was later generalized by G. Higman ([8]). For the description of the Higman-Thompson groups, we refer to [2]. See also [4].

Recall that a finite n -ary rooted planar tree is a finite tree T with root x realized in the oriented plane such that

– If T is not reduced to x , the valence of x is equal to n .

– The valence of a vertex $v \neq x$ is equal to 1 or $n+1$: if the valence of v is 1, we call v a leaf of the tree; if it is equal to $n+1$, v has n adjacent edges not contained in the geodesic joining the root x to v . We realize them by drawing them down from the vertex v . We order them from the left to the right and label their terminal vertices (opposite to v) $\alpha_0(v), \dots, \alpha_{n-1}(v)$.

The set of leaves of a finite n -ary rooted tree T is called a *basis* and is denoted B_T .



DEFINITION 2.1. — A simple expansion of a finite n -ary rooted tree T is any finite n -ary rooted tree T' obtained by the following procedure:

- Choose a vertex v in the base B_T .
- Make an expansion of v by drawing n edges down from it.

We get a new tree T' whose basis $B_{T'}$ is deduced from B_T by replacing v by $\alpha_0(v), \dots, \alpha_{n-1}(v)$.

An expansion T' of T is a tree obtained from T by making finitely many successive simple expansions. Any two trees T_1 and T_2 always possess a common expansion.

The elements of the Higman-Thompson groups will be represented by “symbols”:

DEFINITION 2.2 (symbols). — Consider a pair (T_1, T_2) of finite n -ary rooted trees with basis having the same cardinality. Let $\sigma : B_{T_1} \rightarrow B_{T_2}$ be a bijection from the basis of the first tree to the basis of the second one. We call the triple (T_1, T_2, σ) a symbol.

A simple expansion of a symbol (T_1, T_2, σ) is any symbol (T'_1, T'_2, σ') thus obtained:

- T'_1 is a simple expansion of T_1 , deduced from T_1 by expanding a vertex $v \in B_{T_1}$.
- Then T'_2 is the expansion of T_2 realized from the vertex $\sigma(v)$.

- $\sigma' : B_{T'_1} \rightarrow B_{T'_2}$ is defined by

$$\begin{aligned}\sigma'_{|B_{T'_1} \setminus \{v\}} &= \sigma_{|B_{T_1} \setminus \{v\}}, \\ \sigma'(\alpha_i(v)) &= \alpha_i(\sigma(v)), i = 0, \dots, n-1.\end{aligned}$$

An expansion (T'_1, T'_2, σ') of the symbol (T_1, T_2, σ) is obtained from the latter by making finitely many simple expansions.

Declare now that (T_1, T_2, σ) and (T'_1, T'_2, σ') are equivalent if they possess a common expansion.

All the necessary vocabulary has been introduced to set the following:

DEFINITION 2.3 (Higman-Thompson groups). — The set of equivalence classes of symbols $[(T_1, T_2, \sigma)]$ form a set G_n endowed with the following group structure:

Two elements $[(T_1, T, \sigma)]$ and $[(T', T_2, \sigma')]$ being given, at the price of making expansions of their representing symbols, it may be supposed that $T = T'$. Then $\sigma'\sigma : B_{T_1} \rightarrow B_{T_2}$ can be defined, and we set

$$[(T_1, T, \sigma)][(T, T_2, \sigma')] = [(T_1, T_2, \sigma'\sigma)],$$

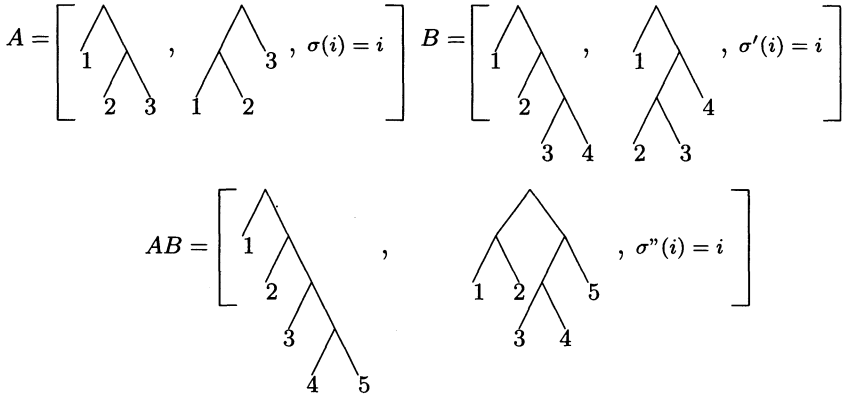
since it is easy to check that this definition is independent of the chosen symbols.

The neutral element is $[(T, T, \sigma = \text{id})]$ represented by any symbol $(T, T, \sigma = \text{id})$.

The inverse of $[(T_1, T_2, \sigma)]$ is $[(T_2, T_1, \sigma^{-1})]$.

The group G_n belongs to the family of Higman-Thompson groups.

Example ($n = 2$).



Recall that the leaves of a tree T (i.e. the vertices in B_T) are always labelled from the left to the right. Let (T, T', σ) be a symbol, and $\sigma : B_T = \{v_1, \dots, v_k\} \rightarrow B_{T'} = \{v'_1, \dots, v'_k\}$. There exists a unique permutation $\tau \in \mathcal{S}_k$ such that

$$\sigma(v_i) = v'_{\tau(i)} \quad \forall i = 1, \dots, k.$$

Then define $\theta(\sigma) = \epsilon(\tau)$ the signature of τ . An easy calculation shows that if $(\tilde{T}, \tilde{T}', \tilde{\sigma})$ is a simple expansion of the symbol (T, T', σ) , then

$$\theta(\tilde{\sigma}) = \theta(\sigma)(-1)^{n-1},$$

so that when n is an odd integer, $\theta(\sigma)$ is independent of the chosen symbol, and we get the group epimorphism

$$\begin{aligned} \theta : G_n &\rightarrow \mathbb{Z}/2\mathbb{Z} \\ \theta([(T, T', \sigma)]) &= \epsilon(\tau). \end{aligned}$$

Generalization. Let $r \geq 1$ be a fixed integer. First consider pairs of r -uplets of finite n -ary rooted trees $((T_1, \dots, T_r), (T'_1, \dots, T'_r))$, and bijections σ from $B_{T_1} \cup \dots \cup B_{T_r}$ to $B_{T'_1} \cup \dots \cup B_{T'_r}$ (We do not ask σ to map B_{T_i} onto $B_{T'_i}$). We always suppose the r -uplet of trees to be ordered from the left (T_1) to the right (T_r). Any triple $((T_1, \dots, T_r), (T'_1, \dots, T'_r), \sigma)$ is called an r -symbol. Similarly to the case $r = 1$, we define the group $G_{n,r}$ where the elements are represented by r -symbols. Of course, $G_{n,1} = G_n$.

As in the case $r = 1$, the morphism $\theta : G_{n,r} \rightarrow \mathbb{Z}/2\mathbb{Z}$ can be defined provided n is odd. We set $G'_{n,r} = \text{Ker } \theta$. If n is even, we agree that $G'_{n,r} = G_{n,r}$. We are now ready to cite the simplicity theorem:

THEOREM 2.1 ([2]). — *The group $G'_{n,r}$ is the commutator subgroup of $G_{n,r}$, and every non-trivial subgroup normalized by $G'_{n,r}$ contains it. In particular, $G_{n,r}$ is simple if n is even, and if n is odd, $G_{n,r}$ contains a simple group of index 2, namely $G'_{n,r} = [G_{n,r}, G_{n,r}]$.*

2.2. Embedding of $G_{n,1} = G_n$ and $G_{n,2}$ into the Neretin group N_n . The finite n -ary rooted trees we used in the definition of the Higman-Thompson groups may be canonically embedded in a chosen branch L of the regular tree \mathcal{T}_n , by simply completing the finite tree to an infinite n -ary rooted tree and then, identifying it to the branch L . Denote by L' the branch opposite to L in \mathcal{T}_n (linked to L by an edge). Each $g \in G_{n,1}$, defined by a symbol (T_1, T_2, σ) , induces a spheromorphism \tilde{g} in an obvious way: if (v_i^1) (resp. (v_i^2)) are the leaves of T_1 (resp. T_2), denote by L_i^1 (resp. L_i^2) the subbranch of L whose root is v_i^1 (resp. v_i^2). Then \tilde{g} is induced on ∂L by the collection $(L_i^1 \xrightarrow{\cong} L_{\sigma_i}^2)_i$, the isomorphisms respecting the left-to-right order of the edges of the branches. On $\partial L'$, one imposes $\tilde{g}|_{\partial L'} = \text{id}|_{\partial L'}$. The embedding

$$G_{n,1} \hookrightarrow N_n$$

is now obtained.

On the other hand, we need the two branches L and L' like above to realize $G_{n,2}$ in N_n . Each $g \in G_{n,2}$ will induce a spheromorphism by a procedure analogous to the previous one. It will appear in the following that, as far as we are concerned with the Neretin group N_n , $G_{n,2}$ is more relevant than the group $G_{n,1} = G_n$ itself.

3. The group $\text{Aut } \mathcal{T}_n$ of automorphisms of the tree \mathcal{T}_n , $n \geq 2$.

3.1. Simplicity theorem. In [15], the author gave a theorem of simplicity of a class of groups of automorphisms of a tree:

DEFINITION 3.1. — *Let A be a tree, G be a group of automorphisms of A , C be a (finite or infinite) chain of A , and F the fixator of C in G . For each vertex x of A , let $\pi(x)$ be the nearest vertex from x in C . For each vertex s of C , the set $\pi^{-1}(s)$ (which constitutes a subtree of A) is invariant under the action of F ; denote by F_s the group of permutations of this set induced by F . There is a natural homomorphism*

$$(1) \quad F \longrightarrow \prod_{s \in \text{Vert}(C)} F_s,$$

where $\text{Vert}(C)$ denotes the set of vertices of C .

We say that the group G possesses the property (P) if the homomorphism (1) is an isomorphism for all chains C (i.e. the actions of F on the sets $\pi^{-1}(s)$ are independent from each other).

For example the group of all automorphisms of A possesses the property (P) .

THEOREM 3.1 (J. Tits). — *Let A be a tree, G be a group of automorphisms of A , and G^+ be the subgroup generated by the stabilizers of the edges of A in G . Suppose that G possesses the property (P) , conserves no proper non-empty subtree of A and fixes no end of A . Then each subgroup of G normalized by G^+ and not reduced to the identity contains G^+ . In particular, G^+ is a simple group or is reduced to the identity.*

Example 1. — $A = \mathcal{T}_n, n \geq 2, G = \text{Aut } \mathcal{T}_n$. It happens that $G^+ = \text{Aut}^+ \mathcal{T}_n$ coincides with the group of type-preserving automorphisms of the tree. So $\text{Aut}^+ \mathcal{T}_n$ is a simple group, of index 2 in $\text{Aut } \mathcal{T}_n$.

Example 2. — Equipped Bruhat-Tits trees.

Let $p \geq 2$ be a prime integer. In [13], the author defines an *equipment* on the tree \mathcal{T}_p as the specification, for each vertex v , of a labelling of its adjacent edges $(l^v_0, \dots, l^v_{p-1}, l^v_\infty)$ by the points of $\mathbb{F}_p P^1$. If v and v' are linked by an edge $l = l^v_i = l^{v'}_j$, there is no reason that $i = j$.

We denote by $\widetilde{\mathcal{T}}_p$ such an equipped tree, and define the subgroup $\text{Aut } \widetilde{\mathcal{T}}_p$ of $\text{Aut } \mathcal{T}_p$ as the set of tree automorphisms such that their restrictions to the adjacent edges of a vertex belong to $PSL_2(\mathbb{F}_p)$. Since $\text{Aut } \widetilde{\mathcal{T}}_p$ obviously satisfies property (P) , conserves no proper non-empty subtree of \mathcal{T}_p and fixes no end, the group $(\text{Aut } \widetilde{\mathcal{T}}_p)^+$ is simple.

Two equipped trees $\widetilde{\mathcal{T}}_p^1$ and $\widetilde{\mathcal{T}}_p^2$ being given, we use the transitivity of $SL_2(\mathbb{F}_p)$ on $\mathbb{F}_p P^1$ to construct a tree isomorphism $\widetilde{\mathcal{T}}_p^1 \rightarrow \widetilde{\mathcal{T}}_p^2$ respecting the equipments. Such an isomorphism conjugates $\text{Aut } \widetilde{\mathcal{T}}_p^1$ and $\text{Aut } \widetilde{\mathcal{T}}_p^2$.

3.2. A family of subgroups of N_n .

DEFINITION 3.2. — *If G is a subgroup of $\text{Aut } \mathcal{T}_n$ we define*

$$(N_n)_G := \langle G_{n,2}, G^+ \rangle$$

the subgroup of N_n generated by $G_{n,2}$ and G^+ .

Example 1. — If $G = \text{Aut } \mathcal{T}_n, (N_n)_G = N_n$. In this case, we can even show:

PROPOSITION 3.1. — *The subgroups $[G_n, G_n]$ and $\text{Aut}^+ \mathcal{T}_n$ of the group $N_n, n \geq 2$, generate the group N_n .*

Proof. — Let us denote by L the chosen branch of the tree \mathcal{T}_n where we realized the Higman-Thompson group G_n . If L' is the branch opposite to L (i.e., linked with L by an edge), then the boundaries of L and L' partition the whole boundary of the tree: $\partial L \cup \partial L' = \partial \mathcal{T}_n$.

First case. — Suppose that $\phi \in N_n$ possesses a broom $(L_i)_{i=1, \dots, l}$ such that $\phi|_{\partial L_1} = \text{id}|_{\partial L_1}$. At the price of making an expansion of L_1 , one can suppose that L_1 and L' have the same type (i.e. their roots have the same type). Then there exists $k \in \text{Aut}^+ \mathcal{T}_n$ such that $k(L') = L_1$. So $k^{-1}\phi k|_{\partial L'} = \text{id}|_{\partial L'}$. Let us now consider $k^{-1}\phi k|_{\partial L}$. It may be seen as the composite

$$\partial L \xrightarrow{\tau} \partial L \xrightarrow{\sigma} \partial L$$

with $\tau \in G_n$ and $\sigma \in \text{Aut}^+ \mathcal{T}_n, \sigma|_{L'} = \text{id}|_{L'}$. Then on the whole boundary $\partial \mathcal{T}_n, k^{-1}\phi k = \sigma\tau$.

When n is odd, $\text{Aut}^+ \mathcal{T}_n \cap (G_n \setminus [G_n, G_n]) \neq \emptyset$, so that it can be supposed that $\tau \in [G_n, G_n]$.

Second case: general case. — (a) Suppose there exists L_i in the broom adapted to ϕ such that ∂L_i and $\phi(\partial L_i) = \partial L'_i$ have the same type. Then there exists $k \in \text{Aut}^+ \mathcal{T}_n$ such that $k\phi(\partial L_i) = \partial L_i$ and $k \circ \phi|_{\partial L_i} = \text{id}|_{\partial L_i}$. The first case enables to conclude.

(b) If not, for all i , the types of ∂L_i and $\phi(\partial L_i)$ are opposite. Then we use an element τ_0 of G_n (it is possible to find it of the form $[\tau_1, \tau_2]$) such that for some branch $L_0, \tau_0(L_0)$ and L_0 have opposite types. At the price of making an expansion of L_1 to make $\phi(\partial L_1)$ and ∂L_0 have the same type, there exists some $k \in \text{Aut}^+ \mathcal{T}_n$ such that $k\phi(\partial L_1) = \partial L_0$. The types of L_1 and L_0 are still opposite. Then $\tau_0 k \phi(\partial L_1) = \tau_0(\partial L_0) = \partial L'_0$, and the types of L_1 and L'_0 coincide. Hence $\tau_0 k \phi$ satisfies the condition of case (a).

It follows that ϕ may be written as a product of elements of G_n and $\text{Aut}^+ \mathcal{T}_n$.

Example 2. — Now p is a prime integer. Let $\tilde{\mathcal{T}}_p$ be any equipment on the tree \mathcal{T}_p such that the elements of $G_{p,2}$ are induced by piecewise tree

automorphisms of $\text{Aut } \widetilde{\mathcal{T}}_p$ (cf. §3.1, Example 2).

If $G = \text{Aut } \widetilde{\mathcal{T}}_p$, then we claim that $(N_p)_G$ is the group denoted $\text{Diff}^+(\widetilde{\mathcal{T}}_p)$ in [13]:

$$\begin{aligned} \text{Diff}^+(\widetilde{\mathcal{T}}_p) &= \{ \phi = (\phi_j : L_j \rightarrow L'_j)_j, \\ &\quad \phi_j = \text{restriction of some element of } \text{Aut } \widetilde{\mathcal{T}}_p \}. \end{aligned}$$

Indeed, $\text{Diff}^+(\widetilde{\mathcal{T}}_p)$ contains G , and because of the condition on the equipment, it contains $G_{p,2}$. So, $\langle G, G_{p,2} \rangle \subset \text{Diff}^+(\widetilde{\mathcal{T}}_p)$. On the other hand, every $\phi \in \text{Diff}^+(\widetilde{\mathcal{T}}_p)$ can be written $\phi = \psi \circ \tau$, where $\tau = (L_j \rightarrow L'_j)_j$ belongs to $G_{p,2}$, and $\psi = (\psi_j = L'_j \rightarrow L'_j)_j$, with ψ_j induced by some element of G , which can be modified to be supported in the branch L'_j . It follows that $\psi_j \in G^+$, and $\psi = \prod_j \psi_j \in G^+$. Thus

$$\langle G, G_{p,2} \rangle \subset \text{Diff}^+(\widetilde{\mathcal{T}}_p) \subset \langle G^+, G_{p,2} \rangle,$$

and the inclusions are equalities. Then $(N_p)_{\text{Aut } \widetilde{\mathcal{T}}_p} = \text{Diff}^+(\widetilde{\mathcal{T}}_p)$ as claimed.

Remarks. — 1) Any isomorphism of equipped trees $\widetilde{\mathcal{T}}_p \rightarrow \widetilde{\mathcal{T}}_p'$ conjugates $\text{Diff}^+(\widetilde{\mathcal{T}}_p)$ and $\text{Diff}^+(\widetilde{\mathcal{T}}_p')$.

2) If $p = 2$, the group $PSL_2(\mathbb{F}_2)$ is the full symmetric group \mathcal{S}_3 , so that $\text{Diff}^+(\widetilde{\mathcal{T}}_2) = N_2$.

4. Simplicity of $(N_p)_G$.

We now give the main theorem of the article, valid for any integer $p \geq 2$:

THEOREM 4.1. — *Let G be a subgroup of $\text{Aut } \mathcal{T}_p$ such that*

1. G^+ is simple (e.g. G satisfies the conditions of Theorem 3.1),
2. If p is odd, $G^+ \cap (G_{p,2} \setminus [G_{p,2}, G_{p,2}])$ is non-empty,
3. G^+ possesses two non-commuting elements supported in a branch of the tree.

Then the group $(N_p)_G$ is simple.

Condition 2. implies that $(N_p)_G$ is generated by G^+ and $G_{p,2}' = [G_{p,2}, G_{p,2}]$, since $G_{p,2}$ is generated by $G_{p,2}'$ together with any element in $G_{p,2} \setminus G_{p,2}'$.

COROLLARY 4.1. — For each integer $p \geq 2$, the group N_p of all spheromorphisms is simple.

For each prime number $p \geq 3$ and for any choice of equipment of the tree \mathcal{T}_p , the commutator subgroup $[\text{Diff}^+(\widetilde{\mathcal{T}}_p), \text{Diff}^+(\widetilde{\mathcal{T}}_p)]$ is simple, and there is a short exact sequence

$$1 \rightarrow [\text{Diff}^+(\widetilde{\mathcal{T}}_p), \text{Diff}^+(\widetilde{\mathcal{T}}_p)] \longrightarrow \text{Diff}^+(\widetilde{\mathcal{T}}_p) \xrightarrow{\tilde{\theta}} \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$

In other words, $H_1(\text{Diff}^+(\widetilde{\mathcal{T}}_p), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof of Corollary 4.1. — $G = \text{Aut } \mathcal{T}_p$ obviously satisfies all the conditions of the theorem above.

As for the statements about $\text{Diff}^+(\widetilde{\mathcal{T}}_p)$, they can be proven by using a particular equipment, since for different equipments the groups are conjugated. So, remembering that \mathcal{T}_p is obtained by gluing by an edge the two branches L and L' appearing in the definition of $G_{p,2}$, define the equipment $\widetilde{\mathcal{T}}_p^0$ in the following way: label the p edges drawn down from a vertex from 0 (on the left) to $p - 1$ (on the right), whereas the edge pointing towards the root of the branch (L or L') is labelled ∞ . Then setting $G = \text{Aut } \widetilde{\mathcal{T}}_p^0$, we have $(N_p)_G = \text{Diff}^+(\widetilde{\mathcal{T}}_p^0)$ (cf. §3.2 Example 2). But condition 2 of Theorem 4.1 fails for such G . We recalled in Section 2 that when p is odd, there is an epimorphism

$$\theta : G_{p,2} \rightarrow \mathbb{Z}/2\mathbb{Z}$$

whose kernel is the simple group $[G_{p,2}, G_{p,2}]$. It happens that θ may be extended to the group $\text{Diff}^+(\widetilde{\mathcal{T}}_p^0)$: if $\phi = (\phi_j : L_j \rightarrow L'_{\sigma(j)})_j$, where the indices of the branches label their roots from the left to the right (suppose the branches involved to be subbranches of L or L'), $\tilde{\theta}(\phi)$ will be the signature of σ . Indeed, if we refine some branch L_j into $L_{j_0} \cup L_{j_1} \cup \dots \cup L_{j_{p-1}}$, then ϕ_j induces

$$\phi_{j_i} : L_{j_i} \rightarrow L'_{\sigma(j)_{k_i}} \quad i = 0, 1, \dots, p - 1,$$

with $i \in \mathbb{F}_p \rightarrow k_i \in \mathbb{F}_p$ in $B \subset \text{PSL}_2(\mathbb{F}_p)$, the stabilizer of ∞ . Since B lies in the alternating group \mathcal{A}_p on a set with p elements, the permutation deduced from σ has the same signature as in the case $k_i = i \forall i \in \mathbb{F}_p$. But then we saw (cf. §2) that, since p is odd, the signature remains unchanged. So

$$\tilde{\theta} : \text{Diff}^+(\widetilde{\mathcal{T}}_p^0) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

is a well-defined homomorphism.

It is clear that the kernel of $\tilde{\theta}$ is generated by $[G_{p,2}, G_{p,2}]$ and $(\text{Aut } \widetilde{\mathcal{T}}_p^0)^+$, and the proof of the theorem will show that this group is simple. Now the kernel contains the commutator subgroup $[\text{Diff}^+(\widetilde{\mathcal{T}}_p^0), \text{Diff}^+(\widetilde{\mathcal{T}}_p^0)]$, which is normal and non-trivial, consequently it coincides with the kernel.

Proof of Theorem 4.1. — Let $H \triangleleft (N_p)_G$ be a non-trivial normal subgroup of $(N_p)_G$. Then $H \cap G^+$ is normal in G^+ and $H \cap [G_{p,2}, G_{p,2}]$ is normal in $[G_{p,2}, G_{p,2}]$. Hence either $H \supset G^+$ or $H \cap G^+ = \{\text{id}\}$, and either $H \supset [G_{p,2}, G_{p,2}]$ or $H \cap [G_{p,2}, G_{p,2}] = \{\text{id}\}$.

So we will prove that the cases $H \cap G^+ = \{\text{id}\}$ and $H \cap [G_{p,2}, G_{p,2}] = \{\text{id}\}$ do not occur. We will use some arguments of a theorem of Epstein ([5] and [1]):

THEOREM 4.2 (Epstein, 1970). — *Let X be a paracompact Hausdorff topological space, Γ a group of homeomorphisms of X , and \mathcal{U} a basis of open sets for the topology of X . The Epstein axioms for the triple (X, Γ, \mathcal{U}) are:*

1. Axiom 1: *If $U \in \mathcal{U}$ and $g \in \Gamma$, then $gU \in \mathcal{U}$.*
2. Axiom 2: *Γ acts transitively on \mathcal{U} .*
3. Axiom 3: *Let $g \in \Gamma$, $U \in \mathcal{U}$ and \mathcal{B} an open covering of X ; then there exists an integer n and $g_1, \dots, g_n \in \Gamma$ and $V_1, \dots, V_n \in \mathcal{B}$ such that*

- (i) $g = g_n g_{n-1} \dots g_1$,
- (ii) $\text{supp}(g_i) \subset V_i$,
- (iii) $\text{supp}(g_i) \cup (g_{i-1} \dots g_1 \bar{U}) \neq X, 1 \leq i \leq n$.

Suppose the triple (X, Γ, \mathcal{U}) as above satisfies the Epstein axioms. Then if H is a non-trivial subgroup of Γ that is normalized by $[\Gamma, \Gamma]$, then $[\Gamma, \Gamma] \subset H$. In particular, the group $[\Gamma, \Gamma]$ is simple.

The simplicity of $[\text{Diff}^+(S^1), \text{Diff}^+(S^1)]$ was an easy corollary of this theorem. M.R. Herman finally proved $\text{Diff}^+(S^1)$ was perfect, hence simple ([7]). For more details, we suggest the reader to refer to the very interesting book [1].

In the case of a non-connected topological space and a non trivial group Γ , axiom 3 can never be satisfied (see [5]). Consequently, we will not be able to use the preceding theorem directly to prove the simplicity of $(N_p)_G$. However, setting $X = \partial \mathcal{T}_p$, $\mathcal{U} = \{\partial L : L \text{ branch of } \mathcal{T}_p\}$ and

$\Gamma = (N_p)_G$, it is easy to see that the triple $(\partial\mathcal{T}_p, (N_p)_G, \mathcal{U})$ satisfies axiom 2 and a

“modified axiom 1”: If $U \in \mathcal{U}$ and $g \in \Gamma$, then there exists $U' \in \mathcal{U}$, $U' \subset U$, such that $gU' \in \mathcal{U}$.

Then we can show that two lemmas, which are steps in the proof of the Epstein theorem, still hold in our case:

LEMMA 4.1 (from 1.4.2 in [5], or Lemma 2.2.5 in [1]). — Let (X, Γ, \mathcal{U}) be a triple satisfying the modified axiom 1 and axiom 2. Let $V_0 \in \mathcal{U}$ and $h \in \Gamma$ with $\text{supp } h \subset V_0$, and suppose that $H \triangleleft \Gamma$ is a non-trivial normal subgroup of Γ . Then there exists some $\rho \in H$ such that $\rho|_{V_0} = h|_{V_0}$.

Proof. — Choose any $\alpha \in H$ with $\alpha \neq \text{id}$, and find $x \in X$ such that $\alpha(x) \neq x$. Choose a small neighborhood $U \in \mathcal{U}$ of x such that $U \cap \alpha^{-1}(U) = \emptyset$. Next, take $V, W \in \mathcal{U}$ such that $V \cap W = \emptyset$, $\bar{V} \cup \bar{W} \subset U$, $x \in V$. Suppose first that $V_0 = V$. By axiom 2, there exists $g \in \Gamma$ with $gW = V$. Define

$$\rho = [\alpha, [g, h]] = \alpha^{-1}[g, h]^{-1}\alpha[g, h].$$

Then $\rho \in \Gamma$ since $H \triangleleft \Gamma$. We can verify that

$$\rho = \begin{cases} h & \text{on } V, \\ g^{-1}h^{-1}g & \text{on } W, \\ \alpha^{-1}h\alpha & \text{on } \alpha^{-1}V, \\ \alpha^{-1}g^{-1}h^{-1}g\alpha & \text{on } \alpha^{-1}W, \\ \text{id} & \text{elsewhere.} \end{cases}$$

Now if $V_0 \neq V$, choose $k \in \Gamma$ (by axiom 2) such that $k(V) = V_0$. Then $\text{supp } k^{-1}hk = k^{-1}(\text{supp } h) \subset V$, and by the previous case, there exists $\rho \in H$ such that $k^{-1}hk|_V = \rho|_V$, so that $h|_{V_0} = k\rho k^{-1}$. Since $k\rho k^{-1} \in H$, the proof is done.

LEMMA 4.2 (variation of 1.4.6 in [5] or Lemma 2.2.7 in [1]). — Γ still satisfies the modified axiom 1 and axiom 2. Moreover, it is supposed 2-transitive:

$$\forall(x_1, x_2), \forall(y_1, y_2), x_1 \neq x_2 \text{ and } y_1 \neq y_2 \Rightarrow \exists \phi \in \Gamma \phi(x_i) = y_i, i = 1, 2.$$

Let $h_1, h_2 \in \Gamma$ be such that there exists $V_0 \in \mathcal{U}$ with $\text{supp } h_i \subset V_0, i = 1, 2$. Then $[h_1, h_2]$ belongs to H .

Proof. — Let x be in X . There exist α_1, α_2 in H such that $x, \alpha_1^{-1}(x)$ and $\alpha_2^{-1}(x)$ are pairwise distinct. Indeed, since $\alpha \neq \text{id} \in H$,

there exists some $x \in X$ with $\alpha(x) \neq x$. So, in a neighborhood of x there exists $y \neq x$ such that $\alpha(y) \neq y$. Now one can find $\phi \in \Gamma$ with $\phi(x) = y$ and $\phi^{-1}\alpha\phi(x) \neq \alpha(x)$ (which is equivalent to $\alpha(y) \neq \phi\alpha(x)$). As for the condition $\alpha(y) \neq y$, it is equivalent to $\phi^{-1}\alpha\phi(x) \neq x$. Then one sets $\alpha_1^{-1} = \alpha$, $\alpha_2^{-1} = \phi^{-1}\alpha\phi$. So α_1 and α_2 belong to H , x , $\alpha_1^{-1}(x)$ and $\alpha_2^{-1}(x)$ are pairwise distinct. Then choose $U \in \mathcal{U}$ a neighborhood of x such that U , $\alpha_1^{-1}(U)$ and $\alpha_2^{-1}(U)$ are pairwise disjoint. One can also find g_1, g_2 in Γ , and a neighborhood $V \in \mathcal{U}$ of x such that V , $g_1^{-1}(V)$ and $g_2^{-1}(V)$ are pairwise disjoint and included in U . Suppose first that $\text{supp } h_i \subset V$, $i = 1, 2$. Then apply the previous lemma to $(\alpha_i, g_i, h_i, V, W_i = g_i^{-1}V)$, $i = 1, 2$. One gets $\rho_{i|V} = h_{i|V}$. The support of ρ_i is included in $V \cup g_i^{-1}(V) \cup \alpha_i^{-1}(V) \cup \alpha_i^{-1}g_i^{-1}(V)$. The seven sets involved are disjoint, so that

$$[h_1, h_2] = [\rho_1, \rho_2].$$

To conclude, we may assume $V = V_0$, at the price of making some conjugation.

End of the proof of Theorem 4.1. — Choose $V_0 = \partial L_0$ where L_0 is some branch of the tree, and by condition 3, find two non-commuting elements h_1 and h_2 in G^+ with supports in ∂L_0 . Apply Lemma 4.2 to $\Gamma = (N_p)_G$, which is 2-transitive on $\partial \mathcal{T}_p$, since $G_{p,2}$ itself is 2-transitive. Then $[h_1, h_2] \in G^+ \cap H$, so $H \supset G^+$.

Similarly, choose two non-commuting elements h'_1 and h'_2 in $G_p = G_{p,1} \subset G_{p,2}$ (they are supported in a branch), so that $[h'_1, h'_2] \in [G_p, G_p] \cap H \subset [G_{p,2}, G_{p,2}] \cap H$, and $H \supset [G_{p,2}, G_{p,2}]$. Finally, H contains two groups that generate $(N_p)_G$, so $H = (N_p)_G$.

5. Concluding remarks.

The question of the simplicity of the group N_n is a preamble of a series of homological problems. First the result implies $H_1(N_n, \mathbb{Z}) = 0$. As for the second homology group $H_2(N_n, \mathbb{Z})$, though its complete computation could not be achieved (because the group N_n is very huge), we know it is non trivial. Indeed, the group N_n possesses a non-trivial central extension by $\mathbb{Z}/2\mathbb{Z}$, called the “Central Geometric Extension” in [9] and [10], a sort of analogue of the Bott-Virasoro extension of $\text{Diff}^+(S^1)$.

On the other hand, K. Brown proved that the groups G_n are all \mathbb{Q} -acyclic, i.e. $H_i(G_n, \mathbb{Q}) = 0$ for all $i > 0$ (cf. [3]). By using a description

of N_n as the automorphism group of a free object of some appropriate category, it becomes possible to define an N_n -simplicial complex, and to use it to prove the \mathbb{Q} -acyclicity of N_n (cf. [9] and [10]).

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Christophe KAPOUDJIAN,
Université Claude Bernard Lyon-I
Institut Girard Desargues – UPRES-A 5028 du CNRS
43, boulevard du 11 novembre 1918
69622 Villeurbanne Cedex (France).
ckapoudj@desargues.univ-lyon1.fr