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WELLEDA BALDONI
PIERLUIGI MÖSENEDER FRAJRIA

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THE HARISH-CHANDRA HOMOMORPHISM FOR A QUANTIZED CLASSICAL HERMITIAN SYMMETRIC PAIR

by W. BALDONI and P. MÖSENEDER FRAJRIA

1. Introduction.

If G/K is a noncompact symmetric space and KAN is an Iwasawa decomposition of G then the Harish-Chandra homomorphism is an explicit homomorphism between the algebra $D(G/K)$ of invariant differential operators on G/K and the algebra of polynomials on A that are invariant for the Weyl group $\mathcal{W}_{\mathbb{R}}$ of the pair (G, A).

If G/K is an hermitian symmetric space then the complexification $\mathfrak{k}$ of the Lie algebra of K is a Levi component of a parabolic subalgebra of the complexification $\mathfrak{g}$ of the Lie algebra of G. This implies that there is a natural analog $(\mathbf{U}, \mathbf{K})$ of the pair (G, K) with $\mathbf{U}$ and $\mathbf{K}$ quantized enveloping algebras. In this quantized setting there is also an analog $D_q(G/K)$ of the algebra $D(G/K)$.

The main result of this paper (Theorem 3.3) is the construction of an isomorphism between the algebra $D_q(G/K)$ and an algebra of Laurent polynomials that are invariant for the group $\mathcal{W}_{\mathbb{R}}$. We feel that ours is the correct generalization to the quantum group setting of the Harish-Chandra homomorphism.

The major difficulty in generalizing the classical result comes from the fact that there is no obvious analog of the group A, we solve this problem by

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substituting the algebra of polynomials on A with an algebra of functions on the lattice of integral weights of the system of restricted roots. Then we identify an element X of $D_q(G/K)$ with the function F_X defined by setting $F_X(\lambda)$ to be $\Phi_\lambda(X)$ where Φ_λ is a spherical function. This identification followed by an appropriate ρ-shift gives the desired isomorphism.

To carry out this program we need to compute precisely the values $\Phi_\lambda(X)$ of spherical functions and we are able to do so only when X is a central element of $\mathbf{U}$. For this reason we can not prove our result in full generality, but only for the classical cases.

2. Notations.

Let G/K be an irreducible hermitian symmetric space of the noncompact type. Let $T \subset K$ be a torus and set $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{t}_0$ to be the Lie algebras of G, K, and T respectively. Set $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{t}$ to be the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{t}_0$. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and we denote by R the root system of $(\mathfrak{g}, \mathfrak{t})$. This is a root system in the euclidean real space $(E, (\ ,\ ))$, where E is the real dual of $\sqrt{-1}\mathfrak{t}_0$ and $(\ ,\ )$ is any (nonzero) multiple of the form induced by the Killing form.

Since G/K is hermitian symmetric, it is well known that we can find a positive system R^+ for R such that, if $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is the corresponding set of simple roots, then there is a simple root α_{i_0} such that the Lie algebra $[\mathfrak{k}, \mathfrak{k}]$ is the algebra corresponding to the subsystem R_c of R generated by $\Pi \backslash \{\alpha_{i_0}\}$ and $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}] + \mathfrak{t}$. In other words the algebra $\mathfrak{k}$ is the Levi component of a parabolic subalgebra of $\mathfrak{g}$. Moreover, if $\tilde{\alpha}$ is the highest root in R^+ and we write

$$\tilde{\alpha} = \sum_j n_j \alpha_j,$$

then $n_{i_0} = 1$. We assume that such a positive system has been chosen once and for all.

We denote by $Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$ the root lattice for R, and we let P denote the lattice of integral weights. We write P^+ for the set of dominant integral weights for R with respect to R^+. We also set $R_c^+ = R^+ \cap R_c$ and Q_c the root lattice for R_c.

We now describe the quantum analog of the pair (G, K): let $\mathbb{F}$ be a field and assume that $\operatorname{char}\mathbb{F} \neq 2$. Fix $q \in \mathbb{F}$ and assume that q is **not** a root of unity.

We normalize $(\ , \ )$ so that $d_i = (1/2)(\alpha_i, \alpha_i) \in \mathbb{N}$ and the d_i are relatively prime. Set $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$, and let A denote the matrix (a_{ij}). We let $\mathbf{U}$ denote the quantized enveloping algebra associated to the matrix A. This is the algebra over $\mathbb{F}$ with generators E_i, F_i, K_i and K_i^{-1} $(i = 1, \ldots, n)$ satisfying the relations described in §1.1 of [12]. We denote by $\mathbf{U}^+$, $\mathbf{U}^-$, and $\mathbf{U}^0$ the subalgebras of $\mathbf{U}$ generated by E_i, F_i, and K_i, K_i^{-1} respectively. We set $\mathbf{U}^{\geqslant}$ (resp. $\mathbf{U}^{\leqslant}$) to be the Hopf subalgebra of $\mathbf{U}$ generated by the elements E_i, $K_i^{\pm}$ (F_i, $K_i^{\pm}$).

As it is well known, $\mathbf{U}$ is also a Hopf algebra and we let Δ denote the comultiplication, S the antipode, and ϵ the counit. We recall that ϵ is the homomorphism from $\mathbf{U}$ to $\mathbb{F}$ defined by

$$\epsilon(E_i) = \epsilon(F_i) = 0, \quad \epsilon(K_i) = \epsilon(K_i^{-1}) = 1$$

and we set $\mathcal{A} = \operatorname{Ker}\epsilon$. For the precise definition of the comultiplication and the antipode we refer again to §1.1 of [12].

If $X \in \mathbf{U}$, then $\Delta(X) = \sum_i X_{1i} \otimes X_{2i}$. We write for short

$$\Delta(X) = \sum X_{(1)} \otimes X_{(2)}.$$

We let Ad denote the adjoint action of $\mathbf{U}$ on itself. This is defined as follows: if $X \in \mathbf{U}$ and $\Delta(X) = \sum X_{(1)} \otimes X_{(2)}$, then

$$\operatorname{Ad}(X)(Y) = \sum X_{(1)} Y S(X_{(2)}).$$

If $\mathbf{M}$ is a Hopf subalgebra of $\mathbf{U}$, then $\mathbf{M}$ acts via the adjoint action on $\mathbf{U}$; this action will still be denoted by Ad.

Set $\mathbf{K}$ to be the subalgebra of $\mathbf{U}$ generated by *all* the $K_r^{\pm}$ and by E_r, F_r with $r \neq i_0$. We note that, in the classical limit $q = 1$, the algebra $\mathbf{K}$ turns out to be the enveloping algebra of $\mathfrak{k}$. As already observed the algebra $\mathfrak{k}$ is the Levi component of a parabolic subalgebra. An analogous situation holds in the quantum case: as shown in [§2] we can find a subalgebra $\mathbf{U}_n$ of $\mathbf{U}^+$ such that $\mathbf{P} = \mathbf{K}\mathbf{U}_n$ is a subalgebra of $\mathbf{U}$ and $\mathbf{U}_n$ is $\operatorname{Ad}(\mathbf{K})$-stable. We recall that the algebra $\mathbf{U}_n$ is the algebra generated by the root vectors X_α with $\alpha \in R^+ \backslash R_c^+$ provided that a convex ordering for R^+ is carefully chosen.

Let τ be the involutive antihomomorphism of $\mathbf{U}$ defined by

$$\tau(E_i) = -F_iK_i, \quad \tau(F_i) = -K_i^{-1}E_i, \quad \tau(K_i) = K_i. \tag{2.1}$$

It is easy to check (see [1, Lemma 3.1]) that $\tau(\mathbf{U}_n)$ is $\mathrm{Ad}(\mathbf{K})$-stable. The PBW theorem for quantized enveloping algebras implies that the multiplication

$$m : \mathbf{U}_n \otimes \tau(\mathbf{U}_n) \otimes \mathbf{K} \to \mathbf{U} \tag{2.2}$$

is an isomorphism of vector spaces (see [§3] for the details).

If M is a $\mathbf{U}^0$–module and $\mu \in P$, then we define

$$M_\mu = \{v \in M \mid K_i v = q^{(\alpha_i,\mu)}v \text{ for all } i\}.$$

If $v \in M_\mu$, we say that v has weight μ and that M_μ is the weight space of M of weight μ. As a particular case we consider the action of $\mathbf{U}^0$ on $\mathbf{U}$ via Ad: in this case $\mathbf{U}_\mu \neq \{0\}$ if and only if $\mu \in Q$ and

$$\mathbf{U} = \bigoplus_{\mu\in Q} \mathbf{U}_\mu.$$

If $X \in \mathbf{U}$ has weight μ we write for short $\lambda(X) = \mu$.

In this paper all $\mathbf{U}$-modules are meant to be finite dimensional and of type $\mathbf{1}$. We recall that a $\mathbf{U}$-module is said to be of type $\mathbf{1}$ if it is the direct sum of its weight spaces.

If $\lambda \in P^+$ then we denote by $V(\lambda)$ the irreducible finite dimensional $\mathbf{U}$-module of highest weight λ as defined in [9]. We denote by P_c^+ the set of weights in P that are dominant for $\Pi\backslash\{\alpha_{i_0}\}$. If $\lambda \in P_c^+$ let $V_c(\lambda)$ be the finite dimensional representation of $\mathbf{K}$ with highest weight λ.

If V is a $\mathbf{U}$-module, $v \in V$ and $f \in V^*$, then we denote by $c_{f,v}$ the matrix coefficient defined by

$$c_{f,v}(X) = f(Xv) \quad X \in \mathbf{U}.$$

Let $\mathbb{F}_q(G)$ denote the subalgebra of $\mathbf{U}^*$ generated by the matrix coefficients of all the representations $V(\lambda)$ with $\lambda \in P^+$. Set

$$\begin{aligned}\mathbb{F}_q(G//K) = \{&f \in \mathbb{F}_q(G) \mid f(k_1Xk_2) \\ &= \epsilon(k_1)f(X)\epsilon(k_2);\ k_1,k_2 \in \mathbf{K},\ X \in \mathbf{U}\}.\end{aligned}$$

This is the q-analog of the algebra of bi-K-invariant functions on G.

If V is a finite dimensional representation we say that a vector $v \in V$ is a spherical vector if $kv = \epsilon(k)v$ for all $k \in \mathbf{K}$. We denote by P_0 the set of all dominant integral weights λ such that $V(\lambda)$ has a nonzero spherical vector. The space of spherical vectors in $V(\lambda)$ has dimension at most one, as follows easily from [11, Theorem 4.12], thus, if $\lambda \in P_0$, we fix a nonzero spherical vector $v_\lambda \in V(\lambda)$ so that the space of spherical vectors in $V(\lambda)$ is precisely $\mathbb{F}v_\lambda$.

Let $f_\lambda \in V(\lambda)^*$ be the spherical vector in $V(\lambda)^*$ such that $f_\lambda(v_\lambda) = 1$. We call $\Phi_\lambda = c_{f_\lambda, v_\lambda}$ the spherical function of weight λ.

3. The algebra $D_q(G/K)$.

We now introduce the quantum version $D_q(G/K)$ of the algebra of invariant differential operators on the symmetric space G/K: set

$$\mathbf{U}^{\mathbf{K}} = \{X \in \mathbf{U} \mid \mathrm{Ad}(k)(X) = \epsilon(k)X, k \in \mathbf{K}\},$$

and $\mathcal{J} = \mathbf{U}(\mathbf{K} \cap \mathcal{A})$. It is natural to define

$$D_q(G/K) = \mathbf{U}^{\mathbf{K}} / \mathcal{J} \cap \mathbf{U}^{\mathbf{K}}.$$

Observe that $D_q(G/K)$ is an algebra since $J \cap \mathbf{U}^{\mathbf{K}}$ is a two sided ideal in $\mathbf{U}^{\mathbf{K}}$.

Recall that R_c is the root system of $(\mathfrak{k}, \mathfrak{t})$ and that $R_c^+ = R^+ \cap R_c$. Let $\gamma_1 \leqslant \gamma_2 \leqslant \ldots \leqslant \gamma_r \in R^+ \backslash R_c^+$ be the strongly orthogonal roots associated to the pair (R^+, R_c^+) (see [5, Proposition 7.4]). Let $\mathcal{W}_c$ denote the Weyl group of R_c. If w_c is the longest element of $\mathcal{W}_c$ with respect to R_c^+ we set $\delta_j = w_c(\gamma_j)$ and observe that $\delta_j \in R^+$.

Set $\mathfrak{a}' = \sum \mathbb{R}\delta_i$ and let $\pi : E \to \mathfrak{a}'$ denote the orthogonal projection. Set $\mathfrak{h}^- = \{X \in \mathfrak{t}_0 \mid \alpha(\sqrt{-1}X) = 0 \text{ for all } \alpha \in \mathfrak{a}'\}$ and $\mathfrak{h}^+$ to be the orthogonal complement of $\sqrt{-1}\mathfrak{h}^-$ in $\sqrt{-1}\mathfrak{t}_0$. We observe that the roots δ_i are the strongly orthogonal roots for the positive system $w_c(R^+) = -R_c^+ \cup R \backslash R_c^+$. Therefore, by [5, Corollary 7.6], we can choose root vectors $X_{\pm\delta_i}$ and an Iwasawa decomposition $G = KAN$ of G so that the space

$$\mathfrak{a}_0 = \sum \mathbb{R}(X_{\delta_i} + X_{-\delta_i})$$

is the Lie algebra $\mathfrak{a}_0$ of A. By standard Lie theory we can find an element $u \in \mathrm{Int}(\mathfrak{g})$ and such that $u_{|\mathfrak{h}^-} = \mathrm{Id}$ and $u(\mathfrak{h}^+) = \mathfrak{a}_0$.

If α is an element of the real dual of $\sqrt{-1}\mathfrak{t}_0$, then we set α^u to be the element of the real dual of $\mathfrak{a}_0 + \sqrt{-1}\mathfrak{h}^+$ defined by $\alpha^u = \alpha \circ u^{-1}$. It follows that $\pi(R)^u \backslash \{0\}$ is the set of restricted roots for $(\mathfrak{g}_0, \mathfrak{a}_0)$, hence $\pi(R)\backslash\{0\}$ is a root system (which is known to be of type BC_r). We denote by $Q_{\mathbb{R}}$, $P_{\mathbb{R}}$ respectively the root and the weight lattice of $\pi(R)\backslash\{0\}$. We also set $\mathcal{W}_{\mathbb{R}}$ to be the Weyl group of $\pi(R)\backslash\{0\}$.

Obviously $(R^+)^u$ is a positive system for the root system R^u. We set $\Sigma^+ = \pi(R^+)^u\backslash\{0\}$.

LEMMA 3.1. — *Σ^+ is a positive system for $\pi(R)^u\backslash\{0\}$.*

Proof. — Set $\mathfrak{a}_0'$ to be the real dual of $\mathfrak{a}_0$. The set $\{\delta_1^u, \delta_2^u, \ldots, \delta_r^u\}$ is a basis for $\mathfrak{a}_0'$. We order this basis by setting $\delta_1^u \geqslant \delta_2^u \geqslant \ldots \geqslant \delta_r^u$ and we extend this order lexicographically to all of $\mathfrak{a}_0'$. This defines a positive system Λ^+ for $\pi(R)^u\backslash\{0\}$. We claim that $\Lambda^+ = \Sigma^+$.

As observed above the roots δ_i are the strongly orthogonal roots for the positive system $w_c(R^+)$, so, by [14, §2], if $\alpha \in R_c^+$ then $\pi(-\alpha)$ is in one of the following possible forms:

$$\pi(-\alpha) = 0 \qquad \pi(-\alpha) = -\frac{1}{2}\delta_i \qquad \pi(-\alpha) = \frac{1}{2}(\delta_i - \delta_j) \text{ with } j < i$$

hence, if $\alpha \in R_c^+$ and $\pi(\alpha) \neq 0$ then $\pi(\alpha) \in \Lambda^+$.

On the other hand, if $\alpha \in R\backslash R_c^+$ then, by [14, §2], $\pi(\alpha)$ is in one of the following possible forms:

$$\pi(\alpha) = \frac{1}{2}\delta_i \qquad \pi(\alpha) = \frac{1}{2}(\delta_i + \delta_j)$$

thus $\pi(\alpha) \in \Lambda^+$.

This shows that $\Sigma^+ \subset \Lambda^+$. In particular we have that Σ^+ and $-\Sigma^+$ are disjoint. Finally we observe that $\Lambda^+ \cup (-\Lambda^+) = \pi(R)^u\backslash\{0\} = \Sigma^+ \cup (-\Sigma^+)$ proving our claim. □

As pointed out in the Introduction the main result of this paper is the quantum version of the Harish-Chandra isomorphism between the algebra of invariant differential operators on G/K and the algebra of invariant polynomials on A. We now describe our result precisely.

We begin with the following observation:

LEMMA 3.2. — *If $\lambda \in P$ then $\pi(\lambda) \in P_{\mathbb{R}}$.*

Proof. — Clearly $\pi(\lambda) = \sum_i \frac{(\lambda, \delta_i)}{(\delta_i, \delta_i)} \delta_i$. If $\lambda \in P$, then $2\frac{(\lambda, \delta_i)}{(\delta_i, \delta_i)} \in \mathbf{Z}$, so we need only to prove that

$$P_{\mathbb{R}} = \sum_i \frac{1}{2} \mathbf{Z} \delta_i. \tag{3.1}$$

This can be checked directly using (iii) and (iv) of [14, §2]. □

Notice that, by (3.1),

$$2P_{\mathbb{R}} = \sum_{j=1}^{r} \mathbf{Z} \delta_j$$

so, given $\lambda \in P_{\mathbb{R}}$, we can define a function e^λ on $2P_{\mathbb{R}}$ with values in $\mathbb{F}$ by setting

$$e^\lambda(\mu) = q^{(\lambda,\mu)}.$$

We denote the algebra of functions generated by $\{e^{2\mu} \mid \mu \in P_{\mathbb{R}}\}$ by $\mathbb{F}_q[2P_{\mathbb{R}}]$. There is a natural homomorphism from the group algebra $\mathbb{F}[2P_{\mathbb{R}}]$ of $2P_{\mathbb{R}}$ onto $\mathbb{F}_q[2P_{\mathbb{R}}]$. Since we are assuming that q is not a root of unity this homomorphism is actually an isomorphism. The natural action of $\mathcal{W}_{\mathbb{R}}$ on $2P_{\mathbb{R}}$ extends to an action on $\mathbb{F}_q[2P_{\mathbb{R}}]$.

If $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$, then we set $\rho_{\mathfrak{a}'} = \pi(\rho)$. We already observed that, if $\lambda \in 2P_{\mathbb{R}}$, then $(\lambda, \mu) \in \mathbf{Z}$ for all $\mu \in P_{\mathbb{R}}$. Combining this last observation with Lemma 3.2 we obtain that we can define an automorphism $\gamma_{-\rho_{\mathfrak{a}'}}$ of $\mathbb{F}_q[2P_{\mathbb{R}}]$ by setting $\gamma_{-\rho_{\mathfrak{a}'}}(e^\lambda) = q^{-(\lambda, \rho_{\mathfrak{a}'})} e^\lambda$.

Suppose now that $\overline{X} \in D_q(G/K)$. We define a function $f_{\overline{X}} : P_0 \to \mathbb{F}$ by setting $f_{\overline{X}}(\mu) = \Phi_\mu(X)$. Here is our main result:

THEOREM 3.3. — *Suppose that the root system R is irreducible and of type A_n, B_n, C_n, D_n. Then the function $f_{\overline{X}} : P_0 \to \mathbb{F}$ has a unique extension $F_{\overline{X}} : 2P_{\mathbb{R}} \to \mathbb{F}$ that belongs to $\mathbb{F}_q[2P_{\mathbb{R}}]$. Moreover*

1. $\gamma_{-\rho_{\mathfrak{a}'}}(F_{\overline{X}}) \in \mathbb{F}_q[2P_{\mathbb{R}}]^{\mathcal{W}_{\mathbb{R}}}$.
2. *The map*

$$\Gamma : D_q(G/K) \to \mathbb{F}_q[2P_{\mathbb{R}}]^{\mathcal{W}_{\mathbb{R}}},$$
$$\Gamma : \overline{X} \mapsto \gamma_{-\rho_{\mathfrak{a}'}}(F_{\overline{X}})$$

is an isomorphism of algebras.

The rest of this paper is devoted to the proof of Theorem 3.3. The reason why we need to restrict to the classical cases will soon be apparent.

We conclude this section by describing briefly the various steps in the proof: in §4 we construct a basis $\{\overline{X}_\lambda \mid \lambda \in P_0\}$ of $D_q(G/K)$ using the decomposition (2.2). Unfortunately we are not able to compute directly the functions $f_{\overline{X}_\lambda}$. To overcome this difficulty, we construct a different basis for $D_q(G/K)$ using the central elements z_V defined in [7, Lemma 6.23]. This is done because the functions $f_{\overline{z}_V}$ are easily computable. To carry out the proof of Theorem 3.3. we need to relate the elements $\overline{z}_V$ with our basis $\{\overline{X}_\lambda\}$. The key result in this direction is Corollary 5.6 which is proved in §5 using the so called L-operators.

Finally, in §6, we show how to reduce Theorem 3.3 to a computation in $\mathbb{F}[2P_{\mathbb{R}}]^{W_{\mathbb{R}}}$. The actual calculations are performed in §7 for the classical cases. It should be noticed that we essentially show that the center of $\mathbf{U}$ completely describes $D_q(G/K)$. It is known that in the $q = 1$ case this fact holds for the classical cases while it is not always true in the exceptional ones (see exercise D3 of ch.II and the remark on page 326 of [6] with the reference cited there). This observation explains why we cannot prove Theorem 3.3 in the exceptional cases.

4. A basis for $D_q(G/K)$.

In this section we construct a basis for $D_q(G/K)$. We start by decomposing $\mathbf{U}_n$ as a $\mathbf{K}$-module:

THEOREM 4.1. — *As a* $\mathbf{K}$*-module*

$$\mathbf{U}_n = \oplus_{\lambda \in P_0} V_c(\lambda). \tag{4.1}$$

Proof. — It is well known in the $q = 1$ case that $V_c(\lambda)$ occurs in $\mathbf{U}_n$ if and only if $\lambda = n_1\delta_1 + n_2\delta_2 + \ldots + n_r\delta_r$ with $n_j \in \mathbf{Z}$ and $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r \geqslant 0$. Moreover such a $V_c(\lambda)$ occurs with multiplicity one (see the remarks after Theorem 0 of [10] for the correct attribution of this result). It follows in particular that, if $n_j \in \mathbf{Z}$ and $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r \geqslant 0$, then $n_1\delta_1 + n_2\delta_2 + \ldots + n_r\delta_r \in P^+$. By construction $\mathbf{U}_n$ has the same character of its classical counterpart, so, by [11, Theorem 4.12], it decomposes in the same way.

To conclude the proof we need only to show that

$$P_0 = \left\{ \sum_{j=1}^{r} n_j\delta_j \mid n_j \in \mathbf{Z} \text{ and } n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r \geqslant 0 \right\}, \tag{4.2}$$

If $\lambda \in \left\{ \sum_{j=1}^{r} n_j\delta_j \mid n_j \in \mathbf{Z} \text{ and } n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r \geqslant 0 \right\}$, then λ^u is a weight on $\mathfrak{a}_0 + \mathfrak{h}^+$ that is dominant integral with respect to $(R^+)^u$. Set $F(\lambda^u)$ to be the finite dimensional irreducible $\mathfrak{g}$-module of highest weight λ^u. By Lemma 3.1 we can apply Theorem 4.1 of [6] and find that $F(\lambda^u)$ has a nonzero $\mathfrak{k}$-invariant vector. We know that in the $q = 1$ case $V(\lambda) = F(\lambda^u)$, therefore, by applying Theorem 4.1 of [11] again , we find that, if q is not a root of unity, then $V(\lambda)$ has a nonzero $\mathbf{K}$-invariant vector. This shows that

$$\left\{ \sum_{j=1}^{r} n_j\delta_j \mid n_j \in \mathbf{Z} \text{ and } n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r \geqslant 0 \right\} \subset P_0.$$

On the other hand, if $\lambda \in P_0$ then any highest weight vector v^+ of $V(\lambda)$ belongs to $\mathbf{U}^+v_\lambda$. By the PBW theorem we know that $\mathbf{U}^+ \simeq \mathbf{U}_n \otimes (\mathbf{K} \cap \mathbf{U}^+)$, so, since v_λ is spherical, we have that $v^+ \in \mathbf{U}_n v_\lambda$. Since the map $\mathbf{U}_n \to \mathbf{U}_n v_\lambda$ $(X \mapsto Xv_\lambda)$ is clearly $\mathbf{K}$-equivariant, we can conclude that $V_c(\lambda)$ occurs in $\mathbf{U}_n$. □

It follows immediately that, as a $\mathbf{K}$-module,

$$\tau(\mathbf{U}_n) = \bigoplus_{\lambda \in P_0} \tau(V_c(\lambda)), \tag{4.3}$$

hence

$$\mathbf{U}_n\tau(\mathbf{U}_n) = \bigoplus_{\lambda,\mu \in P_0} V_c(\lambda)\tau(V_c(\mu)).$$

As shown in [1] it is not hard to check that

$$V_c(\lambda)\tau(V_c(\mu)) \simeq \mathrm{Hom}_{\mathbb{F}}(V_c(\lambda), V_c(\mu))$$

as a $\mathbf{K}$-module. This depends on the fact that there is a nondegenerate $\mathbf{K}$-invariant pairing between $\mathbf{U}_n$ and $\tau(\mathbf{U}_n)$; such a pairing will be exhibited later in this paper.

It follows that

$$(\mathbf{U}_n\tau(\mathbf{U}_n))^{\mathbf{K}} = \bigoplus_{\lambda \in P_0} (V_c(\lambda)\tau(V_c(\lambda)))^{\mathbf{K}}$$

and

$$\dim(V_c(\lambda)\tau(V_c(\lambda)))^{\mathbf{K}} = 1.$$

For each $\lambda \in P_0$ we choose a nonzero element $X_\lambda \in (V_c(\lambda)\tau(V_c(\lambda)))^{\mathbf{K}}$; we have thus shown that $\{X_\lambda\}$ is a basis of $(\mathbf{U}_n\tau(\mathbf{U}_n))^{\mathbf{K}}$.

By (2.2) we have that

$$\mathbf{U} = \mathbf{U}_n\tau(\mathbf{U}_n) \oplus \mathcal{J} \tag{4.4}$$

and therefore since both $\mathbf{U}_n\tau(\mathbf{U}_n)$ and $\mathcal{J}$ are $\mathbf{K}$-stable,

$$\mathbf{U}^{\mathbf{K}} = (\mathbf{U}_n\tau(\mathbf{U}_n))^{\mathbf{K}} \oplus (\mathcal{J})^{\mathbf{K}}.$$

It follows at once that

$$\{\overline{X}_\lambda \mid \lambda \in P_0\} \tag{4.5}$$

is a basis of $D_q(G/K)$.

5. L-operators.

In this section we set up the machinery of L-operators. The result we are aiming at is Corollary 5.6 below. We recall briefly the theory of L-operators: references are [13] and [8, Chapter 9]. A quick exposition can be found in [4].

Let (,) denote the unique bilinear pairing between $\mathbf{U}^{\leqslant}$ and $\mathbf{U}^{\geqslant}$ such that, for all $y, y' \in \mathbf{U}^{\leqslant}$, all $x, x' \in \mathbf{U}^{\geqslant}$ and all $\mu, \lambda \in \mathbb{Z}\Phi$,

$$(y, xx') = (\Delta(y), x' \otimes x) \qquad (yy', x) = (y \otimes y', \Delta(x)) \tag{5.1}$$

$$(K_\mu, K_\nu) = q^{-(\mu,\nu)} \qquad (F_i, E_j) = -\delta_{ij}(q_i - q_i^{-1})^{-1} \tag{5.2}$$

$$(K_\mu, E_i) = (F_i, K_\mu) = 0 \tag{5.3}$$

(see [7, Proposition 6.12]).

We denote by $\check{\mathbf{U}}^0$ the group algebra of P. If $\mathbf{H}$ is a Hopf subalgebra of $\mathbf{U}$ containing $\mathbf{U}^0$ then we can extend the adjoint action Ad of $\mathbf{U}^0$ on $\mathbf{H}$ to $\check{\mathbf{U}}^0$: if $X \in \mathbf{H}$ and $\lambda(X) = \mu$ then we set $\mathrm{Ad}(K_\lambda)(X) = q^{(\lambda,\mu)}X$. We denote by $\check{\mathbf{H}}$ the Hopf algebra $\check{\mathbf{H}} = \mathbf{H} \otimes_{\mathbf{U}^0} \check{\mathbf{U}}^0$ with multiplication defined by

$$(X \otimes K_\lambda)(Y \otimes K_\mu) = X\,\mathrm{Ad}(K_\lambda)(Y) \otimes K_\lambda K_\mu$$

and comultiplication defined by

$$\Delta(X \otimes K_\lambda) = \tau_{23}(\Delta(X) \otimes K_\lambda \otimes K_\lambda)$$

(here τ_{23} is the "flip" $\tau_{23}(X_1 \otimes X_2 \otimes X_3 \otimes X_4) = X_1 \otimes X_3 \otimes X_2 \otimes X_4$).

By Lemma 4.1 of [4], we can extend the pairing (,) to a pairing between $\mathbf{U}^{\leqslant}$ and $\check{\mathbf{U}}^{\geqslant}$ and to a pairing between $\check{\mathbf{U}}^{\leqslant}$ and $\mathbf{U}^{\geqslant}$ by defining

$$(5.4) \qquad (YK_\lambda, XK_\mu) = q^{-(\lambda,\mu)}(Y, X) \quad X \in \mathbf{U}^+,\ Y \in \mathbf{U}^-.$$

We denote both pairings by (,) and observe that the same argument of Corollary 4.3 of [4] shows that the pairings are both nondegenerate.

We can now introduce the L-operators. See [4] for a proof of the following result:

THEOREM 5.1. — *Let M be a finite dimensional $\mathbf{U}$-module. Fix $v \in M$ and $f \in M^*$. Then*

1. *There is a unique element $\ell^+_{f,v} \in \check{\mathbf{U}}^{\geqslant}$ such that $c_{f,v}(y) = (y, \ell^+_{f,v})$ for all $y \in \mathbf{U}^{\leqslant}$.*

2. *There is a unique element $\ell^-_{f,v} \in \check{\mathbf{U}}^{\leqslant}$ such that $c_{f,v}(S(x)) = (\ell^-_{f,v}, x)$ for all $x \in \mathbf{U}^{\geqslant}$.*

The elements $\ell^\pm_{f,v}$ are the so called L-operators.

Let M be a finite dimensional $\mathbf{U}$-module, $\{v_i\}$ a basis of M, and $\{f_i\}$ its dual basis. If $v \in M$, and $f \in M^*$, we set

$$(5.5) \qquad \mathcal{L}_M(f \otimes v) = \sum_i \ell^+_{f,v_i} S(\ell^-_{f_i,v}).$$

We also set

$$(5.6) \qquad z_M = \mathcal{L}_M\left(\sum_i f_i \otimes K_{2\rho} v_i\right).$$

A proof of the following result can be found in §7 of [4]:

THEOREM 5.2. — 1. *$\mathcal{L}_M : M^* \otimes M \to \check{\mathbf{U}}$ is $\mathbf{U}$-equivariant. (The action of $\mathbf{U}$ on $\check{\mathbf{U}}$ is given by* Ad*).*

2. *If $\lambda \in P$ and $2\lambda \in Q$, then*

$$\mathcal{L}_{V(\lambda)}(V(\lambda)^* \otimes V(\lambda)) \subset \mathbf{U}.$$

3. *z_M is a central element in $\check{\mathbf{U}}$.*

By hitting with z_M the lowest weight of $V(\lambda)$ we find by a direct calculation that

$$z_M \cdot v = \operatorname{tr}(K_{-2(\lambda+\rho)}, M)v \tag{5.7}$$

for all $v \in V(\lambda)$. In particular, by applying Proposition 5.11 of [7] or by a direct calculation, we find that

$$z_M z_N = z_{M\otimes N}. \tag{5.8}$$

We set $z_\lambda = z_{V(\lambda)}$ for $\lambda \in P^+$.

Our next task is to find a $\mathbf{K}$-equivariant version of $\mathcal{L}_M$, this time into $\mathbf{U}_n\tau(\mathbf{U}_n)$: set

$$\mathbf{K}^- = \mathbf{K} \cap \mathbf{U}^- \quad \mathbf{K}^+ = \mathbf{K} \cap \mathbf{U}^+$$

and

$$\check{\mathbf{K}}^{\leqslant} = \check{\mathbf{K}} \cap \check{\mathbf{U}}^{\leqslant} \quad \check{\mathbf{K}}^{\geqslant} = \check{\mathbf{K}} \cap \check{\mathbf{U}}^{\geqslant}.$$

We let $\check{\mathcal{A}}$ denote the augmentation ideal of $\check{\mathbf{U}}$. It is the two sided ideal generated by $\mathcal{A}$ and $\{K_\lambda - 1 \mid \lambda \in P\}$. We set

$$\check{\mathcal{I}}^{\leqslant} = \check{\mathbf{U}}^{\leqslant}(\check{\mathbf{K}}^{\leqslant} \cap \check{\mathcal{A}}) \quad \check{\mathcal{I}}^{\geqslant} = (\check{\mathbf{K}}^{\geqslant} \cap \check{\mathcal{A}})\check{\mathbf{U}}^{\geqslant}.$$

By the PBW theorem we have that

$$\check{\mathbf{U}}^{\leqslant} = \tau(\mathbf{U}_n) \oplus \check{\mathcal{I}}^{\leqslant} \quad \check{\mathbf{U}}^{\geqslant} = \mathbf{U}_n \oplus \check{\mathcal{I}}^{\geqslant}.$$

Let $p^+ : \check{\mathbf{U}}^{\geqslant} \to \mathbf{U}_n$ and $p^- : \check{\mathbf{U}}^{\leqslant} \to \tau(\mathbf{U}_n)$ be the projections corresponding to the decompositions above.

We now observe that

$$(\check{\mathcal{I}}^{\leqslant}, \mathbf{U}_n) = 0. \tag{5.9}$$

Indeed, notice that if $Y \in \check{I}^{\leqslant}$ and $X \in \mathbf{U}_n$, then we can assume that $Y = Y'K_\lambda$ with $Y' \in \mathbf{U}^-(\mathbf{K}^- \cap \mathcal{A})$ or $Y = Y'(K_\lambda - 1)$ with $Y' \in \check{\mathbf{U}}^{\leqslant}$ and $\lambda \in P$. In the first case we obtain

$(Y, X) = 0$ by (5.4) and §8.30 of [7]; in the second, $(Y, X) = 0$ by applying (5.4).

Next we prove that

$$(\tau(\mathbf{U}_n), \check{\mathcal{I}}^{\geqslant}) = 0. \tag{5.10}$$

Let ω be the involutive automorphism of $\check{\mathbf{U}}$ defined by

$$\omega(F_i) = E_i \quad \omega(K_\lambda) = K_{-\lambda}.$$

We observe that $S(\omega(X)) = \tau(X)$ for all $X \in \mathbf{U}$, hence we can extend τ to $\check{\mathbf{U}}$. Now, if $X \in \mathbf{U}^+$ and $Y \in \mathbf{U}^-$, then, as shown in [7, Lemma 6.16], we know that

$$(Y, X) = (\omega(X), \omega(Y)). \tag{5.11}$$

Using (5.4) it is easy to check that (5.11) holds also if $Y \in \check{\mathbf{U}}^{\leqslant}$ and $X \in \mathbf{U}^+$. Since $(S(Y), S(X)) = (Y, X)$ for all $Y \in \mathbf{U}^{\leqslant}$ and $X \in \check{\mathbf{U}}^{\geqslant}$ (see Exercise 6.16 of [7]), we deduce that $(Y, X) = (\tau(X), \tau(Y))$ if $X \in \mathbf{U}^+$ and $Y \in \check{\mathbf{U}}^{\leqslant}$. In order to prove (5.10) it is now sufficient to observe that $\check{\mathcal{I}}^{\geqslant} = \tau(\check{I}^{\leqslant})$.

Notice also that (5.9) and (5.10) imply that the pairing between $\mathbf{U}_n$ and $\tau(\mathbf{U}_n)$ given by restricting (,) is nondegenerate.

THEOREM 5.3. — *Let V be a finite dimensional* $\mathbf{U}$*-module.*

1. *The map $\mathcal{L}^+_{n,V} : V \otimes V^* \to \mathbf{U}_n$ defined by*

$$v \otimes f \mapsto p^+(\ell^+_{f, K_{-2\rho} v})$$

is $\mathbf{K}$*-equivariant.*

2. *The map $\mathcal{L}^-_{n,V} : V \otimes V^* \to \tau(\mathbf{U}_n)$ defined by*

$$v \otimes f \mapsto p^-(S(\ell^-_{K_{2\rho} f, v}))$$

is $\mathbf{K}$*-equivariant.*

Proof. — We first show that for all $X_1, X_2 \in \mathbf{U}_n$ and $k \in \mathbf{K}$.

$$(\mathrm{Ad}(S(k))(\tau(X_1)), X_2) = (\tau(X_1), \mathrm{Ad}(k)(X_2)) \tag{5.12}$$

This can most easily checked using the Rosso form $<\ ,\ >$ as defined in [7, §6.20]. Indeed, by equation (1) of [7, §6.20] we have that

$$(\tau(X_1), X_2) = < \mathrm{Ad}(K_{2\rho})(\tau(X_1)), X_2 >$$

hence 5.12) follows readily from the Ad-invariance of the Rosso form.

As observed above, the pairing between $\mathbf{U}_n$ and $\tau(\mathbf{U}_n)$ given by restricting (,) is nondegenerate. It follows that in order to prove our statements it is enough to show that, for all $k \in \mathbf{K}$ and $X \in \mathbf{U}_n$,

$$(\tau(X), \mathcal{L}^+_{n,V}(k \cdot (v \otimes f))) = (\tau(X), \mathrm{Ad}(k)(\mathcal{L}^+_{n,V}(v \otimes f))) \tag{5.13}$$

and that

$$(\mathcal{L}^-_{n,V}(k\cdot(v\otimes f)),X)=(\mathrm{Ad}(k)(\mathcal{L}^-_{n,V}(v\otimes f)),X). \tag{5.14}$$

We now prove (5.13): write $\Delta(k)=\sum_i k_i'\otimes k_i''$ then

$$\begin{aligned}
(\tau(X),\mathcal{L}^+_{n,V}(k\cdot(v\otimes f))) &= \sum_i(\tau(X),p^+(\ell^+_{k_i''\cdot f,(K_{-2\rho}k_i')\cdot v}))\\
&= \sum_i(\tau(X),\ell^+_{k_i''\cdot f,(K_{-2\rho}k_i')\cdot v}) && \text{by (5.10)}\\
&= \sum_i k_i''\cdot f(\tau(X)K_{-2\rho}k_i'\cdot v) && \text{by Theorem 5.1}\\
&= \sum_i f(S(k_i'')\tau(X)K_{-2\rho}k_i'\cdot v)\\
&= \sum_i f(Ad(S(k)(\tau(X))K_{-2\rho}\cdot v)\\
&= (\mathrm{Ad}(S(k)(\tau(X)),\ell^+_{f,K_{-2\rho}\cdot v}) && \text{by Theorem 5.1}\\
&= (\mathrm{Ad}(S(k)(\tau(X)),p^+(\ell^+_{f,K_{-2\rho}\cdot v})) && \text{by (5.10)}\\
&= (\tau(X),\mathrm{Ad}(k)(\mathcal{L}^+_n(v\otimes f))) && \text{by (5.12).}
\end{aligned}$$

The proof of (5.14) is completely analogous. □

Suppose now that $\lambda\in P_0$ so that $V(\lambda)$ has a nonzero spherical vector. This implies that the weights of $V(\lambda)$ are all in Q. In particular we can define an action of $\check{\mathbf{U}}$ on $V(\lambda)$ by $K_\mu v=q^{(\mu,\nu)}v$ for $v\in V(\lambda)_\nu$.

Consider the left ideal $\mathcal{I}$ of $\check{\mathbf{U}}$

$$\mathcal{I}=\{X\in\check{\mathbf{U}}\mid Xv_\lambda=0 \text{ for all } \lambda\in P_0\}.$$

The following result was proven in [3]:

THEOREM 5.4. — 1. $\mathcal{I}=\check{\mathbf{U}}(\check{\mathbf{K}}\cap\check{\mathcal{A}})$.

2. *The decomposition*

$$\check{\mathbf{U}}=(\mathbf{U}_n\tau(\mathbf{U}_n))^{\mathbf{K}}\oplus(\check{\mathbf{U}}(\check{\mathbf{K}}\cap\check{\mathcal{A}})+(\check{\mathbf{K}}\cap\check{\mathcal{A}})\check{\mathbf{U}})$$

holds.

3. *The pairing*

$$<\,,\,>:\mathbb{F}_q(G//K)\times D_q(G/K)\to\mathbb{F}$$

given by $<f,\overline{X}>=f(X)$ *is nondegenerate.*

Let $p : \check{\mathbf{U}} \to (\mathbf{U}_n \tau(\mathbf{U}_n))^{\mathbf{K}}$ be the projection associated with the decomposition given in 2 of Theorem 5.4. If V is a $\mathbf{U}$-module, then we set

$$P_0(V) = \{\lambda \in P_0 \mid V_c(\lambda) \text{ occurs in } V\}.$$

A consequence of Theorem 5.3 is the following result:

Proposition 5.5. — *Let V, W be $\mathbf{U}$-modules. If $v \in V$, $f \in V^*$, $w \in W$, and $g \in W^*$, then*

$$p(\ell^+_{f,v} S(\ell^-_{g,w})) = \sum_\mu c_\mu X_\mu,$$

with $c_\mu = 0$ unless $\mu \in P_0(V \otimes V^*) \cap P_0(W \otimes W^*)$.

Proof. — Obviously,

$$\begin{aligned} p(\ell^+_{f,v} S(\ell^-_{g,w})) &= p(p^+(\ell^+_{f,v}) p^-(S(\ell^-_{g,w}))) \\ &= p(\mathcal{L}^+_{n,V}(K_{2\rho} v \otimes f) \mathcal{L}^-_{n,W}(w \otimes K_{-2\rho} g)). \end{aligned}$$

If $\mu \in P_c^+$ and V is a $\mathbf{U}$-module, we let $C(\mu, V)$ denote the isotypic component of V of type μ. By (4.1), we know that

$$\begin{cases} C(\mu, \mathbf{U}_n) = \{0\} & \text{if } \mu \notin P_0, \\ C(\mu, \mathbf{U}_n) = V_c(\mu) & \text{if } \mu \in P_0; \end{cases}$$

and, by (4.3),

$$\begin{cases} C(\mu, \tau(\mathbf{U}_n)) = \{0\} & \text{if } -w_c(\mu) \notin P_0, \\ C(\mu, \tau(\mathbf{U}_n)) = \tau(V_c(-w_c(\mu))) & \text{if } -w_c(\mu) \in P_0. \end{cases}$$

By the $\mathbf{K}$-equivariance of $\mathcal{L}^+_{n,V}$ we deduce that

$$\mathcal{L}^+_{n,V}(K_{2\rho} v \otimes f)) \in \sum_{\mu \in P_0(V \otimes V^*)} V_c(\mu)$$

and, by the $\mathbf{K}$-equivariance of $\mathcal{L}^-_{n,W}$,

$$\mathcal{L}^-_{n,W}(w \otimes K_{-2\rho} g) \in \sum_{\mu \in P_0(W \otimes W^*)} \tau(V_c(\mu))$$

(here we used the fact that $(W \otimes W^*)^* \simeq W \otimes W^*$).

It follows that

$$p(\ell^+_{f,v} S(\ell^-_{g,w})) \in p\Big(\sum_{\mu_1, \mu_2} V_c(\mu_1) \tau(V_c(\mu_2)) \Big)$$

with $\mu_1 \in P_0(V \otimes V^*)$ and $\mu_2 \in P_0(W \otimes W^*)$. If we set $\check{\mathcal{J}}_{\mathbf{K}} = (\check{\mathbf{U}}(\check{\mathbf{K}} \cap \check{\mathcal{A}}) + (\check{\mathbf{K}} \cap \check{\mathcal{A}})\check{\mathbf{U}})$, it is easy to check that

$$V_c(\mu_1)\tau(V_c(\mu_2)) = (V_c(\mu_1)\tau(V_c(\mu_2)))^{\mathbf{K}} \oplus (V_c(\mu_1)\tau(V_c(\mu_2))) \cap \check{\mathcal{J}}_{\mathbf{K}}$$

hence

$$p(\ell^+_{f,v} S(\ell^-_{g,w})) \in \sum_{\mu_1,\mu_2} (V_c(\mu_1)\tau(V_c(\mu_2)))^{\mathbf{K}}.$$

To conclude the proof it is enough to recall that

$$\begin{cases} (V_c(\mu_1)\tau(V_c(\mu_2)))^{\mathbf{K}} = \{0\} & \text{if } \mu_1 \neq \mu_2, \\ (V_c(\mu)\tau(V_c(\mu)))^{\mathbf{K}} = \mathbb{F}X_\mu. \end{cases}$$ □

Analogous to the decomposition (4.4), we have a decomposition

$$\check{\mathbf{U}} = \mathbf{U}_n\tau(\mathbf{U}_n) \oplus \check{\mathbf{U}}(\check{\mathbf{K}} \cap \check{\mathcal{A}})$$

hence, if we set $\check{\mathcal{J}} = \check{\mathbf{U}}(\check{\mathbf{K}} \cap \check{\mathcal{A}})$, then

$$\check{\mathbf{U}}^{\mathbf{K}} = (\mathbf{U}_n\tau(\mathbf{U}_n))^{\mathbf{K}} \oplus \check{\mathcal{J}}^{\mathbf{K}}$$

thus the embedding $\mathbf{U}^{\mathbf{K}} \subset \check{\mathbf{U}}^{\mathbf{K}}$ induces an isomorphism

$$\check{\mathbf{U}}^{\mathbf{K}} / \check{\mathcal{J}}^{\mathbf{K}} \simeq D_q(G/K).$$

If $X \in \check{\mathbf{U}}^{\mathbf{K}}$ we still denote $\overline{X} = X + \check{\mathcal{J}}^{\mathbf{K}} \in D_q(G/K)$. A consequence of the previous proposition is the following result:

COROLLARY 5.6. — *If V is a $\mathbf{U}$-module, then*

$$\overline{z}_V = \sum_\mu c_\mu \overline{X}_\mu$$

with $c_\mu = 0$ unless $\mu \in P_0(V \otimes V^*)$.

Proof. — We already observed that if $\lambda \in P_0$ then $\check{\mathbf{U}}$ acts on $V(\lambda)$. It follows that we can define $\Phi_\lambda(z_V)$ for all $\lambda \in P_0$. Let $< \, , \, >$ be the pairing defined in Theorem 5.4. Since $\check{\mathbf{K}} \cap \check{\mathcal{A}}$ acts trivially on spherical vectors we have that

$$< \Phi_\lambda, \overline{z}_V >= \Phi_\lambda(z_V) \text{ for all } \lambda \in P_0$$

and that

$$\Phi_\lambda(z_V) = \Phi_\lambda(p(z_V)) =< \Phi_\lambda, \overline{p(z_V)} > \text{ for all } \lambda \in P_0.$$

By the nondegeneracy of the pairing $<\ ,\ >$ we obtain that

$$\overline{z}_V = \overline{p(z_V)}$$

hence our result.

6. The proof of Theorem 3.3.

In this section we show how to reduce the proof of Theorem 3.3 to a calculation in $\mathbb{F}[2P_{\mathbb{R}}]^{\mathcal{W}_{\mathbb{R}}}$.

First of all we observe that it is obvious that $f_{\overline{XY}} = f_{\overline{X}} f_{\overline{Y}}$, so if we can prove the existence and uniqueness of $F_{\overline{X}}$ and $F_{\overline{Y}}$ then clearly $F_{\overline{XY}} = F_{\overline{X}} F_{\overline{Y}}$. Since $\gamma_{-\rho_{\mathfrak{a}'}}$ is obviously an automorphism of $\mathbb{F}_q[2P_{\mathbb{R}}]$, we have that $\Gamma(\overline{XY}) = \Gamma(\overline{X})\Gamma(\overline{Y})$.

We now show that the function $F_{\overline{X}}$ of Theorem 3.3 exists when $X = z_V$: by (5.7)

$$f_{\overline{z}_V}(\lambda) = tr(K_{-2(\lambda+\rho)}, V) = \sum_{\mu \in P_{\mathbb{R}}} \left(\sum_{\pi(\nu)=\mu} \dim V_\nu q^{-2(\nu,\lambda+\rho)} \right).$$

Since $\lambda \in P_0 \subset \mathfrak{a}'$ then we can rewrite $f_{\overline{z}_V}$ as

$$f_{\overline{z}_V}(\lambda) = \sum_{\mu \in P_{\mathbb{R}}} \left(\sum_{\pi(\nu)=\mu} \dim V_\nu q^{-2(\nu,\rho)} \right) q^{-2(\mu,\lambda)}$$

so, if we set

$$F_{\overline{z}_V} = \sum_{\mu \in P_{\mathbb{R}}} \left(\sum_{\pi(\nu)=\mu} \dim V_\nu q^{-2(\nu,\rho)} \right) e^{-2\mu}$$

then clearly $F_{\overline{z}_V} \in \mathbb{F}_q[2P_{\mathbb{R}}]$ and $(F_{\overline{z}_V})_{|P_0} = f_{\overline{z}_V}$. The uniqueness of $F_{\overline{z}_V}$ follows immediately from the following observation:

THEOREM 6.1. — *If $F \in \mathbb{F}_q[2P_{\mathbb{R}}]$ is such that $F_{|P_0} = 0$ then $F = 0$.*

Proof. — Let $\mathbb{F}[x_i, x_i^{-1}]$ be the algebra of Laurent polynomials in r variables. We can define a map $\phi : \mathbb{F}[x_i, x_i^{-1}] \to \mathbb{F}_q[2P_{\mathbb{R}}]$ by setting $\phi(x_i) = e^{\delta_i}$. The map ϕ is clearly onto. If $F = \phi(p)$ is such that $F_{|P_0} = 0$ then, arguing as in Lemma 4.2 of [4], we find that $p = 0$ thus $F = 0$. □

Since $F_{\overline{z}_V}$ exists we can also define $\Gamma(\overline{z}_V) = \gamma_{-\rho_{\mathfrak{a}'}}(F_{\overline{z}_V})$. We now compute it: set $\rho_M = \rho - \rho_{\mathfrak{a}'}$, then

$$\Gamma(\overline{z}_V) = \gamma_{-\rho_{\mathfrak{a}'}}(F_{\overline{z}_V}) \tag{6.1}$$

$$= \sum_{\mu \in P_{\mathbb{R}}} \left(\sum_{\pi(\nu)=\mu} \dim V_\nu q^{-2(\nu,\rho)} \right) q^{2(\rho_{\mathfrak{a}'},\mu)} e^{-2\mu} \tag{6.2}$$

$$= \sum_{\mu \in P_{\mathbb{R}}} \left(\sum_{\pi(\nu)=\mu} \dim V_\nu q^{-2(\nu,\rho_M)} \right) e^{-2\mu}. \tag{6.3}$$

Our next task is to prove that $\Gamma(\overline{z}_V)$ is $\mathcal{W}_{\mathbb{R}}$-invariant. Set

$$R_0 = \{\alpha \in R \mid \pi(\alpha) = 0\}.$$

It is known that R_0 is a root system.

It is well known that Lemma 3.1 implies that $\Pi \cap R_0$ is a set of simple roots for R_0 whose corresponding system of positive roots is $R_0^+ = R^+ \cap R_0$.

LEMMA 6.2. — *Given $w \in \mathcal{W}_{\mathbb{R}}$, there is $\tilde{w} \in \mathcal{W}$ such that*

$$\pi \circ \tilde{w} = w \circ \pi \tag{6.4}$$

and

$$\tilde{w}(\rho_M) = \rho_M. \tag{6.5}$$

Proof. — Set $W_{\mathfrak{a}'} = \{w \in \mathcal{W} \mid w(\mathfrak{a}') = \mathfrak{a}'\}$ and let $\mathcal{W}_0$ be the Weyl group of R_0. Then, by Proposition 8.10 of [5], the map Res : $\mathcal{W}_{\mathfrak{a}'} \to \mathcal{W}_{\mathbb{R}}$ ($\mathrm{Res}(w) = w_{|\mathfrak{a}'}$) is onto and its kernel is $\mathcal{W}_0$.

Given $w \in \mathcal{W}_{\mathbb{R}}$ then we choose $\tilde{w} \in \mathcal{W}_{\mathfrak{a}'}$ such that $\mathrm{Res}(\tilde{w}) = w$ and such that $\tilde{w}$ is an element of minimal length in the coset $\tilde{w}\mathcal{W}_0$ of $\mathcal{W}_{\mathfrak{a}'}/\mathcal{W}_0$. It is clear that $\tilde{w}$ satisfies (6.4).

We now show that $\tilde{w}$ satisfies (6.5) : it is well known that Lemma 3.1 implies that $\rho_M = \frac{1}{2} \sum_{\alpha \in R_0^+} \alpha$, so it is enough to prove that $\tilde{w}(R_0^+) = R_0^+$. Since $\tilde{w}(\mathfrak{a}') = \mathfrak{a}'$, then $\tilde{w}(R_0) = R_0$. Since R_0^+ is the positive system whose set of simple roots is $\Pi \cap R_0$, it follows that we need only to check that $\tilde{w}(\alpha) \in R_0^+$ for all $\alpha \in \Pi \cap R_0$. Suppose that $\alpha \in \Pi \cap R_0$ and $\tilde{w}(\alpha) \in -R_0^+$. Then $\tilde{w}s_\alpha \in \tilde{w}\mathcal{W}_0$ and $\ell(\tilde{w}s_\alpha) < \ell(\tilde{w})$ contrary to the choice of $\tilde{w}$. □

We are now ready to prove that $\Gamma(\overline{z}_V)$ is invariant:

PROPOSITION 6.3.

$$\Gamma(\overline{z}_V) \in \mathbb{F}[2P_{\mathbb{R}}]^{\mathcal{W}_{\mathbb{R}}}.$$

Fix $w \in \mathcal{W}_{\mathbb{R}}$. We need to check that the coefficient of $e^{-2\mu}$ in $\Gamma(\overline{z}_V)$ is the same as the coefficient of $e^{w^{-1}(-2\mu)}$, i.e. that

$$\sum_{\pi(\nu)=\mu} \dim V_\nu q^{-2(\nu,\rho_M)} = \sum_{\pi(\nu)=w^{-1}(\mu)} \dim V_\nu q^{-2(\nu,\rho_M)}.$$

Choose $\tilde{w}$ as in Lemma 6.2. Then $\pi(\nu) = w^{-1}(\mu)$ if and only if $\pi(\tilde{w}(\nu)) = \mu$ so we can write

$$\begin{aligned}\sum_{\pi(\nu)=w^{-1}(\mu)} \dim V_\nu q^{-2(\nu,\rho_M)} &= \sum_{\pi(\tilde{w}(\nu))=\mu} \dim V_\nu q^{-2(\nu,\rho_M)} \\ &= \sum_{\pi(\nu)=\mu} \dim V_{\tilde{w}^{-1}(\nu)} q^{-2(\tilde{w}^{-1}(\nu),\rho_M)} \\ &= \sum_{\pi(\nu)=\mu} \dim V_\nu q^{-2(\nu,\tilde{w}(\rho_M))} \\ &= \sum_{\pi(\nu)=\mu} \dim V_\nu q^{-2(\nu,\rho_M)} \qquad \text{by (6.5)}\end{aligned}$$

as desired. □

Recall that $r = \dim \mathfrak{a}'$ and for $i = 1, \dots, r$ set $\omega_i = \sum_{j=1}^{i} \delta_j$.

If $I = (i_1, \dots, i_r) \in \mathbf{N}^r$ then we set $|I| = \sum_{j=1}^{r} j i_j$. If $\lambda \in P_0$ then $\lambda = \sum_{j=1}^{r} n_j \delta_j$ with $n_j \in \mathbf{Z}$ and $n_1 \geqslant n_2 \geqslant \dots \geqslant n_r \geqslant 0$. We set $|\lambda| = \sum n_j$.

LEMMA 6.4. — *Suppose that there are $\lambda_1, \dots, \lambda_r \in P^+$ such that*

$$(\lambda_j, \omega_r) \leqslant \frac{1}{2} j(\delta_1, \delta_1).$$

Set $z_i = z_{V(\lambda_i)}$ and, if $I = (i_1, \dots, i_r) \in \mathbf{N}^r$, set $\overline{z}^I = \prod_j \overline{z}_j^{i_j}$.

Then

$$\overline{z}^I = \sum_{|\lambda| \leqslant |I|} c_\lambda \overline{X}_\lambda.$$

Proof. — Set

$$M = (\otimes^{i_1} V_1) \otimes (\otimes^{i_2} V_2) \otimes \ldots \otimes (\otimes^{i_r} V_r).$$

By (5.8) we have that

$$\overline{z}^I = \overline{z^I} = \overline{z}_M.$$

Applying Corollary 5.6 we find that

$$\overline{z}_M = \sum_{\lambda \in P_0(M \otimes M^*)} c_\lambda \overline{X}_\lambda.$$

If $\lambda \in P_0(M \otimes M^*)$, then λ must be a weight of $M \otimes M^*$, thus $\lambda = \mu + \nu$ with μ a weight of M and ν a weight of M^*, so

$$\mu = \sum_{j=1}^{r} i_j \lambda_j - \sum n_i \alpha_i \text{ and } \nu = \sum_{j=1}^{r} i_j(-w_0(\lambda_j)) - \sum m_i \alpha_i$$

with $m_i, n_i \in \mathbf{N}$. (Here w_0 is the longest element of the Weyl group of R with respect to R^+).

Since ω_r is dominant for R^+, we have that

$$(\mu, \omega_r) \leqslant \sum_j i_j(\lambda_j, \omega_r) \text{ and } (\nu, \omega_r) \leqslant \sum_j i_j(-w_0(\lambda_j), \omega_r).$$

We are assuming that $(\lambda_j, \omega_r) \leqslant \frac{1}{2} j(\delta_1, \delta_1)$. Arguing as in [1, Lemma 6.2] we see that $-w_0(\omega_r) = \omega_r$ so $(-w_0(\lambda_j), \omega_r) \leqslant \frac{1}{2} j(\delta_1, \delta_1)$. We can conclude that

$$(\lambda, \omega_r) = (\mu + \nu, \omega_r) \leqslant |I|(\delta_1, \delta_1).$$

On the other hand, by (v) of [14, §2], $(\delta_j, \delta_j) = (\delta_1, \delta_1)$ for all j, hence, since $\lambda = \sum n_j \delta_j$, we find that

$$(\lambda, \omega_r) = |\lambda|(\delta_1, \delta_1). \tag{6.6}$$

The result follows. □

By Lemma 6.4, Theorem 3.3 will follow if we prove the following result:

THEOREM 6.5. — *Under the hypothesis of Lemma 6.4 suppose also that the set*

$$\{\Gamma(\overline{z}_1), \dots, \Gamma(\overline{z}_r)\}$$

is an algebraically independent set of generators of $\mathbb{F}_q[2P_{\mathbb{R}}]^{\mathcal{W}_{\mathbb{R}}}$.

Then the set $\{\overline{z}^I\}$ *is a basis of* $D_q(G/K)$, $D_q(G/K) = \mathbb{F}[\overline{z}_i]$, *and*

$$\Gamma : \mathbb{F}[\overline{z}_i] \to \mathbb{F}_q[2P_{\mathbb{R}}]^{\mathcal{W}_{\mathbb{R}}}$$

is an isomorphism of algebras.

Proof. — Fix $N \in \mathbf{N}$ and set $D_q(G/K)[N]$ to be the space generated by $\{\overline{X}_\lambda \mid |\lambda| \leqslant N\}$. By Lemma 6.4, $\{\overline{z}^I \mid |I| \leqslant N\} \subset D_q(G/K)[N]$.

If $I = (i_1, \dots, i_r)$ we set $\lambda(I) = \sum_{j=1}^{r} i_j \omega_j$. By (6.6), using the fact that $(\omega_j, \omega_r) = j(\delta_1, \delta_1)$, we deduce that $|\lambda(I)| = |I|$, so we can define a map

$$\begin{aligned} \{\overline{z}^I \mid |I| \leqslant N\} &\to \{\overline{X}_\lambda \mid |\lambda| \leqslant N\} \\ \overline{z}^I &\mapsto \overline{X}_{\lambda(I)} \end{aligned}$$

that is bijective.

Since the set $\{\Gamma(\overline{z}_1), \dots, \Gamma(\overline{z}_r)\}$ is algebraically independent, the set $\{\overline{z}^I \mid |I| \leqslant N\}$ is linearly independent. Since $\{\overline{X}_\lambda \mid |\lambda| \leqslant N\}$ is a basis of $D_q(G/K)[N]$, and $D_q(G/K)[N]$ is finite dimensional we deduce that $\{\overline{z}^I \mid |I| \leqslant N\}$ is a basis of $D_q(G/K)[N]$. This proves the fact that $\{\overline{z}^I\}$ is a basis of $D_q(G/K)$ and, therefore, $D_q(G/K) = \mathbb{F}[\overline{z}_i]$.

Moreover, since Γ maps an algebraically independent set of generators onto an algebraically independent set of generators, it is clearly an isomorphism. □

7. The classical cases.

In this section we prove that if R is a root system of type A_n, B_n, C_n, D_n then the hypotheses of Theorem 6.5 hold. Our computations will be based on the following easy observation:

LEMMA 7.1. — *Suppose that*

$$\phi : \mathbb{F}[x_1, \dots, x_n] \to \mathbb{F}[x_1, \dots, x_n]$$

is a surjective homomorphism. Then ϕ *is an isomorphism.*

Proof. — The ring $\mathbb{F}[x_1, \dots, x_n]$ is noetherian. □

If $\lambda \in 2P_{\mathbb{R}}$, then we denote by $m(\lambda)$ the $\mathcal{W}_{\mathbb{R}}$-orbit of e^λ. It is known by classical invariant theory that $\mathbb{F}[2P_{\mathbb{R}}]^{\mathcal{W}_{\mathbb{R}}}$ is the polynomial algebra in the free generators $X_i = m(-\omega_i)$.

Suppose that, for each $i = 1, \dots, r$, we can find $\lambda_i \in P^+$ that satisfies the hypothesis of Lemma 6.4. Let z_i be as in Lemma 6.4, and set $Y_i = \Gamma(\overline{z}_i)$. We need to prove that $\{Y_i\}$ is an algebraically independent set of generators for $\mathbb{F}[2P_{\mathbb{R}}]^{\mathcal{W}_{\mathbb{R}}}$. By Lemma 7.1, in order to prove this fact, we need only show that the homomorphism

$$\phi : \mathbb{F}[X_1, \dots, X_r] \to \mathbb{F}[X_1, \dots, X_r]$$

defined by setting $\phi(X_i) = Y_i$ is onto, or, equivalently, that there are polynomials $P_1, P_2, \dots, P_r$ such that $X_i = P_i(Y_1, \dots, Y_r)$. The proof of the existence of the weights λ_i as well of the polynomials $P_1, \dots, P_r$ requires a certain amount of case by case checking, so we prove it separately for each case. In all of our cases we use the description of roots and weights given in [2, Planches I–IV]. The various cases are listed according to Table V at page 518 of [5].

7.1. Case CI.

In this case R is of type C_n, $\alpha_{i_0} = 2\epsilon_n$, and $\delta_i = 2\epsilon_i$, $i = 1, \dots, n$. It follows that $r = n$ and $E = \mathfrak{a}'$, so π is the identity and $\rho_M = 0$. For $i = 1, \dots, n$, we choose $\lambda_i = \sum_{j=1}^{i} \epsilon_j$; it is easily checked that the weights λ_i satisfy the hypothesis of Lemma 6.4. It follows from (6.3) that in this case

$$Y_i = \Gamma(\overline{z}_i) = \sum_{\mu \in P} \dim V(\lambda_i)_\mu e^{-2\mu}. \tag{7.1}$$

Let $F(\lambda_i)$ be the finite dimensional irreducible representation of $\mathfrak{sp}(n, \mathbb{C})$ of highest weight λ_i. Theorem 4.12 of [11] assures that $\dim V(\lambda_i)_\mu = \dim F(\lambda_i)_\mu$. Since $F(\lambda_i)$ is a subrepresentation of $\wedge^i \mathbb{C}^{2n}$ with the action of $\mathfrak{sp}(n, \mathbb{C})$ on $\mathbb{C}^{2n}$ given by the standard representation, then it is easily seen that the weights of $V(\lambda_i)$ are in the $\mathcal{W}$-orbit of λ_j, with $j \leqslant i$ and $i - j$ even. In turns, this implies that, by the $\mathcal{W}$-invariance of Y_i,

$$Y_i = \sum_{j \leqslant i} c_{ij}\, m(-2\lambda_j) = \sum_{j \leqslant i} c_{ij}\, m(-\omega_j) = \sum_{j \leqslant i} c_{ij} X_j \tag{7.2}$$

where, clearly, c_{ij} is the coefficient of $e^{-2\lambda_j}$ when we write Y_i as in (7.1).

We deduce that $c_{ij} = \dim V(\lambda_i)_{\lambda_j}$, hence $c_{ii} = 1$. It follows that we can invert the matrix (c_{rs}) and write

$$X_i = \sum_{j \leqslant i} d_{ij} Y_j.$$

This concludes the proof.

7.2. Case AIII.

In this case R is of type A_{n-1}. We can choose $\alpha_{i_0} = \epsilon_p - \epsilon_{p+1}$ and assume also that $p \geqslant q$ where $q = n - p$. We have that $\delta_i = \epsilon_i - \epsilon_{n+1-i}$ for $i = 1, \dots, q$, hence $r = q$. If $i = 1, \dots, r$, then we set $\lambda_i = \sum_{j=1}^{i} \epsilon_i - \frac{i}{n} \sum_{j=1}^{n} \epsilon_j$. Arguing as in §7.1, we see that the weights occurring in $V(\lambda_i)$ are in the $\mathcal{W}$-orbit of λ_j, with $j \leqslant i$. If $\alpha \in R$, then it is easy to check that $(\lambda_j, \alpha) \in \{1, 0, -1\}$. This implies that, if μ is a weight occurring in $V(\lambda_i)$, then

$$(\mu, \delta_h) = (w(\lambda_j), \delta_h) = (\lambda_j, w^{-1}(\delta_h)) \in \{1, 0, -1\},$$

hence $\pi(2\mu) = \sum_h a_h \delta_h$ with $a_h \in \{1, 0, -1\}$. It follows that $\pi(2\mu)$ is in the $\mathcal{W}_{\mathbb{R}}$-orbit of ω_j for some j. By the $\mathcal{W}_{\mathbb{R}}$-invariance of Y_i we obtain that

$$Y_i = \sum_j c_{ij} m(-\omega_j).$$

Moreover, μ is a weight occurring in $V(\lambda_i)$ only if $\mu = \lambda_i - \sum n_\alpha \alpha$, with $n_\alpha \in \mathbb{N}$ and $\alpha \in R^+$. Since ω_r is dominant for R^+ we see that, if $\pi(2\mu) = \omega_j$, then

$$2j = (\pi(2\mu), \omega_r) = (2\mu, \omega_r) \leqslant (2\lambda_i, \omega_r) = 2i,$$

hence

$$Y_i = \sum_{j \leqslant i} c_{ij} X_j.$$

To conclude the argument we need only check that $c_{ii} \neq 0$. Indeed we prove

LEMMA 7.2.

$$Y_i = \sum_{j \leqslant i} c_{ij} X_j$$

with $c_{ii} = 1$.

Proof. — By (6.3)

$$Y_i = \Gamma(\overline{z}_i) = \sum_{\mu \in P_{\mathbb{R}}} \left(\sum_{\pi(\nu)=\mu} \dim V(\lambda_i)_\nu q^{-2(\nu,\rho_M)} \right) e^{-2\mu},$$

so c_{ii} is the coefficient of $e^{-\omega_i}$ in the expression above.

Suppose that ν is a weight of $V(\lambda_i)$ such that $\pi(2\nu) = \omega_i$. We notice that $\pi(2\lambda_i) = \omega_i$ so we have that $(\nu, \delta_j) = (\lambda_i, \delta_j)$ for $j = 1, \ldots, r$.

We write $\nu = \lambda_i - \sum n_h \alpha_h$, with $n_h \in \mathbb{N}$. Hence

$$(\nu, \delta_j) = (\lambda_i, \delta_j) - \left(\sum n_h \alpha_h, \delta_j\right) = (\nu, \delta_j) - \left(\sum n_h \alpha_h, \delta_j\right)$$

so $(\sum n_h \alpha_h, \delta_j) = 0$ for any j, or, equivalently,

$$\left(\sum n_h \alpha_h, \omega_j\right) = 0$$

for any j. Since ω_j is dominant with respect to R^+, we obtain that if $n_h \neq 0$ then $(\alpha_h, \omega_j) = 0$, hence $\alpha_h \in R_0$. Set $\Pi_0 = \Pi \cap R_0$. We have proven that

$$\nu = \lambda_i - \sum_{\alpha_h \in \Pi_0} n_h \alpha_h. \tag{7.3}$$

Let $\mathbf{M}^-$ be the subalgebra of $\mathbf{U}^-$ generated by the set $\{F_h \mid \alpha_h \in \Pi_0\}$. Fix v^+ to be a highest weight vector for $V(\lambda_i)$ and set $W = \mathbf{M}^- v^+$. By (7.3) the weight ν can only occur in W. On the other hand, since $(\lambda_i, \alpha) = 0$ for any $\alpha \in R_0$, by the standard theory of finite dimensional representations of $\mathbf{U}$ we see that W is one dimensional, so $\nu = \lambda_i$. We can conclude that

$$c_{ii} = \dim V(\lambda_i)_{\lambda_i} q^{(-2\lambda_i, \rho_M)} = 1$$

as desired. □

7.3. Case DIII.

This case is more difficult and somewhat surprising. Here R is of type D_n and we can choose $\alpha_{i_0} = \epsilon_{n-1} + \epsilon_n$. The roots δ_i are given by $\delta_i = \epsilon_{2i-1} + \epsilon_{2i}$ for $i = 1, \ldots, [n/2]$, so $r = [n/2]$.

In this case we choose $\lambda_i = \sum_{j=1}^{i} \epsilon_j$. Arguing as in §7.1 we see that the weights of $V(\lambda_i)$ are in the $\mathcal{W}$-orbit of λ_j, with $j \leqslant i$ and $i - j$ even. It follows that, if μ is a weight of $V(\lambda_i)$ then

$$(\mu, \delta_h) = (w(\lambda_j), \delta_h) = (\lambda_j, w^{-1}(\delta_h)) \in \{2, 1, 0, -1, -2\}.$$

By the $\mathcal{W}_{\mathbb{R}}$ invariance of Y_i, we can write

$$Y_i = \sum a_{ilf} m(-\mu_{lf})$$

where $\mu_{lf} = \omega_l + \omega_f$ with $l \leqslant f$ $(\omega_0 = 0)$ and

$$a_{ilf} = \sum_{\pi(\nu)=\mu_{lf}} \dim V(\lambda_i)_\nu q^{-2(\nu,\rho_M)}.$$

Moreover, μ is a weight occurring in $V(\lambda_i)$ only if $\mu = \lambda_i - \sum n_\alpha \alpha$, with $n_\alpha \in \mathbb{N}$ and $\alpha \in R^+$. Since ω_r is dominant for R^+ we see that, if $\pi(2\mu) = \mu_{lf}$, then

$$2(l+f) = (\pi(2\mu), \omega_r) = (2\mu, \omega_r) \leqslant (2\lambda_i, \omega_r) = 2i,$$

hence

$$Y_i = \sum_{l+f \leqslant i} a_{ilf} m(-\mu_{lf}).$$

We need to compute the coefficients a_{ilf} more precisely:

LEMMA 7.3.

$$Y_i = \sum_{l+f=i} (q+q^{-1})^{(f-l)} m(-\mu_{lf}) + \sum_{l+f<i} a_{ilf} m(-\mu_{lf}).$$

Proof. — If a weight μ occurs in $V(\lambda_i)$ then we can write

$$\mu = \sum_s a_s \epsilon_s$$

with $a_s \in \{1, 0, -1\}$ and $\sum |a_s| \leqslant i$. If moreover $\pi(2\mu) = \mu_{lf}$ with $l+f = i$, then we have that $a_{2s-1} + a_{2s} = 2$ for $s \leqslant l$ and $a_{2s-1} + a_{2s} = 1$ for $l+1 \leqslant s \leqslant f$. We deduce that $a_s = 1$ for $s \leqslant 2l$ and either $a_{2s} = 1,\ a_{2s-1} = 0$ or $a_{2s} = 0,\ a_{2s-1} = 1$ for $l+1 \leqslant s \leqslant f$. Since we are assuming that $l+f = i$, the fact that $\sum |a_s| \leqslant i$ implies that $a_s = 0$ for $s > 2f$. We can therefore write

$$\mu = \sum_{s=1}^{2l} \epsilon_s + \epsilon_{s_1} + \ldots + \epsilon_{s_{f-l}}$$

where $s_h \in \{2l+2h-1, 2l+2h\}$ for $h = 1, \dots, f-l$. Since such a weight μ is in the $\mathcal{W}$-orbit of λ_i we see that $\dim V(\lambda_i)_\mu = 1$ while $q^{-2(\mu,\rho_M)} = q^{t-p}$ where

$$p = \sharp\{h \mid s_h = 2l+2h-1\} \quad t = \sharp\{h \mid s_h = 2l+2h\}.$$

Clearly $t = f-l-p$, so we have that $q^{-2(\mu,\rho_M)} = q^{f-l-2p}$. It follows that the coefficient of μ_{lf} in Y_i when $l+f=i$ is given by

$$a_{ilf} = \sum_{p=0}^{f-l} \binom{f-l}{p} q^{f-l-2p} = (q+q^{-1})^{f-l}$$

as desired. □

LEMMA 7.4.

$$X_iX_j = \sum_{l+f=i+j} \binom{f-l}{i-l} m(-\mu_{lf}) + \sum_{l+f<i+j} k_{lf} m(-\mu_{lf}).$$

Proof. — We observe that, if $\lambda \in P$ then $m(-\lambda)$ occurs in X_iX_j if and only if there is a weight ν_1 in the $\mathcal{W}_\mathbb{R}$-orbit of $-\omega_i$, and a weight ν_2 in the $\mathcal{W}_\mathbb{R}$-orbit of $-\omega_j$ such that $\nu_1+\nu_2 = -\lambda$. If ν is in the $\mathcal{W}_\mathbb{R}$-orbit of $-\omega_i$ then

$$\nu = \sum a_s\delta_s$$

with $a_s \in \{1,0,-1\}$ and $\sum |a_s| = i$. Hence, if we write

$$\nu_1 = \sum a_s\delta_s \quad \nu_2 = \sum b_s\delta_s \quad \lambda = \sum c_s\delta_s$$

then $-c_s = a_s+b_s \in \{2,1,0,-1,-2\}$. This implies that λ is in the $\mathcal{W}_\mathbb{R}$-orbit of μ_{lf} for some l, f, hence $m(-\lambda) = m(-\mu_{lf})$ and we can write

$$X_iX_j = \sum k_{lf} m(-\mu_{lf}).$$

To compute the coefficient k_{lf} we need only compute the coefficient of $e^{-\mu_{lf}}$ when we write X_iX_j as $\sum k_\mu e^\mu$. By taking $\lambda = \mu_{lf}$ in the argument above, then we find that

$$l+f = \sum c_s \leqslant \sum |a_s| + \sum |b_s| = i+j$$

hence

$$X_iX_j = \sum_{l+f\leqslant i+j} k_{lf} m(-\mu_{lf}).$$

It remains only to find the coefficient k_{lf} of $m(-\mu_{lf})$ when $l+f=i+j$. We need to count the number of pairs (ν_1,ν_2) such that ν_1 is in the $\mathcal{W}_{\mathbb{R}}$-orbit of ω_i, ν_2 is in the $\mathcal{W}_{\mathbb{R}}$-orbit of ω_j, and $\nu_1+\nu_2=\mu_{lf}$. The fact that $\nu_1+\nu_2=\mu_{lf}$ implies that $a_s=b_s=1$ if $s\leqslant l$, and either $a_s=0,\ b_s=1$ or $a_s=1,\ b_s=0$ if $l+1\leqslant s\leqslant f$. Since $l+f=i+j$ we see that $a_s=b_s=0$ if $f+1\leqslant s\leqslant r$. Hence

$$\nu_1=\sum_{s=1}^{l}\delta_s+\delta_{s_1}+\ldots\delta_{s_{i-l}}\qquad \nu_2=\mu_{lf}-\nu_1$$

with $l+1\leqslant s_1<s_2<\ldots<s_{i-l}\leqslant f$.

It follows that the number of pairs (ν_1,ν_2) as above is equal to the number of choices for the indices $s_1<s_2<\ldots<s_{i-l}$ that is clearly equal to $\binom{f-l}{i-l}$. □

LEMMA 7.5.

$$m(-\mu_{lf})=Q_{lf}(X_1,\ldots,X_{l+f})$$

where Q_{lf} is a polynomial.

Proof. — Obviously the point of this result is the fact that $m(-\mu_{lf})$ does not depend on $X_{l+f+1},\ldots,X_r$. This fact can be derived from Lemma 7.4. Here we give a more natural proof.

If $\lambda\in 2P_{\mathbb{R}}$ then $\lambda=\sum n_i\delta_i$ with $n_i\in\mathbf{Z}$. We set $|\lambda|=\sum n_i$. Since q is not a root of unity the set $\{e^{\lambda}\mid\lambda\in 2P_{\mathbb{R}}\}$ is a basis of $\mathbb{F}_q[2P_{\mathbb{R}}]$. Hence we can define a degree d on $\mathbb{F}_q[2P_{\mathbb{R}}]$ by setting $d(e^{-\lambda})=|\lambda|$. Since $\mathbb{F}_q[2P_{\mathbb{R}}]$ is a domain we see that if $f,g\in\mathbb{F}_q[2P_{\mathbb{R}}]$ then $d(fg)=d(f)+d(g)$.

We compute that $d(m(-\mu_{lf}))=l+f$ and that $d(X_i)=i$.

Since $m(-\mu_{lf})\in\mathbb{F}_q[2P_{\mathbb{R}}]^{\mathcal{W}_{\mathbb{R}}}$ then there is a polynomial Q_{lf} such that

$$m(-\mu_{lf})=Q_{lf}(X_1,\ldots,X_r),$$

and Q_{lf} does not depend on $X_{l+f+1},\ldots,X_r$ otherwise the degree of $m(-\mu_{lf})$ would be bigger than $l+f$. □

Our result in case **DIII** depends on the following formulas:

LEMMA 7.6. — *Fix $i\in\{1,\ldots,r\}$ and set $m=[i/2]$. If $i=1$, then*

$$Y_1-(q+q^{-1})X_1=k\ \textit{with}\ k\in\mathbb{F}. \tag{7.4}$$

If $i > 1$, then for each $t = 0, \dots, m-1$ there is a polynomial $p_t(X_1, \dots, X_{i-1})$ such that, if i is even,

$$(7.5)\quad Y_i + p_t(X_1, \dots, X_{i-1}) - X_m^2 - \sum_{j=1}^{t}(q^{2j} + q^{-2j})X_{m-j}X_{m+j} = \sum_{s=t+1}^{m}\Big(\sum_{h=0}^{s-t-1}\binom{2s}{h}(q^{2(s-h)} + q^{-2(s-h)})\Big)m(-\mu_{m-s,m+s})$$

while, if i is odd,

$$(7.6)\quad Y_i + p_t(X_1, \dots, X_{i-1}) - \sum_{j=0}^{t}(q^{2j+1} + q^{-2j-1})X_{m-j}X_{m+1+j} = \sum_{s=t+1}^{m}\Big(\sum_{h=0}^{s-t-1}\binom{2s+1}{h}(q^{2(s-h)+1} + q^{-2(s-h)-1})\Big)m(-\mu_{m-s,m+1+s}).$$

Proof. — The proof is by induction on t. First of all we prove (7.5): if i is even and $t = 0$ then by Lemma 7.3 we can write

$$Y_i = \sum_{l+f=i}(q + q^{-1})^{(f-l)}m(-\mu_{lf}) + \sum_{l+f<i} a_{ilf}m(-\mu_{lf}).$$

By Lemma 7.5 there is a polynomial $p'(X_1, \dots, X_{i-1})$ such that

$$Y_i + p'(X_1, \dots, X_{i-1}) = \sum_{l+f=i}(q + q^{-1})^{(f-l)}m(-\mu_{lf}) = \sum_{s=0}^{m}(q + q^{-1})^{2s}m(-\mu_{m-s,m+s}).$$

By Lemma 7.4 we have that

$$X_m^2 = \sum_{s=0}^{m}\binom{2s}{s}m(-\mu_{m-s,m+s}) + \sum_{l+f<i} k_{lf}m(-\mu_{lf})$$

so, by applying Lemma 7.5 we can find a polynomial $p''(X_1, \dots, X_{i-1})$ such that

$$X_m^2 = \sum_{s=0}^{m}\binom{2s}{s}m(-\mu_{m-s,m+s}) + p''(X_1, \dots, X_{i-1}).$$

Substituting we find that, setting $p_0 = p' + p''$,

$$Y_i + p_0(X_1, \dots, X_{i-1}) - X_m^2 = \sum_{s=1}^{m}((q+q^{-1})^{2s} - \binom{2s}{s})m(-\mu_{m-s,m+s})$$
$$= \sum_{s=1}^{m}(\sum_{h=0}^{s-1}\binom{2s}{h}(q^{2(s-h)} + q^{-2(s-h)}))m(-\mu_{m-s,m+s}).$$

The inductive step is similar: by Lemma 7.4 and Lemma 7.5

$$X_{m-t}X_{m+t} = \sum_{l+f=i}\binom{f-l}{m-t-l}m(-\mu_{lf}) + \sum_{l+f<i} k_{lf}m(-\mu_{lf})$$
$$= \sum_{s=t}^{m}\binom{2s}{s-t}m(-\mu_{m-s,m+s}) + p'(X_1, \dots, X_{i-1})$$

and, by the induction hypothesis,

$$Y_i + p_{t-1}(X_1, \dots, X_{i-1}) - X_m^2 - \sum_{j=1}^{t-1}(q^{2j} + q^{-2j})X_{m-j}X_{m+j} =$$
$$= \sum_{s=t}^{m}(\sum_{h=0}^{s-t}\binom{2s}{h}(q^{2(s-h)} + q^{-2(s-h)}))m(-\mu_{m-s,m+s}).$$

Substituting in (7.5) and setting $p_t = p_{t-1} + (q^{2t} + q^{-2t})p'$ we find that

$$Y_i + p_t(X_1, \dots, X_{i-1}) - X_m^2 - \sum_{j=1}^{t}(q^{2j} + q^{-2j})X_{m-j}X_{m+j}$$
$$= \sum_{s=t}^{m}(\sum_{h=0}^{s-t}\binom{2s}{h}(q^{2(s-h)} + q^{-2(s-h)}) - \binom{2s}{s-t}(q^{2t} + q^{-2t}))m(-\mu_{m-s,m+s})$$
$$= \sum_{s=t+1}^{m}(\sum_{h=0}^{s-t-1}\binom{2s}{h}(q^{2(s-h)} + q^{-2(s-h)}))m(-\mu_{m-s,m+s})$$

as desired.

The case i is odd is proved in the same way. □

Finally we are ready to prove that the hypotheses of Theorem 6.5 hold in this case:

PROPOSITION 7.7. — *There are polynomials P_i such that*

$$X_i = P_i(Y_1, \dots, Y_r).$$

Proof. — We prove this result by induction on i: if $i = 1$ then (7.4) says that

$$Y_1 = (q + q^{-1})X_1 + cost.$$

so, since q is not a root of unity, we can write

$$X_1 = (q + q^{-1})^{-1}(Y_1 - cost.).$$

The inductive step is similar: by setting $t = m - 1$ in the formulas of Lemma 7.6 we find that there is a polynomial Q_i such that

$$Y_i + Q_i(X_1, \dots, X_{i-1}) = (q^i + q^{-i})X_i$$

so, since q is not a root of unity, we can write

$$X_i = (q^i + q^{-i})^{-1}(Y_i + Q_i(X_1, \dots, X_{i-1})).$$

The result follows immediately from the induction hypothesis. □

7.4. Case BDI and R of type B_{n+1}.

In this case R is of type B_{n+1}. The case when $n = 1$ has already been discussed in case **CI** hence we can assume $n \geqslant 2$. We can choose $\alpha_{i_0} = \epsilon_1 - \epsilon_2$, thus $r = 2$ and $\delta_1 = \epsilon_1 + \epsilon_2$, $\delta_2 = \epsilon_1 - \epsilon_2$. If we define $\lambda_1 = \sum_{i=1}^{n+1} \frac{1}{2}\epsilon_i$ and $\lambda_2 = \epsilon_1$, then this choice satisfies the hypothesis of Lemma 6.4. Since λ_1 is the highest weight of the spin representation, the weights of $V(\lambda_1)$ are all the expressions $1/2(\pm\epsilon_1 + \ldots \pm \epsilon_{n+1})$ while the weights of λ_2 are $\pm\epsilon_j$ with $j = 1, \dots, n+1$. In both cases the multiplicity of a weight is one.

We now proceed to compute Y_i, $i = 1, 2$.

Suppose ν is a weight of $V(\lambda_1)$ and write

$$\nu = \frac{1}{2}\sum_{i=1}^{n+1} c_i\epsilon_i$$

with $c_i = \pm 1$. Recall that we normalized the form $(\ ,\ )$ so that $(\epsilon_i, \epsilon_i) = 2$.

Then

$$\pi(\nu) = \frac{1}{4}(\nu,\delta_1)\delta_1 + \frac{1}{4}(\nu,\delta_2)\delta_2 = \frac{1}{2}(\frac{c_1+c_2}{2})\delta_1 + \frac{1}{2}\left(\frac{c_1-c_2}{2}\right)\delta_2.$$

By inspection one easily sees that $\pi(\nu)$ is either equal to $\pm\frac{1}{2}\delta_1$ or to $\pm\frac{1}{2}\delta_2$, and thus $\pi(2\nu)$ is conjugate to $\omega_1 = \delta_1$. Therefore by the $\mathcal{W}_{\mathbb{R}}$-invariance of Y_1 we obtain that

$$Y_1 = \left(\sum_{\pi(2\nu)=\omega_1} q^{-2(\nu,\rho_M)}\right) m(-\omega_1).$$

In order to compute the coefficient we argue as before: suppose that $\pi(2\nu) = \omega_1$, then $c_1+c_2 = 2$, i. e. $c_1 = c_2 = 1$. Since $\rho_M = \sum_{j=3}^{n+1}\left(n-j+\frac{3}{2}\right)\epsilon_j$, we find that

$$q^{-2(\nu,\rho_M)} = q^{-\sum_{j=3}^{n+1} c_j(2n-2j+3)}$$

and

$$\sum_{\pi(2\nu)=\omega_1} q^{-2(\nu,\rho_M)} = \sum_{c_3,\dots,c_{n+1}} q^{-\sum_{j=3}^{n+1} c_j(2n-2j+3)}$$
$$= \sum_{a_1,\dots,a_{n-1}} q^{-\sum_{j=1}^{n-1} a_j(2n-2j-1)}$$

where $a_i = c_{i+2}$, $a_i = \pm 1$.

Set

$$A_n = \{(a_1,\dots,a_n) \mid a_i = \pm 1\}.$$

We claim that

$$\sum_{(a_1,\dots,a_{n-1})\in A_{n-1}} q^{-\sum_{j=1}^{n-1} a_j(2(n-j)-1)} = \prod_{j=0}^{n-2}(q^{(2j+1)} + q^{-(2j+1)}).$$

We prove it by induction: if $n = 2$ we have

$$\sum_{a=\pm 1} q^{-a} = q + q^{-1}.$$

Suppose it is true up to $n-2$ and let us prove it for $n-1$:

$$\begin{aligned}
\sum_{(a_1,\dots,a_{n-1})\in A_{n-1}} & q^{-\sum_{j=1}^{n-1} a_j(2(n-j)-1)} \\
&= \sum_{(a_2,\dots,a_{n-1})\in A_{n-2}} q^{-2(n-2)-1} q^{-\sum_{j=2}^{n-1} a_j(2(n-j)-1)} \\
&+ \sum_{(a_2,\dots,a_{n-1})\in A_{n-2}} q^{2(n-2)+1} q^{-\sum_{j=2}^{n-1} a_j(2(n-j)-1)} \\
&= q^{-2(n-2)-1} \sum_{(b_1,\dots,b_{n-2})\in A_{n-2}} q^{-\sum_{j=1}^{n-2} b_j(2(n-1-j)-1)} \\
&+ q^{2(n-2)+1} \sum_{(b_1,\dots,b_{n-2})\in A_{n-2}} q^{-\sum_{j=1}^{n-2} b_j(2(n-1-j)-1)} \\
&= (q^{-2(n-2)-1} + q^{2(n-2)+1}) \prod_{j=0}^{n-3} (q^{(2j+1)} + q^{-(2j+1)})
\end{aligned}$$

as desired.

In particular

$$Y_1 = \left(\prod_{j=0}^{n-2} (q^{(2j+1)} + q^{-(2j+1)})\right) m(-\omega_1) = p(q)m(-\omega_1).$$

We now compute Y_2. In this case if ν is a weight of $V(\lambda_2)$, then $\nu = \sum_{i=1}^{n+1} c_i\epsilon_i$ with $\sum |c_i| = 1$; thus $\pi(2\nu)$ is conjugate to ω_2 and

$$Y_2 = \left(\sum_{\pi(2\nu)=\omega_2} q^{-2(\nu,\rho_M)}\right) m(-\omega_2).$$

If $\pi(2\nu) = \omega_2$ then $\nu = \lambda_2$ and so $Y_2 = X_2$.

Therefore $Y_1 = p(q)X_1$ and $Y_2 = X_2$. Since q is not a root of unity we have that $p(q) \neq 0$, hence our result.

7.5. Case BDI and R of type D_{n+1}.

In this case R is of type D_{n+1}. We can assume $n \geqslant 3$ for the case $n = 2$ has already been discussed in §7.2. We can choose $\alpha_{i_0} = \epsilon_1 - \epsilon_2$, thus $r = 2$

and $\delta_1 = \epsilon_1 + \epsilon_2$, $\delta_2 = \epsilon_1 - \epsilon_2$. If we define $\lambda_1 = \sum_{i=1}^{n+1} \frac{1}{2}\epsilon_i$ and $\lambda_2 = \epsilon_1$, then this choice satisfies the hypothesis of Lemma 6.4. We recall that since λ_1 is the highest weight of the spin plus representation, the weights of $V(\lambda_1)$ are all the expressions $1/2(\pm\epsilon_1 + \ldots \pm \epsilon_{n+1})$ with an even number of minus signs while the weights of λ_2 are $\pm\epsilon_j$ with $j = 1, \ldots, n+1$. In both cases the multiplicity of a weight is one.

We now proceed to compute Y_i, $i = 1, 2$.

Suppose ν is a weight of $V(\lambda_1)$ and write

$$\nu = \frac{1}{2}\sum_{i=1}^{n+1} c_i \epsilon_i$$

with $c_i = \pm 1$, then

$$\pi(\nu) = \frac{1}{2}(\nu, \delta_1)\delta_1 + \frac{1}{2}(\nu, \delta_2)\delta_2 = \frac{1}{2}\left(\frac{c_1 + c_2}{2}\right)\delta_1 + \frac{1}{2}\left(\frac{c_1 - c_2}{2}\right)\delta_2.$$

By inspection one easily sees that $\pi(\nu)$ is either equal to $\pm\frac{1}{2}\delta_1$ or to $\pm\frac{1}{2}\delta_2$; thus $\pi(2\nu)$ is conjugate to $\omega_1 = \delta_1$. Therefore by the $\mathcal{W}_{\mathbb{R}}$-invariance of Y_1 we obtain

$$Y_1 = \left(\sum_{\pi(2\nu)=\omega_1} q^{-2(\nu,\rho_M)}\right) m(-\omega_1).$$

In order to compute the coefficient we argue as follows: suppose that $\pi(2\nu) = \omega_1$, then $c_1 + c_2 = 2$, i. e. $c_1 = c_2 = 1$. Since $\rho_M = \sum_{j=3}^{n+1}(n-j+1)\epsilon_j$, we find that

$$q^{-2(\nu,\rho_M)} = q^{-\sum_{j=3}^{n+1} c_j(n-j+1)}$$

and

$$\sum_{\pi(2\nu)=\omega_1} q^{-2(\nu,\rho_M)} = \sum_{c_3,\ldots,c_{n+1}} q^{-\sum_{j=3}^{n+1} c_j(n-j+1)} = \sum_{a_1,\ldots,a_{n-1}} q^{-\sum_{j=1}^{n-1} a_j(n-j-1)}$$

where $a_i = c_{i+2}$, $a_i = \pm 1$, and we have an even number of minus signs.

Set

$$A_n^+ = \{(a_1, \dots, a_n) \mid a_i = \pm 1, \text{ an even number of } -1\}$$

$$A_n^- = \{(a_1, \dots, a_n) \mid a_i = \pm 1, \text{ an odd number of } -1\}.$$

We claim that

$$\sum_{(a_1,\dots,a_{n-1}) \in A_{n-1}^{\pm}} q^{-\sum_{j=1}^{n-1} a_j(n-j-1)} = \prod_{j=1}^{n-2} (q^j + q^{-j}).$$

We prove this by induction: if $n = 3$ we have $A_2^+ = \{(1,1), (-1,-1)\}$, so

$$\sum_{(a_1,a_2) \in A_2^+} q^{-\sum_{j=1}^{2} a_j(2-j)} = q + q^{-1}$$

while $A_2^- = \{(1,-1), (-1,+1)\}$, so

$$\sum_{(a_1,a_2) \in A_2^-} q^{-\sum_{j=1}^{2} a_j(2-j)} = q + q^{-1}.$$

Suppose true up to $n-2$ and let us prove it for $n-1$:

$$\begin{aligned}
\sum_{(a_1,\dots,a_{n-1}) \in A_{n-1}^+} q^{-\sum_{j=1}^{n-1} a_j(n-j-1)} = & \sum_{(a_2,\dots,a_{n-1}) \in A_{n-2}^+} q^{-(n-2)} q^{-\sum_{j=2}^{n-1} a_j(n-j-1)} \\
& + q^{(n-2)} \sum_{(a_2,\dots,a_{n-1}) \in A_{n-2}^-} q^{-\sum_{j=2}^{n-1} a_j(n-j-1)} \\
& = q^{-(n-2)} \sum_{(b_1,\dots,b_{n-2}) \in A_{n-2}^+} q^{-\sum_{j=1}^{n-2} b_j(n-j-2)} \\
& + q^{(n-2)} \sum_{(b_1,\dots,b_{n-2}) \in A_{n-2}^-} q^{-\sum_{j=1}^{n-2} b_j(n-j-2)} \\
& = (q^{(n-2)} + q^{-(n-2)}) \prod_{j=1}^{n-3} (q^j + q^{-j}),
\end{aligned}$$

while

$$\sum_{(a_1,\dots,a_{n-1})\in A^-_{n-1}} q^{-\sum_{j=1}^{n-1} a_j(n-j-1)} = \sum_{(a_2,\dots,a_{n-1})\in A^-_{n-2}} q^{-(n-2)} q^{-\sum_{j=2}^{n-1} a_j(n-j-1)}$$

$$+ \sum_{(a_2,\dots,a_{n-1})\in A^+_{n-2}} q^{(n-2)} q^{-\sum_{j=2}^{n-1} a_j(n-j-1)}$$

$$= q^{-(n-2)} \sum_{(b_1,\dots,b_{n-2})\in A^-_{n-2}} q^{-\sum_{j=1}^{n-2} b_j(n-j-2)}$$

$$+ q^{(n-2)} \sum_{(b_1,\dots,b_{n-2})\in A^+_{n-2}} q^{-\sum_{j=1}^{n-2} b_j(n-j-2)}$$

$$= (q^{(n-2)} + q^{-(n-2)}) \prod_{j=1}^{n-3} (q^j + q^{-j}).$$

In particular

$$Y_1 = \left(\prod_{j=1}^{n-2} (q^j + q^{-j}) \right) m(-\omega_1) = p(q) m(-\omega_1).$$

We now compute Y_2. In this case if ν is a weight of $V(\lambda_2)$, then $\nu = \sum_{i=1}^{n+1} c_i \epsilon_i$ with $\sum |c_i| = 1$, thus $\pi(2\nu)$ is conjugate to ω_2 and

$$Y_2 = \left(\sum_{\pi(2\nu)=\omega_2} q^{-2(\nu,\rho_M)} \right) m(-\omega_2).$$

If $\pi(2\nu) = \omega_2$ then $\nu = \lambda_2$ and so $Y_2 = X_2$.

Therefore $Y_1 = p(q) X_1$ and $Y_2 = X_2$. Since q is not a root of unity we have $p(q) \neq 0$, hence our result.

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W. BALDONI,
Università degli Studi di Roma "Tor Vergata"
Dipartimento di Matematica
Via della Ricerca Scientifica
00133 Roma (Italy).
&
P. MÖSENEDER FRAJRIA,
Università degli Studi di Trento
Dipartimento di Matematica
38050 Povo (Trento) (Italy).