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ON COMPACT KÄHLER SURFACES

by Nicholas BUCHDAHL

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0. Introduction.

As a consequence of the Hodge identities [GH], a compact complex manifold which admits a Kähler metric must have all of its odd-dimensional Betti numbers even. In [MK], p. 85, Kodaira conjectured that for complex surfaces the converse is true; i.e., that every compact complex surface X with $b_1(X)$ even admits a Kähler metric.

By virtue of Kodaira's own classification of compact complex surfaces, many special cases of the conjecture automatically follow, leaving only the cases of elliptic surfaces and K3 surfaces. Miyaoka [M] proved that the conjecture holds for elliptic surfaces, and this was reproved by Harvey and Lawson [HL] using the theory of currents; this left only the K3 surfaces.

In [T], Todorov presented a proof that every K3 surface admits a Kähler metric using Yau's then recently-proved results [Y] on the Calabi conjecture. Unfortunately, as pointed out by Siu in his review [S2], Todorov's proof contained some serious errors. In his own paper [S3], Siu was able to overcome the difficulties in Todorov's proof to finally prove that every K3 surface is Kähler, thereby completing the proof of Kodaira's conjecture.

A unified proof of the conjecture which does not invoke the classification of compact complex surfaces has hitherto remained lacking. Furthermore, if a given surface X does admit a Kähler metric, in general it is

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not known *how many* such metrics there are; i.e., which classes in $H^{1,1}_{\mathbb{R}}(X)$ can be represented by positive closed $(1,1)$-forms. The well-known Nakai-Moishezon criterion provides an answer in the case of integral cohomology classes: a class $\rho \in H^{1,1}(X) \cap H^2(X,\mathbb{Z})$ can be represented by such a form if and only if it satisfies $\rho \cdot \rho > 0$ and $\rho \cdot [D] > 0$ for every effective divisor D on X; (see e.g., [BPV]). Campana and Peternell [CP] have generalised this result to the case of real cohomology classes on projective algebraic varieties, but a general characterisation remains lacking.

In this paper, the following results are proved, respectively given in Theorem 11, Corollary 15 and Theorem 16 of §4:

THEOREM. — *A compact complex surface X with $b_1(X) \equiv 0 (\mathrm{mod}\, 2)$ admits a Kähler metric.*

THEOREM. — *Let X be a compact complex surface with $b_1(X) \equiv 0\,(\mathrm{mod}\, 2)$ and let $\rho \in H^{1,1}_{\mathbb{R}}(X)$ be a class satisfying $\rho \cdot \rho > 0$, $\rho \cdot [D] > 0$ and $\rho \cdot [\omega] > 0$ for every effective divisor $[D]$ on X and some positive closed $(1,1)$-form ω on X. Then ρ can be represented by a positive closed $(1,1)$-form.*

THEOREM. — *Suppose $b_1(X)$ is even and $\omega \in \Lambda^{1,1}_{\mathbb{R}}(X)$ is $\bar\partial\partial$-closed and positive. If $\varphi \in \Lambda^{1,1}_{\mathbb{R}}(X)$ is $\bar\partial\partial$-closed and satisfies $\int_X \varphi^2 > 0$, $\int_X \omega \wedge \varphi > 0$ and $\int_D \varphi > 0$ for every effective divisor $D \subset X$, then φ is homologous to a smooth $\bar\partial\partial$-closed positive $(1,1)$-form modulo the image of $\bar\partial\partial$.*

The key ingredients in the proofs are Gauduchon's results [G] on the existence of $\bar\partial\partial$-closed positive $(1,1)$-forms on compact surfaces, Siu's theorem [S1] on the analyticity of the sets associated with the Lelong numbers of closed positive currents, and Demailly's results [D1], [D2] on the smoothing of positive closed $(1,1)$-currents.

1. Preliminaries.

The purpose of this section is to establish notation, review some well-known facts concerning Kähler metrics (details of which can be found in [GH] §7 Chapter 0) and to establish some basic results.

Let X be a compact complex manifold. Denote by $\Lambda^{p,q}$ the sheaf of germs of smooth (p,q)-forms on X, and set $\Lambda^{p,q}(X) := \Gamma(X, \Lambda^{p,q})$.

A hermitian metric h on X corresponds to a positive $(1,1)$-form $\omega \in \Lambda^{1,1}_{\mathbb{R}}(X)$, where the subscript denotes invariance under complex conjugation. In local holomorphic coordinates $\{z^a\}$, ω is given by $\omega = \frac{i}{2} h_{a\bar{b}} dz^a \wedge dz^{\bar{b}}$ if $h = h_{a\bar{b}} dz^a \otimes dz^{\bar{b}}$. The form ω is the Kähler form associated to the metric, and the metric itself is Kähler if $d\omega = 0$.

The Kähler form ω determines a volume form on X given by $dV = \frac{1}{n!}\omega^n$ where n is the dimension of X. The adjoint of the map $\Lambda^{p,q} \ni f \mapsto \omega \wedge f \in \Lambda^{p+1,q+1}$ is denoted by Λ; up to a combinatorial factor, Λ is contraction with the inverse $h^{a\bar{b}}$ of $h_{a\bar{b}}$. If $d\omega = 0$, the formal adjoints of the operators ∂ and $\bar{\partial}$ on $\Lambda^{p,q}$ are found to be $\partial^* = i\Lambda\bar{\partial} - i\bar{\partial}\Lambda$ and $\bar{\partial}^* = -i\Lambda\partial + i\partial\Lambda$, from which it follows that the $\bar{\partial}$ Laplacian $\triangle'' = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ agrees with the ∂ Laplacian $\triangle'$, each being one half of the full Laplacian $\triangle = dd^* + d^*d$. As a consequence, there is an $\mathbb{R}$-linear isomorphism between $H^p(X, \Omega^q)$ and $H^q(X, \Omega^p)$ determined by complex conjugation of harmonic representatives, and the decomposition of r-forms into forms of type $(p, r-p)$, which is preserved by the Laplacian, shows that the Betti numbers of X of odd degree must be even.

Although X need not admit a Kähler metric, Gauduchon [G] has shown that there is a conformal rescaling of the metric h, unique up to a positive constant, such that the associated form satisfies $\bar{\partial}\partial(\omega^{n-1}) = 0$. Given such a form, the Maximum Principle implies the adjoint of the elliptic operator $P := *(\omega^{n-1} \wedge i\bar{\partial}\partial)$ on functions (i.e., $P^* : f \mapsto *i\bar{\partial}\partial(\omega^{n-1} f)$) has only the constants as kernel since P^* annihilates the constants. It follows from standard L^2 harmonic theory that for each $f \in L^2(X)$ satisfying $\int_X f\, dV = 0$ there is a function $u \in L^2_2(X)$ satisfying $Pu = f$, with u unique up the addition of a constant; here $L^p_k(X)$ denotes the functions in $L^p(X)$ with weak derivatives up to and including order k also in $L^p(X)$. Moreover, standard regularity arguments apply to P, so for example u is smooth if f is smooth.

In the case of a compact complex surface, the splitting of 2-forms into types is compatible with the splitting of forms into self-dual and anti-self-dual parts from the underlying Riemannian metric induced by h; namely, $\Lambda^2_+ \otimes \mathbb{C} = \Lambda^{0,1} \oplus \Lambda^0\omega \oplus \Lambda^{1,0}$ and $\Lambda^2_- \otimes \mathbb{C} = \ker \omega\wedge : \ \Lambda^{1,1} \to \Lambda^{2,2}$. Hence for a real $(1,1)$-form ψ,

$$\begin{aligned} \omega \wedge \psi &= (\Lambda\psi)\, \omega^2/2 \\ *\psi &= (\Lambda\psi)\omega - \psi \\ |\psi|^2\, dV = \psi \wedge *\psi &= (\Lambda\psi)^2\, \omega^2/2 - \psi^2. \end{aligned}$$

The Hilbert space of (p,q)-forms on X with coefficients in L^2 is denoted throughout by $\Lambda^{p,q}\otimes L^2(X)$. Unless otherwise indicated, $\|\cdot\|$ always denotes the L^2 norm induced by ω on this space.

Let $d^{1,1}$ be the composition of $d\colon \Lambda^1 \to \Lambda^2$ with the projection $\Lambda^2 \to \Lambda^{1,1}$.

LEMMA 1. — *Let X be a compact complex surface equipped with a smooth positive $\bar\partial\partial$-closed $(1,1)$-form ω. Then $d^{1,1}\colon \Lambda^1_{\mathbb{R}} \otimes L^2_1(X) \to \Lambda^{1,1}_{\mathbb{R}} \otimes L^2(X)$ has closed range.*

Proof. — Let $\{v_i\}$ be a sequence of real 1-forms on X with coefficients in L^2_1 such that $\psi_i := d^{1,1}v_i$ is converging in L^2 to some $\psi \in \Lambda^{1,1}_{\mathbb{R}} \otimes L^2(X)$. Write $v_i = u_i + \bar u_i$ for some $(0,1)$-form u_i, so $\psi_i = \partial u_i + \bar\partial \bar u_i$.

By smoothing and diagonalising, it can be assumed without loss of generality that u_i is smooth for each i. Using Stokes' Theorem,

$$\begin{aligned}\|\psi_i\|^2 &= \int_X (\Lambda\psi_i)^2\omega^2 - \int_X \psi_i^2 \\ &= \int_X (\Lambda\psi_i)^2\omega^2 + 2\int_X \bar\partial u_i \wedge \partial \bar u_i = 2\|\Lambda\psi_i\|^2 + 2\|\bar\partial u_i\|^2\end{aligned}$$

so it follows that $dv_i = d^{1,1}v_i + \bar\partial u_i + \partial \bar u_i$ is bounded in L^2.

Let $\tilde v_i$ be the L^2 projection of v_i perpendicular to the kernel of d, so $d^*\tilde v_i = 0$ and $\tilde v_i$ is perpendicular to the harmonic 1-forms. Hence there is a constant C such that $\|\tilde v_i\|_{L^2_1} \le C(\|d\tilde v_i\| + \|d^*\tilde v_i\|) \le$ Const., so a subsequence of the sequence $\{\tilde v_i\}$ converges weakly in L^2_1 to some $\tilde v \in \Lambda^1_{\mathbb{R}} \otimes L^2_1(X)$. Since $d^{1,1}\tilde v_i = \psi_i$ it follows $d^{1,1}\tilde v = \psi$, proving the claim. □

LEMMA 2. — *If $\psi \in \Lambda^{1,1}_{\mathbb{R}} \otimes L^2(X)$ is weakly $\bar\partial\partial$-closed there exists $u \in \Lambda^{0,1} \otimes L^2_1(X)$ such that $\psi + \partial u + \bar\partial \bar u$ is smooth.*

Proof. — Let $\tilde\psi$ be the L^2 projection of ψ perpendicular to the image of $d^{1,1}$, so $\tilde\psi = \psi + \partial u + \bar\partial u$ for some $u \in \Lambda^{0,1} \otimes L^2_1(X)$ by Lemma 1. Then $\int_X \tilde\psi \wedge *(\partial v + \bar\partial \bar v) = 0 = \int_X \tilde\psi \wedge *(i\partial v - i\bar\partial v)$ for every $(0,1)$-form v with coefficients in L^2_1, implying $\partial(*\tilde\psi) = 0 = \bar\partial(*\tilde\psi)$ weakly. Hence $\bar\partial\partial(*\tilde\psi) = 0 = \bar\partial\partial\psi$ weakly, so $\bar\partial\partial((\Lambda\tilde\psi)\omega) = 0$ in the sense of distributions. Standard regularity theorems imply that $\Lambda\tilde\psi$ is constant, and then the equations $d^*\tilde\psi = 0$, $d\tilde\psi = (\Lambda\tilde\psi)\, d\omega$ together with elliptic regularity imply $\tilde\psi$ is smooth. □

Remark. — Note that this argument shows that any weakly d-closed $\chi \in \Lambda^{1,1}_{\mathbb{R}} \otimes L^2(X)$ satisfying $\omega \wedge \chi = c\omega^2$ for some constant c is in fact smooth.

LEMMA 3. — *If $\psi \in \Lambda^{1,1}_{\mathbb{R}} \otimes L^2(X)$ is weakly $\bar\partial\partial$-closed there is a sequence of smooth $\bar\partial\partial$-closed real $(1,1)$-forms ψ_i converging to ψ in L^2.*

Proof. — By Lemma 2, $\psi = \tilde\psi - \partial u - \bar\partial\bar u$ for some smooth $\tilde\psi$ and some $(0,1)$-form u with coefficients in L^2_1. Approximating u by smooth $(0,1)$-forms then gives the desired result. □

2. The intersection form on the kernel of $\bar\partial\partial$.

If $b_1(X)$ is even and $h \in H^{1,1}_{\mathbb{R}}(X)$ satisfies $h^2 > 0$, the intersection form on $H^2(X,\mathbb{R})$ restricted to $H^{1,1}_{\mathbb{R}}(X)$ is negative definite on the subspace $\{v \in H^{1,1}_{\mathbb{R}}(X) \mid h \cdot v = 0\}$ (Corollary IV.2.14 [BPV]). Interestingly, on an arbitrary compact complex surface X a similar statement applies to the induced form on the $\bar\partial\partial$-closed $(1,1)$-forms modulo the $\bar\partial\partial$-exact forms, as shown in Proposition 5 below.

LEMMA 4. — *If $\psi \in \Lambda^{1,1}_{\mathbb{R}} \otimes L^2(X)$ is weakly $\bar\partial\partial$-closed, $(\int_X \omega\wedge\psi)^2 \geq (\int_X \omega^2)(\int_X \psi^2)$ with equality if and only if $\psi = c\omega + i\bar\partial\partial g$ for some constant c and some $g \in L^2_2(X)$.*

Proof. — Let $c := (\int_X \omega \wedge \psi)/(\int_X \omega^2)$. If ψ is smooth there is a smooth solution g to the equation $\omega \wedge (\psi - c\omega - i\bar\partial\partial g) = 0$, and $\|\psi - c\omega - i\bar\partial\partial g\|^2 = -\int_X(\psi - c\omega - i\bar\partial\partial g)^2 = -\int_X \psi^2 + (\int_X \psi\wedge\omega)^2/\int_X \omega^2$.

If ψ is not smooth, the inequality follows from the smooth case after approximating ψ using Lemma 3.

If $(\int_X \omega \wedge \psi)^2 = (\int_X \omega^2)(\int_X \psi^2)$, choose a sequence of smooth $\bar\partial\partial$-closed forms $\{\psi_j\}$ converging to ψ in L^2, and let g_j satisfy $\omega \wedge (\psi_j - c_j\omega - i\bar\partial\partial g_j) = 0$ where c_j is defined as above. Then $\|\psi_j - c_j\omega - i\bar\partial\partial g_j\|^2 = -\int_X \psi_j^2 + c_j^2 \int_X \omega^2$ is converging to $-\int_X \psi^2 + c^2 \int_X \omega^2 = 0$. The sequence $\{i\bar\partial\partial g_j\}$ is therefore uniformly bounded in L^2, so after normalising g_j so that $\int_X g_j\, dV = 0$, a subsequence of $\{g_j\}$ can be found which converges weakly in L^2_2 to some $g \in L^2_2(X)$. By semi-continuity of $\|\cdot\|$ under weak limits, $\|\psi - c\omega - i\bar\partial\partial g\| \leq \lim \|\psi_j - c_j\omega - i\bar\partial\partial g_j\| = 0$, completing the proof. □

Proposition 5. — *Let $\varphi_1, \varphi_2 \in \Lambda^{1,1}_{\mathbb{R}} \otimes L^2(X)$ be weakly $\bar{\partial}\partial$-closed and satisfy $\int_X \varphi_j^2 \geq 0$ and $\int_X \omega \wedge \varphi_j \geq 0$ for $j = 1, 2$. Then*

$$\int_X \varphi_1 \wedge \varphi_2 \geq \Big(\int_X \varphi_1^2\Big)^{1/2} \Big(\int_X \varphi_2^2\Big)^{1/2}$$

with equality if and only if φ_1 and φ_2 are linearly dependent modulo the image of $i\bar{\partial}\partial$.

Proof. — By Lemma 4 it can be assumed that $a_j := \int_X \omega \wedge \varphi_j$ is strictly positive for $j = 1, 2$ else φ_j is $\bar{\partial}\partial$-exact.

To prove the inequality, after replacing φ_j by $\varphi_j + \epsilon\omega$ and taking the limit as $\epsilon \searrow 0$ it can be assumed that $\int_X \varphi_j^2 > 0$ for $j = 1, 2$, and then by Lemma 3 it can be assumed without loss of generality that these forms are both smooth.

Since $\int_X \omega \wedge (a_2\varphi_1 - a_1\varphi_2) = 0$, it follows that $a_2\varphi_1 - a_1\varphi_2 + i\bar{\partial}\partial g$ is anti-self-dual for some function $g \in \Lambda^0_{\mathbb{R}}(X)$, so

$$\begin{aligned} 0 \geq \int_X (a_2\varphi_1 - a_1\varphi_2)^2 &= a_2^2 \int_X \varphi_1^2 + a_1^2 \int_X \varphi_2^2 - 2a_1a_2 \int_X \varphi_1 \wedge \varphi_2 \\ &\geq 2a_1a_2 \Big(\int_X \varphi_1^2\Big)^{\frac{1}{2}} \Big(\int_X \varphi_2^2\Big)^{\frac{1}{2}} - 2a_1a_2 \int_X \varphi_1 \wedge \varphi_2, \end{aligned}$$

giving the desired inequality.

Now suppose $\int_X \varphi_1 \wedge \varphi_2 = (\int_X \varphi_1^2)^{1/2} (\int_X \varphi_2^2)^{1/2}$ (no longer assuming smoothness or $\int_X \varphi_j^2 > 0$). Take sequences $\{\varphi_{1,k}\}$, $\{\varphi_{2,k}\}$ of smooth $\bar{\partial}\partial$-closed $(1,1)$-forms converging to φ_1, φ_2 respectively in L^2 and solve the equations $\omega \wedge (a_{2,k}\varphi_{1,j} - a_{1,k}\varphi_{2,j} + i\bar{\partial}\partial g_k) = 0$ for $g_k \in \Lambda^0_{\mathbb{R}}(X)$ satisfying $\int_X g_k \, dV = 0$ (with $a_{j,k} = \int_X \omega \wedge \varphi_{j,k}$). Arguing as in the proof of Lemma 4 then yields a weakly convergent subsequence whose limit $g \in L^2_2(X)$ satisfies $a_2\varphi_1 - a_1\varphi_2 + i\bar{\partial}\partial g = 0$. □

Corollary 6. — *If $\varphi \in \Lambda^{1,1}_{\mathbb{R}} \otimes L^2(X)$ is weakly $\bar{\partial}\partial$-closed and satisfies $\int_X \varphi^2 > 0$ and $\int_X \varphi \wedge \omega > 0$, then $\int_X \varphi \wedge \chi > 0$ for any other such form χ satisfying $\int_X \chi^2 \geq 0$ and $\int_X \chi \wedge \omega > 0$.* □

Lemma 7. — *Suppose $\chi \in \Lambda^{1,1}_{\mathbb{R}} \otimes L^2(X)$ is weakly $\bar{\partial}\partial$-closed and satisfies $\int_X \chi^2 \geq 0$ and $\int_X \chi \wedge \omega \geq 0$. For each $\epsilon > 0$ there is a positive $(1,1)$-form p_ϵ and a function g_ϵ such that $\|\chi + i\bar{\partial}\partial g_\epsilon - p_\epsilon\|_{L^2(X,\omega)} < \epsilon$. Moreover, p_ϵ and g_ϵ can be assumed to be smooth.*

Proof. — If $\int_X \chi \wedge \omega = 0$ it follows from Lemma 4 that χ is $\bar{\partial}\partial$-exact, in which case the result follows from the denseness of the smooth functions in $L_2^2(X)$. Assume therefore that $\int_X \chi \wedge \omega > 0$, and after rescaling χ if necessary, it can be supposed that $\int_X \chi \wedge \omega = 1$.

Let

$$\mathcal{P} := \Big\{p \in \Lambda_{\mathbb{R}}^{1,1}(X) \otimes L^2(X) \mid p \geq 0 \; a.e., \int_X \omega \wedge p = 1\Big\},$$

$$\mathcal{P}_\epsilon := \big\{\rho \in \Lambda_{\mathbb{R}}^{1,1}(X) \otimes L^2(X) \mid \|\rho - p\| < \epsilon \text{ for some } p \in \mathcal{P}\big\},$$

and

$$\mathcal{H} := \big\{\chi + i\bar{\partial}\partial g \mid g \in L_2^2(X)\big\}.$$

Then $\mathcal{P}_\epsilon$ is an open convex subset of the Hilbert space $H := \Lambda_{\mathbb{R}}^{1,1} \otimes L^2(X)$ and $\mathcal{H}$ is a closed convex subset. If $\mathcal{P}_\epsilon \cap \mathcal{H} = \emptyset$, the Håhn-Banach Theorem implies there exists $\varphi \in H$ and a constant $c \in \mathbb{R}$ such that $\int_X \varphi \wedge h \leq c$ and $\int_X \varphi \wedge p > c$ for every $h \in \mathcal{H}$ and every $p \in \mathcal{P}_\epsilon$. It follows immediately that φ is weakly $\bar{\partial}\partial$-closed.

Let $\varphi_0 := \varphi - c\omega$. Then φ_0 is weakly $\bar{\partial}\partial$-closed, $\int_X \varphi_0 \wedge \chi \leq c - c = 0$ and $\int_X \varphi_0 \wedge p_0 > 0$ for every $p_0 \in \mathcal{P}$, so φ_0 is strictly positive almost everywhere. Hence $\int_X \varphi_0^2 > 0$ and $\int_X \varphi_0 \wedge \omega > 0$, so it follows from Corollary 6 that $\int_X \varphi_0 \wedge \chi > 0$, a contradiction.

Therefore $\mathcal{P}_\epsilon \cap \mathcal{H}$ cannot be empty proving the existence of p_ϵ and g_ϵ. The last statement of the lemma follows from the denseness of the smooth positive $(1,1)$-forms in the L^2 positive $(1,1)$-forms and of the smooth functions in $L_2^2(X)$. □

3. Surfaces with even first Betti number.

Throughout this section, ω will be a fixed smooth positive $\bar{\partial}\partial$-closed $(1,1)$-form on the compact complex surface X. All norms and adjoints are computed using the corresponding hermitian metric.

LEMMA 8. — *Let X be a compact complex surface. Then $b_1(X)$ is even if and only for each $\psi \in \Lambda^{1,1}(X)$ satisfying $\bar{\partial}\partial\psi = 0$ the equation $\bar{\partial}\psi = \partial\bar{\partial}u$ can be solved for $u \in \Lambda^{0,1}(X)$.*

Proof. — Suppose $\psi \in \Lambda^{1,1}(X)$ satisfies $\bar{\partial}\partial\psi = 0$. The equation $\bar{\partial}\psi = \partial\bar{\partial}u$ has a solution $u \in \Lambda^{0,1}(X)$ if and only if $\bar{\partial}\psi$ is perpendicular to the kernel of the adjoint of $\partial\bar{\partial}$; that is, if and only if $\int_X \bar{w} \wedge *\bar{\partial}\psi = 0$

for every $w \in \Lambda^{1,2}(X)$ such that $\partial\bar{\partial}(*w) = 0$; equivalently, if and only if $\int_X v \wedge \bar{\partial}\psi = 0$ for every $v\,(= *\bar{w}) \in \Lambda^{1,0}(X)$ satisfying $\bar{\partial}\partial v = 0$. By Stokes' Theorem, such a v must satisfy $0 = \int_X \bar{v} \wedge \bar{\partial}\partial v = \int_X \bar{\partial}\bar{v} \wedge \partial v = \|\partial v\|^2$, so the kernel of $\bar{\partial}\partial$ on $\Lambda^{1,0}(X)$ is the same as that of ∂. If $b_1(X)$ is even, Theorem IV.2.9 of [BPV] implies any such v is ∂-homologous to a d-closed $(1,0)$-form, so if $d(v + \partial g) = 0$ for some function $g \in \Lambda^{0,0}(X)$ it follows $\int_X v \wedge \bar{\partial}\psi = \int_X (v + \partial g) \wedge \bar{\partial}\psi = 0$; hence $\bar{\partial}\psi$ is in the image of $\partial\bar{\partial}\colon \Lambda^{0,1}(X) \to \Lambda^{1,2}(X)$.

Conversely, given $v \in \Lambda^{1,0}(X)$ satisfying $\partial v = 0$, the equation $0 = d(v + \partial g) = \bar{\partial}v + \bar{\partial}\partial g$ can be solved for $g \in \Lambda^{0,0}(X)$ if and only if $\int_X \bar{\partial}v \wedge \psi = 0$ for every $\psi \in \Lambda^{1,1}(X)$ satisfying $\bar{\partial}\partial\psi = 0$. If $\bar{\partial}\psi = \partial\bar{\partial}u$ it follows $\int_X \bar{\partial}v \wedge \psi = \int_X v \wedge \bar{\partial}\psi = \int_X v \wedge \partial\bar{\partial}u = 0$. Thus if the equation $\bar{\partial}\psi = \bar{\partial}\partial u$ can be solved for each $\bar{\partial}\partial$-closed ψ it follows that every ∂-closed $v \in \Lambda^{1,0}(X)$ is ∂-homologous to a d-closed $(1,0)$-form; this implies that the natural $\mathbb{R}$-linear map $H^0(X,\Omega^1) \ni h \mapsto \bar{h} \in H^1(X,\mathcal{O})$ is an isomorphism. Hence $h^{0,1}(X) = h^{1,0}(X)$ implying that $b_1(X) = h^{0,1}(X) + h^{1,0}(X)$ is even. □

If $b_1(X)$ is even, complex conjugation of the above argument shows that for each $\bar{\partial}\partial$-closed $\psi \in \Lambda^{1,1}(X)$ there exists $u' \in \Lambda^{1,0}(X)$ such that $\bar{\partial}\psi = \bar{\partial}\partial u'$; if $\bar{\psi} = \psi$, u' can be taken to be $\bar{u}$. The $(1,1)$-form $\psi + \partial u + \bar{\partial}u'$ is d-closed, as is the 2-form $\psi - \bar{\partial}u - \partial u'$, with ψ being the $(1,1)$ component of the latter. It follows from the proof of Lemma 8 that the form $\bar{\partial}u \in \Lambda^{0,2}(X)$ is uniquely determined by ψ, whereas the form $\psi + \partial u + \bar{\partial}u'$ is determined up to the addition of a $\bar{\partial}\partial$-exact term which can be uniquely fixed by requiring (for example) that $\omega \wedge (\psi + \partial u + \bar{\partial}u')$ be a constant multiple of ω^2.

Lemma 8 combined with Lemma 2 yields the following useful result:

Corollary 9. — *If $b_1(X)$ is even and $\psi \in \Lambda^{1,1}_{\mathbb{R}} \otimes L^2(X)$ is weakly $\bar{\partial}\partial$-closed, there exists $u \in \Lambda^{0,1} \otimes L^2_1(X)$ such that $\psi + \partial u + \bar{\partial}\bar{u}$ is d-closed and smooth.* □

Globally, $\bar{\partial}\partial$-exactness of d-closed forms holds in the same way that it does on any Kähler manifold:

Lemma 10. — *If $b_1(X)$ is even and if $\psi \in \Lambda^{1,1} \otimes L^2(X)$ is d-exact then $\varphi = \bar{\partial}\partial g$ for some $g \in L^2_2(X)$.*

Proof. — Suppose $\psi = dv$ for some $v \in \Lambda^1 \otimes L^2_1(X)$. Then the $(1,0)$ component of v is ∂-closed and the $(0,1)$ component is $\bar{\partial}$-closed. From the second paragraph of the proof of Lemma 8 it follows $\int_X \omega \wedge \psi = 0$, so $\omega \wedge (dv + \bar\partial\partial g) = 0$ for some $g \in L^2_2(X)$. The $(1,1)$-form $d(v + \partial g)$ is therefore anti-self-dual, so $\|d(v+\partial g)\|^2 = -\int_X d(v+\partial g) \wedge d(\bar v + \bar\partial \bar g) = 0$, giving $\psi = -\bar\partial\partial g$. $\square$

Suppose $b_1(X)$ is even and $u \in \Lambda^{0,1}(X)$ is such that $\tilde\omega = \omega + \partial u + \bar\partial \bar u$ is d-closed and satisfies $\omega \wedge \tilde\omega = c\omega^2$ for some constant c. Then $\int_X \omega \wedge \partial u = -\int_X \partial\omega \wedge u = \int_X \partial\bar\partial\bar u \wedge u = -\int_X \bar\partial\partial\bar u \wedge u = \int_X \partial\bar u \wedge \bar\partial u = \|\bar\partial u\|^2$, so $c = 1 + \|\bar\partial u\|^2/V$ for $V := \int_X \omega^2/2$. Furthermore, $\int_X \tilde\omega^2 = \int_X \tilde\omega \wedge \omega = \int_X \omega \wedge \omega + 2\|\bar\partial u\|^2$. Thus $\tilde\omega$ defines a d-closed element of $H^{1,1}_{\mathbb{R}}(X)$ which satisfies $[\tilde\omega]\cdot[\tilde\omega] > 0$ in $H^2(X,\mathbb{R})$ and $[\tilde\omega]\cdot[D] > 0$ for every effective divisor $D \subset X$; such a form is a very natural candidate to be $\bar\partial\partial$-homologous to a positive closed $(1,1)$-form on X.

4. The main results.

Suppose X is a compact complex surface with $b_1(X)$ even, and let ω be a fixed smooth positive $\bar\partial\partial$-closed $(1,1)$-form on X, normalised so that $\int_X \omega^2 = 2$. As in §3, let u_0 be a smooth $(0,1)$-form on X such that $\tilde\omega = \omega + \partial u_0 + \bar\partial \bar u_0$ is d-closed, with $\int_X \tilde\omega^2 = 2(1 + \|\bar\partial u_0\|^2) =: 2(1 + a_0)$ and $\omega \wedge \partial u_0 = a_0\, \omega^2/2$.

Let χ be a smooth real $\bar\partial\partial$-closed $(1,1)$-form satisfying $\int_X \chi^2 > 0$ and $\int_X \chi \wedge \omega > 0$, and assume that χ has been normalised so that $\int_X \chi^2 = 2$. In addition, let u be a $(0,1)$-form on X such that $\tilde\chi = \chi + \partial u + \bar\partial\bar u$ is d-closed. Since $\int_X \tilde\chi^2 = 2 + 2\|\bar\partial u\|^2 =: 2(1+a)$, it follows from Proposition 5 that the number $b := (1/2)\int_X \omega \wedge \tilde\chi = (1/2)\int_X \tilde\omega \wedge \chi$ is positive and satisfies $b^2 \geq (1+a)(1+a_0)$ with equality iff $\tilde\chi$ is $\bar\partial\partial$-homologous to a multiple of $\tilde\omega$.

For $t_0 := b - \sqrt{b^2 - (1+a)}$, the form $\varphi := \tilde\chi - t_0\omega$ satisfies $\bar\partial\partial\varphi = 0$, $\int_X \varphi^2 = 0$ and $\int_X \omega \wedge \varphi = 2(b - t_0) = 2\sqrt{b^2 - (1+a)} \geq 0$, with equality iff ω is d-closed and $\tilde\chi$ is $\bar\partial\partial$-homologous to a multiple of ω.

Assume that $b - t_0 = \sqrt{b^2 - (1+a)}$ is strictly positive. By Lemma 7, for each $n \in \mathbb{N}$ there is a smooth positive $(1,1)$-form p_n and a smooth function g_n such that $\|\varphi + i\bar\partial\partial g_n - p_n\| < 1/n$.

Since $\int_X p_n \wedge \omega$ is converging to $2(b - t_0)$ the positive functions $(\Lambda p_n)^{1/2}$ are uniformly bounded in L^2 so a subsequence can be found converging weakly in L^2. The forms $p_n/\Lambda p_n$ are bounded in L^∞ so a subsequence of these forms can also be found converging weakly in L^4 say. The sequence $\{i\bar{\partial}\partial g_n\}$ is uniformly bounded in L^1 so after adding a constant so that $\int_X g_n \, dV = 0$, by the Sobolev embedding theorem a subsequence of $\{g_n\}$ can be assumed to converge weakly in $L_1^{4/3} \cap L^2$ and strongly in L^q for some fixed $q \in (1,2)$ to some function g. Thus the subsequence of positive forms p_n converges in the sense of currents to define a positive $(1,1)$-current p with $\varphi + i\bar{\partial}\partial g = p$, and it follows that the current $P := p + t_0\omega = \tilde{\chi} + i\bar{\partial}\partial g$ is a *closed* positive $(1,1)$-current satisfying $P \geq t_0\omega$. Note that since $\Lambda p \in L^1$ and $p/|\Lambda p| \in L^\infty$, the current P lies in L^1.

Let $\nu(P,x)$ denote the Lelong number of P at x; ([GH], p. 391). For $c > 0$, the sublevel set $E_c(P) = \{x \in X \mid \nu(P,x) \geq c\}$ is a proper analytic subset of X by Siu's Theorem [S1]. By Lemmas 6.2 and 6.3 of that same paper, if D is an irreducible 1-dimensional component of $E_c(P)$ and $\nu_0 := \inf\{\nu(P,x) \mid x \in D\}$, $\nu(P,x) = \nu_0$ for almost all $x \in D$ and $P - \nu_0[D]$ is positive. (Although Siu's lemmas consider the case of smooth D, upper semicontinuity of the Lelong number implies the same results even if D has singularities.)

Fix a number $K \geq 0$ such that the curvature Θ of the Chern connection on T_X induced by ω satisfies $\Theta \geq -K\omega \otimes \mathrm{Id}_{T_X}$ and let $c > 0$ be such that $t_0 - cK > 0$. If $D_1, \ldots, D_n$ are the irreducible 1-dimensional components of $E_c(P)$ and $\nu_i := \inf\{\nu(P,x) \mid x \in D_i\}$, the closed $(1,1)$-current $T := P - \sum \nu_i[D_i]$ is positive and the c-sublevel set $E_c(T)$ of this current is 0-dimensional. By Theorem 6.1 of [D2] (see also [D1] for more complete results), there is a 1-parameter family $T_{c,\epsilon}$ of closed almost positive $(1,1)$-currents in the same cohomology class as T which is converging weakly to T as $\epsilon \searrow 0$, with $T_{c,\epsilon}$ smooth off $E_c(T)$, $T_{c,\epsilon} \geq (t_0 - \min\{\nu_\epsilon, c\}K - \delta_\epsilon)\omega$ for some continuous functions ν_ϵ on X and constants δ_ϵ satisfying $\nu_\epsilon(x) \searrow \nu(T,x)$ for each $x \in X$ and $\delta_\epsilon \searrow 0$ as $\epsilon \searrow 0$. Moreover, $\nu(T_{c,\epsilon}, x) = (\nu(T,x) - c)_+$ at each point of x. For ϵ sufficiently small therefore, $T_{c,\epsilon} \geq t_1\omega$ for some $t_1 > 0$, where t_1 can be chosen arbitrarily close to t_0 if c and ϵ are small enough.

The current $T_{c,\epsilon}$ is smooth off the 0-dimensional set $E_c(T)$; that is, off a finite set of points. In a neighbourhood of any such point x_0, $T_{c,\epsilon}$ can be represented by a strictly plurisubharmonic function f say: $T_{c,\epsilon} = i\partial\bar{\partial} f$ with f smooth off x_0. Using a standard mollifying function as in [GT], p. 147, f can be smoothed in a neighbourhood of x_0 to a family f_t of strictly plurisubharmonic functions converging to f, and on an annular region surrounding x_0 the convergence of this sequence is uniform in $\mathcal{C}^k$ for any k (by Lemma 4.1 and the accompanying discussion in [GT]). If ρ is a standard cutoff function which is 1 on the exterior of the annulus and 0 on the interior region, $\rho f + (1-\rho)f_t$ is a smooth plurisubharmonic function for t sufficiently small which agrees with f outside the annulus.

Hence the current $T_{c,\epsilon}$ is $\bar{\partial}\partial$-homologous to a smooth positive closed form $\tau_{c,\epsilon}$ for $\epsilon > 0$ sufficiently small; moreover for any $t_1 < t_0$, there is some c and ϵ such that $\tau_{c,\epsilon} \geq t_1\omega$. Thus:

THEOREM 11. — *A compact complex surface X with $b_1(X)$ even admits a Kähler metric.* □

Suppose now that $\chi = \omega$, so $\tilde{\chi} = \tilde{\omega}$, $b = 1 + a_0$ and $t_0 = 1 + a_0 - \sqrt{a_0(1+a_0)} = \left(1 + \sqrt{\frac{a_0}{1+a_0}}\right)^{-1}$. Assume that ω is not d-closed, so $a_0 \neq 0$ and therefore $t_0 < 1$.

The cohomology class of $\tau_{c,\epsilon}$ is the same as that of T which is in turn the same as that of $\tilde{\omega} - D$ for $D := \sum \nu_i[D_i]$. Here and subsequently the notation is abused in the standard way by identifying a d-closed $(1,1)$-form with its image in $H^{1,1}(X)$ and vice versa; unless otherwise stated, the $(1,1)$-form representing a given class will always be that which has self-dual component a constant multiple of ω. In the same vein, the notation $\psi \cdot \chi$ will be used to denote $\int_X \psi \wedge \chi$ for d-closed $(1,1)$-forms ψ, χ; thus $H^{1,1}_{\mathbb{R}}(X) \ni \psi \mapsto 2(\tilde{\omega}\cdot\psi)^2(\tilde{\omega}\cdot\tilde{\omega})^{-1} - \psi\cdot\psi = (1+a_0)^{-1}(\tilde{\omega}\cdot\psi)^2 - \psi\cdot\psi =: \|\psi\|^2_{\tilde{\omega}}$ defines a norm on $H^{1,1}_{\mathbb{R}}(X)$ since the cup-product pairing is negative definite on the orthogonal complement of $\tilde{\omega}$.

If $t_1 < t_0$ is such that $\tau_{c,\epsilon} \geq t_1\omega$, a short calculation using the fact that $\tau_{c,\epsilon} = \tilde{\omega} - D$ in $H^{1,1}_{\mathbb{R}}(X)$ yields

$$0 \leq \int_X (\tau_{c,\epsilon} - t_1\omega)^2 = [2(1+a_0) - 4t_1(1+a_0) + 2t_1^2] + D\cdot D - 2(\tilde{\omega} - t_1\omega)\cdot D.$$

Here $\omega \cdot D := \int_X \omega \wedge D = \sum_i \nu_i \int_{D_i} \omega$, a non-negative term which is 0 iff $D = 0$ in $H^{1,1}_{\mathbb{R}}(X)$. Since D is d-closed, $\tilde{\omega} \cdot D = \omega \cdot D$, so the last term on the right of the inequality is $-2(1-t_1)\omega \cdot D$.

Since $D \cdot D = -\|D\|^2_{\tilde{\omega}} + (\omega \cdot D)^2(1+a_0)^{-1}$, the inequality can be rewritten

$$\|D\|^2_{\tilde{\omega}} \leq \left[2(1+a_0) - 4t_1(1+a_0) + 2t_1^2\right] + \frac{\omega \cdot D}{1+a_0}\left[\omega \cdot D - 2(1+a_0)(1-t_1)\right].$$

By Proposition 5, $0 \leq \tilde{\omega} \cdot (\tau_{c,\epsilon} - t_1\omega) = 2(1+a_0)(1-t_1) - \omega \cdot D$, implying

$$\|D\|^2_{\tilde{\omega}} \leq 2(1+a_0) - 4t_1(1+a_0) + 2t_1^2 = 2(t_0 - t_1)[2(1+a_0) + t_0 + t_1];$$

(equality will hold only if D is homologous to a multiple of $\tilde{\omega}$).

Now choose a sequence of constants t_i increasing to t_0 and corresponding constants c_i and ϵ_i so that $\tau_{c_i,\epsilon_i} \geq t_i\omega$, with $\tau_{c_i,\epsilon_i} = \tilde{\omega} - D_{(i)}$ say in $H^{1,1}_{\mathbb{R}}(X)$. It follows immediately from the last inequality above that the corresponding sequence of classes $D_{(i)}$ is converging to 0 in $H^{1,1}_{\mathbb{R}}(X)$ and therefore the corresponding representative $(1,1)$-forms (which have self-dual component a constant multiple of ω) are converging to 0 in $L^2(X)$; by elliptic regularity and choice of representatives, these forms are converging to 0 in $C^0(X)$. Since $\tau_{c_i,\epsilon_i} \geq t_i\omega$, it follows that for i large enough the form $\tau_{c_i,\epsilon_i} + D_{(i)}$ is positive; i.e., $\tilde{\omega}$ is $\bar{\partial}\partial$-homologous to a smooth positive closed form. This proves:

THEOREM 12. — *If X is a compact complex surface with $b_1(X)$ even and ω is a positive $\bar{\partial}\partial$-closed $(1,1)$-form on X, there exists $u \in \Lambda^{0,1}(X)$ such that $\tilde{\omega} = \omega + \partial u + \bar{\partial}\bar{u}$ is d-closed and positive.* □

COROLLARY 13. — *If X is a compact complex surface with $b_1(X)$ even and P is a $\bar{\partial}\partial$-closed positive $(1,1)$-current, there is a $(0,1)$-current U on X such that $P + \partial U + \bar{\partial}\bar{U}$ is a d-closed positive current.*

Proof. — The $(2,1)$-current ∂P is $\bar{\partial}$-closed. By smoothing of cohomology ([GH], p. 385), there is a $(2,0)$-current V on X such that $\partial P - \bar{\partial}V$ is a smooth $\bar{\partial}$-closed $(2,1)$-form. Then V is a ∂-closed current on X so by the same result there is a $(1,0)$-current W such that $V - \partial W$ is a smooth $(2,0)$-form. Then $\partial P - \bar{\partial}\partial W$ is a smooth $\bar{\partial}$-closed $(2,1)$-form. As in the proof of Lemma 8, the fact that $b_1(X)$ is even implies that $\partial P - \bar{\partial}\partial W = \bar{\partial}\partial w$ for some smooth $(1,0)$-form w, so for some $(0,1)$-current U the $(1,1)$-current $\tilde{P} := P + \partial U + \bar{\partial}\bar{U}$ is d-closed.

If there is no real $(0,0)$-current G such that $\tilde{P} + i\bar{\partial}\partial G$ is positive, then as in the proof of Theorem 14 of [HL], there is a smooth positive $\bar{\partial}\partial$-closed $(1,1)$-form ψ with $\tilde{P}(\psi) \leq 0$. But by Theorem 12, ψ is homologous to a positive closed $(1,1)$-form $\tilde{\psi}$ modulo the image of $d^{1,1}$ so $0 < P(\tilde{\psi}) = \tilde{P}(\tilde{\psi}) = \tilde{P}(\psi) \leq 0$, a contradiction. □

THEOREM 14. — *Suppose $b_1(X)$ is even and $\omega \in \Lambda^{1,1}_{\mathbb{R}}(X)$ is $\bar{\partial}\partial$-closed and positive. If $\varphi \in \Lambda^{1,1}_{\mathbb{R}}(X)$ is $\bar{\partial}\partial$-closed and satisfies $\int_X \varphi^2 > 0$, $\int_X \omega \wedge \varphi > 0$ and $\int_D \varphi > 0$ for every irreducible divisor $D \subset X$, then φ is homologous to a smooth closed positive $(1,1)$-form modulo the image of $\partial + \bar{\partial}$.*

Remark. — The hypothesis $\int_X \omega \wedge \varphi > 0$ can not be omitted in general even if X has at least one curve D. For example, if Y is a K3 surface with no divisors at all and $y_0 \in Y$, let X be the blowup of Y at y_0 with blowing down map $\pi: X \to Y$ and exceptional divisor $D = \pi^{-1}(y_0)$. If ω_0 is a positive closed $(1,1)$-form and ρ is a smooth closed $(1,1)$-form on X with $[\rho] = [D]$ in $H^{1,1}_{\mathbb{R}}(X)$, $\varphi := -\pi^*\omega_0 - \epsilon\rho$ satisfies $\int_X \varphi^2 > 0$ for $\epsilon > 0$ sufficiently small and $\int_D \varphi > 0$, but φ can never be homologous to a positive $(1,1)$-form modulo the image of $\partial + \bar\partial$ since $\int_X \pi^*\omega_0 \wedge \varphi < 0$.

Proof of Theorem 14. — If φ is not homologous to a positive $(1,1)$-form modulo the image of $\partial + \bar\partial$ (i.e., $\varphi + \partial u + \bar\partial \bar u$ is not positive for any $u \in \Lambda^{0,1}(X)$), by the Hahn-Banach theorem there is a positive *closed* $(1,1)$-current P such that $P(\varphi) \leq 0$.

Arguing as in the proof of Theorem 11, for any $\epsilon > 0$ there is a real effective divisor $D = \sum \nu_i D_i$ such that $P - D$ is homologous modulo the image of $\bar\partial\partial$ to a smooth real closed $(1,1)$-form τ with $\tau \geq -\epsilon\omega$. By Corollary 6, $0 \leq \int_X \varphi \wedge (\tau + \epsilon\omega) = P(\varphi) - \sum \nu_i \int_{D_i} \varphi + \epsilon \int_X \varphi \wedge \omega$. If $P(\varphi) < 0$, a contradiction results by choosing ϵ sufficiently small so it may be assumed that $P(\varphi) = 0$.

Now choose a sequence $\epsilon_i \searrow 0$ and smooth forms $\tau_i \geq -\epsilon_i\omega$ with τ_i homologous to $P - D_{(i)}$ for some real effective divisor $D_{(i)}$. By Proposition 5, $0 \leq (\tau_i + \epsilon_i\omega) \cdot \varphi = -D_{(i)} \cdot \varphi + \epsilon_i\omega \cdot \varphi$, so $\varphi \cdot D_{(i)} \searrow 0$.

If $(\omega \cdot D_{(i)})^2 - D_{(i)} \cdot D_{(i)} = \|D_{(i)}\|^2$ is not bounded independent of i, a subsequence can be found with $\|D_{(i_j)}\| \to \infty$. The cohomology classes $D_{(i_j)}/\|D_{(i_j)}\|$ can be assumed to converge to some $D \in H^{1,1}_{\mathbb{R}}(X)$ of norm 1, with $\varphi \cdot D = 0$; hence $D \cdot D < 0$ by Proposition 5 since D is non-zero in $H^2(X, \mathbb{R})$. But since $(P - D_{(i_j)} + \epsilon_{i_j}\omega) \cdot (P - D_{(i_j)} + \epsilon_{i_j}\omega) > 0$, it follows that $D \cdot D \geq 0$, a contradiction. Therefore $\{\|D_{(i)}\|\}$ is bounded, and by passing to a subsequence it can be assumed that $D_{(i)}$ converges to some D with $D \cdot D \leq 0$, $\omega \cdot D \geq 0$ and $\varphi \cdot D = 0$. By Proposition 5 again, the inequality $(P - D) \cdot (P - D) \geq 0$ and identity $\varphi \cdot (P - D) = 0$ now imply $P = D$ in $H^2(X, \mathbb{R})$.

Recall now that the real divisor $D_{(i)}$ is of the form $D_{(i)} = \sum\limits_j \nu_{ij} D_{(i)j}$ where $D_{(i)j}$ is the j-th irreducible 1-dimensional component of the c_i-sublevel set of $\nu(P, -)$, with ν_{ij} the Lelong number of P at the generic point of $D_{(i)j}$. As ϵ_i tends to 0, corresponding constants c_i can also be assumed to converge monotonically to 0, so $D_{(i)} \subset D_{(j)}$ for $i < j$ and the constants ν_{ik} appearing in $D_{(i)}$ also appear as the coefficients in $D_{(j)}$. Since

$\varphi \cdot E > 0$ for every effective divisor E, it follows $\varphi \cdot D_{(i)} \leq \varphi \cdot D_{(j)}$ for $i < j$ with equality iff $D_{(i)} = D_{(j)}$.

Since $\varphi \cdot D = 0$, it therefore follows that $D_{(i)}$ is homologous to 0 for every i, as is the limit D. Therefore $P = 0$ in $H^2(X, \mathbb{R})$ implying $\omega \cdot P = 0$, a contradiction. It therefore follows that φ is homologous to a positive $(1,1)$-form modulo the image of $\partial + \bar{\partial}$, so from Theorem 12 it follows that the form is actually homologous to a closed positive $(1,1)$-form. □

Theorem 14 thus yields the following real version of the classical Nakai criterion, and indeed gives an independent proof of that result:

COROLLARY 15. — *A class $\sigma \in H^{1,1}_{\mathbb{R}}(X)$ can be represented by a smooth positive $(1,1)$-form iff $\sigma \cdot \sigma > 0$, $\omega \cdot \sigma > 0$ and $\sigma \cdot D > 0$ for every effective divisor $D \subset X$.* □

Remarks. — A canonical choice for the representative of σ is given by a smooth closed $(1,1)$-form ρ with $[\rho] = \sigma$ and with $\rho \wedge \rho = c\omega^2$, where $c = \sigma \cdot \sigma / \int_X \omega^2$. For if ρ_0 is some positive representative of σ, the complex Monge-Ampère equation $(\rho_0 + i\bar{\partial}\partial g)^2 = c\omega^2$ can be solved by following Yau's proof in [Y], which is an equation of precisely this kind.

More generally, the techniques for solving the Monge-Ampère equation $(\rho_0 + i\bar{\partial}\partial g)^2 = e^u \rho_0^2$ where $\int_X e^u \rho_0^2 = \int_X \rho_0^2$ for g are easily adapted to the case when ρ_0 is $\bar{\partial}\partial$-closed and positive. Considerable effort was put into attempting to solve the equation $((1-t)\omega + t\tilde{\omega} + i\bar{\partial}\partial g_t)^2 = c_t\omega^2$ (for the appropriate constant c_t) where $t \in [0,1]$ but the requisite a priori estimates on the solution g_t eluded the author. It would be interesting to be able to find a direct analytical proof of Theorem 12.

Hitherto, the full strength of Demailly's results have not be used in that they apply to *almost* positive closed $(1,1)$-currents (i.e., positive modulo the addition of smooth forms), not just positive currents. Using this fact, the following strengthening of Theorem 14 is obtained:

THEOREM 16. — *Suppose $b_1(X)$ is even and $\omega \in \Lambda^{1,1}_{\mathbb{R}}(X)$ is $\bar{\partial}\partial$-closed and positive. If $\varphi \in \Lambda^{1,1}_{\mathbb{R}}(X)$ is $\bar{\partial}\partial$-closed and satisfies $\int_X \varphi^2 > 0$, $\int_X \omega \wedge \varphi > 0$ and $\int_D \varphi > 0$ for every irreducible divisor $D \subset X$, then φ is homologous to a smooth $\bar{\partial}\partial$-closed positive $(1,1)$-form modulo the image of $\bar{\partial}\partial$.*

Proof. — If $d\varphi = 0$, the result follows from Theorem 14 and

Lemma 10, so assume $d\varphi \neq 0$.

Let u, $u_0 \in \Lambda^{0,1}(X)$ be such that $\tilde{\omega} = \omega + \partial u_0 + \bar{\partial}\bar{u}_0$ and $\tilde{\varphi} = \varphi + \partial u + \bar{\partial}\bar{u}$ are both d-closed; by Theorem 12 it can be assumed that $\tilde{\omega}$ is positive. Since $\int_X \tilde{\varphi}^2 = \int_X \varphi^2 + 2\|\bar{\partial}u\|^2 > 0$ and $\int_D \tilde{\varphi} = \int_D \varphi > 0$ for every effective divisor $D \subset X$ and $\int_X \tilde{\varphi} \wedge \omega = \int_X \varphi \wedge \tilde{\omega} > 0$, it follows from Theorem 14 that u can be chosen so that $\tilde{\varphi}$ is positive; this is henceforth assumed.

For $t_0 = 1 - \sqrt{1 - \int_X \varphi^2 / \int_X \tilde{\varphi}^2}$ the form $\psi := \varphi - t_0\tilde{\varphi}$ satisfies $\int_X \psi^2 = 0$ and $\int_X \tilde{\varphi} \wedge \psi > 0$ so by the same arguments as those leading to Theorem 11 there is a sequence of smooth functions g_n and positive $(1,1)$-forms p_n such that $\|\psi + i\bar{\partial}\partial g_n - p_n\| \to 0$, with g_n converging weakly in $L_1^{4/3} \cap L^2$ and strongly in L^q for some $q \in (1,2)$ to a function g and with p_n converging in the sense of currents to some positive $(1,1)$-current $p \in \Lambda^{1,1} \otimes L^1(X)$. The current $P := i\bar{\partial}\partial g = p - \psi$ is then closed and almost positive with $P \geq -\psi$.

Arguing exactly as in the proof of Theorem 11, for some constant K depending on the curvature of $\tilde{\varphi}$, given $c > 0$ with $t_0 - cK > 0$ there is an $\mathbb{R}$-linear combination D_c of effective divisors on X with positive coefficients and a family of smooth functions $g_{c,\epsilon}$ with $i\bar{\partial}\partial g_{c,\epsilon} - D_c \geq -\psi - (cK + \delta_\epsilon)\tilde{\varphi}$ such that $\delta_\epsilon \searrow 0$ as $\epsilon \searrow 0$; (as before D_c is here identified with a representative smooth closed $(1,1)$-form).

Since

$$
\begin{aligned}
\int_X (\psi + i\bar{\partial}\partial g_{c,\epsilon} - D_c)^2 &= D_c \cdot D_c - 2(1 - t_0) D_c \cdot \varphi \\
&= -\|D_c\|^2_{\tilde{\varphi}} - 2\frac{\varphi \cdot D_c}{\tilde{\varphi} \cdot \tilde{\varphi}} \tilde{\varphi} \cdot \big(\psi - D_c + (cK + \delta_\epsilon)\tilde{\varphi}\big) \\
&\quad + 2(cK + \delta_\epsilon)
\end{aligned}
$$

is converging to a non-negative number as c, $\epsilon \to 0$, it follows exactly as in the proof of Theorem 12 that (representative $(1,1)$ forms for) D_c must be converging to 0 in $\mathcal{C}^0(X)$. Consequently the inequality $\varphi + i\bar{\partial}\partial g_{c,\epsilon} \geq (t_0 - cK - \delta_\epsilon)\tilde{\varphi} + D_c$ implies $\varphi + i\bar{\partial}\partial g_{c,\epsilon}$ is positive for c, ϵ sufficiently small. $\square$

BIBLIOGRAPHY

[BPV] W. BARTH, C. PETERS and A. VAN de VEN, Compact Complex Surfaces, Berlin–Heidelberg–New York, Springer, 1984.

[CP] F. CAMPANA and T. PETERNELL, Algebraicity of the ample cone of projective varieties, J. reine angew. Math., 407 (1990), 160–166.

[D1] J.-P. DEMAILLY, Regularization of closed positive currents and intersection theory, J. Alg. Geom., 1 (1992), 361–409.

[D2] J.-P. DEMAILLY, Regularization of closed positive currents of type $(1,1)$ by the flow of a Chern connection, in: Contributions to complex analysis and analytic geometry: dedicated to Pierre Dolbeault, ed. H. Skoda and J. M. Trépreau, Wiesbaden, Vieweg 1994.

[G] P. GAUDUCHON, Le théorème de l'excentricité nulle, C. R. Acad. Sci. Paris, 285 (1977), 387–390.

[GT] D. GILBARG and N. TRUDINGER, Elliptic partial differential equations of second order, 2nd ed. Berlin–Heidelberg–New York, Springer, 1983.

[GH] P. A. GRIFFITHS and J. HARRIS, Principles of Algebraic Geometry, New York, Wiley, 1987.

[HL] R. HARVEY and H. B. LAWSON, An intrinsic characterisation of Kähler manifolds, Invent. Math., 74 (1983), 169–198.

[MK] J. MORROW and K. KODAIRA, Complex Manifolds, Holt-Rinehart & Wilson, New York, 1971.

[M] Y. MIYAOKA, Kähler metrics on elliptic surfaces, Proc. Japan Acad., 50 (1974), 533–536.

[S1] Y.-T. SIU, Analyticity of sets associated to Lelong numbers and the extension of closed positive currents, Invent. Math., 27 (1974), 53–156.

[S2] Y.-T. SIU, Review of [T] in Mathematical Reviews, (1982) MR#82k:32065.

[S3] Y.-T. SIU, Every K3 surface is Kähler, Invent. Math., 73 (1983), 139–150.

[T] A. N. TODOROV, Applications of the Kähler-Einstein-Calabi-Yau metric to moduli of $K3$ surfaces, Invent. Math., 61 (1980), 251–265.

[Y] S.-T. YAU, On the Ricci curvature of a complex Kähler manifold and the complex Monge-Ampère equation, Comm. Pure Appl. Math., 31 (1978), 339–411.

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