ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 48, nº 4 (1998), p. 1167-1188
<http://www.numdam.org/item?id=AIF_1998__48_4_1167_0>

Ann. Inst. Fourier, Grenoble
48, 4 (1998), 1167–1188

CONTACT TOPOLOGY AND THE STRUCTURE OF 5-MANIFOLDS WITH $\pi_1 = \mathbf{Z}_2$

by **H. GEIGES and C.B. THOMAS**

1. Introduction.

One of the fundamental problems in contact topology is the question as to which manifolds admit a contact structure. A classical theorem of Lutz and Martinet [15] asserts that every closed, orientable 3-manifold admits a contact structure; indeed, any given tangent 2-plane field is homotopic to a contact structure. The generalization of this result to highly connected manifolds of arbitrary (odd) dimension was begun in [17], using explicit realizations of such manifolds as Brieskorn manifolds and a connected sum (*i.e.* 0-surgery) theorem for contact manifolds due to Meckert. General surgical methods for contact manifolds were developed by Eliashberg [3] and Weinstein [19], and in [4] these methods were used to show that every closed, orientable and simply connected 5-manifold which admits an almost contact structure actually admits a contact structure; again it is true that in fact every homotopy class of almost contact structures contains a contact structure (see below for the definition of these concepts). In [6] this result was extended to essentially all highly connected manifolds.

In [6] an example is given for the construction (based on contact surgery) of contact structures on certain non-linear quotients of S^5 under the action of a metacyclic group, and various ad hoc methods have been employed by Lutz and others to construct special examples like the 5-torus (*cf.* [5]). But the following result is arguably the first general existence

Key words : Surgery – Contact structures – Contact surgery – Involutions – Brieskorn manifolds – Characteristic submanifold – Pin cobordism.
Math. classification : 57R65 – 57R90 –53C15 – 58F05.

statement about contact structures on non-simply connected manifolds (of dimension greater than three).

THEOREM 1. — *Let M be a closed, orientable 5-manifold with fundamental group $\mathbb{Z}_2$ and second Stiefel-Whitney class equal to zero on homology. Then M admits a contact structure.*

The condition on $w_2(M)$ will be discussed in the next section. Our manifolds M are always assumed connected.

Theorem 1 is derived as a corollary of the following structure theorem for 5-manifolds. First we need to describe certain model manifolds. Let V_q^6 be the complex hypersurface in $\mathbb{C}^4$ given by the equation

$$z_0^q + z_1^2 + z_2^2 + z_3^2 = 0,$$

where $q \in \mathbb{N}_0$ (the natural numbers including 0). Define the Brieskorn manifold Σ_q^5 as the intersection of V_q^6 with the unit sphere $S^7 \subset \mathbb{C}^4$. The orientation preserving involution $T : \Sigma_q^5 \to \Sigma_q^5$ given by

$$T(z_0, z_1, z_2, z_3) = (z_0, -z_1, -z_2, -z_3)$$

is fixed point free on Σ_q^5. The manifolds Σ_q^5/T with $0 \leq q \leq 8$ provide nine of our model manifolds. As we shall see, only the congruence class of $\pm q$ in $\mathbb{Z}_{16}/\pm$ is of importance.

The tenth and final model manifold that we shall need is the quotient Q_0 of $S^2 \times S^3$ under the free and orientation preserving involution

$$\big((x_0, x_1, x_2), (y_0, y_1, y_2, y_3)\big) \longmapsto \big((-x_0, -x_1, -x_2), (-y_0, y_1, y_2, y_3)\big),$$

where

$$\big((x_0, x_1, x_2), (y_0, y_1, y_2, y_3)\big) \in S^2 \times S^3 \subset \mathbb{R}^3 \times \mathbb{R}^4.$$

The map $Q_0 \to \mathbb{R}P^2$ induced by projection onto the x-coordinates gives Q_0 the structure of an S^3-bundle over $\mathbb{R}P^2$.

More information about the structure of the manifolds Σ_q^5/T and Q_0 will be given in Section 4.

THEOREM 2. — *Let M be a closed, orientable 5-manifold with fundamental group $\mathbb{Z}_2$ and $w_2(M)$ equal to zero on $H_2(M)$. Then M*

is obtained from exactly one of the ten model manifolds Q_0 *or* Σ^5_q/T, $0 \leq q \leq 8$, *by surgery along a link of 2-spheres. More precisely,* M *is obtained in this way from*

(i) Q_0 *if* $w_2(M) = 0$,

(ii) *one of* Σ^5_q/T, q *odd, if* $w_2(M) \neq 0$ *and the rank of* $\pi_2(M)$ *is even,*

(iii) *one of* Σ^5_q/T, q *even, if* $w_2(M) \neq 0$ *and the rank of* $\pi_2(M)$ *is odd.*

The outline of this paper is as follows. In Section 2 we discuss some basic notions of contact geometry, as well as the condition on $w_2(M)$ in the preceding theorems. In Section 3 we collect together various elementary remarks about surgery, and we describe the contact surgeries that we shall need later on. In Section 4 we give explicit descriptions of the ten model manifolds, show that each of them admits a contact structure, and deduce Theorem 1 (assuming Theorem 2). Finally, Theorem 2 is proved in Section 5 by analyzing Pin cobordisms of 4-dimensional characteristic submanifolds of 5-manifolds with a free involution.

2. Basic concepts.

A *contact structure* ξ on a manifold M of dimension $2n+1$ is a maximally non-integrable hyperplane distribution $\xi \subset \tau M$, which means that any (locally defined) 1-form α with $\xi = \ker \alpha$ satisfies $\alpha \wedge (\mathrm{d}\alpha)^n \neq 0$.

If ξ is co-orientable it can be defined by a global 1-form α, which is then called a *contact form.* For n even and M orientable this is always the case since the orientation of ξ defined by the symplectic form $\mathrm{d}\alpha \mid \xi$ is independent of the choice of a local 1-form α defining ξ. We then have a splitting of the tangent bundle τM into the Whitney sum of a trivial line bundle transverse to ξ and the symplectic and hence complex vector bundle $(\xi, d\alpha)$, which induces a reduction of the structure group of τM to $U(n) \times 1$. Such a reduction of the structure group is called an *almost contact structure.*

The first and, in the case of 5-manifolds, only obstruction to the existence of an almost contact structure is the third integral Stiefel-Whitney class W_3 (*cf.* [4]), *i.e.* the image of w_2 under the Bockstein operator β of the coefficient sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$. The vanishing of W_3 is equivalent to the existence of an integral lift $c_1 \in H^2(M;\mathbb{Z})$ of $w_2 \in H^2(M;\mathbb{Z}_2)$, so the condition $W_3 = 0$ is evidently necessary for the existence of an almost contact structure : for c_1 take the first Chern class of the $(\mathrm{U}(n) \times 1)$-bundle.

In the following discussion and throughout this paper, (co)homology is understood with integer coefficients unless a different coefficient group is specified. The universal coefficient theorem gives a natural epimorphism $H^2(M;\mathbb{Z}_2) \to \mathrm{Hom}(H_2(M),\mathbb{Z}_2)$. Saying that $w_2(M)$ is zero on $H_2(M)$ means that it maps to zero under this epimorphism. In that case (and with $\pi_1(M) = \mathbb{Z}_2$) we may think of $w_2(M)$ as an element in $\mathrm{Ext}(H_1(M),\mathbb{Z}_2) = \mathbb{Z}_2$. The ten model manifolds M_0 all have $H_2(M_0) = 0$ as our detailed description below will show, so for these we have

$$H^2(M_0;\mathbb{Z}_2) \cong \mathrm{Ext}\big(H_1(M_0),\mathbb{Z}_2\big) = \mathbb{Z}_2.$$

The Σ_q^5/T for q odd are homotopy equivalent to $\mathbb{R}P^5$, and the diffeomorphism type is determined by the congruence class of q modulo 8, the standard $\mathbb{R}P^5$ being represented by $q = 1$, *cf.* [7], [10], p. 332, [13]. In particular, the Σ_q^5/T with q odd all have non-zero second Stiefel-Whitney class (since it is non-zero for the standard $\mathbb{R}P^5$ and because of the homotopy invariance of Stiefel-Whitney classes). However, in this paper we do not need to rely on these facts. In Section 4 we shall prove directly that $w_2(\Sigma_q^5/T) \neq 0$ for all q.

Observe that if $w_2(M)$ is zero on $H_2(M)$, then the fact that $W_3(M) = \beta w_2(M) = 0$ follows from exactness of the Bockstein sequence and naturality (with respect to group homomorphisms) of the splitting in the universal coefficient sequence. So all these manifolds are candidates to admit contact structures.

For the proof of Theorem 2 it would suffice to require that $w_2(M)$ be zero on the *spherical* elements in $H_2(M)$. This condition, however, is equivalent to the previous one by a theorem of Hopf [9], *cf.* [2], which states that for any path-connected space X with fundamental group π there is an exact sequence

$$\pi_2(X) \longrightarrow H_2(X) \longrightarrow H_2(\pi) \to 0,$$

where the first map is the Hurewicz homomorphism. Since

$$H_2(\mathbb{Z}_2) = H_2(\mathbb{R}P^\infty) = 0,$$

we see that in our situation the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$ is surjective. Furthermore, by the Whitney embedding theorem, any element in $\pi_2(M)$ and thus, *a fortiori*, any element in $H_2(M)$ is represented by an embedded sphere. Notice, however, that the isotopy class of this embedding need not be unique.

3. Contact surgery.

Let M be a closed, orientable 5-manifold with fundamental group $\mathbb{Z}_2$. By performing surgery along an embedded 1-sphere representing the generator of $\pi_1(M)$, we obtain a simply connected 5-manifold M'. Provided $W_3(M') = 0$, we know that M' admits a contact structure. However, this information is no help in trying to construct a contact structure on M, for M is obtained from M' by surgery along an embedded 3-sphere, but contact surgery in the sense of [3], [19] is only possible along *isotropic spheres*, *i.e.* spheres tangent to the contact distribution ξ, which are necessarily of dimension at most n in a $(2n+1)$-dimensional contact manifold. For isotropic embeddings there is an h-principle (*cf.* [3], [4]), which means that under an obvious necessary condition it is possible to isotope a given embedding to an isotropic one. In principle it is feasible to perform contact surgery above the middle dimension, provided the sphere along which surgery is carried out is 'coisotropic' in a suitably defined sense, but for such embeddings we lack an h-principle.

So instead of killing $\pi_1(M)$ our strategy will be to turn M into one of the ten model manifolds M_0 by a sequence of surgeries along 2-spheres. Then the reverse sequence of surgeries leading from M_0 to M is again along 2-spheres, and by handle reordering this is the same as surgery along a link of 2-spheres in M_0. Thus we need to show that each of the ten model manifolds M_0 admits a contact structure and that 2-surgeries on M_0 can be performed as contact surgeries.

First some remarks about 2-surgeries in general. Clearly the fundamental group $\mathbb{Z}_2$ is preserved under 2-surgeries by the Seifert-van Kampen theorem.

Further, the condition that $w_2(M)$ be zero on $H_2(M)$ is equivalent to saying that every embedded 2-sphere in M has trivial normal bundle, so surgery is possible along any 2-sphere in M. Indeed, if $S^2 \subset M$ is an embedded 2-sphere with normal bundle ν and i denotes the inclusion of S^2 in M (and τ the tangent bundle), then

$$\begin{aligned}\langle w_2(\tau M), i_*[S^2]\rangle &= \langle i^* w_2(\tau M), [S^2]\rangle \\ &= \langle w_2(i^*\tau M), [S^2]\rangle \\ &= \langle w_2(\tau S^2) + w_2(\nu), [S^2]\rangle \\ &= \langle w_2(\nu), [S^2]\rangle.\end{aligned}$$

Because of $\pi_1(\mathrm{SO}_3) = \mathbb{Z}_2$, there is only one non-trivial $\mathbb{R}^3$-bundle over S^2, and it is detected by $w_2 \neq 0$, since w_2 is the primary obstruction to finding a 2-frame (hence, by orientability, a trivialization) over the 2-skeleton, i.e. the whole of S^2.

By representing a basis for $H_2(M)$ by embedded 2-spheres, we see, by a general position argument, that this condition is preserved under surgeries along 2-spheres (the belt sphere of the 3-handle corresponding to such a surgery automatically comes with trivial normal bundle).

More is true, though. Namely, by that same general position argument we see that the triviality or non-triviality of $w_2(M)$, which is detected by evaluation on 2-cycles over $\mathbb{Z}_2$, is not affected by a 2-surgery. So if $w_2(M)$ is zero on homology and M' is obtained from M by a sequence of surgeries along 2-spheres, then both M and M' have $w_2 = 0$ or both have $w_2 \neq 0$ (recall that if w_2 is equal to zero on homology we may regard it as an element of $\mathrm{Ext}(H_1(M), \mathbb{Z}_2)$, which in the case $H_1 = \pi_1 = \mathbb{Z}_2$ is the group $\mathbb{Z}_2$). This observation explains the distinction in Theorem 2 between (i) on the one hand and (ii), (iii) on the other, for we shall see in Section 4 that Q_0, in contrast to the manifolds Σ_q^5/T, has $w_2 = 0$.

The distinction between (ii) and (iii) is a consequence of [18], p. 260, where it is shown that the parity of the rank of $\pi_2(M)$ is a 2-surgery invariant (for $\pi_1(M) = \mathbb{Z}_2$), and the description of the Σ_q^5/T in Section 4, where the rank of $\pi_2(\Sigma_q^5/T)$ is shown to be zero or one, depending on q being odd or even.

We note in passing that as a consequence of Theorem 2 and these observations there are no 5-manifolds with π_2 of even rank, $\pi_1 = \mathbb{Z}_2$, and $w_2 = 0$.

We now turn to the contact surgeries relevant to our subsequent applications.

LEMMA 3.

(i) *Any* 0*-surgery* (*this includes connected sums*) *can be performed as a contact surgery.*

(ii) *Let* $i : S^1 \to (M, \xi)$ *be an embedding into an arbitrary contact manifold with co-orientable contact structure* ξ*. Then this embedding is isotopic to an isotropic embedding* i' (*i.e. one that is tangent to* ξ) *and thus contact surgery is possible along* $i'(S^1)$.

(iii) *Let* $i : S^2 \to (M, \xi)$ *be an embedding with* ξ *as in* (ii), $\dim M \geq 5$, *and* $H^2(M)$ *finite. Then the same conclusion as in* (i) *holds.*

Proof. — Contact surgery in the sense of [3], [19] is possible along any isotropic sphere $S^k \subset (M, \xi)$ with trivial *conformal symplectic normal bundle*

$$CSN(M, S^k) \stackrel{\text{def}}{=} (\tau S^k)^{\perp'}/\tau S^k,$$

where $(\tau S^k)^{\perp'} \subset \xi$ denotes the symplectic orthogonal bundle to $\tau S^k \subset \xi$ with respect to the (conformal) symplectic structure $d\alpha$ on $\xi = \ker \alpha$ (*cf.* [6], Section 3). Notice that $CSN(M^{2n+1}, S^k)$ is an $\mathrm{Sp}(2n-2k)$-bundle (or $\mathrm{U}(n-k)$-bundle). Necessary and sufficient for the existence of an isotropic embedding i' isotopic to $i : S^k \to (M, \xi)$ is that i be covered by a fibrewise injective complex bundle homomorphism $\tau S^k \otimes \mathbb{C} \to \xi | i(S^k)$, see [3], [4], [6].

• For case (i) we observe that any embedding of S^0 is evidently isotropic.

• In case (ii) this bundle homomorphism exists since both $\tau S^1 \otimes \mathbb{C}$ and $i^*\xi$ are trivial complex bundles over S^1. The symplectic bundle $CSN(M^{2n+1}, S^1)$ is trivial because its structure group $\mathrm{Sp}(2n-2)$ is connected.

• In case (iii) the assumption that $H^2(M)$ is finite implies that the induced homomorphism $i^* : H^2(M) \to H^2(S^2)$ is trivial, so $i^*\xi$ is a $\mathrm{U}(n)$-bundle over S^2 with vanishing first Chern class and hence a trivial bundle. The complexified tangent bundle $\tau S^2 \otimes \mathbb{C}$ is likewise a trivial bundle, so again there is the required fibrewise injective bundle homomorphism $\tau S^2 \otimes \mathbb{C} \to \xi \mid i(S^2)$ covering i. Notice that $i^*\xi$ being trivial implies in particular that the normal bundle of $i(S^2)$ in M is trivial (this is detected by w_2), so the topological condition for surgery is satisfied. The condition that $H^2(M)$ be finite also guarantees that the $\mathrm{Sp}(2n-4)$-bundle $CSN(M^{2n+1}, S^2)$ is trivial. □

Remark. — In general it is a subtle question which framings can be realized by contact surgeries. However, we shall apply (ii) only in a situation where the choice of framing can be controlled, and in (iii) no questions of framing arise because $\pi_2(\mathrm{SO}_{m-2}) = 0$ ($m = \dim M$).

4. The model manifolds.

In this section we show that each of the ten model manifolds admits a contact structure and determine enough of their homological data to prove that the lemma on contact surgery of the preceding section applies (which will prove Theorem 1), and to explain the distinction of the three cases in Theorem 2.

We begin with the manifold Q_0. Clearly we have $\pi_2(Q_0) \cong \mathbb{Z}$. The bundle $\pi : Q_0 \to \mathbb{R}P^2$ may be thought of as being obtained by gluing the trivial S^3-bundle over D^2 with the non-trivial S^3-bundle over the Möbius band. Then a straightforward application of the Mayer-Vietoris sequence shows $H_2(Q_0) = 0$.

Write η for the canonical line bundle over $\mathbb{R}P^2$ with total Stiefel-Whitney class $w(\eta) = 1 + a$, where a denotes the generator of $H^1(\mathbb{R}P^2; \mathbb{Z}_2) = \mathbb{Z}_2$, and write ε for a trivial line bundle. By construction, Q_0 is the unit sphere bundle of the $\mathbb{R}^4$-bundle $\eta \oplus 3\varepsilon$ over $\mathbb{R}P^2$, so the tangent bundles τQ_0 satisfies

$$\tau Q_0 \oplus \varepsilon = \pi^*(\tau \mathbb{R}P^2 \oplus \eta \oplus 3\varepsilon),$$

where the ε an the left hand side is to be thought of as the trivial line bundle complementary to τQ_0 in the tangent bundle of the $\mathbb{R}^4$-bundle $\eta \oplus 3\varepsilon$. Hence, computing modulo a^3,

$$1 + w_1(Q_0) + w_2(Q_0) = (1 + a)^4 = 1,$$

i.e. we find $w_2(Q_0) = 0$.

We also need a surgery description of Q_0 that will allow us to use contact surgery to find a contact structure on Q_0. We claim that Q_0 is obtained from S^5 by first performing a 0-surgery (which yields $S^1 \times S^4$) and then a 1-surgery along twice the generator of $\pi_1(S^1 \times S^4)$. The manifold obtained this way has $w_2 = 0$ and the same π_1 and H_2 as Q_0, which in connection with the arguments below would be sufficient for our purposes. But we can actually give a direct geometric identification of Q_0 with the result of this surgery. For this it is sufficient to show that $S^1 \times S^4$ can be obtained from Q_0 by a surgery along an S^3-fibre. Indeed, by removing a neighbourhood $S^3 \times D^2$ of a fibre we obtain the non-trivial S^3-bundle over the Möbius band, and gluing a copy of $D^4 \times S^1$ along the boundary produces an orientable S^4-bundle over the spine S^1 of the Möbius band, *i.e.* the desired $S^1 \times S^4$.

Because S^5 admits a contact structure, Lemma 3 shows that the same is true for Q_0, provided we can realize the framing of the 1-surgery on $S^1 \times S^4$ (described in the preceding paragraph) by a contact surgery.

To see that this is indeed the case we notice that $CSN(S^1 \times S^4, S^1)$ is a U(1)-bundle, so the choices of trivialization are given by $\pi_1(\mathrm{U}(1)) \cong \mathbb{Z}$, which maps onto the group $\pi_1(\mathrm{SO}(4)) = \mathbb{Z}_2$ (under the homomorphism induced by inclusion $\mathrm{U}(1) \to \mathrm{SO}(4)$), the group classifying topologically possible framings.

Next we describe the model manifolds Σ^5_q/T. Let $V^6_q(\varepsilon)$ be the complex hypersurface in $\mathbb{C}^4$ given by the equation

$$z_0^q + z_1^2 + z_2^2 + z_3^2 = \varepsilon,$$

where $\varepsilon \in \mathbb{R}^+_0$ (the non-negative real numbers) and $q \in \mathbb{N}_0$. For $\varepsilon \neq 0$ this hypersurface is non-singular. Define the Brieskorn manifold $\Sigma^5_q(\varepsilon)$ as the intersection of $V^6_q(\varepsilon)$ with $S^7(2) \subset \mathbb{C}^4$, the sphere of radius two. (Usually one considers the intersection with the unit sphere S^7, but some of the formulae below take a neater form if we take $S^7(2)$ instead. In [8], Section 14.2, it is shown that $V^6_q(\varepsilon) \cap S^7(2)$ is diffeomorphic to $V^6_q(\varepsilon) \cap S^7$ for $\varepsilon > 0$ sufficiently small.) We abbreviate $\Sigma^5_q(0)$ to Σ^5_q. For $\varepsilon > 0$ small enough, $\Sigma^5_q(\varepsilon)$ is diffeomorphic to Σ^5_q, see [8], Satz 14.3.

The orientation preserving involution $T : \Sigma^5_q(\varepsilon) \to \Sigma^5_q(\varepsilon)$ given by

$$T(z_0, z_1, z_2, z_3) = (z_0, -z_1, -z_2, -z_3)$$

is fixed point free on $\Sigma^5_q(\varepsilon)$ for small ε. The argument used in the proof of [8], Satz 14.3, also applies to show that $\Sigma^5_q(\varepsilon)/T$ is diffeomorphic to Σ^5_q/T for small ε.

LEMMA 4. — *The manifolds Σ^5_q are simply connected. For q odd, $H_2(\Sigma^5_q) = 0$ (so Σ^5_q is diffeomorphic to S^5). For q even, $H_2(\Sigma^5_q) \cong \mathbb{Z}$.*

Remark. — Below we shall see that Σ^5_q is diffeomorphic to $S^2 \times S^3$ for q even.

Proof. — For the case $q = 0$ see [8], p. 36. For $q \geq 1$ we observe that Σ^5_q is homotopy equivalent to

$$X_q = \{z \in \mathbb{C}^4 \mid z_0^q + z_1^2 + z_2^2 + z_3^2 = 0\} - \{0\},$$

since the function $f : X_q \to \mathbb{R}$ defined by

$$f(z_0, z_1, z_2, z_3) = \sum_{j=0}^{3} |z_j|^2$$

has no critical points on X_q, as can be seen by a simple Lagrange multiplier method (see [8], Satz 5.5). Hence $\Sigma_1^5 \simeq X_1 \cong \mathbb{C}^3 - \{0\}$, since X_1 may be regarded as the graph of a function over $\mathbb{C}^3 - \{0\}$.

For $q = 2$ we have an explicit diffeomorphism between Σ_2^5 and $S^2 \times S^3$. Here the formulae are very simple. Write $z_j = x_j + iy_j$, $j = 0, \ldots, 3$. Then the defining equations for Σ_2^5 become

$$|x|^2 - |y|^2 + 2i\langle x, y\rangle = 0 \quad \text{and} \quad |x|^2 + |y|^2 = 2,$$

i.e. $|x| = |y| = 1$ and $\langle x, y\rangle = 0$. So Σ_2^5 is the tangent sphere bundle of S^3, which is the trivial bundle because S^3 is a Lie group.

For the remainder of this proof we assume $q \geq 3$. By Satz 11.2 of [8], Σ_q^5 is diffeomorphic to the tree manifold $M^5(A_{q-1}^{(2)})$. That is, let $\mathcal{M}^6(A_{q-1}^{(2)})$ be the manifold with boundary obtained by plumbing $q-1$ copies of the tangent disc bundle $D\tau S^3$ of S^3 according to the graph A_{q-1}, which means that the i-th copy of $D\tau S^3$ is plumbed together with the $(i+1)$-st, $1 \leq i \leq q-2$. Then $\Sigma_q^5 = M^5(A_{q-1}^{(2)})$ is defined as the boundary of $\mathcal{M}^6(A_{q-1}^{(2)})$.

Then, according to Section 8.1 of [8], $H_2(\Sigma_q^5)$ can be computed from the sequence

$$\mathbb{Z}^{\oplus(q-1)} \xrightarrow{S} \mathbb{Z}^{\oplus(q-1)} \longrightarrow H_2(\Sigma_q^5) \to 0,$$

where S is the $(q-1) \times (q-1)$ matrix

$$S = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix}$$

determined by the intersection product in $H_3(\mathcal{M})$ (notice that $\mathcal{M}$ is homotopy equivalent to a wedge of $(q-1)$ 3-spheres).

Congruence modulo the image of S is generated by $e_2 \equiv e_{q-2} \equiv 0$ and $e_i \equiv e_{i+2}$ for $i = 1, \ldots, q-3$, where $e_1, \ldots, e_{q-1}$ denotes the standard basis for $\mathbb{Z}^{\oplus(q-1)}$. So for q odd, S defines an isomorphism, hence $H_2(\Sigma_q^5) = 0$; for q even, $H_2(\Sigma_q^5) \cong \mathbb{Z}^{\oplus(q-1)}/\operatorname{im} S$ is isomorphic to $\mathbb{Z}$, generated by e_1.

Since $\mathcal{M}$ with its spine $S^3 \vee \ldots \vee S^3$ removed retracts onto the boundary $\partial\mathcal{M} = \Sigma_q^5$ and this spine has codimension three in $\mathcal{M}$, we find

$$\pi_1(\Sigma_q^5) = \pi_1(\partial\mathcal{M}) = \pi_1\big(\mathcal{M} - S^3 \vee \ldots \vee S^3\big) = \pi_1(\mathcal{M}) = 0. \qquad \square$$

As mentioned in Section 2, the Σ^5_q/T, $q \equiv 1,3,5,7 \bmod 8$ represent the four diffeomorphism types of homotopy projective 5-spaces. In particular, the group $H^2(\Sigma^5_q/T) = \mathbb{Z}_2$ is finite, $\pi_2(\Sigma^5_q/T) = 0$, and $w_2(\Sigma^5_q/T) \neq 0$. These statements are also easy to prove directly (for all q), as we shall now see.

We begin with a preparatory lemma for the case q even.

LEMMA 5. — *The involution $T : \Sigma^5_q \to \Sigma^5_q$ induces the non-trivial isomorphism on $H_2(\Sigma^5_q) \cong \mathbb{Z}$ (q even).*

Proof. — Consider the embedded 2-sphere in Σ^5_q defined by

$$S^2 = \{x_0 = 1,\ x_1 = x_2 = x_3 = 0,\ y_0 = 0,\ y_1^2 + y_2^2 + y_3^2 = 1\}.$$

We observe that T maps this S^2 orientation reversingly onto itself, so it suffices to show that the fundamental cycle $[S^2]$ generates $H_2(\Sigma^5_q)$.

Define a 3-dimensional submanifold of Σ^5_q by

$$N^3 = S^7(2) \cap \left\{z_1 = iz_0^{\frac{1}{2}q},\ z_2 = iz_3\right\}.$$

This is a submanifold of Σ^5_q which is diffeomorphic to S^3 by an argument analogous to that employed in the proof of Lemma 4. Observe that we can regard N^3 as the intersection of $S^7(2)$ with the graph of the homogeneous polynomial $(z_0, z_2, z_3) \mapsto z_1 = iz_0^{q/2}$, defined on the hyperplane $z_2 = iz_3$.

The intersection of S^2 and $S^3 = N^3$ consists of the single point

$$p = \{x_0 = y_1 = 1,\ x_1 = x_2 = x_3 = y_0 = y_2 = y_3 = 0\},$$

and this intersection is transverse since T_pS^2 is spanned by ∂_{y_2} and ∂_{y_3}, and T_pS^3 is spanned by $\partial_{x_2} - \partial_{y_3}$, $\partial_{x_3} + \partial_{y_2}$ and $\partial_{y_0} +$ a linear combination of $\partial_{x_1}, \partial_{y_1}$.

So the intersection product $[S^2] \bullet [S^3]$ is equal to ± 1, which proves that $[S^2]$ generates $H_2(\Sigma^5_q)$. □

PROPOSITION 6. — *The manifolds Σ^5_q/T satisfy $H_2(\Sigma^5_q/T) = 0$ and $w_2(\Sigma^5_q/T) \neq 0$.*

Proof. — For q odd we have $\Sigma^5_q \cong S^5$, hence

$$\pi_2(\Sigma^5_q/T) = \pi_2(\Sigma^5_q) = 0.$$

With the theorem of Hopf mentioned in Section 2 we conclude $H_2(\Sigma^5_q/T) = 0$.

To deal with the case q even we consider the following short exact sequence of chain complexes (with integer coefficients) :

$$0 \to (1-T)C_*(\Sigma^5_q) \longrightarrow C_*(\Sigma^5_q) \longrightarrow C_*(\Sigma^5_q/T) \to 0,$$

the first map being given by inclusion and the second by projection. The relevant part of the induced long exact sequence in homology is

$$H_2\big((1-T)C_*(\Sigma^5_q)\big) \longrightarrow H_2(\Sigma^5_q) \longrightarrow H_2(\Sigma^5_q/T).$$

Again by Hopf's theorem we know that there is a surjection $\pi_2(\Sigma^5_q/T) \to H_2(\Sigma^5_q/T)$, and a generator of $\pi_2(\Sigma^5_q) \cong H_2(\Sigma^5_q) \cong \mathbb{Z}$ maps to a generator of $\pi_2(\Sigma^5_q/T)$ under the homomorphism induced by the projection $\Sigma^5_q \to \Sigma^5_q/T$. So it suffices to show that a generator of $H_2(\Sigma^5_q)$ maps to zero in $H_2(\Sigma^5_q/T)$. But this follows from the homology exact sequence above and the proof of Lemma 5, because there we have seen that we can find a generator of $H_2(\Sigma^5_q)$ of the form $(1-T)\Delta$ with Δ a suitable 2-disc in Σ^5_q, for instance

$$\Delta = \left\{x_0 = 0,\, x_1 = x_2 = x_3 = 0,\, y_0 = 0,\, y_1^2 + y_2^2 + y_3^2 = 1,\, y_3 \geq 0\right\}.$$

It remains to compute $w_2(\Sigma^5_q/T)$. The 2-sphere described in the proof of Lemma 5 descends to an embedded $\mathbb{R}P^2$ in Σ^5_q/T (for q even or odd). To prove $w_2(\Sigma^5_q/T) \neq 0$ it suffices to show that the restriction of the tangent bundle to this $\mathbb{R}P^2$ has non-trivial w_2.

Along $S^2 \subset \Sigma^5_q$ the differentials of the defining equations of Σ^5_q are

$$q\,\mathrm{d}z_0 + 2iy_1\,\mathrm{d}z_1 + 2iy_2\,\mathrm{d}z_2 + 2iy_3\,\mathrm{d}z_3 = 0$$

and

$$2\,\mathrm{d}x_0 + 2y_1\,\mathrm{d}y_1 + 2y_2\,\mathrm{d}y_2 + 2y_3\,\mathrm{d}y_3 = 0,$$

which simplifies to

$$\begin{aligned} \mathrm{d}x_0 &= 0, \\ y_1\,\mathrm{d}y_1 + y_2\,\mathrm{d}y_2 + y_3\,\mathrm{d}y_3 &= 0, \\ q\,\mathrm{d}y_0 + 2y_1\,\mathrm{d}x_1 + 2y_2\,\mathrm{d}x_2 + 2y_3\,\mathrm{d}x_3 &= 0. \end{aligned}$$

We see that the normal bundle νS^2 of S^2 in Σ_q^5 is spanned by

$$X_0 = 2\partial_{y_0} - q(y_1\partial_{x_1} + y_2\partial_{x_2} + y_3\partial_{x_3})$$

and vectors of the form

$$a_1\partial_{x_1} + a_2\partial_{x_2} + a_3\partial_{x_3}$$

with

$$\langle a_1, a_2, a_3\rangle\langle y_1, y_2, y_3\rangle = 0,$$

so we can naturally identify νS^2 with $\tau S^2 \oplus \varepsilon$. The differential T_* of T acts trivially on ε and like the antipodal map on τS^2, so we find $\nu \mathbb{R}P^2 \cong \tau \mathbb{R}P^2 \oplus \varepsilon$.

Write i for the inclusion of $\mathbb{R}P^2$ in Σ_q^5/T. Then for the total Stiefel-Whitney class we have

$$\begin{aligned} i^* w(\Sigma_q^5/T) &= w(\tau\mathbb{R}P^2 \oplus \nu\mathbb{R}P^2) \\ &= w(\tau\mathbb{R}P^2 \oplus \tau\mathbb{R}P^2) \\ &= (1+a)^6 = 1 + a^2, \end{aligned}$$

where a denotes the generator of $H^1(\mathbb{R}P^2; \mathbb{Z}_2)$. Therefore $i^* w_2(\Sigma_q^5/T) \neq 0$ and hence $w_2(\Sigma_q^5/T) \neq 0$.

This concludes the proof of Proposition 6. □

For completeness we mention the following simple proposition.

Proposition 7. — *For q even, Σ_q^5 is diffeomorphic to $S^2 \times S^3$.*

Proof. — In the proof of Lemma 5 we have described embedded copies of S^2 and S^3 which have trivial normal bundle, generate $H_2(\Sigma_q^5) \cong \mathbb{Z}$ and $H_3(\Sigma_q^5) \cong \mathbb{Z}$, respectively, and which intersect transversely in one point. Hence, after removing a tubular neighbourhood U of $S^2 \vee S^3 \hookrightarrow \Sigma_q^5$ we are left with a contractible 5-manifold with boundary S^4. By Smale's classical work (*cf.* [12], Cor. VIII.4.7), $\Sigma_q^5 - U$ is diffeomorphic to D^5, and because of the non-existence of exotic spheres in dimension 5 the result of gluing U with D^5 has to be diffeomorphic to $S^2 \times S^3$. □

Remarks.

(1) Using the discussion in Section 3 about the triviality of normal bundles of 2-spheres being detected by w_2, it is easy to show that any

5-manifold with $\pi_1 = \mathbb{Z}_2$, $H_2 = 0$ and $\pi_2 \cong \mathbb{Z}$ has $S^2 \times S^3$ as universal cover.

(2) As pointed out by the referee, the Σ^5_q/T with q even are not all pairwise homotopy equivalent (in contrast with the case q odd). This can be seen by looking at the action of T on $\pi_3(\Sigma^5_q) \cong \pi_3(S^2) \oplus \pi_3(S^3) \cong \mathbb{Z} \oplus \mathbb{Z}$. For different values of q one may obtain actions on $\mathbb{Z} \oplus \mathbb{Z}$ which are not conjugate in $\mathrm{GL}(2, \mathbb{Z})$.

Finally, we still have to exhibit a contact structure on Σ^5_q/T.

PROPOSITION 8. — *The manifolds Σ^5_q/T admit a contact structure.*

Proof. — By a result of Lutz and Meckert [14] the real 1-form

$$\alpha = \frac{i}{2} \sum_{j=0}^{3} \frac{1}{a_j} (z_j d\bar{z}_j - \bar{z}_j dz_j)$$

with $a_0 = q$, $a_1 = a_2 = a_3 = 2$, defines a contact form on Σ^5_q. Obviously α is T-invariant, so it induces a contact form on Σ^5_q/T. □

The discussion in this section, together with Lemma 3 and Theorem 2, completes the proof of Theorem 1.

5. Proof of Theorem 2.

The distinction between the three cases in Theorem 2 has already been explained in Section 3. We only have to show that a manifold M as in Theorem 2 can be obtained from one and only one of the ten model manifolds M_0 by surgery along a link of 2-spheres, or, equivalently, that there is a cobordism between M and M_0 containing 3-handles only.

With every M we can associate a 4-dimensional characteristic submanifold, as will be explained in a moment. We shall then see that the separation of the manifolds M into ten "2-surgery classes" is directly linked with the existence of ten 4-dimensional Pin cobordism classes (modulo the action of w_1) of such characteristic 4-manifolds.

Throughout this section, M will denote a closed, orientable 5-manifold with $\pi_1(M) = \mathbb{Z}_2$ and $w_2(M)$ zero on homology. Let $\widetilde{M}$ be the universal cover of M and T the free involution on $\widetilde{M}$ such that $M = \widetilde{M}/T$. Let $\widetilde{P}$ be a *characteristic submanifold for* $(\widetilde{M}, T)$, *i.e.* a 4-manifold such that $\widetilde{P} = A \cap TA$ and $\widetilde{M} = A \cup TA$, where A is a compact submanifold of $\widetilde{M}$ with boundary $\partial A = \widetilde{P}$. We also call $P = \widetilde{P}/T$ a characteristic submanifold for M.

Such characteristic submanifolds always exist : consider a classifying map $f : M \to \mathbb{R}P^n$, n large, for the double cover $\pi : \widetilde{M} \to M$ (recall $B\mathbb{Z}_2 = \mathbb{R}P^\infty$), make f transverse to $\mathbb{R}P^{n-1}$, and define $\widetilde{P} = \pi^{-1}f^{-1}(\mathbb{R}P^{n-1})$. Furthermore, by equivariant surgery we may assume that $\widetilde{P}$ is simply connected and hence $\pi_1(P) = \mathbb{Z}_2$, with the inclusion $i : P \to M$ inducing an isomorphism on fundamental groups [13], pp. 11–12.

Before formulating the next lemma we recall a few basic facts about Spin and Pin structures (*cf.* [11]). The group Spin(n) is the double cover of SO(n), the groups Pin$^+(n)$ and Pin$^-(n)$ are double covers of O(n) which are topologically the same but algebraically different as central $\mathbb{Z}_2$-extensions of O(n). The obstruction to putting a Spin(n) or Pin$^+(n)$ structure on an $\mathbb{R}^n$-bundle ζ (oriented in the former case) over a base space B is $w_2(\zeta) \in H^2(B; \mathbb{Z}_2)$; for a Pin$^-(n)$ structure the obstruction is $w_2(\zeta) + w_1^2(\zeta)$. In all cases $H^1(B; \mathbb{Z}_2)$ acts simply transitively on the set of structures on ζ, so a one-to-one correspondence between this set of structures and $H^1(B; \mathbb{Z}_2)$ is given by this action and a choice of structure.

LEMMA 9. — *If* $w_2(M) = 0$, *then* P *(with* $\pi_1(P) = \mathbb{Z}_2$*) has a pair of* Pin$^-(4)$ *structures, and if* $w_2(M) \neq 0$, *then* P *has a pair of* Pin$^+(4)$ *structures. The* Pin *cobordism class of this pair of structures is independent of the choice of characteristic submanifold* P *(with* $\pi_1(P) = \mathbb{Z}_2$*) and remains unchanged under 2-surgeries on* M.

Proof. — Write $i : P \to M$ for the inclusion of P in M. The normal bundle of P in M is non-orientable, because T acts orientation reversingly on the normal bundle of $\widetilde{P}$ in $\widetilde{M}$. Hence $i^*(\tau M) = \tau P \oplus \eta$, where η denotes the unique non-trivial line bundle over P, characterized by $w_1(\eta) \neq 0$ (since $H^1(P; \mathbb{Z}_2) = \mathbb{Z}_2$). Furthermore we can identify $w_1(\eta)$ with $w_1(P)$. Hence

$$\begin{aligned} i^* w_2(M) &= w_2(\tau P \oplus \eta) \\ &= w_2(P) + w_1(P) w_1(\eta) \\ &= w_2(P) + w_1^2(P). \end{aligned}$$

So if $w_2(M) = 0$, then P (or to be precise, τP) admits a pair of Pin$^-$ structures (but no Pin$^+$ structure, as the following arguments will show).

If $w_2(M)$ is non-zero (but vanishes on homology), then $w_2(M)$ may be regarded as the non-zero element in $\mathrm{Ext}(H_1(M), \mathbb{Z}_2) = \mathbb{Z}_2$. Since the inclusion $i : P \to M$ induces an isomorphism on π_1, we get an induced isomorphism

$$i^* : \mathrm{Ext}\big(H_1(M), \mathbb{Z}_2\big) \longrightarrow \mathrm{Ext}\big(H_1(P), \mathbb{Z}_2\big),$$

so $i^* w_2(M) = 1 \in \mathbb{Z}_2 = \mathrm{Ext}(H_1(P), \mathbb{Z}_2)$. Furthermore, $w_1(P)$ is the pull-back of the generator a of $H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ under the classifying map $f \circ i : P \to \mathbb{R}P^n$. Since a^2 is the generator of

$$H^2(\mathbb{R}P^n; \mathbb{Z}_2) = \mathrm{Ext}\big(H_1(\mathbb{R}P^n), \mathbb{Z}_2\big) = \mathbb{Z}_2,$$

its pull-back $w_1^2(P)$ is the generator of $\mathrm{Ext}(H_1(P), \mathbb{Z}_2)$. We conclude that, in the case $w_2(M) \neq 0$, we have

$$w_2(P) = i^* w_2(M) + w_1^2(P) = 0$$

(but $w_2(P) + w_1^2(P) \neq 0$), so P admits a pair of Pin^+ structures (but no Pin^- structure).

Different characteristic submanifolds can be joined by a characteristic cobordism (obtained from a homotopy between classifying maps f_0 and f_1), which gives a $\mathrm{Pin}^{\pm}$ cobordism in the case under consideration.

If M' is the result of performing 2-surgeries on M, the preceding arguments (and the considerations in Section 3), applied to the cobordism between M and M' (which consists of 3-handles only), show the existence of a $\mathrm{Pin}^{\pm}$ cobordism between respective characteristic submanifolds P and P'. □

We now prove the converse of this lemma. Let M, M' be manifolds as in Theorem 2 and P, P' characteristic submanifolds with $\pi_1 = \mathbb{Z}_2$ and a pair of $\mathrm{Pin}^{\pm}$ structures as described above. By slight abuse of language we say P and P' are $\mathrm{Pin}^{\pm}$ cobordant if for matching choices of $\mathrm{Pin}^{\pm}$ structures on P and P' we find a $\mathrm{Pin}^{\pm}$ cobordism between them.

LEMMA 10. — *If P and P' are* $\mathrm{Pin}^{\pm}$ *cobordant, then M' can be obtained from M by surgery along a link of 2-spheres.*

Proof. — Let V be a $\mathrm{Pin}^{\pm}$ cobordism between P and P'. Choose a generator g_0 of $\pi_1(V)$ on which $w_1(V)$ is non-zero. By performing surgery on g_0^2 we may assume that g_0 has order two. By replacing each element g_i, $1 \leq i \leq s$, of a generating set $\{g_0, g_1, \dots, g_s\}$ of $\pi_1(V)$ by $g_0 g_i$, if necessary, we may assume further that $w_1(V)$ is zero on $g_1, \dots, g_s$. After surgery on $g_1, \dots, g_s$ we arrive at a $\mathrm{Pin}^\pm$ cobordism (still denoted V) between P and P' with $\pi_1(V) = \mathbb{Z}_2$ (and the inclusions of P resp. P' in V inducing isomorphisms on π_1, as can be seen by looking at the first Stiefel-Whitney class). Notice that this V admits exactly two $\mathrm{Pin}^\pm$ structures, restricting to the pairs of $\mathrm{Pin}^\pm$ structures on P resp. P'.

This $\mathrm{Pin}^\pm$ cobordism V lifts to a simply connected Spin cobordism $\widetilde{V}$ between $\widetilde{P}$ and $\widetilde{P}'$. We obtain a closed 5-manifold V_0 by gluing

$$A \cup_{\widetilde{P}} \widetilde{V} \cup_{\widetilde{P}'} A',$$

where A, A' are as at the beginning of this section. This manifold admits a Spin structure since there is a unique Spin structure on each of the simply connected manifolds $\widetilde{P}$ and $\widetilde{P}'$, so the Spin structures given on the three constituent pieces $A, \widetilde{V}$, and A' match along the boundaries.

Since the 5-dimensional Spin cobordism group Ω_5^{Spin} is zero, we find a 6-dimensional Spin manifold W_0 with boundary $\partial W_0 = V_0$. By performing 1-surgery we may assume $\pi_1(W_0) = 0$. The cobordism $\widetilde{V}$ was obtained by lifting a cobordism between P and P', so there is an involution T_0 on $\widetilde{V}$ extending the involutions T on $\widetilde{P}$ and T' on $\widetilde{P}'$. Now think of W_0 as having corners at $\widetilde{P}$ and $\widetilde{P}'$ and glue it with a diffeomorphic copy W_0^* of W_0, thought of as cobounding

$$TA \cup_{\widetilde{P}} \widetilde{V} \cup_{\widetilde{P}'} T'A',$$

by identifying $x \in \widetilde{V}$ with $T_0 x \in \widetilde{V}$ (see Figure 1 for a schematic illustration). This yields a Spin cobordism $W_0 \cup_{\widetilde{V}} W_0^*$ between $\widetilde{M}$ and $\widetilde{M}'$ which descends to a cobordism W between M and M' with $w_2(W)$ zero on homology and $\pi_1(W) = \mathbb{Z}_2$, and the inclusions of M and M' in W inducing isomorphisms on π_1.

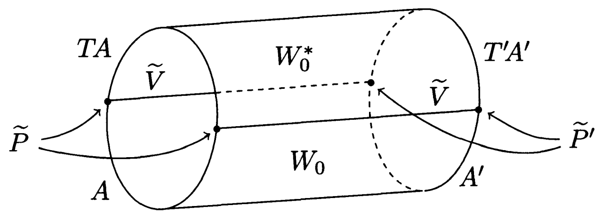

Figure 1. Constructing the cobordism.

The cobordism W has a handle decomposition on M from which all 0- and 6-handles can be removed using [16], Corollary 6.14. Furthermore we can eliminate all 1-handles (see [16], Lemma 6.15) at the price of introducing extra 3-handles. In essence this is because the vanishing of the relative homotopy group $\pi_1(W, M)$ allows us to introduce a complementary pair of 2- and 3-handles in such a way that the former cancels a 1-handle in the original decomposition. Rourke and Sanderson work in the piecewise linear category, but the smooth argument is identical. By considering the dual handle decomposition on M' we may also remove all 5-handles (of the handle decomposition on M), again at the price of introducing additional 3-handles.

To finish the proof of the lemma, we need to show that W can further be changed into a cobordism without 2- and 4-handles.

Geometrically this can be seen as follows. To each 2-handle (resp. 4-handle) there corresponds a descending core 2-disc (resp. ascending cocore 2-disc). If we choose a Morse function on W corresponding to its handle decomposition, these are the stable (resp. unstable) manifolds of the critical points of index 2 (resp. 4). Since the inclusions of the boundary components M, M' in W are injective on π_1, these 2-discs close off to 2-spheres. Surgery along these 2-spheres, which is possible because of $w_2(W)$ being zero on homology, removes these 2- and 4-handles.

More algebraically one may argue that by 2-surgery on W we can kill $H_2(W, M)$ and $H_2(W, M')$, because every element of these groups comes from $H_2(W)$ (the homomorphisms $H_1(M) \to H_1(W)$ and $H_1(M') \to H_1(W)$ being isomorphisms) and $w_2(W)$ is zero on homology. The groups $H_*(W, M)$ can be computed from a cellular chain complex whose generators are in one-to-one correspondence with the handles of W on M. The fact that $H_2(W, M) = 0$ implies that to every 2-handle we find a complementary 3-handle, and such complementary handles cancel in pairs [16], Lemma 6.4. By considering the 4-handles as 2-handles in the dual handle decomposition of W on M', we can remove those as well. □

To complete the proof of Theorem 2 we now only have to show that our ten model manifolds represent all possible $\mathrm{Pin}^{\pm}(4)$ cobordism classes of characteristic submanifolds. Recall our abuse of language explained before the preceding lemma.

LEMMA 11. — *The ten model manifolds Q_0 and Σ_q^5/T, $0 \leq q \leq 8$ (resp. their characteristic submanifolds) represent the ten different* $\mathrm{Pin}^{\pm}(4)$

cobordism classes of pairs of $\mathrm{Pin}^{\pm}(4)$ *structures on non-orientable* 4-*manifolds* P *with* $\pi_1(P) = \mathbb{Z}_2$.

Proof. — The group $\Omega_4^{\mathrm{Pin}^-}$ is the zero group [11], so Q_0 is the only representative we need (recall that Pin^- corresponds to $w_2(M) = 0$).

The group $\Omega_4^{\mathrm{Pin}^+}$ is isomorphic to $\mathbb{Z}_{16}$ (generated by $\pm\mathbb{R}P^4$), and the action of $w_1(\mathbb{R}P^4)$ exchanges these two generators [11]. So we need to represent exactly $|\mathbb{Z}_{16}/\pm| = 9$ cobordism classes of pairs of Pin^+ structures. Following the dictum of Kirby-Siebenmann [10], p. 337, that it is "wise to seek several proofs" when working with the involutions T on the Σ_q^5, we offer two alternative arguments for showing that the Σ_q^5/T, $0 \le q \le 8$, represent the required 9 cobordism classes. The first proof is based on a Lefschetz fixed point formula, the second on a more direct cobordism argument. In either proof the subtle point is to establish correct signs, be it those of contributions of fixed points to the Lefschetz formula, or those of Pin^+ structures on copies of $\mathbb{R}P^4$ in the boundary of a certain cobordism.

First argument. — It suffices to show that there can be no cobordism W between Σ_q^5/T and Σ_r^5/T of the kind described above (in particular $\pi_1(W) = \mathbb{Z}_2$ and $w_2(W)$ zero on homology) unless $q \equiv \pm r \bmod 16$.

Assume there is such a cobordism for some $q, r \in \mathbb{N}$. The free involution T on $\Sigma_q^5(\varepsilon)$ extends to an involution with q isolated fixed points

$$\left(\sqrt[q]{\varepsilon}\, e^{2\pi i k/q}, 0, 0, 0\right), \quad 0 \le k \le q-1,$$

on $W_q^6 = V_q^6(\varepsilon) \cap D^8(2)$. Let X be the closed Spin manifold obtained by gluing

$$W_q^6 \cup_{\Sigma_q^5(\varepsilon)} \widetilde{W} \cup_{\Sigma_r^5(\varepsilon)} W_r^6.$$

The given involutions on the three constituent parts glue together to give an involution on X with (geometrically) $q + r$ fixed points. Applying a Lefschetz fixed point formula as in [1] one finds that $q \equiv \pm r \bmod 16$. More specifically, the Lefschetz fixed point formula yields divisibility by 16 of the total number of fixed points (counted with sign), and so the key issue is to show that the q fixed points of T on W_q^6 all contribute with the same sign.

Notice that the proof of Theorem 9.8 in [1] only shows $q \equiv \pm r \bmod 8$; the statement $q \equiv \pm r \bmod 16$ in that theorem is a misprint. But in a short addendum published in [10], p. 338, Atiyah proves the stronger statement.

Notice further that the argument in [1] for showing that all q fixed points contribute with the same sign is (apparently) only valid for q odd. But the alternative argument provided by Satz 15.7 of [8] applies in all cases.

Second argument. — There is a $\mathbb{Z}_q$-action on $W_q^6 = V_q^6(\varepsilon) \cap D^8(2)$, generated by

$$S(z_0, z_1, z_2, z_3) = (e^{2\pi i/q} z_0, z_1, z_2, z_3),$$

which cyclically permutes the q fixed points of T. Notice that the characteristic submanifold of $\Sigma_q^5(\varepsilon)/T$ can be realized as

$$P_q = \big(\Sigma_q^5(\varepsilon) \cap \{\operatorname{Im} z_1 = 0\}\big)/T,$$

the intersection of $\Sigma_q^5(\varepsilon)$ with $\{\operatorname{Im} z_1 = 0\}$ being transverse. (Again, this is a straightforward check using the Lagrange multiplier method.) Remove from W_q^6 a T- and S-invariant union of q disjoint small balls $B_1, \ldots, B_q$ around the q fixed points of T. The intersection of W_q^6 with the hyperplane $H := \{\operatorname{Im} z_1 = 0\}$ fails to be transverse in the two points $(0, \pm\sqrt{\varepsilon}, 0, 0)$ only. We can perturb H around these points to obtain a T- and S-invariant hyperplane H' transverse to W_q^6, so that

$$Y := \Big(\Big(W_q^6 - \bigcup_{k=1}^{q} B_k\Big) \cap H'\Big)/T$$

is a cobordism from the characteristic submanifold P_q to q copies of $\mathbb{R}P^4$ (since the coordinate description shows that restricted to $\partial B_k \cap \{\operatorname{Im} z_1 = 0\}$ the involution T is equivalent to the antipodal map on S^4).

We have $\pi_1(Y) = \mathbb{Z}_2$, so Y admits a pair of Pin^+ structures (by Lemma 9, since $w_2(\Sigma_q^5/T) \neq 0$ and hence $w_2((W_q^6 - \cup B_k)/T) \neq 0$). The argument will be complete if we can show that for a fixed choice of Pin^+ structure on Y, the induced Pin^+ structures on the q copies of $\mathbb{R}P^4$ in the boundary ∂Y are the same. For that, in turn, it suffices to prove that the action of S on Y preserves the chosen Pin^+ structure.

Consider the loop $\widetilde{C}$ in the universal cover $\widetilde{Y}$ of Y defined by

$$\big(0, 0, \sqrt{\varepsilon}\cos\varphi, \sqrt{\varepsilon}\sin\varphi\big), \quad 0 \leq \varphi \leq 2\pi.$$

One half of this loop, say that corresponding to $0 \leq \varphi \leq \pi$, descends to an embedded loop C in Y representing the generator of $\pi_1(Y) = \mathbb{Z}_2$, and C

is pointwise fixed under the action of S. Along $\widetilde{C}$ the defining equation for the complex hypersurface V_q^6 has differential

$$\sqrt{\varepsilon}\cos\varphi\,\mathrm{d}z_2 + \sqrt{\varepsilon}\sin\varphi\,\mathrm{d}z_3 = 0,$$

so the normal bundle of $\widetilde{C}$ in $\widetilde{Y}$ is spanned by

$$\partial_{x_0}, \partial_{y_0}, \partial_{x_1}, \text{ and } \cos\varphi\,\partial_{y_3} - \sin\varphi\,\partial_{y_2},$$

and the tangent bundle by $\cos\varphi\,\partial_{x_3} - \sin\varphi\,\partial_{x_2}$. The involution T acts non-trivially on ∂_{x_1} and trivially on the other vector fields. Hence

$$\tau Y_{|C} \cong \eta \oplus 4\varepsilon,$$

where, as before, η denotes the canonical (non-trivial) line bundle on $C = \mathbb{R}P^1$ and ε the trivial line bundle. The action of S on the trivial subbundle ε^2 of $\tau Y_{|C}$ spanned by $\partial_{x_0}, \partial_{y_0}$ is a rotation, hence homotopic to the identity, and the action on the complementary subbundle $\eta \oplus 2\varepsilon$ is trivial. So we obtain the desired result that S fixes the chosen Pin^+ structure on Y (which is determined by its restriction to $\tau Y_{|C}$). □

Acknowledgement. — We are most grateful to the referee for detecting a grave error in an earlier manuscript. The idea to use Pin cobordism theory for proving Theorem 2 was kindly suggested by the referee. In fact, this new argument allowed us to remove a weak but superfluous condition on the second Stiefel-Whitney class in Theorem 1.

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Manuscrit reçu le 1er septembre 1997,
révisé le 11 décembre 1997,
accepté le 6 mai 1998.

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