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Hart F. Smith

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A PARAMETRIX CONSTRUCTION FOR WAVE EQUATIONS WITH $C^{1,1}$ COEFFICIENTS

by Hart F. SMITH

1. Introduction.

We introduce in this paper a new parametrix construction for variable coefficient wave equations, under the assumption that the coefficients of the principal term possess two bounded derivatives in the space variables, and one bounded derivative in the time variable. This regularity condition is the weakest of its sort under which the bicharacteristic flow is well-posed, and under which energy estimates hold. As a consequence of our construction, we obtain the Strichartz and Pecher estimates for solutions to such equations in space dimensions $n = 2, 3$. We remark that the assumption of two bounded derivatives in the space variables is minimal for the validity of the Strichartz estimates, as shown by counterexamples of the author and Sogge [10].

Throughout this paper, $A(t,x) = \{a_{ij}(t,x)\}_{i,j=1}^n$ denotes a matrix valued function of the variables $(t,x) \in [-t_0, t_0] \times \mathbb{R}^n$, which takes values in the real, symmetric, $n \times n$ matrices, such that, for some $c > 0$,

$$c^{-1}|\xi|^2 \geq \sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j \geq c|\xi|^2, \quad \forall (t,x,\xi) \in [-t_0, t_0] \times \mathbb{R}^n \times \mathbb{R}^n.$$

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We assume that the coefficients $a_{ij}(t,x)$ satisfy a Lipschitz condition in t,

$$|a_{ij}(t,x) - a_{ij}(t',x)| \le C|t-t'|.$$

We also assume that the first derivatives in x satisfy a Lipschitz condition in x,

$$|\nabla_x a_{ij}(t,x) - \nabla_x a_{ij}(t,x')| \le C|x-x'|.$$

This is equivalent to assuming that the partial derivatives of second order with respect to x (in the distribution sense) of the coefficients $a_{ij}(t,x)$, as well as the partial derivative of first order with respect to t, belong to $L^\infty([-t_0,t_0]\times\mathbb{R}^n)$.

We set

$$A(t,x,\partial_x) = \sum_{i,j=1}^{n} a_{ij}(t,x)\partial_{x_i}\partial_{x_j}.$$

Let $H^\alpha(\mathbb{R}^n)$ denote the Sobolev space of functions with α derivatives in $L^2(\mathbb{R}^n)$.

DEFINITION 1.1. — *Suppose that u belongs to*

$$C([-t_0,t_0];H^{\alpha+1}(\mathbb{R}^n)) \cap C^1([-t_0,t_0];H^\alpha(\mathbb{R}^n)),$$

where $-1 \le \alpha \le 2$. We say that u is a weak solution to the Cauchy problem

$$\begin{aligned}(\partial_t^2 - A(t,x,\partial_x))u(t,x) &= F(t,x),\\ u(t,x)|_{t=0} &= f(x),\\ \partial_t u(t,x)|_{t=0} &= g(x),\end{aligned}$$

if the initial conditions are satisfied in the vector valued sense at $t=0$, and if, in the sense of distributions, it holds that $(\partial_t^2 - A(t,x,\partial_x))u(t,x) = F(t,x)$.

The assumption that $\alpha \ge -1$ implies that $u \in L^2([-t_0,t_0]\times\mathbb{R}^n)$, so that $(\partial_t^2 - A(t,x,\partial_x))u(t,x)$ makes sense weakly.

For $-1 \le \alpha \le 2$, given data $f \in H^{\alpha+1}(\mathbb{R}^n), g \in H^\alpha(\mathbb{R}^n)$, and $F \in L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))$, we establish existence and uniqueness of a weak solution u to the Cauchy problem in the above sense. The key step is constructing a family of operators $\mathbf{s}(t,s)$, for $-t_0 \le s,t \le t_0$, which are essentially Fourier integral operators of order -1, such that

$$\mathbf{s}(t,s)|_{t=s} = 0, \qquad \partial_t\mathbf{s}(t,s)|_{t=s} = \mathbf{I},$$

and such that $(\partial_t^2 - A(t,x,\partial_x))\mathbf{s}(t,s)$ is a bounded operator (for each fixed t,s) on the Sobolev spaces $H^\alpha(\mathbb{R}^n)$, for α in the above range. We then pose

$$u(t,x) = \int_0^t (\mathbf{s}(t,s)G(s,\cdot))(x)ds,$$

and observe that

$$(\partial_t^2 - A(t,x,\partial_x))u(t,x) = G(t,x) + \int_0^t [(\partial_t^2 - A(t,x,\partial_x))\mathbf{s}(t,s)G(s,\cdot)](x)ds.$$

The right hand side is a Volterra equation, and may be solved by a convergent expansion for G in terms of F.

The operators $\mathbf{s}(t,s)$ are constructed as matrices in terms of a frame of functions on $L^2(\mathbb{R}^n)$. This frame consists of "coherent wave packets", similar to the wave packets of Cordoba-Fefferman [3], which are sufficiently localised in phase space that the action of the wave group on a wave packet may be approximated by a rigid motion corresponding roughly to translation of the center of the packet along the Hamiltonian (bicharacteristic) flow.

The limited differentiability of the coefficients $a_{ij}(t,x)$ is handled by adapting a technique from the multilinear Fourier analysis/paraproduct theory of Coifman and Meyer [2], and Bony [1]. Precisely, for $k \geq 0$, we take $a_{ij}^k(t,x)$ to be a sequence of smooth functions satisfying

$$\|\partial_t^m \partial_x^\beta a_{ij}^k(t,x)\|_{L^\infty([-t_0,t_0]\times\mathbb{R}^n)} \leq C_{m,\beta} 2^{\frac{k}{2}(2m+|\beta|-2)}, \quad 2m+|\beta| \geq 2,$$

which approximates $a_{ij}(t,x)$ in the following sense:

$$\|a_{ij}(t,x) - a_{ij}^k(t,x)\|_{L^\infty([-t_0,t_0]\times\mathbb{R}^n)} \leq C2^{-k},$$
$$\|\nabla_x(a_{ij}(t,x) - a_{ij}^k(t,x))\|_{L^\infty([-t_0,t_0]\times\mathbb{R}^n)} \leq C2^{-\frac{k}{2}}.$$

For large k, it follows that the corresponding matrix function $A_k(t,x)$ is uniformly elliptic with lower bound $\frac{c}{2}$. Without loss of generality we assume that this holds for all $k \geq 0$.

We also assume that the partial Fourier transform of $a_{ij}^k(t,x)$ in the x variables vanish for frequencies of modulus greater than $2^{\frac{k}{2}}$,

$$\widehat{a_{ij}^k}(t,\xi) = 0, \quad |\xi| \geq 2^{\frac{k}{2}}.$$

Such a sequence may be obtained by smoothly truncating the Fourier transform in x of $a_{ij}(t,x)$ to frequencies of modulus less than $2^{\frac{k}{2}}$, followed by convolution with respect to t against dilates by 2^{-k} of a compactly supported test function.

For $k \geq 0$, we set

$$A_k(t,x,\partial_x) = \sum_{i,j=1}^{n} a_{ij}^k(t,x)\partial_{x_i}\partial_{x_j}.$$

We then construct $\mathbf{s}(t,s)$ as a sum of operators $\mathbf{s}_k(t,s)$, which are localised to frequencies of size $|\xi| \approx 2^k$, such that

$$\sum_{k=0}^{\infty}(\partial_t^2 - A_k(t,x,\partial_x))\mathbf{s}_k(t,s)$$

is a Fourier integral operator of order 0.

This replacement of $A(t,x,\partial_x)$ by $A_k(t,x,\partial_x)$ at the Littlewood-Paley localisation of level k represents a balance between the requirement of a suitable operator theory for $\mathbf{s}(t,s)$, and the requirement of a bounded error term. On the one hand, it yields a family of operators $\mathbf{s}(t,s)$ which are essentially Fourier integral operators with symbols of class $S_{\frac{1}{2},\frac{1}{2}}$. This is the largest class of symbols for which the standard theory of Fourier integral operators goes through. On the other hand, since

$$\|a_{ij}(t,x) - a_{ij}^k(t,x)\|_{L^\infty([-t_0,t_0]\times\mathbb{R}^n)} \leq C2^{-k},$$

the operator $A(t,x,\partial_x) - A_k(t,x,\partial_x)$ behaves as an operator of order 1 against $\mathbf{s}_k(t,s)$. As a result, the error term

$$\sum_{k=1}^{\infty}(A(t,x,\partial_x) - A_k(t,x,\partial_x))\mathbf{s}_k(t,s)$$

is a bounded mapping on $H^\alpha(\mathbb{R}^n)$, for $-1 \leq \alpha \leq 2$. Balancing these requirements is one of several places where the methods of this paper break down if the coefficients $a_{ij}(t,x)$ have less than two bounded derivatives in the spatial variables.

The organisation of this paper is as follows. In Section 2, we introduce the frame of coherent wave packets, and a class of Fourier integral operators in terms of decay properties on their matrix representation in this frame. In Theorem 2.7, we establish composition and mapping properties for this class, under very weak conditions on the associated canonical transformation. In Section 3, we develop the main idea of the parametrix construction, which is that, with an error term that is one derivative smoother, the action of the wave group on an element of the frame can be approximated by a rigid translation along the flow of the Hamiltonian field of A. In Section 4, this idea is further developed to produce an approximate inverse for the wave operator, and an exact solution to the Cauchy problem by iteration.

In Sections 4a and 4b, these results are adapted to operators in divergence and Laplace-Beltrami form. In Section 5, we establish uniqueness of the weak solutions produced in Section 4, 4a, and 4b. Finally, in Section 6, we use the approximate inverse of Section 4 to establish the Strichartz and Pecher estimates for weak solutions to the Cauchy problem.

2. The frame of functions.

In this section, we introduce the frame of "coherent" wave packets that are used to realise the construction of the parametrix. The expansion of a function f in this frame corresponds to a dyadic-parabolic decomposition of the Fourier transform of f. This dyadic-parabolic decomposition of phase space has been used in many papers to understand the $L^p(\mathbb{R}^n)$ behaviour of oscillatory integrals. We mention here the work of Fefferman [4] and Seeger-Sogge-Stein [8]; see also the presentation in chapter 9 of [12], where it is referred to as the second dyadic decomposition. We also mention the development by the author in [9] of a Hardy space based on this decomposition of phase space.

We start with a smooth partition of unity on $\mathbb{R}^n$ of the form

$$1 = |h_0(\xi)|^2 + \sum_{k=1}^{\infty} \sum_{\omega} |h_k^{\omega}(\xi)|^2.$$

The index ω varies over a set, the set depending on k, of approximately $2^{\frac{(n-1)k}{2}}$ unit vectors evenly distributed over the surface of the unit sphere. The functions $h_k^{\omega}(\xi)$ are smooth functions, which vanish outside the set

$$2^{k-\frac{1}{2}} < |\xi| < 2^{k+\frac{3}{2}}, \quad \left| \frac{\xi}{|\xi|} - \omega \right| \le 2^{-\frac{k}{2}}.$$

We also require the following estimates on the derivatives of $h_k^{\omega}(\xi)$,

$$|\langle \omega, \partial_{\xi} \rangle^j \partial_{\xi}^{\alpha} h_k^{\omega}(\xi)| \le C_{j,\alpha} 2^{-k\left(j+\frac{|\alpha|}{2}\right)},$$

where the constants $C_{j,\alpha}$ are independent of ω and k. For the construction of such a partition of unity, see Chapter 9, §4.4 of [12].

The function $h_k^{\omega}(\xi)$ is thus supported in a rectangle, with one sidelength, in the ω direction, equal to $2\pi \cdot 2^k$, and the orthogonal sidelengths equal to $2\pi \cdot 2^{\frac{k}{2}}$. For each pair (ω, k), we now let Ξ_k^{ω} be a rectangular lattice in $\mathbb{R}^n$ with spacing 2^{-k} in the direction ω, and spacing $2^{-\frac{k}{2}}$ in directions orthogonal to ω. Let Γ denote the corresponding set of triples $\{\gamma = (x, \omega, k) : x \in \Xi_k^\omega\}$, and set

$$\widehat{\varphi_\gamma}(\xi) = (2\pi)^{-\frac{n}{2}} 2^{-\frac{k(n+1)}{4}} e^{-i\langle x,\xi\rangle} h_k^\omega(\xi).$$

Then

$$(2.1) \quad |\langle \omega, \partial_y\rangle^j \langle \omega^\perp, \partial_y\rangle^\alpha \varphi_\gamma(y)| \\ \leq C_{j,\alpha,N} 2^{k\left(\frac{(n+1)}{4}+j+\frac{|\alpha|}{2}\right)} (1 + 2^k|\langle \omega, y-x\rangle| + 2^k|y-x|^2)^{-N},$$

where $\langle \omega^\perp, \partial_y\rangle$ denotes differentation in directions normal to ω, and where the constants $C_{j,\alpha,N}$ are independent of the index γ. The functions $\varphi_\gamma(y)$ form a frame of functions on $L^2(\mathbb{R}^n)$, in the sense that if

$$c(\gamma) = \int_{\mathbb{R}^n} \overline{\varphi_\gamma(y)} f(y)\,dy,$$

then

$$f(y) = \sum_\gamma c(\gamma)\varphi_\gamma(y), \qquad \int_{\mathbb{R}^n} |f(y)|^2 dy = \sum_\gamma |c(\gamma)|^2.$$

The frame $\varphi_\gamma(y)$ is not orthogonal, nor even independent; however, it also holds for any sequence of coefficients $d(\gamma)$ that

$$\int_{\mathbb{R}^n} \Big|\sum_\gamma d(\gamma)\varphi_\gamma(y)\Big|^2 dy \lesssim \sum_\gamma |d(\gamma)|^2.$$

To clarify the relation between a function $f(y)$ and the corresponding sequence of coefficients $c(\gamma)$, we define maps

$$U_1 : L^2(\mathbb{R}^n) \to \ell^2(\Gamma), \quad U_1(f) = \{c(\gamma)\}_{\gamma\in\Gamma}$$
$$U_2 : \ell^2(\Gamma) \to L^2(\mathbb{R}^n), \quad U_2(d(\gamma)) = \sum_\gamma d(\gamma)\varphi_\gamma(x).$$

These are continuous mappings, and U_2 is a left inverse for U_1. The mapping

$$\Pi = U_1 \circ U_2$$

is a non-orthogonal projection of $\ell^2(\Gamma)$ onto the range of U_1. If T is an operator from Schwartz functions to tempered distributions, it has a naturally associated matrix $a = U_1 \circ T \circ U_2$, or

$$a(\gamma, \gamma') = \int \overline{\varphi_\gamma(y)}(T\varphi_{\gamma'})(y)\,dy.$$

This gives an algebraic homomorphism from bounded operators on $L^2(\mathbb{R}^n)$ into the space of bounded operators on $\ell^2(\Gamma)$.

Alternatively, if $a(\gamma,\gamma')$ is any matrix, one can formally associate the operator $T = U_2 \circ a \circ U_1$, or

$$Tf(y) = \sum_{\gamma,\gamma'} a(\gamma,\gamma')c(\gamma')\varphi_\gamma(y).$$

This map, which takes bounded operators on $\ell^2(\Gamma)$ to bounded operators on $L^2(\mathbb{R}^n)$, is not a homomorphism. However, it is a left inverse for the first map. We remark also that the map from matrices to operators on $L^2(\mathbb{R}^n)$ back to matrices takes the form

$$a \to \Pi \circ a \circ \Pi.$$

The following is an immediate consequence of the preceeding results.

LEMMA 2.1 (Schur's Lemma). — *Suppose that $a(\gamma,\gamma')$ is a matrix, and that there exists a strictly positive function $\rho(\gamma)$, such that for all γ one has*

$$\sum_{\gamma'} |a(\gamma,\gamma')|\rho(\gamma') \le C_a\rho(\gamma),$$
$$\sum_{\gamma'} |a(\gamma',\gamma)|\rho(\gamma') \le C_a\rho(\gamma).$$

Then the operator determined by a is continuous on $L^2(\mathbb{R}^n)$, with operator norm bounded by a multiple of C_a.

There is a natural algebra of operators associated to the above frame. We first recall the pseudodistance function on the cosphere bundle $S^*(\mathbb{R}^n)$, which was introduced in [9],

$$d(x,\omega;\tilde{x},\tilde{\omega}) = |\langle\omega, x-\tilde{x}\rangle| + |\langle\tilde{\omega}, x-\tilde{x}\rangle| + \min(|x-\tilde{x}|, |x-\tilde{x}|^2) + |\omega-\tilde{\omega}|^2.$$

A useful estimate to be observed is that

$$d(x,\omega;\tilde{x},\tilde{\omega}) \approx |\langle\omega, x-\tilde{x}\rangle| + \min(|x-\tilde{x}|, |x-\tilde{x}|^2) + |\omega-\tilde{\omega}|^2.$$

It was shown in Lemma 2.2 of that paper that this pseudodistance is invariant (up to constants) under canonical transformations; the proof required that the transformation have 2 bounded derivatives. The canonical transformations determined by a $C^{1,1}$ Hamiltonian need not meet this criterion, so we provide the following lemma.

LEMMA 2.2 (Invariance of the pseudodistance). — *Suppose that $\mathrm{H}(t,x,\xi)$ is real and homogeneous of degree 1 in ξ. Suppose also that $\nabla_x\mathrm{H}(t,x,\xi)$ and $\nabla_\xi\mathrm{H}(t,x,\xi)$ satisfy a Lipschitz condition in (x,ξ), with* *uniform Lipschitz constant over the set* $|\xi| = 1$. *Let* χ_t *be the transformation on the cosphere bundle* $S^*(\mathbb{R}^n)$ *induced by the projected Hamiltonian flow at time* t,

$$\frac{dx}{ds} = \mathrm{H}_\xi(s,x,\omega), \qquad \frac{d\omega}{ds} = -\mathrm{H}_x(s,x,\omega) + \langle \omega, \mathrm{H}_x(s,x,\omega)\rangle\omega.$$

Then

$$d(\chi_t(x,\omega);\chi_t(\tilde{x},\tilde{\omega})) \approx d(x,\omega;\tilde{x},\tilde{\omega}),$$

where the constant of proportionality is bounded by $e^{K|t|}$ *for some* K.

Proof. — Let (x,ω) and $(\tilde{x},\tilde{\omega})$ denote the integral curves with initial condition (y,η) and $(\tilde{y},\tilde{\eta})$ respectively. Since H_x and H_ξ are Lipschitz in (x,ω), the curves are uniquely defined, and are Lipschitz functions of the initial parameters. The flow is thus a bilipschitz mapping, so that

$$|x-\tilde{x}| + |\omega-\tilde{\omega}| \approx |y-\tilde{y}| + |\eta-\tilde{\eta}|,$$

where the constant of proportionality is bounded by $e^{K|t|}$. We next show that

$$|\langle \omega, x-\tilde{x}\rangle - \langle \eta, y-\tilde{y}\rangle| \le e^{K|t|}(|y-\tilde{y}|^2 + |\eta-\tilde{\eta}|^2). \tag{2.2}$$

Consider

$$\frac{d}{dt}\langle \omega, x-\tilde{x}\rangle = \langle \omega, \mathrm{H}_\xi(t,x,\omega)\rangle - \langle \omega, \mathrm{H}_\xi(t,\tilde{x},\tilde{\omega})\rangle \\ -\langle \mathrm{H}_x(t,x,\omega), x-\tilde{x}\rangle + \langle \omega, \mathrm{H}_x(t,x,\omega)\rangle\langle \omega, x-\tilde{x}\rangle.$$

By homogeneity, the first three terms on the right hand side can be rewritten as

$$\mathrm{H}(t,x,\omega) - \mathrm{H}(t,\tilde{x},\tilde{\omega}) - \langle \omega-\tilde{\omega}, \mathrm{H}_\xi(t,\tilde{x},\tilde{\omega})\rangle \\ -\langle \mathrm{H}_x(t,\tilde{x},\tilde{\omega}), x-\tilde{x}\rangle + \langle \mathrm{H}_x(t,\tilde{x},\tilde{\omega}) - \mathrm{H}_x(t,x,\omega), x-\tilde{x}\rangle.$$

The first four terms of this latter expression combine to give the second order error in the Taylor expansion of $\mathrm{H}(t,x,\omega)$ in (x,ω) about $(\tilde{x},\tilde{\omega})$, and hence are bounded by $|x-\tilde{x}|^2 + |\omega-\tilde{\omega}|^2 \lesssim |y-\tilde{y}|^2 + |\eta-\tilde{\eta}|^2$. The last term is similarly bounded since $\mathrm{H}_x(t,x,\omega)$ is Lipschitz in (x,ω). The inequality (2.2) now follows by Gronwall's Lemma. □

The following weight function $\mu_\delta(\gamma,\gamma')$ is closely related to the weight function that characterises the matrix of a Calderon-Zygmund operator in the wavelet bases (for example, see page 282 of [6]).

DEFINITION 2.3. — *If* $\gamma = (x,\omega,k)$ *and* $\gamma' = (x',\omega',k')$, *we set*

$$\mu_\delta(\gamma,\gamma') = (1+|k'-k|^2)^{-1}2^{-(\delta+\frac{n}{2})|k'-k|}\Big(1 + \frac{d(x,\omega;x',\omega')}{2^{-k}+2^{-k'}}\Big)^{-n-\delta}.$$

LEMMA 2.4. — *If $\delta > 0$, then there is a constant $C(\delta)$ such that*

$$\sum_{\gamma'} \mu_\delta(\gamma, \gamma') 2^{-\frac{nk'}{2}} \le C(\delta) 2^{-\frac{nk}{2}}.$$

Proof. — To establish the estimate, we will show that, for each $\gamma = (x, \omega, k)$, and each k',

$$\sum_{\omega'} \sum_{x' \in \Xi_{k'}^{\omega'}} \left(1 + \frac{d(x, \omega; x', \omega')}{2^{-k} + 2^{-k'}}\right)^{-n-\delta} \le C(\delta)(1 + 2^{n(k'-k)}). \tag{2.3}$$

It is then straightforward to verify that

$$\sum_{k'=0}^{\infty} (1 + 2^{n(k'-k)}) 2^{-\frac{nk'}{2}} 2^{-(\delta+\frac{n}{2})|k'-k|} \le C(\delta) 2^{-\frac{nk}{2}}.$$

We first verify the inequality (2.3) in case $k \ge k'$. The part of the sum involving $|x - x'| \le 1$ is dominated by

$$\sum_{\omega'} (1 + 2^{k'} |\omega' - \omega|^2)^{-\frac{n-1}{2} - \frac{\delta}{2}} \sum_{x' \in \Xi_{k'}^{\omega'}} (1 + 2^{k'} d(x, \omega; x', \omega'))^{-\frac{n+1}{2} - \frac{\delta}{2}}.$$

This sum converges with bounds independent of k'. The part of the sum with $|x - x'| \ge 1$ is controlled by noting that

$$\sum_{\substack{x' \in \Xi_{k'}^{\omega'} \\ |x - x'| \ge 1}} (1 + 2^{k'} |x - x'|)^{-n-\delta} \le C(\delta) 2^{-\frac{k(n-1)}{2}}.$$

In case $k \le k'$, the proof of (2.3) is similar; the factor of $2^{n(k'-k)}$ arises by comparing the density of points (x, ω) to the density of the (x', ω'). □

LEMMA 2.5. — *If $\delta > 0$, there is a constant $C(\delta)$ such that*

$$\sum_{\gamma'} \mu_\delta(\gamma, \gamma') \mu_\delta(\gamma', \gamma_0) \le C(\delta) \mu_\delta(\gamma, \gamma_0).$$

Proof. — By symmetry, we assume $k \le k_0$. We divide the sum into three parts, for $k' \ge k_0$, $k' \le k$, and $k < k' < k_0$.

If $k' \ge k_0$, then we dominate

$$\begin{aligned}(1 + 2^k d(x', \omega'; x, \omega))^{-n-\delta} &(1 + 2^{k_0} d(x', \omega'; x_0, \omega_0))^{-n-\delta} \\ &\le C(\delta)(1 + 2^k d(x, \omega; x_0, \omega_0))^{-n-\delta}(1 + 2^{k_0} \cdot \\ &\qquad \min[d(x', \omega'; x, \omega), d(x', \omega'; x_0, \omega_0)])^{-n-\delta}.\end{aligned}$$

By (2.3), the sum of the latter quantity over (x', ω') is dominated by

$$2^{n(k'-k_0)}(1+2^k d(x,\omega;x_0,\omega_0))^{-n-\delta}.$$

The desired bounds on the sum over $k' \geq k_0$ follow since

$$\sum_{k'\geq k_0} 2^{-(\delta+\frac{n}{2})(2k'-k-k_0)+n(k'-k_0)} \leq C(\delta)2^{-(\delta+\frac{n}{2})(k_0-k)}.$$

For the sum over $k' \leq k$, similar steps lead us to consider

$$\sum_{k'\leq k} 2^{-(\delta+\frac{n}{2})(k+k_0-2k')}(1+2^{k'}d(x,\omega;x_0,\omega_0))^{-n-\delta},$$

which is dominated by

$$C(\delta)2^{-(\delta+\frac{n}{2})(k_0-k)}(1+2^k d(x,\omega;x_0,\omega_0))^{-n-\delta}.$$

For the sum over $k < k' < k_0$, we are led to consider

$$\sum_{k<k'<k_0} (1+|k'-k|^2)^{-1}(1+|k'-k_0|^2)^{-1}2^{-(\delta+\frac{n}{2})(k_0-k)} \cdot (1+2^k d(x,\omega;x_0,\omega_0))^{-n-\delta},$$

which is dominated by

$$(1+|k-k_0|^2)^{-1}2^{-(\delta+\frac{n}{2})(k_0-k)}(1+2^k d(x,\omega;x_0,\omega_0))^{-n-\delta}. \qquad \square$$

DEFINITION 2.6. — *If χ is a mapping on $S^*(\mathbb{R}^n)$, we say that a matrix $a(\gamma,\gamma')$ belongs to the class $\mathcal{M}^r_\delta(\chi)$ if*

$$|a(\gamma,\gamma')| \leq C_a 2^{rk'}\mu_\delta(\gamma,\chi(\gamma')).$$

Here, $\chi(\gamma') = (\chi(x',\omega'),k')$. We also set

$$\mathcal{M}^r(\chi) = \bigcap_{\delta>0} \mathcal{M}^r_\delta(\chi).$$

THEOREM 2.7. — *Suppose that χ is an invertible mapping on $S^*(\mathbb{R}^n)$ such that for some constant C*

$$C^{-1}d(x,\omega;x',\omega') \leq d(\chi(x,\omega);\chi(x',\omega')) \leq Cd(x,\omega;x',\omega'). \tag{2.4}$$

If $a(\gamma,\gamma') \in \mathcal{M}^r(\chi)$, then the operator determined by $a(\gamma,\gamma')$ is a continuous mapping from $H^\alpha(\mathbb{R}^n)$ to $H^{\alpha-r}(\mathbb{R}^n)$.

If $a_j(\gamma,\gamma') \in \mathcal{M}^{r_j}(\chi_j)$, where $\chi_j, j=1,2$ are invertible mappings on $S^*(\mathbb{R}^n)$ satisfying (2.4), then $a_1 \circ a_2 \in \mathcal{M}^{r_1+r_2}(\chi_1 \circ \chi_2)$.

Proof. — These are simple consequences of Lemmas 2.1, 2.4 and 2.5, together with the fact that

$$\mu_\delta(\gamma,\chi(\gamma')) \approx \mu_\delta(\chi^{-1}(\gamma),\gamma'),$$

and the relation

$$\|f\|_{H^\alpha(\mathbb{R}^n)} \approx \Big(\sum_\gamma 2^{2k\alpha}|c(\gamma)|^2\Big)^{\frac{1}{2}}. \qquad \square$$

DEFINITION 2.8. — *We say that an operator T, defined as a map from Schwartz functions to tempered distributions, belongs to the class $\mathcal{I}^r(\chi)$, if the matrix*

$$a(\gamma,\gamma') = \int \overline{\varphi_\gamma(y)}(T\varphi_{\gamma'})(y)dy$$

belongs to $\mathcal{M}^r(\chi)$.

It is a simple matter to show that if T is a standard Fourier integral operator of order r, associated to a smooth canonical transformation χ, then $T \in \mathcal{I}^r(\chi)$. The advantage of the classes $\mathcal{I}^r(\chi)$ is that they require only limited regularity of χ.

One can also define classes $\mathcal{I}^r_\delta(\chi)$, with only finite order decay conditions on the coefficients, with the result that $\mathcal{I}^0_\delta(\mathrm{I})$ forms a Banach algebra of continuous operators on $L^2(\mathbb{R}^n)$, provided $\delta > 0$. For our purposes, however, we only need the immediate consequences of the above theorem, that

$$\mathcal{I}^r(\chi) : H^\alpha(\mathbb{R}^n) \to H^{\alpha-r}(\mathbb{R}^n),$$
$$\mathcal{I}^{r_1}(\chi_1)\circ\mathcal{I}^{r_2}(\chi_2) \subseteq \mathcal{I}^{r_1+r_2}(\chi_1\circ\chi_2).$$

LEMMA 2.9. — *Suppose that χ satisfies (2.4). If $a(\gamma,\gamma') \in \mathcal{M}^r(\chi)$, then the operator determined by $a(\gamma,\gamma')$ belongs to $\mathcal{I}^r(\chi)$. In other words,*

$$a \to \Pi\circ a\circ\Pi$$

preserves $\mathcal{M}^r(\chi)$.

Proof. — By the comments preceeding Lemma 2.1, it suffices to check that $\Pi \in \mathcal{M}^0(\mathrm{I})$, where I denotes the identity transformation. The matrix of Π has the form

$$\Pi(\gamma,\gamma') = \int_{\mathbb{R}^n} \overline{\varphi_\gamma(y)}\varphi_{\gamma'}(y)dy.$$

The integral vanishes identically unless both $|k-k'| \le 1$ and $|\omega-\omega'|^2 \le 2^{-k+4}$. That $\Pi \in \mathcal{M}^0(\mathrm{I})$ then follows by simple absolute value estimates on the integral. $\square$

3. The ansatz.

In this section, for an element $\varphi_\gamma(y)$ of the frame introduced in Section 2, $\gamma = (x,\omega,k)$, we introduce a family $\varphi_\gamma(t,y)$, where $\varphi_\gamma(t,y)$ is related to $\varphi_\gamma(y)$ by a rigid motion that corresponds roughly to flowing the point (x,ω) for time t along the projected Hamiltonian flow of $\sqrt{A_k(t,x,\xi)}$. The construction is such that the second order terms in the expansion of

$$(\partial_t^2 - A_k(t,x,\partial_x))\varphi_\gamma(t,y)$$

cancel against each other. As a consequence, we show that if $\mathbf{e}_k(t)$ is defined as the operator which takes $f = \sum_\gamma c_\gamma \varphi_\gamma(y)$ to $\sum_{k_\gamma = k} c_\gamma \varphi_\gamma(t,y)$, then

$$\sum_{k=0}^{\infty} (\partial_t^2 - A_k(t,x,\partial_x))\mathbf{e}_k(t)$$

is an operator of order one, in the class of operators of the previous section. In the next section, this idea is used to produce an approximate inverse for $\partial_t^2 - A(t,x,\partial_x)$, where the error is one derivative better than expected. We remark that the family $\varphi_\gamma(t,y)$ will be denoted by $\varphi_\gamma^+(t,0,y)$ in the next section, where it is necessary to consider initial manifolds $t = s$ for nonzero s, and also the flow along $-\sqrt{A_k(t,x,\xi)}$. For the purposes of this section the consideration of $\varphi_\gamma(t,y)$ is sufficient.

Let $\mathrm{H}_k(t,x,\xi) = \sqrt{A_k(t,x,\xi)}$. The Hamiltonian flow of H_k on $T^*(\mathbb{R}^n)$ commutes with dilation in ξ, and hence induces a flow on the cosphere bundle $S^*(\mathbb{R}^n)$, which is given by

$$\frac{dx}{dt} = (\mathrm{H}_k)_\xi(t,x,\omega), \quad \frac{d\omega}{dt} = -(\mathrm{H}_k)_x(t,x,\omega) + \langle \omega, (\mathrm{H}_k)_x(t,x,\omega)\rangle\omega. \tag{3.1}$$

Let Θ denote a $n \times n$ matrix variable. In the following, $v \otimes w$ denotes the matrix $x \to \langle w, x\rangle v$. Consider the flow on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n^2}$ given by

$$\begin{aligned} \frac{dx}{dt} &= (\mathrm{H}_k)_\xi(t,x,\omega), \\ \frac{d\omega}{dt} &= -(\mathrm{H}_k)_x(t,x,\omega) + \langle \omega, (\mathrm{H}_k)_x(t,x,\omega)\rangle\omega, \\ \frac{d\Theta}{dt} &= -\Theta[\omega \otimes (\mathrm{H}_k)_x(t,x,\omega) - (\mathrm{H}_k)_x(t,x,\omega)\otimes\omega]. \end{aligned} \tag{3.2}$$

It is easily verified that the flow preserves $S^*(\mathbb{R}^n) \times O(n)$. The fact that the orthogonal group is conserved follows from antisymmetry of the term in braces.

Now let $(x_\gamma(t), \omega_\gamma(t), \Theta_\gamma(t))$ denote the solution to (3.2) with the initial condition

$$(x_\gamma(0), \omega_\gamma(0), \Theta_\gamma(0)) = (x_\gamma, \omega_\gamma, \mathrm{I}).$$

It is verified that

$$\frac{d}{dt}[\Theta_\gamma(t)\omega_\gamma(t)] = 0,$$

so that $\Theta_\gamma(t)\omega_\gamma(t) = \omega_\gamma$ for all t. Next, let

$$\varphi_\gamma(t,y) = \varphi_\gamma(\Theta_\gamma(t)(y - x_\gamma(t)) + x_\gamma). \tag{3.3}$$

DEFINITION 3.1. — *If* $c(\gamma') = \int_{\mathbb{R}^n} \overline{\varphi_{\gamma'}(y)} f(y)dy$, *we set*

$$\mathbf{e}_k(t)f(y) = \sum_{\gamma':k'=k} c(\gamma')\varphi_{\gamma'}(t,y).$$

Let $\chi_t(x,\omega)$ denote the value at time t of the solution to

$$\frac{dx}{ds} = \mathrm{H}_\xi(s,x,\omega), \quad \frac{d\omega}{ds} = -\mathrm{H}_x(s,x,\omega) + \langle \omega, \mathrm{H}_x(s,x,\omega)\rangle\omega, \tag{3.4}$$

with $x(0) = x, \omega(0) = \omega$. We then have the following.

THEOREM 3.2. — *The following hold:*

$$\sum_k \mathbf{e}_k(t) \in \mathcal{I}^0(\chi_t),$$

$$\sum_k (\partial_t^2 - A_k(t,y,\partial_y))\mathbf{e}_k(t) \in \mathcal{I}^1(\chi_t).$$

Furthermore, the constant appearing in Definition 2.6 is uniformly bounded (for each δ) provided s and t vary over compact intervals.

We prepare for the proof of Theorem 3.2 by examining the function

$$\partial_t^2\varphi_\gamma(t,y) - A_k(t,y,\partial_y)\varphi_\gamma(t,y).$$

We first observe that

$$\partial_t\varphi_\gamma(t,y) = -L(t, x_\gamma(t), \omega_\gamma(t), y, \partial_y)\varphi_\gamma(t,y), \tag{3.5}$$

where

$$L(t,x,\omega,y,\partial_y) = \langle (\mathrm{H}_k)_\xi(t,x,\omega), \partial_y\rangle + \langle (\mathrm{H}_k)_x(t,x,\omega), y-x\rangle\langle\omega,\partial_y\rangle \\ -\langle \omega, y-x\rangle\langle(\mathrm{H}_k)_x(t,x,\omega),\partial_y\rangle.$$

DEFINITION 3.3. — *We say that a family of functions* $\{f_\gamma(t)\}_{\gamma\in\Gamma} \subset C^\infty(\mathbb{R})$ *is of type* C^m *if there exists a sequence of coefficients* C_j, *independent of* γ, *such that*

$$|\partial_t^j f_\gamma(t)| \le C_j, \quad j \le m,$$
$$|\partial_t^j f_\gamma(t)| \le C_j 2^{(j-m)k}, \quad j \ge m.$$

An inductive proof shows that any particular coordinate of $x_\gamma(t)$ gives a family of type C^2, whereas the coordinates of $\omega_\gamma(t)$, and $\Theta_\gamma(t)$, give families of type $C^{\frac{3}{2}}$. Also, the family of functions $(\mathrm{H}_k)_x(t, x_\gamma(t), \omega_\gamma(t))$ is of type $C^{\frac{1}{2}}$, while $(\mathrm{H}_k)_\xi(t, x_\gamma(t), \omega_\gamma(t))$ and $\mathrm{H}_k(t, x_\gamma(t), \omega_\gamma(t))$ are of type C^1. In the following Lemmas 3.4–3.5, the $f_\gamma(t)$ that arise are products of these functions and their derivatives, so that the constants C_j in each instance can be taken to depend only on the derivative bounds for $A(t,x)$.

The index γ can be considered fixed for the purposes of the next two lemmas, so for convenience of notation we use $(x_t, \omega_t, \Theta_t)$ to denote $(x_\gamma(t), \omega_\gamma(t), \Theta_\gamma(t))$.

For the given γ, we fix additional vectors $(v_{\gamma,2}, \dots, v_{\gamma,n})$ such that $(\omega_\gamma, v_{\gamma,2}, \dots, v_{\gamma,n})$ is an orthonormal set. Let

$$v_{j,t} = \Theta_\gamma(t)^{-1} v_{\gamma,j},$$

so that $(\omega_t, v_{2,t}, \dots, v_{n,t})$ form an orthonormal frame for all t.

In the subsequent lemmas, $\psi_\gamma(y)$ denotes a generic function of the form

(3.6)
$$\widehat{\psi}_\gamma(\xi) = e^{-i\langle x_\gamma, \xi\rangle} 2^{k(i-j+\frac{|\alpha|}{2}-\frac{|\beta|}{2})} \langle \omega_\gamma, \partial_\xi\rangle^i \langle v_\gamma, \partial_\xi\rangle^\alpha \langle \omega_\gamma, \xi\rangle^j \langle v_\gamma, \xi\rangle^\beta h_k^{\omega_\gamma}(\xi),$$

where v_γ denotes the span of $(v_{\gamma,2}, \dots, v_{\gamma,n})$. The particular form of $\psi_\gamma(y)$ may differ at each occurrence. As in (3.3), we let

$$\psi_\gamma(t,y) = \psi_\gamma(\Theta_\gamma(t)(y - x_\gamma(t)) + x_\gamma).$$

Thus, $\psi_\gamma(t,y)$ is a general function of the same "size" as $\phi_\gamma(t,y)$.

Recall that $A_k(t,y)$ denotes the matrix valued function such that

$$A_k(t, y, \partial_y) = \langle A_k(t,y)\partial_y, \partial_y\rangle.$$

LEMMA 3.4. — *One can write $L(t, x_t, \omega_t, y, \partial_y)^2 \varphi_\gamma(t,y)$ as*

$$[2\langle A_k(t,x_t)\omega_t, \partial_y\rangle\langle \omega_t, \partial_y\rangle - \langle A_k(t,x_t)\omega_t, \omega_t\rangle\langle\omega_t, \partial_y\rangle^2$$
$$+\langle (y - x_t)\cdot \partial_x A_k(t,x_t)\omega_t, \omega_t\rangle\langle \omega_t, \partial_y\rangle^2]\varphi_\gamma(t,y)$$

plus a finite sum of remainder terms, where each remainder term is of the form

$$2^{mk} f_\gamma(t)\psi_\gamma(t,y), \tag{3.7}$$

where $m \le 1$, f_γ is of type $C^{\frac{1}{2}}$, and $\psi_\gamma(y)$ is as in (3.6).

Proof. — We first observe that one can write

$$\begin{aligned}\langle\omega_t, y-x_t\rangle\varphi_\gamma(t,y) &= 2^{-k}\psi_\gamma(t,y),\\ \langle\omega_t, \partial_y\rangle\varphi_\gamma(t,y) &= 2^{k}\psi_\gamma(t,y),\\ \langle v_{j,t}, y-x_t\rangle\varphi_\gamma(t,y) &= 2^{-\frac{k}{2}}\psi_\gamma(t,y),\\ \langle v_{j,t}, \partial_y\rangle\varphi_\gamma(t,y) &= 2^{\frac{k}{2}}\psi_\gamma(t,y),\end{aligned}$$

where ψ_γ is respectively of the form

$$\begin{aligned}\psi_\gamma(y) &= 2^{k}\langle\omega_\gamma, y-x_\gamma\rangle\varphi_\gamma(y)\\ \psi_\gamma(y) &= 2^{-k}\langle\omega_\gamma, \partial_y\rangle\varphi_\gamma(y)\\ \psi_\gamma(y) &= 2^{\frac{k}{2}}\langle v_j, y-x_\gamma\rangle\varphi_\gamma(y)\\ \psi_\gamma(y) &= 2^{-\frac{k}{2}}\langle v_j, \partial_y\rangle\varphi_\gamma(y).\end{aligned}$$

Now write $L(t,x_t,\omega_t,y,\partial_y)$ as a sum of four terms:

$$\begin{aligned}L(t,x_t,\omega_t,y,\partial_y) = {} & \mathrm{H}_k(t,x_t,\omega_t)\langle\omega_t,\partial_y\rangle)\\ & + \langle(\mathrm{H}_k)_\xi(t,x_t,\omega_t) - \mathrm{H}_k(t,x_t,\omega_t)\omega_t, \partial_y\rangle\\ & + \langle(\mathrm{H}_k)_x(t,x_t,\omega_t), y-x_t\rangle\langle\omega_t,\partial_y\rangle\\ & - \langle\omega_t, y-x_t\rangle\langle(\mathrm{H}_k)_x(t,x_t,\omega_t),\partial_y\rangle.\end{aligned}\tag{3.8}$$

We claim that these four terms are respectively of order $1, \frac{1}{2}, \frac{1}{2}, 0$, in the sense that, applied to $\varphi_\gamma(t,x)$, they lead to a sum of expressions of the form (3.7), where the exponent m is less than or equal to the assigned order. This is immediate for the first, third, and fourth term; it holds for the second term since the vector $(\mathrm{H}_k)_\xi(t,x_t,\omega_t) - \mathrm{H}_k(t,x_t,\omega_t)\omega_t$ is normal to ω_t, hence is a linear combination of the $v_{j,t}$. We now observe that, upon applying $L(t,x_t,\omega_t,y,\partial_y)^2$ to $\varphi_\gamma(t,y)$, the only terms that do not immediately lead to functions of the form (3.7) come from combinations of the four terms where the combined order is at least $\frac{3}{2}$.

The cross term between the first and third terms involves

$$\mathrm{H}_k(t,x_t,\omega_t)\langle\omega_t, (\mathrm{H}_k)_x(t,x_t,\omega_t)\rangle\langle\omega_t,\partial_y\rangle$$

which is also of order 1. Thus, modulo terms of order at most 1, $L(t,x_t,\omega_t,y,\partial_y)^2$ equals

$$\begin{aligned}&2\mathrm{H}_k(t,x_t,\omega_t)\langle(\mathrm{H}_k)_\xi(t,x_t,\omega_t),\partial_y\rangle\langle\omega_t,\partial_y\rangle - \mathrm{H}_k^2(t,x_t,\omega_t)\langle\omega_t,\partial_y\rangle^2\\ &\qquad + 2\mathrm{H}_k(t,x_t,\omega_t)\langle(\mathrm{H}_k)_x(t,x_t,\omega_t), y-x_t\rangle\langle\omega_t,\partial_y\rangle^2\\ &= \langle(\mathrm{H}_k^2)_\xi(t,x_t,\omega_t),\partial_y\rangle\langle\omega_t,\partial_y\rangle - \mathrm{H}_k^2(t,x_t,\omega_t)\langle\omega_t,\partial_y\rangle^2\\ &\qquad + \langle(\mathrm{H}_k^2)_x(t,x_t,\omega_t), y-x_t\rangle\langle\omega_t,\partial_y\rangle^2.\end{aligned}$$

Since $\mathrm{H}_k^2(t,x,\xi) = \langle A_k(t,x)\xi, \xi\rangle$, these last three terms are equal to the three terms in braces in the statement of the lemma. □

LEMMA 3.5. — *One can write* $(\partial_t^2 - A_k(t,y,\partial_y))\varphi_\gamma(t,y)$ *as a finite sum of terms of the following form:*

$$2^k f_\gamma(t)\psi_\gamma(t,y),$$
$$2^k a_k(t,y) f_\gamma(t)\psi_\gamma(t,y),$$
$$2^{\frac{3k}{2}}[a_k(t,y) - a_k(t,x_t)] f_\gamma(t)\psi_\gamma(t,y),$$
$$2^{2k}[a_k(t,y) - a_k(t,x_t) - (y-x_t)\cdot(\partial_x a_k)(t,x_t)] f_\gamma(t)\psi_\gamma(t,y),$$

where $a_k(t,x)$ *denotes an element of the matrix* $A_k(t,x)$*. In the first form, the functions* f_γ *are of type* C^0*, and in the other three forms the functions* f_γ *are of type* $C^{\frac{3}{2}}$*.*

Proof. — By (3.5), it holds that

$$\partial_t^2\varphi_\gamma(t,y) = L(t,x_t,\omega_t,y,\partial_y)^2\varphi_\gamma(t,y) - [\partial_t L(t,x_t,\omega_t,y,\partial_y)]\varphi_\gamma(t,y).$$

Consider first the term $[\partial_t L(t,x_t,\omega_t,y,\partial_y)]\varphi_\gamma(t,y)$. It is observed that, after differentiation in t, each of the four terms in (3.8) is of order at most 1, in the sense according to the comments following (3.8). Thus, this term can be written as a sum of terms of the first form.

Next, let $R_k(t,x_t,\omega_t,y,\partial_y)$ denote the operator

$$\begin{aligned} R_k(t,x_t,\omega_t,y,\partial_y) =& A_k(t,y,\partial_y) - 2\langle A_k(t,x_t)\omega_t,\partial_y\rangle\langle\omega_t,\partial_y\rangle \\ &+ \langle A_k(t,x_t)\omega_t,\omega_t\rangle\langle\omega_t,\partial_y\rangle^2 \\ &- \langle (y-x_t)\cdot\partial_x A_k(t,x_t)\omega_t,\omega_t\rangle\langle\omega_t,\partial_y\rangle^2. \end{aligned}$$

Then, by Lemma 3.4 and the preceeding result, one can write

$$(\partial_t^2 - A_k(t,y,\partial_y))\varphi_\gamma(t,y) + R_k(t,x_t,\omega_t,y,\partial_y)\varphi_\gamma(t,y)$$

as a sum of terms of the first form. It remains to express

$$R_k(t,x_t,\omega_t,y,\partial_y)\varphi_\gamma(t,y)$$

as a sum of terms of the other three forms. To do this, we write $R_k(t,x_t,\omega_t,y,\partial_y)$ as a sum of three terms

$$\begin{aligned} &\sum_{2\le i,j\le n} \langle A_k(t,y)v_{i,t},v_{j,t}\rangle\langle v_{i,t},\partial_y\rangle\langle v_{j,t},\partial_y\rangle \\ &\qquad + 2\sum_{2\le j\le n}\langle [A_k(t,y) - A_k(t,x_t)]\omega_t, v_{j,t}\rangle\langle\omega_t,\partial_y\rangle\langle v_{j,t},\partial_y\rangle \\ &\qquad + \langle [A_k(t,y) - A_k(t,x_t) - (y-x_t)\cdot(\partial_x A_k)(t,x_t)]\omega_t,\omega_t\rangle\langle\omega_t,\partial_y\rangle^2. \end{aligned}$$

These lead, respectively, to terms of the last three forms. □

Recall that the operator $\mathbf{e}_k(t)$ is defined using the Hamiltonian flow of $\sqrt{A_k(t,x,\xi)}$, whereas χ_t denotes the Hamiltonian flow of $\sqrt{A(t,x,\xi)}$. We thus require the following lemma.

LEMMA 3.6. — *Given k, let $(x(t),\omega(t))$ be the integral curve of (3.1), with $x(0)=x,\omega(0)=\omega$. Then*

$$d(x(t),\omega(t);\chi_t(x,\omega)) \leq 2^{-k}(e^{C|t|}-1),$$

where C depends on the derivative bounds and lower bounds for $A(t,x)$, but in particular is independent of k.

Proof. — We temporarily denote $\chi_t(x,\omega)$ by $(\tilde{x}(t),\tilde{\omega}(t))$. Observe that, on the set $|\xi|=1$, we have the following estimates:

$$|\partial_\xi^\alpha \partial_x^\beta(\mathrm{H}(t,x,\xi)-\mathrm{H}_k(t,x,\xi))| \leq C2^{k\left(\frac{|\beta|}{2}-1\right)}, \quad |\beta|\leq 2.$$

By Gronwall's Lemma, we conclude that

$$|x(s)-\tilde{x}(s)|+|\omega(s)-\tilde{\omega}(s)| \leq 2^{-\frac{k}{2}}(e^{C|t|}-1).$$

It remains to show that

$$|\langle\tilde{\omega}(t),\tilde{x}(t)-x(t)\rangle| \leq 2^{-k}(e^{C|t|}-1).$$

Consider

$$\begin{aligned}\frac{d}{dt}\langle\tilde{\omega},\tilde{x}-x\rangle &= \langle\tilde{\omega},\mathrm{H}_\xi(t,\tilde{x},\tilde{\omega})-(\mathrm{H}_k)_\xi(t,x,\omega)\rangle - \langle\mathrm{H}_x(t,\tilde{x},\tilde{\omega}),\tilde{x}-x\rangle \\ &\qquad + \langle\tilde{\omega},\mathrm{H}_x(t,\tilde{x},\tilde{\omega})\rangle\langle\tilde{\omega},\tilde{x}-x\rangle \\ &= (\langle\tilde{\omega},\mathrm{H}_\xi(t,\tilde{x},\tilde{\omega})-\mathrm{H}_\xi(t,x,\omega)\rangle - \langle\mathrm{H}_x(t,\tilde{x},\tilde{\omega}),\tilde{x}-x\rangle) \\ &\qquad + \mathcal{O}(2^{-k}) + \mathcal{O}(\langle\tilde{\omega},\tilde{x}-x\rangle).\end{aligned}$$

By homogeneity, the terms in parentheses can be rewritten as

$$\begin{aligned}&\mathrm{H}(t,\tilde{x},\tilde{\omega})-\mathrm{H}(t,x,\omega)-\langle\tilde{\omega}-\omega,\mathrm{H}_\xi(t,x,\omega)\rangle-\langle\mathrm{H}_x(t,x,\omega),\tilde{x}-x\rangle \\ &\qquad\qquad + \langle\mathrm{H}_x(t,x,\omega)-\mathrm{H}_x(t,\tilde{x},\tilde{\omega}),\tilde{x}-x\rangle.\end{aligned}$$

The first four terms combine to give the second order error in the Taylor expansion of $\mathrm{H}(t,\tilde{x},\tilde{\omega})$ in $(\tilde{x},\tilde{\omega})$ about (x,ω), and hence are bounded by $|x-\tilde{x}|^2+|\omega-\tilde{\omega}|^2$. The last term is similarly bounded since $\mathrm{H}_x(t,x,\omega)$ is Lipschitz in x and ω. The desired result now follows by Gronwall's Lemma. □

Proof of Theorem 3.2. — The operator $\sum_k \mathbf{e}_k(t)$ is determined by the matrix

$$a(\gamma',\gamma) = \int_{\mathbb{R}^n} \overline{\varphi_{\gamma'}(y)}\varphi_\gamma(t,y)dy.$$

By Lemma 2.9, the first part of Theorem 3.2 follows by showing that this matrix belongs to $\mathcal{M}^0(\chi_t)$. The coefficients vanish unless both $|k'-k|\le 1$, and $|\omega_{\gamma'}-\omega_\gamma(t)|^2 \le 2^{-k+4}$. Simple size estimates show that, in this case,

$$\int_{\mathbb{R}^n} |\overline{\varphi_{\gamma'}(y)}\varphi_\gamma(t,y)|dy \le C_N(1+2^k d(x_{\gamma'},\omega_{\gamma'};x_\gamma(t),\omega_\gamma(t)))^{-N}, \quad \forall N.$$

By Lemma 3.6,

$$d(x_\gamma(t),\omega_\gamma(t);\chi_t(x_\gamma,\omega_\gamma)) \lesssim 2^{-k}.$$

This completes the proof of the first part of Theorem 3.2.

To establish the second part of Theorem 3.2, it suffices to show that, if $g_\gamma(t,y)$ is any of the four forms in Lemma 3.5, then

$$\int_{\mathbb{R}^n} |\overline{\varphi_{\gamma'}(y)}g_\gamma(t,y)|dy \le C_N 2^k(1+2^k d(x_{\gamma'},\omega_{\gamma'};x_\gamma(t),\omega_\gamma(t)))^{-N} \quad \forall N,$$

in the case that both $|k'-k|\le 1$ and $|\omega_{\gamma'}-\omega_\gamma(t)|^2\le 2^{-k+4}$, since the matrix coefficients vanish otherwise as the function $a_k(t,y)$ has partial Fourier transform in the y variable supported in the set $|\xi|\le 2^{\frac{k}{2}}$. In each case, this estimate follows from simple size estimates on $|\varphi_{\gamma'}(y)|$ and $|g_\gamma(t,y)|$, together with the fact that

$$|a_k(t,y)-a_k(t,x_t)| \le C|y-x_t|,$$
$$|a_k(t,y)-a_k(t,x_t)-(y-x_t)\cdot(\partial_x a_k)(t,x_t)| \le C|y-x_t|^2. \qquad \square$$

4. The parametrix.

We now generalise the results of the preceeding section, by introducing two pairs of families of functions, $\varphi_\gamma^\pm(t,s,y)$ and $\vartheta_\gamma^\pm(t,s,y)$, so that the solution to the Cauchy problem

$$(\partial_t^2 - A_k(t,y,\partial_y))u(t,y) = 0,$$
$$u(t,y)|_{t=s} = a\varphi_\gamma(y),$$
$$\partial_t u(t,y)|_{t=s} = b\varphi_\gamma(y),$$

is well approximated by

$$u(t,y) = \frac{a}{2}(\varphi_\gamma^+(t,s,y)+\varphi_\gamma^-(t,s,y)) + \frac{b}{2}(2^{-k}\vartheta_\gamma^+(t,s,y) - 2^{-k}\vartheta_\gamma^-(t,s,y)).$$

To begin, for $k > 0$ we set

$$\widehat{\vartheta_\gamma}(\xi) = -2^k \langle \omega_\gamma, \xi\rangle^{-1}\widehat{\varphi_\gamma}(\xi).$$

For a given index $\gamma = (x_\gamma, \omega_\gamma, k)$, we let $(x^\pm_\gamma(t,s), \omega^\pm_\gamma(t,s), \Theta^\pm_\gamma(t,s))$ be solutions to

$$\begin{aligned}
\frac{dx}{dt} &= \pm(\mathrm{H}_k)_\xi(t,x,\omega),\\
\frac{d\omega}{dt} &= \mp(\mathrm{H}_k)_x(t,x,\omega) \pm \langle \omega, (\mathrm{H}_k)_x(t,x,\omega)\rangle\omega,\\
\frac{d\Theta}{dt} &= \mp\Theta[\omega\otimes(\mathrm{H}_k)_x(t,x,\omega) - (\mathrm{H}_k)_x(t,x,\omega)\otimes\omega],
\end{aligned}$$

with the initial conditions

$$(x^\pm_\gamma(s,s), \omega^\pm_\gamma(s,s), \Theta^\pm_\gamma(s,s)) = (x_\gamma, \omega_\gamma, \mathrm{I}).$$

We next define transformations $\chi^\pm_{t,s}$ on $S^*(\mathbb{R}^n)$ by setting $\chi^\pm_{t,s}(x,\omega) = (x^\pm(t,s), \omega^\pm(t,s))$, where the latter solve

$$\frac{dx}{dt} = \pm\mathrm{H}_\xi(t,x,\omega), \quad \frac{d\omega}{dt} = \mp\mathrm{H}_x(t,x,\omega) \pm \langle\omega, \mathrm{H}_x(t,x,\omega)\rangle\omega,$$

with initial condition $x^\pm(s,s) = x, \omega^\pm(s,s) = \omega$.

We now set

$$\begin{aligned}
\varphi^\pm_\gamma(t,s,y) &= \varphi_\gamma(\Theta^\pm_\gamma(t,s)(y - x^\pm_\gamma(t,s)) + x_\gamma),\\
\vartheta^\pm_\gamma(t,s,y) &= \frac{1}{\mathrm{H}_k(s,x_\gamma,\omega_\gamma)}\vartheta_\gamma(\Theta^\pm_\gamma(t,s)(y - x^\pm_\gamma(t,s)) + x_\gamma).
\end{aligned}$$

Thus,

$$\begin{aligned}
(4.1) \qquad \partial_t\varphi^\pm_\gamma(t,s,y) &= \mp L(t, x^\pm_\gamma(t,s), \omega^\pm_\gamma(t,s), y, \partial_y)\varphi^\pm_\gamma(t,s,y),\\
\partial_t\vartheta^\pm_\gamma(t,s,y) &= \mp L(t, x^\pm_\gamma(t,s), \omega^\pm_\gamma(t,s), y, \partial_y)\vartheta^\pm_\gamma(t,s,y),
\end{aligned}$$

where $L(t,x,\omega,y,\partial_y)$ is as in (3.5).

DEFINITION 4.1. — *If* $c(\gamma') = \int_{\mathbb{R}^n} \overline{\varphi_{\gamma'}(y)} f(y)dy$, *we set*

$$\mathbf{c}_k(t,s)f(y) = \frac{1}{2}\sum_{\gamma':k'=k} c(\gamma')(\varphi^+_{\gamma'}(t,s,y) + \varphi^-_{\gamma'}(t,s,y)),$$

$$\mathbf{s}_k(t,s)f(y) = \frac{1}{2}\sum_{\gamma':k'=k} c(\gamma')(2^{-k}\vartheta^+_{\gamma'}(t,s,y) - 2^{-k}\vartheta^-_{\gamma'}(t,s,y)).$$

We then have the following two theorems, the proofs of which follow immediately from the proof of Theorem 3.2.

THEOREM 4.2. — *The following hold:*

$$\sum_k \mathbf{c}_k(t,s) \in \mathcal{I}^0(\chi^+_{t,s}) \oplus \mathcal{I}^0(\chi^-_{t,s}),$$

$$\sum_k \mathbf{s}_k(t,s) \in \mathcal{I}^{-1}(\chi^+_{t,s}) \oplus \mathcal{I}^{-1}(\chi^-_{t,s}).$$

Furthermore, the constant appearing in Definition 2.6 is uniformly bounded (for each δ) provided s and t vary over compact intervals.

THEOREM 4.3. — *The following hold:*

$$\sum_k (\partial_t^2 - A_k(t,y,\partial_y))\mathbf{c}_k(t,s) \in \mathcal{I}^1(\chi^+_{t,s}) \oplus \mathcal{I}^1(\chi^-_{t,s}),$$

$$\sum_k (\partial_t^2 - A_k(t,y,\partial_y))\mathbf{s}_k(t,s) \in \mathcal{I}^0(\chi^+_{t,s}) \oplus \mathcal{I}^0(\chi^-_{t,s}).$$

Furthermore, the constant appearing in Definition 2.6 is uniformly bounded (for each δ) provided s and t vary over compact intervals.

We now set

$$\mathbf{c}(t,s) = \sum_{k=0}^{\infty} \mathbf{c}_k(t,s).$$

As operators on $H^\alpha(\mathbb{R}^n)$, the family $\mathbf{c}(t,s)$ is a strongly continuous function of both t and s. This is seen by observing that, for fixed f (hence fixed $c(\gamma)$) the sum $\sum_\gamma c(\gamma)\varphi^\pm_\gamma(t,s,y)$ is the uniform limit of finite sums as s and t vary over bounded intervals. It similarly follows from (4.1) that, in the sense of distributions, $\partial_t^k \mathbf{c}(t,s)$ is a strongly continous family of operators from $H^\alpha(\mathbb{R}^n) \to H^{\alpha-k}(\mathbb{R}^n)$, and that

$$\mathbf{c}(t,s)|_{t=s} = \mathbf{I}, \quad \partial_t \mathbf{c}(t,s)|_{t=s} = 0.$$

The definition of $\mathbf{s}(t,s)$ is more complicated since $\partial_t \vartheta^\pm_\gamma(t,s,y)|_{t=s} \neq \varphi_\gamma(y)$. To define $\mathbf{s}(t,s)$, we let

$$\Delta_k f(y) = \sum_{\gamma':k'=k} \Big(\int \overline{\varphi_{\gamma'}(z)} f(z)\,dz\Big)\varphi_{\gamma'}(y).$$

For a constant k_0, to be determined depending on the derivative estimates and lower bounds for $A(t,x)$, we now set

$$\tilde{\mathbf{s}}(t,s) = \sum_{k\geq k_0} \mathbf{s}_k(t,s) + (t-s)\sum_{k<k_0} \Delta_k.$$

LEMMA 4.4. — *For k_0 sufficiently large, depending only on the lower bounds and derivative estimates for $A(t,x)$, and in particular independent of s, the operator $\partial_t\tilde{\mathbf{s}}(t,s)|_{t=s}$ admits a bounded inverse on $L^2(\mathbb{R}^n)$. The inverse extends to a bounded operator on $H^\alpha(\mathbb{R}^n)$ for all α, and is a continuous function of s in the norm operator topology on $H^\alpha(\mathbb{R}^n)$.*

Proof. — By (4.1) and (3.8), we observe that

$$\begin{aligned}
&2^{-k}\partial_t\vartheta_\gamma^\pm(t,s,y)|_{t=s}\\
&\quad=\pm\,\varphi_\gamma(y)\pm\frac{2^{-k}}{\mathrm{H}_k(s,x_\gamma,\omega_\gamma)}[\langle(\mathrm{H}_k)_\xi(s,x_\gamma,\omega_\gamma)-\mathrm{H}_k(s,x_\gamma,\omega_\gamma)\omega_\gamma,\partial_y\rangle\\
&\qquad+\langle(\mathrm{H}_k)_x(s,x_\gamma,\omega_\gamma),y-x_\gamma\rangle\langle\omega_\gamma,\partial_y\rangle\\
&\qquad-\langle\omega_\gamma,y-x_\gamma\rangle\langle(\mathrm{H}_k)_x(s,x_\gamma,\omega_\gamma),\partial_y\rangle]\vartheta_\gamma(y).
\end{aligned}$$

By the proof of Lemma 3.4, the terms in braces applied to $\vartheta_\gamma(y)$ give terms that are $\frac{1}{2}$ order better than $\varphi_\gamma(y)$. It follows that the operator $\mathrm{R}(s)=\mathbf{I}-\partial_t\tilde{\mathbf{s}}(t,s)|_{t=s}$ belongs to $\mathcal{I}^{-\frac{1}{2}}(\mathrm{I})$, with uniform bounds as s varies. For k_0 sufficiently large, $\mathrm{R}(s)$ has $L^2(\mathbb{R}^n)$ operator norm less than 1, so that $\partial_t\tilde{\mathbf{s}}(t,s)|_{t=s}=\mathbf{I}-\mathrm{R}(s)$ is invertible on $L^2(\mathbb{R}^n)$. By writing

$$(\mathbf{I}-\mathrm{R}(s))^{-1}=\sum_{j=0}^{2N-1}\mathrm{R}(s)^j+\mathrm{R}(s)^N(\mathbf{I}-\mathrm{R}(s))^{-1}\mathrm{R}(s)^N,$$

it is seen that $(\mathbf{I}-\mathrm{R}(s))^{-1}$ is bounded on $H^\alpha(\mathbb{R}^n)$ for all values of α. Since $\mathrm{R}(s)$ is a continuous function of s in the operator norm topology on maps from $H^\alpha(\mathbb{R}^n)\to H^{\alpha+\frac{1}{2}-\epsilon}(\mathbb{R}^n)$ for $\epsilon>0$, it is seen that $(\mathbf{I}-\mathrm{R}(s))^{-1}$ is continuous in s in the operator norm topology on $H^\alpha(\mathbb{R}^n)$. □

We now set

$$\mathbf{s}(t,s)=\tilde{\mathbf{s}}(t,s)(\partial_t\tilde{\mathbf{s}}(t,s)|_{t=s})^{-1}.$$

The family $\mathbf{s}(t,s)$ is then a strongly continuous function of both t and s as operators from $H^\alpha(\mathbb{R}^n)\to H^{\alpha+1}(\mathbb{R}^n)$. Furthermore, in the sense of distributions, $\partial_t^k\mathbf{s}(t,s)$ is a strongly continous family of operators from $H^\alpha(\mathbb{R}^n)\to H^{\alpha+1-k}(\mathbb{R}^n)$, and we have

$$\mathbf{s}(t,s)|_{t=s}=0,\quad \partial_t\mathbf{s}(t,s)|_{t=s}=\mathbf{I}.$$

Now set

$$\begin{aligned}
&\mathrm{T}_0(t,s)=(\partial_t^2-A(t,x,\partial_x))\mathbf{c}(t,s),\\
&\mathrm{T}_1(t,s)=(\partial_t^2-A(t,x,\partial_x))\mathbf{s}(t,s).
\end{aligned}\tag{4.2}$$

THEOREM 4.5. — *If $-1 \le \alpha \le 2$, then for each fixed t, s,*

$$\mathrm{T}_0(t,s) : H^{\alpha+1}(\mathbb{R}^n) \to H^{\alpha}(\mathbb{R}^n),$$
$$\mathrm{T}_1(t,s) : H^{\alpha}(\mathbb{R}^n) \to H^{\alpha}(\mathbb{R}^n).$$

The operator norms are uniformly bounded as $t-s$ varies in any bounded interval.

Proof. — By Theorem 4.3, this follows by showing that

$$\sum_k (a_{ij}(t,x) - a_{ij}^k(t,x))\partial_{x_i}\partial_{x_j}\mathbf{c}_k(t,s) : H^{\alpha+1}(\mathbb{R}^n) \to H^{\alpha}(\mathbb{R}^n),$$
$$\sum_k (a_{ij}(t,x) - a_{ij}^k(t,x))\partial_{x_i}\partial_{x_j}\mathbf{s}_k(t,s) : H^{\alpha}(\mathbb{R}^n) \to H^{\alpha}(\mathbb{R}^n),$$

where $\mathbf{s}_k(t,s)$ is defined in Definition 4.1.

By splitting the sum into even and odd k, it suffices to show that if T is an operator of the form

$$T = \sum_{k \text{ even}} (a_{ij}(t,x) - a_{ij}^k(t,x))\beta_k(D),$$

where $\{\beta_k\}$ is an appropriate family of Littlewood-Paley cutoffs, then

$$T : H^{\alpha-1}(\mathbb{R}^n) \to H^{\alpha}(\mathbb{R}^n), \quad -1 \le \alpha \le 2. \tag{4.3}$$

We first establish (4.3) in the case $\alpha = 1$. We suppress the t and ij. Consider the operators

$$T_{j,k} = \beta_j(D)(a(x) - a^k(x))\beta_k(D).$$

From the fact that

$$\|a(x) - a^r(x)\|_{L^\infty(\mathbb{R}^n)} \le C2^{-r},$$

and that

$$\beta_j(D)(a(x) - a^k(x))\beta_k(D) = \begin{cases} \beta_j(D)(a(x) - a^{2k-3}(x))\beta_k(D), & j \le k-3, \\ \beta_j(D)(a(x) - a^{2j-3}(x))\beta_k(D), & j \ge k+3, \end{cases}$$

we conclude that

$$\|T_{j,k}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le \begin{cases} C2^{-2k}, & j \le k-3, \\ C2^{-k}, & |j-k| \le 2, \\ C2^{-2j}, & j \ge k+3. \end{cases}$$

The estimate (4.3) for $\alpha = 1$ is an easy consequence.

We next observe that

$$[\partial_x, T] = (\partial_x a(x)) \sum_{k \text{ even}} \beta_k(D) - \sum_{k \text{ even}} (\partial_x a_k(x))\beta_k(D).$$

Each of the two terms on the right hand side is bounded on $H^\alpha(\mathbb{R}^n)$, for $-1 \le \alpha \le 1$; indeed, $\partial_x a$ is a multiplier on $H^\alpha(\mathbb{R}^n)$, and the second term is a pseudodifferential operator of type $S^0_{1,\frac{1}{2}}$. Together with the fact that (4.3) holds for $\alpha = 1$, this establishes (4.3) for $-1 \le \alpha \le 2$. □

THEOREM 4.6. — *If $-1 \le \alpha \le 2$, and $f \in H^{\alpha+1}(\mathbb{R}^n)$, $g \in H^\alpha(\mathbb{R}^n)$, and $F \in L^1([-t_0,t_0]; H^\alpha(\mathbb{R}^n))$, then there exists $G \in L^1([-t_0,t_0]; H^\alpha(\mathbb{R}^n))$, with*

$$\|G\|_{L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))} \le C(t_0)(\|f\|_{H^{\alpha+1}(\mathbb{R}^n)} + \|g\|_{H^\alpha(\mathbb{R}^n)} + \|F\|_{L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))}),$$

such that

$$u(t,x) = \mathbf{c}(t,0)f(x) + \mathbf{s}(t,0)g(x) + \int_0^t (\mathbf{s}(t,s)G(s,\cdot))(x)\,ds$$

is a weak solution to the Cauchy problem

$$\begin{aligned}(\partial_t^2 - A(t,x,\partial_x))u(t,x) &= F(t,x),\\ u(t,x)|_{t=0} &= f(x),\\ \partial_t u(t,x)|_{t=0} &= g(x).\end{aligned}$$

If $f = g = 0$, and F vanishes for $t < 0$ (respectively, for $t > 0$) then G, hence u, vanishes for $t < 0$ (respectively, for $t > 0$).

Proof. — Suppose that $G(t,x) \in L^1([-t_0,t_0]; H^\alpha(\mathbb{R}^n))$, and set

$$v(t,x) = \int_0^t \mathbf{s}(t,s)G(s,x)\,ds.$$

From the strong continuity of $\mathbf{s}(t,s)$ and $\partial_t\mathbf{s}(t,s)$, and the fact that $\mathbf{s}(t,t) = 0$, it follows that

$$v(t,x) \in C([-t_0,t_0]; H^{\alpha+1}(\mathbb{R}^n)) \cap C^1([-t_0,t_0]; H^\alpha(\mathbb{R}^n)),$$

with

$$\partial_t v(t,x) = \int_0^t \partial_t\mathbf{s}(t,s)G(s,x)\,ds,$$

and hence that

$$v(0,x) = 0, \qquad \partial_t v(t,x)|_{t=0} = 0.$$

Next, if $\phi(t,x) \in C_c^\infty([-t_0,t_0] \times \mathbb{R}^n_x)$, then the function

$$\langle \phi(t,\cdot), \partial_t\mathbf{s}(t,s)G(s,\cdot)\rangle$$

(where $\langle\cdot,\cdot\rangle$ denotes the distribution pairing on $\mathbb{R}^n$) is smooth in t for each fixed s, and

$$\partial_t\langle\phi(t,\cdot),\partial_t\mathbf{s}(t,s)G(s,\cdot)\rangle$$
$$= \langle\partial_t\phi(t,\cdot),\partial_t\mathbf{s}(t,s)G(s,\cdot)\rangle + \langle\phi(t,\cdot),\partial_t^2\mathbf{s}(t,s)G(s,\cdot)\rangle.$$

integrating both sides that, in the sense of distributions,

$$\partial_t^2 v(t,x) = G(t,x) + \int_0^t \partial_t^2\mathbf{s}(t,s)G(s,x)ds,$$

where the last integral is a vector valued integral in $H^{\alpha-1}(\mathbb{R}^n)$.

If $u(t,x)$ is of the form in the theorem, it follows that u is a weak solution to the Cauchy problem provided that

$$G(t,x)+\int_0^t \mathrm{T}_1(t,s)G(s,x)ds = F(t,x)-(\partial_t^2-A(t,x,\partial_x))(\mathbf{c}(t,0)f(x) + \mathbf{s}(t,0)g(x)), \tag{4.4}$$

where $\mathrm{T}_1(t,s)$ is given by (4.2). By Theorem 4.5, the operator $\mathrm{T}_1(t,s)$ is bounded on $H^\alpha(\mathbb{R}^n)$, for $-1 \le \alpha \le 2$, with norm less than $C(t_0)$ if $|t|,|s| \le t_0$. The Volterra equation (4.4) can thus be solved by iteration. Indeed, if $\widetilde{F}(t,x) \in L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))$, then the equation

$$G(t,x) + \int_0^t \mathrm{T}_1(t,s)G(s,x)ds = \widetilde{F}(t,x)$$

is solved by $G(t,x) = \widetilde{F}(t,x) + \sum_{n=1}^{\infty} G_n(t,x)$, where

$$G_n(t,x) = \int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-1}} \mathrm{T}_1(t,s_1)\mathrm{T}_1(s_1,s_2)\cdots\mathrm{T}_1(s_{n-1},s_n)\widetilde{F}(s_n,x)ds_n\cdots ds_1.$$

The series converges in $L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))$, with norm dominated by $\exp(t_0C(t_0))\|\widetilde{F}\|$. □

THEOREM 4.7. — *Suppose that, in addition to the regularity assumptions of the introduction, it holds that*

$$a(t,x) \in L^\infty([-t_0,t_0];H^{s_0+1}(\mathbb{R}^n)), \quad s_0 > \frac{n}{2}.$$

Then the statement of Theorem 4.6 holds for $s_0+1 \ge \alpha \ge -s_0$.

Proof. — By the proof of Theorem 4.5, it suffices to show that, for $s_0+1 \ge \alpha \ge -s_0$, the operator

$$T = \sum_k (a(x) - a^k(x))\beta_k(D)$$

maps $H^{\alpha-1}(\mathbb{R}^n)$ to $H^\alpha(\mathbb{R}^n)$, whenever $a(x) \in H^{s_0+1}(\mathbb{R}^n) \cap C^{1,1}(\mathbb{R}^n)$, and $\beta_k(D)$ is a Littlewood-Paley decomposition. This holds for $-1 \leq \alpha \leq 2$ by Theorem 4.5. Since $\partial_x a(x)$ is a multiplier on $H^s(\mathbb{R}^n)$ for $-s_0 \leq s \leq s_0$, it holds that $[\partial_x, T]$ is a bounded operator on $H^s(\mathbb{R}^n)$ for $-s_0 \leq s \leq s_0$, which implies the result. □

4a. Operators in divergence form.

In this section, we modify the theorems of Section 4 to the situation where $A(t,x,\partial_x)$ is replaced by an operator in divergence form

$$A^D(t,x,\partial_x)u(t,x) = \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(t,x)\partial_{x_j}u(t,x)), \tag{4.1a}$$

where we assume that the matrix $a_{ij}(t,x)$ satisfies the conditions of the introduction. We take $\mathbf{c}(t,0)$ and $\mathbf{s}(t,0)$ to be the family of operators constructed in Section 4. Thus, the approximate inverse is the same, for a given $a_{ij}(t,x)$, whether the operator is in divergence form or standard form. However, the error is invertible on a different range of Sobolev spaces for the two cases.

THEOREM 4.6A. — *If $-2 \leq \alpha \leq 1$, and $f \in H^{\alpha+1}(\mathbb{R}^n)$, $g \in H^\alpha(\mathbb{R}^n)$, and $F \in L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))$, then there exists $G \in L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))$, with*

$$\|G\|_{L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))} \leq C(t_0)(\|f\|_{H^{\alpha+1}(\mathbb{R}^n)} + \|g\|_{H^\alpha(\mathbb{R}^n)} + \|F\|_{L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))}),$$

such that

$$u(t,x) = \mathbf{c}(t,0)f(x) + \mathbf{s}(t,0)g(x) + \int_0^t (\mathbf{s}(t,s)G(s,\cdot))(x)ds$$

is a weak solution to the Cauchy problem

$$\begin{aligned}(\partial_t^2 - A^D(t,x,\partial_x))u(t,x) &= F(t,x),\\ u(t,x)|_{t=0} &= f(x),\\ \partial_t u(t,x)|_{t=0} &= g(x).\end{aligned}$$

If $f = g = 0$, and F vanishes for $t < 0$ (respectively, for $t > 0$) then G, hence u, vanishes for $t < 0$ (respectively, for $t > 0$).

Proof. — By writing

$$A_k^D(t,x,\partial_x) = A_k(t,x,\partial_x) + \sum_{ij=1}^{n}(\partial_{x_i}a_{ij}^k(t,x))\partial_{x_j},$$

and observing that

$$\sum_{k=0}^{\infty}(\partial_{x_i}a_{ij}^k(t,x))\partial_{x_j}\mathbf{c}_k(t,s) \in \mathcal{I}^1(\chi_{t,s}^+) \oplus \mathcal{I}^1(\chi_{t,s}^-),$$

it follows that

$$\sum_{k=0}^{\infty}(\partial_t^2 - A_k^D(t,y,\partial_y))\mathbf{c}_k(t,s) \in \mathcal{I}^1(\chi_{t,s}^+) \oplus \mathcal{I}^1(\chi_{t,s}^-).$$

Similarly, we have

$$\sum_{k=0}^{\infty}(\partial_t^2 - A_k^D(t,y,\partial_y))\mathbf{s}_k(t,s) \in \mathcal{I}^0(\chi_{t,s}^+) \oplus \mathcal{I}^0(\chi_{t,s}^-).$$

We next note that an operator of the form

$$Tu = \partial_{x_i}\Big(\sum_{k \text{ even}}(a_{ij}(t,x) - a_{ij}^k(t,x))\beta_k(D)u\Big)$$

maps $H^\alpha(\mathbb{R}^n) \to H^\alpha(\mathbb{R}^n)$ for $-2 \le \alpha \le 1$. This follows from the fact that the operator in parentheses maps $H^\alpha(\mathbb{R}^n) \to H^{\alpha+1}(\mathbb{R}^n)$ for α in this range. Thus, we have the analogy of Theorem 4.5, for $-2 \le \alpha \le 1$:

$$(\partial_t^2 - A^D(t,x,\partial_x))\mathbf{c}(t,s) : H^{\alpha+1}(\mathbb{R}^n) \to H^\alpha(\mathbb{R}^n),$$
$$(\partial_t^2 - A^D(t,x,\partial_x))\mathbf{s}(t,s) : H^{\alpha}(\mathbb{R}^n) \to H^\alpha(\mathbb{R}^n).$$

The proof now follows exactly as in Theorem 4.6. □

Similarly, we have

THEOREM 4.7a. — *Suppose that, in addition to the regularity assumptions of the introduction, it holds that*

$$a(t,x) \in L^\infty([-t_0,t_0]; H^{s_0+1}(\mathbb{R}^n)), \quad s_0 > \frac{n}{2}.$$

Then the statement of Theorem 4.6a holds for $s_0 \ge \alpha \ge -s_0 - 1$.

4b. Time independent Laplace-Beltrami operators.

In this section, we assume that the matrix coefficients $a_{ij}(x)$ are independent of the time variable t, in addition to satisfying the conditions of the introduction. We consider the wave equation for an operator of the form

$$A^L(x,\partial_x)u(t,x) = \rho(x)^{-1}\sum_{i,j=1}^{n}\partial_{x_i}(\rho(x)a_{ij}(x)\partial_{x_j}u(t,x)), \tag{4.1b}$$

where $\rho(x) \in C^{1,1}(\mathbb{R}^n)$ is a real scalar function, globally bounded from below. If $g_{ij}(x)$ is a Riemannian metric, and

$$a_{ij}(x) = g^{ij}(x), \qquad \rho(x) = |\det[g_{ij}(x)]|,$$

then $A^L(x,\partial_x)$ is the Laplace-Beltrami operator associated to g.

For general $f \in H^{\alpha+1}(\mathbb{R}^n)$, the expression $A^L(x,\partial_x)f(x)$ makes sense only for $\alpha \geq -1$. On the other hand, we note that if $\alpha \geq -1$, then

$$u(t,x) \in C([-t_0,t_0];H^{\alpha+1}(\mathbb{R}^n)) \cap C^1([-t_0,t_0];H^{\alpha}(\mathbb{R}^n))$$

is a weak solution to the Cauchy problem for $\partial_t^2 - A^L(x,\partial_x)$, with initial data (f,g), precisely when $v(t,x) = \rho(x)u(t,x)$ is a weak solution to the following Cauchy problem:

$$\begin{aligned}\partial_t^2 v(t,x) - \sum_{i,j=1}^{n}\partial_{x_i}(\rho(x)a_{ij}(x)\partial_{x_j}(\rho(x)^{-1}v(t,x))) &= \rho(x)F(t,x),\\ v(t,x)|_{t=0} &= \rho(x)f(x),\\ \partial_t v(t,x)|_{t=0} &= \rho(x)g(x).\end{aligned} \tag{4.2b}$$

As the next theorem shows, our technique yields solutions to (4.2b) for the range $-2 \leq \alpha \leq 0$.

THEOREM 4.6b. — *If $-1 \leq \alpha \leq 1$, and $f \in H^{\alpha+1}(\mathbb{R}^n)$, $g \in H^{\alpha}(\mathbb{R}^n)$, and $F \in L^1([-t_0,t_0];H^{\alpha}(\mathbb{R}^n))$, then there exists $G \in L^1([-t_0,t_0];H^{\alpha}(\mathbb{R}^n))$, with*

$$\|G\|_{L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))} \leq C(t_0)(\|f\|_{H^{\alpha+1}(\mathbb{R}^n)} + \|g\|_{H^\alpha(\mathbb{R}^n)} + \|F\|_{L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))}),$$

such that

$$u(t,x) = \mathbf{c}(t,0)f(x) + \mathbf{s}(t,0)g(x) + \int_0^t (\mathbf{s}(t,s)G(s,\cdot))(x)\,ds$$

is a weak solution to the Cauchy problem

$$\begin{aligned}(\partial_t^2 - A^L(x,\partial_x))u(t,x) &= F(t,x),\\ u(t,x)|_{t=0} &= f(x),\\ \partial_t u(t,x)|_{t=0} &= g(x).\end{aligned}$$

If $-2 \le \alpha \le 0$, *and* $f \in H^{\alpha+1}(\mathbb{R}^n)$, $g \in H^\alpha(\mathbb{R}^n)$, *and* $F \in L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))$, *then there exists* $G \in L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))$, *with*

$$\|G\|_{L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))} \le C(t_0)\big(\|f\|_{H^{\alpha+1}(\mathbb{R}^n)} + \|g\|_{H^\alpha(\mathbb{R}^n)} + \|F\|_{L^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))}\big),$$

such that

$$v(t,x) = \mathbf{c}(t,0)(\rho f)(x) + \mathbf{s}(t,0)(\rho g)(x) + \int_0^t (\mathbf{s}(t,s)G(s,\cdot))(x)ds$$

is a weak solution to the Cauchy problem (4.2b).

In each case, if $f = g = 0$, *and* F *vanishes for* $t < 0$ *(respectively, for* $t > 0$*) then* G, *hence* u, *vanishes for* $t < 0$ *(respectively, for* $t > 0$*).*

Proof. — The proof of the first statement follows by showing that, for $-1 \le \alpha \le 1$, one has

$$(\partial_t^2 - A^L(x,\partial_x))\mathbf{c}(t,s) : H^{\alpha+1}(\mathbb{R}^n) \to H^\alpha(\mathbb{R}^n),$$
$$(\partial_t^2 - A^L(x,\partial_x))\mathbf{s}(t,s) : H^{\alpha}(\mathbb{R}^n) \to H^\alpha(\mathbb{R}^n).$$

This follows by writing

$$A^L(x,\partial_x) = A^D(x,\partial_x) + \sum_{i,j=1}^n \rho(x)^{-1}(\partial_{x_i}\rho(x))a_{ij}(x)\partial_{x_j},$$

and noting that the second term maps $H^{\alpha+1}(\mathbb{R}^n) \to H^\alpha(\mathbb{R}^n)$ for $-1\le\alpha\le1$.

The second statement follows in a similar fashion by writing

$$\rho(x)A^L(x,\partial_x)(\rho^{-1}f)(x) = A^D(x,\partial_x)f(x) - \partial_{x_i}(\rho(x)^{-1}\partial_{x_j}\rho(x)a_{ij}(x)f(x)),$$

and noting that the second term maps $H^{\alpha+1}(\mathbb{R}^n) \to H^\alpha(\mathbb{R}^n)$ for $-2 \le \alpha \le 0$. □

5. Uniqueness of solutions.

In this section, we show that the solutions to the Cauchy problem constructed in Theorems 4.6, 4.6a, and 4.6b, are the unique such weak solutions, in the sense of Definition 1.1. Precisely, we show that, if α lies in the range indicated by the appropriate theorem, then given a weak solution $u \in C([-t_0,t_0];H^{\alpha+1}(\mathbb{R}^n)) \cap C^1([-t_0,t_0];H^\alpha(\mathbb{R}^n))$ to the Cauchy problem with $f = g = F = 0$, necessarily $u = 0$. For a uniqueness theorem under the condition that $u \in H^1_{\mathrm{loc}}((-t_0,t_0)\times\mathbb{R}^n)$, see [5].

We provide the proof in the case of Theorem 4.6, the other versions follow similarly. To prove uniqueness for Theorem 4.6, it suffices to consider the case $\alpha = -1$. Thus, we will suppose that $u \in C([-t_0,t_0];L^2(\mathbb{R}^n)) \cap C^1([-t_0,t_0];H^{-1}(\mathbb{R}^n))$, and that

$$\begin{aligned}(\partial_t^2 - A(t,x,\partial_x))u(t,x) &= 0,\\ u(t,x)|_{t=0} &= 0,\\ \partial_t u(t,x)|_{t=0} &= 0.\end{aligned}$$

We set $u(t,x) = u^+(t,x) + u^-(t,x)$, where

$$u^+(t,x) = \begin{cases} u(t,x), & t \geq 0,\\ 0, & t \leq 0.\end{cases}$$

Then $u^\pm \in C([-t_0,t_0];L^2(\mathbb{R}^n)) \cap C^1([-t_0,t_0];H^{-1}(\mathbb{R}^n))$, and a simple limiting argument shows that, in the distribution sense,

$$(\partial_t^2 - A(t,x,\partial_x))u^\pm(t,x) = 0.$$

We will show that $u^+(t,x) = 0$, the proof that $u^-(t,x) = 0$ being similar.

Thus, we are given that, for $\psi \in C_c^\infty((-t_0,t_0)\times\mathbb{R}^n)$, the following holds:

$$\int u^+(t,x)(\partial_t^2 - A^*(t,x,\partial_x))\psi(t,x)dtdx = 0, \tag{5.1}$$

where

$$A^*(t,x,\partial_x)\psi(t,x) = \sum_{ij=1}^n \partial_{x_i}\partial_{x_j}(a_{ij}(t,x)\psi(t,x)).$$

By the support condition on u^+ and a density argument, (5.1) holds for $\psi \in C^2([-t_0,t_0];H^2(\mathbb{R}^n))$, provided that $\psi(t,x) = 0$ for $t \geq t_0 - \epsilon$, for some $\epsilon > 0$.

We need to show that, for $\phi \in C_c^\infty((-t_0,t_0)\times\mathbb{R}^n)$, the following holds:

$$\int u^+(t,x)\phi(t,x)dtdx = 0. \tag{5.2}$$

We observe that

$$\left|\int u^+(t,x)\phi(t,x)dtdx\right| \leq \|\phi\|_{L^1_t([-t_0,t_0];L^2(\mathbb{R}^n))}.$$

Therefore, (5.2) is a result of (5.1) and the following lemma.

LEMMA 5.1. — *Suppose that $\phi(t,x) \in L^1_t([-t_0,t_0];L^2(\mathbb{R}^n))$, and that $\phi(t,x) = 0$ for $t \geq t_0 - \epsilon$. Then there exists a sequence $\psi_k(t,x) \in C^2([-t_0,t_0];H^2(\mathbb{R}^n))$, with $\psi_k(t,x) = 0$ for $t \geq t_0 - \epsilon$, such that*

$$\lim_{k\to\infty} \|(\partial_t^2 - A^*(t,x,\partial_x))\psi_k - \phi\|_{L^1_t([-t_0,t_0];L^2(\mathbb{R}^n))} = 0.$$

Proof. — We first note that the operator

$$(\partial_t^2 - A^*(t,x,\partial_x))\mathbf{s}(t,s) \tag{5.3}$$

is a uniformly bounded family of operators on $H^\alpha(\mathbb{R}^n)$ for $-3 \le \alpha \le 0$. This is seen, as in the proof of Theorem 4.6a, by noting that

$$\sum_{k=0}^{\infty}(\partial_t^2 - A_k^*(t,y,\partial_y))\mathbf{s}_k(t,s) \in \mathcal{I}^0(\chi_{t,s}^+) \oplus \mathcal{I}^0(\chi_{t,s}^-),$$

and that an operator of the form

$$Tu = \partial_{x_i}\partial_{x_j}\Big(\sum_{k \text{ even}} (a_{ij}(t,x) - a_{ij}^k(t,x))\beta_k(D)u\Big)$$

maps $H^{\alpha+1}(\mathbb{R}^n) \to H^\alpha(\mathbb{R}^n)$ for $-3 \le \alpha \le 0$. In particular, the family of operators (5.3) is uniformly bounded on $L^2(\mathbb{R}^n)$.

Let $G \in L_t^1([-t_0,t_0];L^2(\mathbb{R}^n))$ solve

$$G(t,x) - \int_t^{t_0}((\partial_t^2 - A^*(t,x,\partial_x))\mathbf{s}(t,s)G(s,\cdot))(x)ds = \phi(t,x).$$

Then $G(t,x) = 0$ for $t \ge t_0 - \epsilon$. We choose a sequence $G_k \in C^\infty([-t_0,t_0] \times \mathbb{R}^n)$, with $G_k(t,x) = 0$ for $t \ge t_0 - \epsilon$, such that

$$\lim_{k\to\infty} \|G_k - G\|_{L_t^1([-t_0,t_0];L^2(\mathbb{R}^n))} = 0.$$

Now set

$$\psi_k(t,x) = -\int_t^{t_0}(\mathbf{s}(t,s)G_k(s,\cdot))(x)ds.$$

Then $\psi_k(t,x) \in C^2([-t_0,t_0];H^2(\mathbb{R}^n))$, and

$$(\partial_t^2 - A^*(t,x,\partial_x))\psi_k = G_k(t,x) - \int_t^{t_0}((\partial_t^2 - A^*(t,x,\partial_x))\mathbf{s}(t,s)G_k(s,\cdot))(x)ds.$$

This converges to $\phi(t,x)$ as $k \to \infty$. □

6. Estimates for the wave equation.

In this section, we apply the results of the previous section to obtain certain space-time estimates, in the case of space dimensions $n = 2$ and $n = 3$, for the solution to the Cauchy problem

$$\begin{aligned}(\partial_t^2 - A(t,x,\partial_x))u(t,x) &= F(t,x),\\ u(t,x)|_{t=0} &= f(x),\\ \partial_t u(t,x)|_{t=0} &= g(x).\end{aligned}$$

The estimates we obtain hold also for the solution to the Cauchy problem where $A(t,x,\partial_x)$ is replaced by $A^D(t,x,\partial_x)$ or $A^L(x,\partial_x)$, given respectively by (4.1a) and (4.1b).

We first reduce matters to establishing mapping properties for operators belonging to the class developed in Section 2. In particular, we observe that, by Theorem 4.6 and Lemma 4.4, if $-1 \le \alpha \le 2$, then an estimate of the form

$$\|u\|_{L^p_t([-t_0,t_0];L^q_x(\mathbb{R}^n))} \le C(\|f\|_{H^{\alpha+1}(\mathbb{R}^n)} + \|g\|_{H^\alpha(\mathbb{R}^n)} + \|F\|_{L^1_t([-t_0,t_0];H^\alpha(\mathbb{R}^n))}),$$

is a consequence of the following estimates:

$$\|\mathbf{c}(t,0)f\|_{L^p_t([-t_0,t_0];L^q_x(\mathbb{R}^n))} \le C\|f\|_{H^{\alpha+1}(\mathbb{R}^n)},$$
$$\|\tilde{\mathbf{s}}(t,s)g\|_{L^p_t([-t_0,t_0];L^q_x(\mathbb{R}^n))} \le C\|g\|_{H^{\alpha}(\mathbb{R}^n)}.$$

The same reduction holds for the Cauchy problem in divergence or Laplace-Beltrami form, provided that α is in the appropriate range indicated in the corresponding version of Theorem 4.6. The estimates below correspond to $\alpha = -\dfrac{1}{2}$ and $\alpha = 0$, which lie in the correct range for all three versions of the Cauchy problem. We also note that it suffices to establish such an estimate for some small value of t_0, since by the group property of the wave group, the estimate then holds for any finite value of t_0.

We use this reduction to establish two sets of estimates of importance in nonlinear wave equations, in space dimensions $n = 2$ and $n = 3$. The estimates we establish are, in the case of the standard wave equation on $\mathbb{R}^{n+1}$, due respectively to Pecher [7], and Strichartz [13], [14].

In higher dimensions the proof of the crucial endpoint estimate breaks down, due to the fact that the symbols of $\mathbf{c}(t,0)$ and $\tilde{\mathbf{s}}(t,0)$ are only of class $S_{\frac{1}{2},\frac{1}{2}}$.

Pecher Estimates.

$$\|u\|_{L_t^{\frac{2q}{q-6}}L^q_x([-t_0,t_0]\times\mathbb{R}^3)} \le C(\|f\|_{H^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)} + \|F\|_{L^1_t([-t_0,t_0];L^2_x(\mathbb{R}^3))}),$$
$$6 \le q < \infty.$$

Homogeneous Strichartz Estimates.

$$\|u\|_{L^6([-t_0,t_0]\times\mathbb{R}^2)} \le C(\|f\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} + \|g\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)}),$$
$$\|u\|_{L^4([-t_0,t_0]\times\mathbb{R}^3)} \le C(\|f\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} + \|g\|_{H^{-\frac{1}{2}}(\mathbb{R}^3)}).$$

By the comments above, these estimates are reduced to appropriate mapping properties of $\mathbf{c}(t,0)$ and $\tilde{\mathbf{s}}(t,0)$. These, in turn, are an immediate consequence of the following theorem.

THEOREM 6.1. — *Let χ_t be the canonical transformation given by the solution of (3.4). Suppose that $b(t,\gamma,\gamma')$ is a one parameter family of matrices, such that the coefficients vanish unless $|k-k'|\le 1$, and such that the following estimates are satisfied for all N, with constant C_N independent of $t\in[-t_0,t_0]$:*

$$|b(t,\gamma,\gamma')|\le C_N(1+2^k d(x,\omega;\chi_t(x',\omega')))^{-N}.$$

Let $\mathbf{B}$ be the operator mapping functions on $\mathbb{R}^n$ to functions on $[-t_0,t_0]\times\mathbb{R}^n$, such that the matrix of $f\to\mathbf{B}f(t,\cdot)$ is given by $b(t,\gamma,\gamma')$. Then, if t_0 is sufficiently small, depending on the bounds for the $a_{ij}(t,x)$, the following estimates hold:

$$\begin{aligned}
\|\mathbf{B}f\|_{L^6([-t_0,t_0]\times\mathbb{R}^2)} &\le C\|f\|_{H^{\frac12}(\mathbb{R}^2)},\\
\|\mathbf{B}f\|_{L^4([-t_0,t_0]\times\mathbb{R}^3)} &\le C\|f\|_{H^{\frac12}(\mathbb{R}^3)},\\
\|\mathbf{B}f\|_{L_t^{\frac{2q}{q-6}}L_x^q([-t_0,t_0]\times\mathbb{R}^3)} &\le C\|f\|_{H^1(\mathbb{R}^3)},\quad 6\le q<\infty.
\end{aligned}$$

Proof. — The first step in the proof of Theorem 6.1 is to reduce matters to establishing uniform bounds for the operator $\mathbf{B}$ localised to dyadic frequency shells. Specifically, we set

$$b_k(t,\gamma,\gamma')=\begin{cases} b(t,\gamma,\gamma') & ,\ k_\gamma=k,\\ 0 & ,\ k_\gamma\ne k,\end{cases}$$

and define $\mathbf{B}_k$ to be the operator given by $b_k(t,\gamma,\gamma')$. Then, by Littlewood-Paley theory, together with the fact that the exponents on the left hand side are greater than 2, Theorem 6.1 is a consequence of the following estimates (with constant C independent of k):

$$\begin{aligned}
\|\mathbf{B}_kf\|_{L^6([-t_0,t_0]\times\mathbb{R}^2)} &\le C2^{\frac{k}{2}}\|f\|_{L^2(\mathbb{R}^2)},\\
\|\mathbf{B}_kf\|_{L^4([-t_0,t_0]\times\mathbb{R}^3)} &\le C2^{\frac{k}{2}}\|f\|_{L^2(\mathbb{R}^3)},\\
\|\mathbf{B}_kf\|_{L_t^{\frac{2q}{q-6}}L_x^q([-t_0,t_0]\times\mathbb{R}^3)} &\le C2^{k}\|f\|_{L^2(\mathbb{R}^3)},\qquad 6\le q<\infty.
\end{aligned}\tag{6.1}$$

For example, in the case of the first estimate, we observe that by the vanishing condition on the matrix coefficients, (6.1) implies the stronger estimate

$$\|\mathbf{B}_kf\|_{L^6([-t_0,t_0]\times\mathbb{R}^2)}\le C2^{\frac{k}{2}}\|f_k\|_{L^2(\mathbb{R}^2)},$$

where f_k is an appropriate Littlewood-Paley decomposition of f. The following argument then holds:

$$\begin{aligned}\|\mathbf{B}f\|_{L^6([-t_0,t_0]\times\mathbb{R}^2)} &\approx \Big\|\Big(\sum_{k=0}^{\infty}|\mathbf{B}_k f|^2\Big)^{\frac{1}{2}}\Big\|_{L^6([-t_0,t_0]\times\mathbb{R}^2)}\\ &\le \Big(\sum_{k=0}^{\infty}\|\mathbf{B}_k f\|^2_{L^6([-t_0,t_0]\times\mathbb{R}^2)}\Big)^{\frac{1}{2}}\\ &\le C\Big(\sum_{k=0}^{\infty}2^k\|f_k\|^2_{L^2(\mathbb{R}^2)}\Big)^{\frac{1}{2}} \approx C\|f\|_{H^{\frac{1}{2}}(\mathbb{R}^2)}.\end{aligned}$$

The next step is to observe that, by a duality argument, the estimates (6.1) are a consequence of the following estimates:

$$\begin{aligned}\|\mathbf{B}_k\mathbf{B}_k^*f\|_{L^6([-t_0,t_0]\times\mathbb{R}^2)} &\le C2^k\|f\|_{L^{\frac{6}{5}}([-t_0,t_0]\times\mathbb{R}^2)},\\ \|\mathbf{B}_k\mathbf{B}_k^*f\|_{L^4([-t_0,t_0]\times\mathbb{R}^3)} &\le C2^k\|f\|_{L^{\frac{4}{3}}([-t_0,t_0]\times\mathbb{R}^3)},\\ \|\mathbf{B}_k\mathbf{B}_k^*f\|_{L_t^{\frac{2q}{q-6}}L_x^q([-t_0,t_0]\times\mathbb{R}^3)} &\le C2^{2k}\|f\|_{L_t^{\frac{2q}{q+6}}L_x^{\frac{q}{q-1}}([-t_0,t_0]\times\mathbb{R}^3)},\\ &\qquad 6\le q<\infty.\end{aligned}\tag{6.2}$$

We now write the operator $\mathbf{B}_k\mathbf{B}_k^*$ in the form

$$\mathbf{B}_k\mathbf{B}_k^*F(t,x) = \int_{-t_0}^{t_0}(\mathbf{T}_{t,s}^kF(s,\cdot))(x)ds,$$

where $\mathbf{T}_{t,s}^k \in \mathcal{I}^0(\chi_{t-s})$ has matrix coefficients given by

$$t_{t,s}^k(\gamma,\gamma') = \sum_{\gamma''}b(t,\gamma,\gamma'')\overline{b(s,\gamma',\gamma'')}.$$

By the arguments on page 894 of [11], the estimates (6.2) can be obtained by interpolation of the following pair of estimates:

$$\begin{aligned}&\|\mathbf{T}_{t,s}^kf\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)},\\ &\|\mathbf{T}_{t,s}^kf\|_{L^\infty(\mathbb{R}^n)} \le C2^{kn}(1+2^k|t-s|)^{-\frac{n-1}{2}}\|f\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

The first of these estimates follows from Theorem 2.7. To establish the second, we note that the matrix of $\mathbf{T}_{t,s}^k$ satisfies the same conditions as the matrix $b_k(t-s,\gamma,\gamma')$ of Theorem 6.1. The second estimate, and hence the conclusion of Theorem 6.1, is then a consequence of the following lemma. □

LEMMA 6.2. — *Suppose that $b(t,\gamma',\gamma)$ is as in Theorem 6.1. Set*

$$K_k(t,y,x) = \sum_{\gamma,\gamma'} b_k(t,\gamma',\gamma)\varphi_{\gamma'}(y)\overline{\varphi_\gamma(x)}.$$

Then, for t_0 sufficiently small, depending on the bounds for the $a_{ij}(t,x)$, the following bounds hold, for dimensions $n=2$ and $n=3$:

$$|K_k(t,y,x)| \le C2^{kn}(1+2^k|t|)^{-\frac{n-1}{2}}.$$

Proof. — By Lemma 3.6, the size bounds on the coefficients $b(t,\gamma,\gamma')$ remain valid if $\chi_t(x,\omega)$, the Hamiltonian flow determined by the matrix $A(t,x)$, is replaced by the Hamiltonian flow determined by the smoothed out matrix $A_k(t,x)$. Consequently, there is no loss of generality in assuming $\partial_x^2 A(t,x)$ is continuous.

We write $\gamma = (z,\omega,k), \gamma' = (z',\omega',k)$, and observe the following estimates:

$$|b(t,\gamma',\gamma)| \le C_N(1+2^k d(z',\omega';\chi_t(z,\omega)))^{-N},$$
$$|\varphi_\gamma(x)| \le C_N 2^{\frac{k(n+1)}{4}}(1+2^k d(z,\omega;x,\omega))^{-N},$$
$$|\varphi_{\gamma'}(y)| \le C_N 2^{\frac{k(n+1)}{4}}(1+2^k d(z',\omega';y,\omega'))^{-N}.$$

As a result, $|K_k(t,y,x)|$ is dominated by

$$2^{\frac{k(n+1)}{2}} \sum_{z,\omega,z',\omega'} (1+2^k d(z',\omega';\chi_t(z,\omega)))^{-N}$$
$$(1+2^k d(z,\omega;x,\omega))^{-N}(1+2^k d(z',\omega';y,\omega'))^{-N}.$$

By Lemma 2.5, this is in turn dominated by

$$2^{\frac{k(n+1)}{2}} \sum_{\omega,\omega'} (1+2^k d(y,\omega';\chi_t(x,\omega)))^{-N}.$$

Write $\chi_t(x,\omega) = (x(t),\omega(t))$. The above sum is then dominated (for $t\le 1$) by

$$2^{\frac{k(n+1)}{2}} \sum_{\omega} (1+2^k|\langle\omega(t),y-x(t)\rangle| + 2^k|y-x(t)|^2)^{-N}.$$

We next observe that the summand is essentially constant as ω varies over any set of diameter $2^{-\frac{k}{2}}$ in S^{n-1}. We are thus led to the following estimate:

$$|K_k(t,y,x)| \le C_N 2^{kn}\int_{S^{n-1}} (1+2^k|\langle\omega(t),y-x(t)\rangle| + 2^k|y-x(t)|^2)^{-N} d\omega.$$

The conclusion of Lemma 6.2 is now immediate if $2^k|t| \le 1$, so we henceforth assume that $2^k|t| \ge 1$.

We next make a change of coordinates, so that $x = 0$, and so that $a_{ij}(0,0) = \delta_{ij}$. Set

$$\mathrm{H}^t(s,x,\xi) = \Big(\sum_{ij} a_{ij}(ts,tx)\xi_i\xi_j\Big)^{\frac{1}{2}}.$$

Let $(\tilde{x}(\omega),\tilde{\omega}(\omega))$ denote the image at $s = 1$ of $(0,\omega)$ under the following flow:

$$\begin{cases} \dfrac{d\tilde{x}}{ds} = \mathrm{H}^t_\xi(s,\tilde{x},\tilde{\omega}), \\ \dfrac{d\tilde{\omega}}{ds} = -\mathrm{H}^t_x(s,\tilde{x},\tilde{\omega}) + \langle\tilde{\omega},\mathrm{H}^t_x(s,\tilde{x},\tilde{\omega})\rangle\tilde{\omega}. \end{cases}$$

Then

$$\omega(t) = \tilde{\omega}(\omega), \qquad x(t) = t\tilde{x}(\omega).$$

We next observe that the following estimates hold, uniformly on the set $|s| \le 1, |x| \le 2$.

$$|\partial_x^\alpha(a_{ij}(ts,tx) - \delta_{ij})| \le C|t|, \qquad |\alpha| \le 2.$$

Thus, uniformly on the set $|s| \le 1$, $|x| \le 2$, $\frac{1}{2} \le |\xi| \le 2$, we have

$$\sum_{|\alpha|+|\beta|\le 2} |\partial_x^\alpha\partial_\xi^\beta(\mathrm{H}^t(s,x,\xi) - |\xi|)| \le C|t|.$$

Replacing y by ty, Lemma 6.2 is now a consequence of the following lemma. □

LEMMA 6.3. — *Suppose that the function* $\mathrm{H}(s,x,\xi)$ *is homogeneous of degree* 1 *in* ξ, *twice continuously differentiable with respect to* x *and* ξ, *and that, uniformly on the set* $|s| \le 1$, $|x| \le 2$, $\frac{1}{2} \le |\xi| \le 2$, *the following estimates hold:*

$$\sum_{|\alpha|+|\beta|\le 2} |\partial_x^\alpha\partial_\xi^\beta(\mathrm{H}(s,x,\xi) - |\xi|)| \le \epsilon. \tag{6.3}$$

Let $(\tilde{x}(\omega),\tilde{\omega}(\omega))$ *denote the image of* $(0,\omega)$ *at time* 1 *under the Hamiltonian flow on the cosphere bundle induced by* $\mathrm{H}(s,x,\xi)$. *Then, in dimensions* $n = 2$ *and* $n = 3$, *the following holds:*

$$\sup_{y\in\mathbb{R}^n}\int_{S^{n-1}} (1 + \lambda|\langle\tilde{\omega}(\omega), y - \tilde{x}(\omega)\rangle|)^{-2}d\omega \le C\lambda^{-\frac{n-1}{2}}.$$

Here, ϵ *and* C *are universal constants.*

Proof. — Consider the Hamiltonian flow

$$(6.4)\qquad \begin{cases} \dfrac{dx}{ds}(s,\eta) = \mathrm{H}_\xi(s,x(s,\eta),\xi(s,\eta)), \\ \dfrac{d\xi}{ds}(s,\eta) = -\mathrm{H}_x(s,x(s,\eta),\xi(s,\eta)), \\ x(0,\eta)=0, \quad \xi(0,\eta)=\eta. \end{cases}$$

By (6.3), we see that, uniformly for $|s|\le 1$,

$$\left|x(s,\eta)-s\frac{\eta}{|\eta|}\right| = \mathcal{O}(\epsilon), \qquad |\xi(s,\eta)-\eta| = \mathcal{O}(\epsilon).$$

Differentiating the equations (6.4) and applying Gronwall's Lemma now leads to

$$\frac{\partial x}{\partial\eta}(1,\eta) = \frac{\mathbf{I}}{|\eta|} - \frac{\eta\otimes\eta}{|\eta|^3} + \mathcal{O}(\epsilon), \qquad \frac{\partial\xi}{\partial\eta}(1,\eta) = \mathbf{I} + \mathcal{O}(\epsilon).$$

It follows that, for ϵ sufficiently small, the map $\omega \to \tilde{x}(\omega)$ is a C^1 embedding of S^{n-1} into $\mathbb{R}^n$, and that the map $\omega \to \tilde{\omega}(\omega)$ is a C^1 diffeomorphism of S^{n-1}, with uniform upper and lower bounds on the Jacobian. From the fact that the Hamiltonian flow is a canonical transformation, it follows that

$$\tilde{\omega}(\omega)\cdot\frac{\partial\tilde{x}(\omega)}{\partial\omega} = 0,$$

so that $\tilde{\omega}(\omega)$ is the outer normal map to the surface $\omega \to \tilde{x}(\omega)$.

We next make a change of coordinates in the integral from $\omega \to \tilde{\omega}$, and write $\tilde{x}(\tilde{\omega})$ to express $\tilde{x}$ written in terms of the new variable. We are thus left to demonstrate the following inequality:

$$\int_{S^{n-1}} (1+\lambda|\langle\tilde{\omega}, y-\tilde{x}(\tilde{\omega})\rangle|)^{-2}\,d\tilde{\omega} \le C\lambda^{-\frac{n-1}{2}}.$$

For notational convenience, we relabel $\tilde{x}$ and $\tilde{\omega}$ by x and ω.

We first consider the case $n=3$. Since $\langle\omega, x(\omega)\rangle = 1+\mathcal{O}(\epsilon)$, the estimate is immediate if $|y| \le \frac{7}{8}$. By rotating, we will assume that the minimum of $|y-x(\omega)|$ occurs at $\omega=(0,0,1)$, and $x(\omega)=x_0$. From the fact that ω is the outer normal to the surface, we may write $y = x_0+(0,0,r)$, where $r\ge -\frac{1}{4}$. We take standard coordinates $\omega(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$ on S^2. We will show that, for each fixed value of θ, the following holds:

$$\int_0^\pi (1+\lambda|\langle\omega(\theta,\phi), x(\theta,\phi)-x_0\rangle - r\cos\phi|)^{-2}\sin\phi\,d\phi \le C\lambda^{-1},$$

uniformly over θ. For simplicity, we consider $\theta = 0$ and suppress the θ in what follows. We then have the following:

$$\begin{aligned} x(\phi) - x_0 &= (\sin\phi, 0, \cos\phi - 1) + \mathcal{O}(\epsilon\phi), \\ \omega(\phi) &= (\sin\phi, 0, \cos\phi), \\ \partial_\phi \omega(\phi) &= (\cos\phi, 0, -\sin\phi). \end{aligned}$$

It follows that

$$\langle \omega(\phi), x(\phi) - x_0 \rangle = 1 - \cos\phi + \mathcal{O}(\epsilon\phi). \tag{6.5}$$

The integral over $\frac{3\pi}{4} \le \phi \le \pi$ is easily bounded, since for small ϵ, from the fact that $r \ge -\frac{1}{4}$, the integrand is bounded by $C\lambda^{-2}$ on this interval .

We next consider the integral over $\frac{\pi}{4} \le \phi \le \frac{3\pi}{4}$. By (6.5), for small ϵ,

$$\frac{1}{4} \le \langle \omega(\phi), x(\phi) - x_0 \rangle \le \frac{7}{4} \quad \text{if} \quad \frac{\pi}{4} \le \phi \le \frac{3\pi}{4}.$$

Thus, if $-\frac{1}{4} \le r \le \frac{1}{4}$, then the integrand is bounded by $C\lambda^{-2}$ on this interval. We thus assume $r \ge \frac{1}{4}$. Let

$$f(\phi) = \frac{\cos\phi}{\langle \omega(\phi), x(\phi) - x_0 \rangle}.$$

Since

$$\partial_\phi \langle \omega(\phi), x(\phi) - x_0 \rangle = \langle \partial_\phi \omega(\phi), x(\phi) - x_0 \rangle = \sin\phi + \mathcal{O}(\epsilon\phi), \tag{6.6}$$

we deduce that

$$\frac{\partial f}{\partial \phi} = \frac{-\sin\phi}{(1 - \cos\phi)^2} + \mathcal{O}(\epsilon),$$

so that $-\frac{1}{2} \le f'(\phi) \le -\frac{1}{4}$ on the interval $\frac{\pi}{4} \le \phi \le \frac{3\pi}{4}$. The desired bound on the integral over this interval is now a consequence of the bound

$$\begin{aligned} &\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (1 + \lambda |\langle \omega(\phi), x(\phi) - x_0 \rangle - r\cos\phi|)^{-2} \sin\phi \, d\phi \\ &\qquad\qquad \approx \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (1 + (\lambda r)|\frac{1}{r} - f(\phi)|)^{-2} \, d\phi \le C(\lambda r)^{-1}. \end{aligned}$$

To estimate the integral over $0 \le \phi \le \frac{\pi}{4}$, we consider the function

$$g(\phi) = \frac{\langle \omega(\phi), x(\phi) - x_0 \rangle}{\cos\phi}.$$

By (6.5) and (6.6), we have

$$g'(\phi) = \frac{\sin\phi}{\cos^2\phi} + \mathcal{O}(\epsilon\phi).$$

We can then compare

$$\int_0^{\frac{\pi}{4}} (1 + \lambda|\langle\omega(\phi), x(\phi) - x_0\rangle - r\cos\phi|)^{-2} \sin\phi \, d\phi$$

$$\approx \int_0^{\frac{\pi}{4}} (1 + \lambda|g(\phi) - r|)^{-2} g'(\phi)\, d\phi \le C\lambda^{-1}.$$

This concludes the bounds for the case $n = 3$. For the case $n = 2$, we need to show that

$$\int_0^{\pi} (1 + \lambda|\langle\omega(\phi), x(\phi) - x_0\rangle - r\cos\phi|)^{-2}\, d\phi \le C\lambda^{-\frac{1}{2}}.$$

The integral over $\frac{\pi}{4} \le \phi \le \pi$ is bounded by λ^{-1} as above. To conclude, we dominate the integral over $0 \le \phi \le \frac{\pi}{4}$ by

$$\lambda^{-\frac{1}{2}} + \lambda^{\frac{1}{2}} \int_{\lambda^{-\frac{1}{2}}}^{\frac{\pi}{4}} (1 + \lambda|\langle\omega(\phi), x(\phi) - x_0\rangle - r\cos\phi|)^{-2} \sin\phi\, d\phi,$$

which is dominated by $\lambda^{-\frac{1}{2}}$ by the arguments for $n = 3$. □

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Hart F. SMITH,
University of Washington
Department of Mathematics
Box 354 350
Seattle, WA 98195-4350 (USA).
hart@math.washington.edu