ANNALES DE L'INSTITUT FOURIER

NOBUO TSUZUKI

Slope filtration of quasi-unipotent overconvergent F-isocrystals

Annales de l'institut Fourier, tome 48, nº 2 (1998), p. 379-412

<http://www.numdam.org/item?id=AIF_1998__48_2_379_0>

Ann. Inst. Fourier, Grenoble
48, 2 (1998), 379–412

SLOPE FILTRATION OF QUASI-UNIPOTENT OVERCONVERGENT F-ISOCRYSTALS

by Nobuo TSUZUKI

1. Introduction.

Let X be a smooth curve over a perfect field k with a positive characteristic p. Let $\overline{X}$ and Z be the smooth compactification of X and the complement of X in $\overline{X}$, respectively. In [Cr2] R. Crew defined the notion of quasi-unipotent overconvergent (F-)isocrystals over X around Z and proved some expected properties, finiteness and duality for rigid cohomologies and the global monodromy theorem, of quasi-unipotent overconvergent (F-)isocrystals. However, the problem that what kinds of overconvergent (F-)isocrystals are quasi-unipotent is still open.

In this paper we study local properties of quasi-unipotent F-isocrystals. Let K be a complete valuation field with an absolute value $|\ |$ and let $\mathcal{R}$ be the Robba ring over K (2.2). The Robba ring is a ring of analytic functions on some annulus $\eta < |x| < 1$. We define φ-∇-modules over $\mathcal{R}$ by a free $\mathcal{R}$-module with a connection and Frobenius structures (3.2.1). A φ-∇-module is quasi-unipotent if and only if it is a successive extension of copies of the unit object as differential modules (4.1.1) after a finite etale extension. For φ-∇-modules over $\mathcal{R}$, we define a slope filtration for Frobenius structures (5.1.1). If a φ-∇-module has a slope filtration, then it is unique (5.1.5). We establish

THEOREM 5.2.1. — *A φ-∇-module over $\mathcal{R}$ is quasi-unipotent if and only if it has a slope filtration for Frobenius structures.*

Key words: Quasi-unipotent F-isocrystals – φ-∇-modules – Slope filtration.
Math. classification: 12H25 – 14F30 – 14F40.

Let $\mathcal{M}$ be an overconvergent F-isocrystal on $\overline{X}$ around Z. $\mathcal{M}$ determines a φ-∇-module $i_s^*\mathcal{M}$ over a Robba ring for every closed point $s \in \overline{X}$ canonically. Then $\mathcal{M}$ is quasi-unipotent in the sense of Crew [Cr2, 10.1] if and only if $i_s^*\mathcal{M}$ is quasi-unipotent for any closed point $s \in X$ by (6.1.2) and (6.1.8).

The theorem above is useful since we have known finiteness of irregularities of φ-∇-modules with pure slopes [TN2]. So it implies finiteness of irregularities of quasi-unipotent φ-∇-modules in the sense of [TN2]. We will apply it to the global formula of Euler's number of quasi-unipotent overconvergent F-isocrystals in the future.

It is expected that any φ-∇-module over $\mathcal{R}$ is quasi-unipotent. If this holds, then any overconvergent F-isocrystal is quasi-unipotent (6.1). It is conjectured that an overconvergent F-isocrystal on a curve is quasi-unipotent if it has some geometric origin. (See [Cr2, 10.1].)

Now we explain the contents of this paper. In Section 2 we fix notations and prove some properties of the Robba ring $\mathcal{R}$. In Section 3 we define a φ-∇-module over $\mathcal{R}$. In Section 4 we define a quasi-unipotent φ-∇-module over $\mathcal{R}$ and prove that the category of quasi-unipotent φ-∇-modules over $\mathcal{R}$ is independent of the choice of Frobenius on $\mathcal{R}$. In Section 5 we define the slope filtration for Frobenius structures of φ-∇-modules over $\mathcal{R}$. We prove the existence of the slope filtration for quasi-unipotent φ-∇-modules over $\mathcal{R}$. In Section 6 we apply our local study to overconvergent F-isocrystals on a curve. We define a quasi-unipotent overconvergent F-isocrystal. The definition is a different form from that of Crew. Of course, the two definitions are equivalent to each other. We give some examples of quasi-unipotent overconvergent F-isocrystals.

The author would like to thank F. Baldassarri, B. Chiarellotto, L. Garnier, C. Huyghe and S. Matsuda for helpful conversation and advices. Many of ideas on this work were found during his visit to the Università di Padova. He also thanks members of Università di Padova.

2. The Robba ring $\mathcal{R}$.

2.1. Let p be a prime number. Let k (resp. K) be a perfect field with characteristic p (a complete discrete valuation field of mixed characteristics $(0, p)$ with residue class field k). Fix an algebraic closure K^{alg} of K and denote by k^{alg} the residue class field of K^{alg}. Denote by $|\ |$ (resp. v_p) the absolute value (resp. the additive valuation) of K^{alg} which is normalized by $|p| = p^{-1}$ (resp. $v_p(p) = 1$).

For any valuation field L, we denote by O_L (resp. k_L, resp. L^{unr}, resp. m_L) the valuation ring of L (resp. the residue class field of L, resp. the maximum unramified subfield in the fixed algebraic closure of L whose residue class field is separable over k_L, resp. the maximal ideal of O_L).

Let $F = k((x))$ be the field of fraction of the ring of formal power series with k-coefficients. Fix an algebraic closure F^{alg} of k such that the residue class field of F^{alg} is k^{alg} and denote by F^{sep} the separable closure of F in F^{alg}.

For a matrix (a_{ij}) and for an application f (resp. for a norm N), define

$$f((a_{ij})) = (f(a_{ij})) \quad (\text{resp. } N((a_{ij})) = \sup_{i,j} N(a_{ij})).$$

2.2. For a complete field Ω with a non-Archimedean absolute value $|\ | : \Omega \to \mathbf{R}_{\geqq 0}$ and for an indeterminate x, we define several Ω-algebras as follows:

$$\mathcal{R}_{x,\Omega} = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \;\middle|\; \begin{array}{l} a_n \in \Omega, \sup_{n<0} |a_n|\xi^n < \infty \text{ for some } 0 < \xi < 1, \\ |a_n|\eta^n \to 0 \ (n \to +\infty) \text{ for any } 0 < \eta < 1 \end{array} \right\}$$

$$\mathcal{E}_{x,\Omega} = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \;\middle|\; \begin{array}{l} a_n \in \Omega, \sup_{n} |a_n| < \infty, \\ |a_n| \to 0 \ (n \to -\infty) \end{array} \right\}$$

$$\mathcal{E}^{\dagger}_{x,\Omega} = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \in \mathcal{R}_{x,\Omega} \;\middle|\; \sup_{n} |a_n| < \infty \right\}$$

$$S_{x,\Omega} = \Omega \bigotimes_{O_\Omega} O_\Omega[[x]].$$

Each ring is functorial in Ω. We have natural injections of Ω-algebras:

$$\begin{array}{ccc} & & \mathcal{R}_{x,\Omega} \\ & \nearrow & \\ S_{x,\Omega} \to \mathcal{E}^{\dagger}_{x,\Omega} & & \\ & \searrow & \\ & & \mathcal{E}_{x,\Omega}. \end{array}$$

We call the ring $\mathcal{R}_{x,\Omega}$ Robba ring over Ω and an element of $\mathcal{R}_{x,\Omega}$ is regarded as a function on some annulus $\xi < |x| < 1$ for some $\xi < 1$. We use the notations $\mathcal{R}, \mathcal{E}, \mathcal{E}^{\dagger}$ and S_K instead of $\mathcal{R}_{x,K}, \mathcal{E}_{x,K}, \mathcal{E}^{\dagger}_{x,K}$ and $S_{x,K}$ respectively if there is no ambiguity.

Remark 2.2.1. Our $\mathcal{R}_{x,\Omega}$ coincides with $\mathcal{R}_0(1)$ in [Ro, 2].

For formal Laurent power series $a = \sum a_n x^n$, we define $|a|_G \in \mathbf{R}_{\geqq 0} \cup \{\infty\}$ by $\sup_n |a_n|$. The field $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) is a complete discrete valuation field (resp. a henselian discrete valuation field) under the absolute value $|\ |_G$. $|\ |_G$ is an extension of the absolute value $|\ |$ of K and the residue class field of $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) is F by the natural projection. (See [Cr1, 4.2] [Ma, 3.2].) For a finite separable extension E over F in F^{sep}, denote by $\mathcal{E}_E$ (resp. $\mathcal{E}_E^\dagger$) the unique finite unramified extension of $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) with residue class field E in the fixed algebraic closure of $\mathcal{E}$.

LEMMA 2.2.2 ([Ma, 3.2]). — *Under the notation as above, $\mathcal{E}_E$ (resp. $\mathcal{E}_E^\dagger$) is isomorphic to $\mathcal{E}_{y,K_E}$ (resp. $\mathcal{E}_{y,K_E}^\dagger$) for any lifting y of a uniformizer of E. Here K_E is the unique finite unramified extension of K with residue class field k_E. Moreover the unique extension of the absolute value $|\ |_G$ of $\mathcal{E}$ on $\mathcal{E}_E$ coincides with the map $\sum b_n y^n \mapsto \sup_n |b_n|$.*

Let E be a finite separable extension of F and choose a lifting y of a uniformizer of E in $\mathcal{E}_E^\dagger$. Define a K algebra $\mathcal{R}_E$ by

$$\mathcal{R}_E = \mathcal{R}_{y,K_E}.$$

Since $x = x(y) \in \mathcal{E}_E^\dagger = \mathcal{E}_{y,K_E}^\dagger$, $\mathcal{R}$ is naturally included in $\mathcal{R}_E$.

LEMMA 2.2.3. — (1) *$\mathcal{R}_E$ is independent of the choice of the lifting of the uniformizer of E up to canonical isomorphism.*

(2) *$\mathcal{R}_E$ is free over $\mathcal{R}$ of degree $[E : F]$. Moreover, $\mathcal{R}_E \cong \mathcal{E}_E^\dagger \bigotimes_{\mathcal{E}^\dagger} \mathcal{R}$ and $\mathcal{E}^\dagger = \mathcal{R} \bigcap \mathcal{E}_E^\dagger$.*

Assume that the extension E/F is Galois and denote by $\mathrm{Gal}(E/F)$ the Galois group. Since $\mathcal{E}^\dagger$ is a henselian discrete valuation field, the Galois group $\mathrm{Gal}(\mathcal{E}_E^\dagger/\mathcal{E}^\dagger)$ is canonically isomorphic to $\mathrm{Gal}(E/F)$. The action of $\mathrm{Gal}(E/F)$ on $\mathcal{E}_E^\dagger$ extends naturally on $\mathcal{R}_E$. By [Se1, X.1.Prop.3] and Lemma (2.2.3) we have

LEMMA 2.2.4. — *Under the notation as above,*

(1) $H^0(\mathrm{Gal}(E/F), \mathcal{E}_E^\dagger) = \mathcal{E}^\dagger$ *and* $H^1(\mathrm{Gal}(E/F), GL_r(\mathcal{E}_E^\dagger)) = \{1\}$;

(2) $H^0(\mathrm{Gal}(E/F), \mathcal{R}_E) = \mathcal{R}$.

2.3. For formal Laurent power series $\sum a_n x^n$ of indeterminate x, we define an additive map $\delta_x = x\frac{d}{dx}$ by

$$\delta_x(\sum a_n x^n) = \sum n a_n x^n.$$

Then δ_x is a K-derivation on $\mathcal{R}$ (resp. $\mathcal{E}$, resp. $\mathcal{E}^\dagger$, resp. S_K).

Let R be either $\mathcal{R}$, $\mathcal{E}$, $\mathcal{E}^\dagger$ or S_K. Define a free R-module ω_R of rank one by

$$\omega_R = R\frac{dx}{x}.$$

We define an additive map $d : R \to \omega_R$ by $d(a) = \delta_x(a)\frac{dx}{x}$ for $a \in R$. Then d is a K-derivation on R.

Let E be a finite separable extension of F and choose a lifting y of a uniformizer of E in $\mathcal{E}_E^\dagger$. Then the derivation δ_x extends uniquely on $\mathcal{R}_E$ and we also use the notation δ_x for this extension. We have the relation

$$\delta_x = \frac{x(y)}{\delta_y(x(y))}\delta_y,$$

where $x = x(y) \in \mathcal{E}_E^\dagger$ and δ_x commutes with the action of $\mathrm{Gal}(E/F)$ if E/F is Galois.

LEMMA 2.3.1. — *Under the notation as above, we have*

(1) $\ker(\delta_x : \mathcal{R}_E \to \mathcal{R}_E) = K_E$;

(2) $\operatorname{coker}(\delta_x : \mathcal{R}_E \to \mathcal{R}_E) \cong K_E\overline{\frac{x(y)}{\delta_y(x(y))}}$, *where* $\overline{\frac{x(y)}{\delta_y(x(y))}}$ *is the image of* $\frac{x(y)}{\delta_y(x(y))}$.

Proof. — The assertion easily follows from the fact that $\frac{x(y)}{\delta_y(x(y))}$ is a unit in $\mathcal{R}_E$. □

2.4. Fix a power $q = p^a$ $(a \geqslant 1)$ of p. Denote by K_0 the field of fraction of the Witt vector ring $W(k)$ and Frob is the usual lifting of the q-th power map on K_0. We say that an automorphism $\sigma : K \to K$ is a Frobenius on K if and only if σ is a continuous lifting of the q-th power map on the residue class field k. Since k is perfect, we have $\sigma|_{K_0} = \mathrm{Frob}^a$. Note that, if K has a Frobenius and if L is an unramified extension of K, then the Frobenius σ extends uniquely on L.

For a Frobenius σ on K, put $K^{\sigma=1} = \{u \in K \mid \sigma(u) = u\}$. One can easily see that $K^{\sigma=1}$ is finite over the field $\mathbf{Q}_p$ of p-adic integers.

LEMMA 2.4.1 ([Cr1, 1.8]). — *Let σ be a Frobenius on K. Then there is a finite unramified extension L of K such that $L \cong L^{\sigma=1} \bigotimes_{(L^{\sigma=1})_0} L_0$ and that the unique extension σ on L is $\mathrm{id}_{L^{\sigma=1}} \otimes \mathrm{Frob}^a$. Assume furthermore that the residue class field k is algebraically closed, then one can choose $L = K$.*

Proof. — First we prove the assertion in the case where k is algebraically closed. In this case there exists a uniformizer π of K which is algebraic over $\mathbf{Q}_p$. Then we have $K^{\sigma=1} \cong \mathbf{Q}_q(\pi)$ and $K \cong \mathbf{Q}_q(\pi) \bigotimes_{\mathbf{Q}_q} K_0$, where $\mathbf{Q}_q$ is the unique finite unramified extension of $\mathbf{Q}_p$ with residue class field $\mathbf{F}_q$ of q elements. Now we prove the assertion in the case where k is an arbitrary perfect field. Denote by $\widehat{K^{\mathrm{unr}}}$ the p-adic completion of K^{unr}. Then σ extends uniquely on $\widehat{K^{\mathrm{unr}}}$. Put $L = K(\widehat{K^{\mathrm{unr}}}^{\sigma=1})$ in $\widehat{K^{\mathrm{alg}}}$. Then L is finite over K and is included in $\widehat{K^{\mathrm{unr}}}$. Hence, L is a desired extension of K. □

From now on to the end of this paper we assume that K has a Frobenius σ.

We say a ring endomorphism σ on $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) is a Frobenius on $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) if and only if it is the Frobenius σ on K and $\sigma(a) \equiv a^q (\mathrm{mod}\ m_{\mathcal{E}})$ (resp. $\sigma(a) \equiv a^q (\mathrm{mod}\ m_{\mathcal{E}^\dagger})$) for $a \in O_{\mathcal{E}}$. (resp. $a \in O_{\mathcal{E}}^\dagger$). A Frobenius σ on $\mathcal{E}$ is that on $\mathcal{E}^\dagger$ if and only if $\sigma(x) \in \mathcal{E}^\dagger$. One can easily see that a Frobenius on $\mathcal{E}^\dagger$ extends naturally on $\mathcal{R}$ by $\sum a_n x^n \mapsto \sum \sigma(a_n x^n)$ (adding coefficients in each term of x^n). We call this extension a Frobenius on $\mathcal{R}$. We say a ring endomorphism σ on S_K is a Frobenius if and only if it is the Frobenius σ on $\mathcal{E}$ with $x^{-q}\sigma(x) \in S_K$.

For a Frobenius σ on $\mathcal{E}$, put

$$\mu = \mu(x, \sigma) = \frac{\delta_x(\sigma(x))}{\sigma(x)}.$$

Then $|\mu|_G < 1$. One can easily see that σ is a Frobenius on $\mathcal{E}^\dagger$ (resp. S_K) if and only if $\mu \in \mathcal{E}^\dagger$ (resp. $\mu \in S_K$).

Let R be either $\mathcal{R}$, $\mathcal{E}$, $\mathcal{E}^\dagger$ or S_K and let σ be a Frobenius on R.

LEMMA 2.4.2. — *If we regard R as an R-module through the Frobenius σ, then R is free of rank q.*

Define $\sigma : \omega_R \to \omega_R$ by $a\dfrac{dx}{x} \mapsto \mu\sigma(a)\dfrac{dx}{x}$. Then the diagram below

$$\begin{array}{ccc} R & \xrightarrow{d} & \omega_R \\ \sigma\downarrow & & \downarrow\sigma \\ R & \xrightarrow[d]{} & \omega_R \end{array}$$

is commutative. Equivalently, $\delta \circ \sigma = \mu\sigma \circ \delta$.

Let E be a finite separable extension of F and choose a lifting y of a uniformizer of E in $\mathcal{E}_E^\dagger$. Then the Frobenius σ on R extends uniquely on $\mathcal{R}_E$ and we also use the same notation σ for this extension. The Frobenius σ commutes with the derivation δ_x (resp. the action of $\mathrm{Gal}(E/F)$ if E/F is Galois).

2.5. Fix a Frobenius σ on $\mathcal{E}$ and put $\widetilde{\mathcal{E}} = K^{\sigma=1} \bigotimes_{(K^{\sigma=1})_0} W(F^{\mathrm{alg}})$. Then there is a unique homomorphism

$$i_\sigma : \mathcal{E} \to \widetilde{\mathcal{E}}$$

such that (i) $|u|_G = |i_\sigma(u)|$ for $u \in \mathcal{E}$, where $|\ |$ is the unique valuation on $\widetilde{\mathcal{E}}$ which is the extension of that on K, (ii) the map on residue class field induced by i_σ is the injection $F \subset F^{\mathrm{alg}}$ and (iii) $i_\sigma(\sigma(u)) = (\mathrm{id}_\Lambda \otimes \mathrm{Frob}^a)(i_\sigma(u))$. (See [TN1, 2.5.1].)

3. φ-∇-modules over $\mathcal{R}$.

Assume that the complete discrete valuation field K has a Frobenius σ from this section to the end of this paper.

3.1. Let R be either $\mathcal{R}$, $\mathcal{E}$, $\mathcal{E}^\dagger$ or S_K.

DEFINITION 3.1.1. — (1) A pair (M, ∇) is called a ∇-module over R if and only if it satisfies the conditions as follows:

(i) *M is a free R-module of finite rank.*

(ii) *$\nabla : M \to \omega_R \bigotimes_R M$ is a K-connection over R.*

(2) *A morphism of ∇-modules over R is an R-linear homomorphism which commutes with connections.*

(3) *We denote by $\underline{\mathbf{M}}_R^\nabla$ the category of ∇-modules over R.*

For a ∇-module M over R and for a basis $\{e_1, e_2, \cdots, e_r\}$ of M, define a matrix $C_{M,e} \in M_r(R)$ by

$$\nabla(e_1, e_2, \cdots, e_r) = \frac{dx}{x} \otimes (e_1, e_2, \cdots, e_r) C_{M,e}.$$

The category $\underline{\mathbf{M}}_R^\nabla$ is additive. We can define tensor products and duals for ∇-modules by usual methods and, then, (R, d) is the unit object of the category. We often use the notation M instead of (M, ∇) for simplicity.

Since an $\mathcal{R}$-module of finite presentation with a connection is free over $\mathcal{R}$ by [Cr2, 6.1], we have

Proposition 3.1.2. — *If $R = \mathcal{R}, \mathcal{E}$ or $\mathcal{E}^\dagger$, then the category $\underline{\mathbf{M}}_R^\nabla$ is an abelian category.*

Now fix a Frobenius σ on R.

Definition 3.1.3. — (1) *A pair (M, φ) is called a φ-module over R with respect to σ if and only if it satisfies the conditions as follows:*

(i) *M is a free R-module of finite rank;*

(ii) *$\varphi : M \to M$ is a σ-linear homomorphism such that the induced R-linear map*

$$\varphi_\sigma : \sigma^* M \to M \qquad a \otimes m \mapsto a\varphi(m)$$

is an isomorphism. Here $\sigma^* M$ is the scalar extension of M by σ. We call φ Frobenius.

(2) *A morphism of φ-modules over R is an R-linear homomorphism which commutes with Frobenius.*

(3) *We denote by $\underline{\mathbf{M\Phi}}_{R,\sigma}$ the category of φ-modules over R with respect to σ.*

For a φ-module M over R and for a basis $\{e_1, e_2, \cdots, e_r\}$ of M, define a matrix $A_{M,e} \in M_r(R)$ by

$$\varphi(e_1, e_2, \cdots, e_r) = (e_1, e_2, \cdots, e_r) A_{M,e}.$$

The category $\underline{\mathbf{M\Phi}}_{R,\sigma}$ is additive. We can define tensor products and duals for φ-modules by usual methods and, then, (R,σ) is the unit object. We often use the notation M instead of (M,φ) for simplicity.

PROPOSITION 3.1.4. — *If $R = \mathcal{E}, \mathcal{E}^\dagger$ or S_K, then the category $\underline{\mathbf{M\Phi}}_{R,\sigma}$ is an abelian category.*

Proof. — In the case where $R = \mathcal{E}$ or $\mathcal{E}^\dagger$ the assertion is trivial. Let $R = S_K$. We have only to check that, for a morphism $\eta : M \to N$ of $\underline{\mathbf{M\Phi}}_{S_K,\sigma}$, the cokernel of η is a free S_K-module, and then the rest is easy. Since S_K is a principal ideal domain, the torsion submodules of the cokernel of η is the form $\oplus_i S_K/(a_i)$ for some $a_i \in S_K$ with $|a_i|_G = 1$. Since σ is flat by (2.4.2), the induced S_K-linear map $\sigma^*(\oplus_i S_K/(a_i)) \to \oplus_i S_K/(a_i)$ is isomorphic. However, we have

$$\dim_K \sigma^*\Big(\bigoplus_i S_K/(a_i)\Big) = \dim_K \bigoplus_i S_K/(\sigma(a_i)) = q \dim_K \bigoplus_i S_K/(a_i).$$

Hence, $N/\eta(M)$ is a free S_K-module. □

We recall the notion of slopes for Frobenius structures. Denote by the same notation v_p the additive valuation of $\widetilde{\mathcal{E}}$ which is the unique extension of the valuation on K.

DEFINITION 3.1.5. — (1) *For an object (M,φ) of $\underline{\mathbf{M\Phi}}_{\mathcal{E},\sigma}$ (resp. $\underline{\mathbf{M\Phi}}_{\mathcal{E}^\dagger,\sigma}$), we define the slopes of (M,φ) by those of $(\widetilde{\mathcal{E}} \bigotimes_R M, \varphi)$ as φ-spaces on $\widetilde{\mathcal{E}}$ (resp. by those of $(\mathcal{E} \bigotimes_{\mathcal{E}^\dagger} M, \varphi)$) which are measured using the valuation $\frac{1}{a} v_p$. Here $p^a = q$. We denote by Newton(M) the Newton polygon of slopes of M.*

(2) *For an object (M,φ) of $\underline{\mathbf{M\Phi}}_{S_K,\sigma}$, we define the slopes of M for the Frobenius structure at the generic point by those of $\mathcal{E} \bigotimes_{S_K} M$ and the slopes of M for the Frobenius structure at the special point by those of $(\widehat{K^{\mathrm{unr}}} \bigotimes_S M, \overline{\varphi})$ as φ-spaces on $\widehat{K^{\mathrm{unr}}}$, where $S \to K$ (resp. $\overline{\varphi}$) is the natural reduction modulo x (resp. φ modulo xM). We denote by Newton$_\eta(M)$ (resp. Newton$_s(M)$) the Newton polygon of slopes of M at the generic point (resp. at the special point).*

Since $\mathcal{E}$ is p-adically complete, we have

Proposition 3.1.6. — *Let M be an object of $\underline{\mathbf{M\Phi}}_{\mathcal{E},\sigma}$. Then there is an increasing filtration $\{S_\gamma M\}_{\gamma\in\mathbf{Q}}$ of M such that each $S_\gamma M$ is an object of $\underline{\mathbf{M\Phi}}_{\mathcal{E},\sigma}$ and, for a sufficiently small positive rational number $\epsilon << 1$, $S_\gamma M/S_{\gamma-\epsilon}M$ is pure of slope γ.*

By [Ka1, 2.6.3] we have

Proposition 3.1.7. — *Let M be an object of $\underline{\mathbf{M\Phi}}_{S_K,\sigma}$. Assume that the Newton Polygon both at the generic point and at the special point coincide with each other, that is,* $\mathrm{Newton}_\eta(M) = \mathrm{Newton}_s(M)$. *Then there is an increasing filtration $\{S_\gamma M\}_{\gamma\in\mathbf{Q}}$ of M such that each $S_\gamma M$ is an object of $\underline{\mathbf{M\Phi}}_{S_K,\sigma}$ and, for a sufficiently small positive rational number $\epsilon << 1$, $S_\gamma M/S_{\gamma-\epsilon}M$ is pure of slope γ at both points.*

3.2. Now we define φ-∇-modules over R.

Definition 3.2.1. — (1) *A triple (M, φ, ∇) is called a φ-∇-module over R with respect to σ if and only if it satisfies the conditions as follows:*

(i) *(M, ∇) is a ∇-module over R;*

(ii) *(M, φ) is a φ-module over R with respect to σ;*

(iii) *the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\nabla} & \omega_R \bigotimes_R M \\ \varphi\downarrow & & \downarrow \sigma\otimes\varphi \\ M & \xrightarrow[\nabla]{} & \omega_R \bigotimes_R M \end{array}$$

is commutative.

(2) *A morphism of φ-modules over R is an R-linear homomorphism which commutes with connections and Frobenius.*

(3) *We denote by $\underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma}$ the category of φ-∇-modules over R with respect to σ.*

For a φ-∇-module M and for a basis $\{e_1, e_2, \cdots, e_r\}$, the condition (3.2.1)(1)(iii) is equivalent to the relation

$$\delta_x(A_{M,e}) + C_{M,e}A_{M,e} = \mu(x,\sigma)A_{M,e}\sigma(C_{M,e}). \tag{3.2.2}$$

We can define tensor products and duals for φ-∇-modules by usual methods and, then, (R, σ, d) is the unit object of the category. We often use the notation M instead of (M, φ, ∇) for simplicity.

By Proposition (3.1.2) and Proposition (3.1.4) we have

THEOREM 3.2.3. — *The category $\underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma}$ is an abelian category with tensor products and duals.*

By the extension of scalar there are natural functors

$$\begin{array}{ccccc} & & & & \mathcal{C}_{\mathcal{R}} \\ & & & \nearrow & \\ \mathcal{C}_{S_K} & \rightarrow & \mathcal{C}_{\mathcal{E}^\dagger} & & \\ & & & \searrow & \\ & & & & \mathcal{C}_{\mathcal{E}} \end{array}$$

of categories, where $\mathcal{C}$ is either $\underline{\mathbf{M}}^{\nabla}$, $\underline{\mathbf{M\Phi}}$ or $\underline{\mathbf{M\Phi}}^{\nabla}_{,\sigma}$. For an object M of $\mathcal{C}_{\mathcal{R}}$, a sub $\mathcal{E}^\dagger$-module (resp. a sub S_K-module, resp. a sub K-space) L is an $\mathcal{E}^\dagger$-lattice (an S_K-lattice, a K-lattice) if and only if $M \cong \mathcal{R} \bigotimes_{\mathcal{E}^\dagger} L$ (resp. $M \cong \mathcal{R} \bigotimes_{S_K} L$, resp. $M \cong \mathcal{R} \bigotimes_{K} L$) and $(L, \varphi|_L, \nabla|_L)$ belongs to $\mathcal{C}_{\mathcal{E}^\dagger}$ (resp. $(L, \varphi|_L, \nabla|_L)$ belongs to $\mathcal{C}_{S_K}$, resp. L is stable under φ and ∇).

3.3. In this subsection we define inverse images and direct images of φ-∇-modules.

Let $f : F \rightarrow E$ be a finite separable extension in F^{sep} and let R_F be either $\mathcal{R}_F(= \mathcal{R})$, $\mathcal{E}_F(= \mathcal{E})$ or $\mathcal{E}^\dagger_F(= \mathcal{E}^\dagger)$. Then the extension f determines a unique finite and flat extension R_E over R_F and denote by the same notation f the extension $R_F \rightarrow R_E$. Fix a Frobenius σ on R_F. Then σ extends on R_E and $\omega_{R_E} \cong R_E \bigotimes_R \omega_R$.

Let $\mathcal{C}$ be either the category $\underline{\mathbf{M}}^{\nabla}$, $\underline{\mathbf{M\Phi}}_{\sigma}$ or $\underline{\mathbf{M\Phi}}^{\nabla}_{\sigma}$. Define an inverse image functor

$$f^* : \mathcal{C}_{R_F} \rightarrow \mathcal{C}_{R_E}$$

as follows. For an object M of $\mathcal{C}_{R_F}$, put $f^*M = (M_E, \varphi_E, \nabla_E)$ to be

$$\begin{aligned} M_E &= R_E \bigotimes_R M \\ \varphi_E &= \sigma \otimes \varphi \\ \nabla_E &= d \otimes \mathrm{id}_M + \mathrm{id}_{R_E} \otimes \nabla. \end{aligned}$$

One can easily check that f^*M is an object of $\mathcal{C}_{R_E}$. By the definition f^* is faithful and exact.

Define a direct image functor

$$f_* : \mathcal{C}_{R_E} \to \mathcal{C}_{R_F}$$

as follows. For an object M of $\mathcal{C}_{R_E}$, put $f_*M = (M_F, \varphi_F, \nabla_F)$ to be

$$\begin{aligned} &M_F = M \text{ (we regard it as an } R\text{-module)} \\ &\varphi_F = \varphi \\ &\nabla_F = \nabla : M_F \to \omega_{R_E} \bigotimes_{R_E} M \cong \omega_R \bigotimes_R M_F. \end{aligned}$$

LEMMA 3.3.1. — *For an object M of $\mathcal{C}_{R_E}$, f_*M belongs to $\mathcal{C}_{R_F}$.*

Proof. — It is sufficient to check that the natural map from $\sigma^*(M_F)$ (a pull back by $\sigma : R_F \to R_F$) to σ^*M (a pull back by $\sigma : R_E \to R_E$) is bijective. Since M is free over $\mathcal{R}_E$, it is enough to prove that the natural map $\sigma^*((\mathcal{R}_E)_F) \to \sigma^*\mathcal{R}_E$ is bijective. The following Lemma (3.3.2) implies the assertion by (2.2.3).

LEMMA 3.3.2. — *Under the notation as above, the natural map $\sigma^*((\mathcal{E}_E^\dagger)_F) \to \sigma^*\mathcal{E}_E^\dagger$ is bijective.*

Proof. — Denote by σ_q the q-th power map. Consider the perfections both of F and E, and dimensions over F, then $\sigma_q^*(E_F) \to \sigma_q^*(E)$ is injective, hence bijective. The assertion holds by Nakayama's Lemma. □

We show some properties of inverse images and direct images.

LEMMA 3.3.3. — *Let $f : F \to E_1$ and $g : E_1 \to E_2$ be finite separable extensions over F in F^{sep}. Then, we have $(gf)^* = g^*f^*$ and $(gf)_* = f_*g_*$.*

PROPOSITION 3.3.4. — (1) *The functor f^* (resp. f_*) commutes with natural functors $\mathcal{C}_{\mathcal{E}^\dagger} \to \mathcal{C}_{\mathcal{R}}$ and $\mathcal{C}_{\mathcal{E}^\dagger} \to \mathcal{C}_{\mathcal{E}}$.*

(2) *The functor f^* preserves tensor products and duals.*

(3) *f_* is a right adjoint of f^* and f^* is a left adjoint of f_*.*

We study the behavior of Newton polygons of φ-modules under an inverse image functor (resp. a direct image functor). By the definition of Newton polygon we have

PROPOSITION 3.3.5. — *Let R_F be either $\mathcal{E}_F$ or $\mathcal{E}_F^\dagger$. The Newton polygon of φ-modules is preserved by the inverse image functor f^*. In other words, we have*

$$\mathrm{Newton}(f^*M) = \mathrm{Newton}(M)$$

for any object M of $\underline{\mathbf{M}\Phi}_{R_F}$.

PROPOSITION 3.3.6. — *Let R_F be either $\mathcal{E}_F$ or $\mathcal{E}_F^\dagger$. For an object M of $\underline{\mathbf{M}\Phi}_{R_E,\sigma}$, the Newton polygon* $\mathrm{Newton}(f_*M)$ *of f_*M is $[E:F]$ times* $\mathrm{Newton}(M)$. *In other words, the rank of the slope γ-part of f_*M is $[E:F]$ times the rank of the slope γ-part of M.*

Proof. — One may assume that the extension E over F is Galois by (3.3.5). If we denote by M_τ a scalar extension of M by an $\mathcal{R}_F$-embedding $\tau : R_E \to \widetilde{\mathcal{E}}$, then we have

$$\widetilde{\mathcal{E}} \bigotimes_{R_F} f_*M \cong \bigoplus_{\tau\in\mathrm{Hom}\mathcal{R}_F(\mathcal{R}_E,\widetilde{\mathcal{E}})} M_\tau$$

as φ-modules over $\widetilde{\mathcal{E}}$. Since the action of Galois commutes with Frobenius, we obtain the assertion. □

3.4. Let R be either $\mathcal{E}$, $\mathcal{E}^\dagger$ or S_K. Let M be an object of $\underline{\mathbf{M}}_R^\nabla$ and $\{e_1, e_2, \cdots, e_r\}$ a basis of M. For an element $m = a_1e_1 + \cdots + a_re_r$, define

$$||m||_{M,e} = \max_i |a_i|_G.$$

Then $||\ ||_{M,e}$ is a norm on M which is compatible with the norm $|\ |_G$ of R. The topology which is determined by the norm $||\ ||_{M,e}$ is independent of the choice of the basis of M.

Define a K-linear map $\nabla^{[n]} : M \to M$ by

$$\nabla^{[0]} = \mathrm{id}_M \quad \text{and} \quad \nabla^{[n+1]} = \Big(\nabla(x\frac{d}{dx}) - n\Big)\nabla^{[n]}.$$

for any non-negative integer n. Here the map $\nabla\Big(x\dfrac{d}{dx}\Big)$ is defined by $\nabla(m) = \dfrac{dx}{x} \otimes \nabla\Big(x\dfrac{d}{dx}\Big)(m)$ for $m \in M$. By Leibniz's rules we have

LEMMA 3.4.1. — $\nabla^{[n]}(am) = \sum_{i+j=n} \dfrac{n!}{i!j!}\delta^{[i]}(a)\nabla^{[j]}(m)$ *for $a \in R$, $m \in M$.*

Let M be an object of $\underline{\mathbf{M}}_R^\nabla$. Consider the conditions (C) and (OC) as follows:

$$\text{(C)} \qquad \left\| \frac{1}{n!}\nabla^{[n]}(m)\right\|_{M,e}\eta^n \to 0 \quad (n\to\infty)$$

for any $m \in M$ and any number $0<\eta<1$;

$$\text{(OC)} \qquad \sum_{n=0}^{\infty}\frac{w^n}{n!}\nabla^{[n]}(m) \text{ converges in } M$$

for any $m \in M$ and for any $w \in R$ with $|w|_G < 1$. If $R = \mathcal{E}$ and S_K, the condition (C) implies (OC) since R is complete in the p-adic topology. In the case of $\mathcal{E}^\dagger$, however, the condition (OC) is delicate since $\mathcal{E}^\dagger$ is not complete.

PROPOSITION 3.4.2. — *Any object M of $\underline{\mathbf{M\Phi}}_{R,\sigma}^\nabla$ satisfies the condition (C).*

Proof. — Fix a positive integer k with $\eta < p^{-1/(p^k(p-1))}$. By (3.4.1) we have only to prove the condition (C) for one basis of M. Choose a basis $\{e_1, e_2, \cdots, e_r\}$ of M such that $|C|_G \leqslant p^{-(p^k-1)/(p-1)}$, where we denote $C = C_{M,e}$. We can choose such a basis after changing a basis by $(e_1,e_2,\cdots,e_r) \mapsto (e_1,e_2,\cdots,e_r)A\sigma(A)\cdots\sigma^n(A)$ for a sufficiently large n, where $A = A_{M,e}$. Define matrixes $C^{[n]} \in M_r(R)$ by $\nabla^{[n]}(e_1,e_2,\cdots,e_r) = (e_1,e_2,\cdots,e_r)C^{[n]}$. Since $|C^{[n+1]} - (\delta_x(C^{[n]}) - nC^{[n]})|_G \leqslant |C^{[n]}|_G p^{-(p^k-1)/(p-1)}$, one can easily check that $|C^{[n]}|_G \leqslant p^{-(i+1)(p^k-1)/(p-1)}$ for $n = ip^k + j$ $(i \geqslant 0, 0 < j \leqslant p^k)$. Note that $v_p(n!) < n/(p-1)$ for any positive integer n. Since

$$\begin{aligned}&(i+1)(p^k-1)/(p-1) + n/(p^k(p-1)) - v_p(n!)\\ &\quad = ((p^k-1)/(p-1) - v_p(j!)) + (i/(p-1) - v_p(i!)) + j/(p^k(p-1)) > 0,\end{aligned}$$

we have $|C^{[n+1]}/n!|_G\eta^n \to 0$ if $n\to\infty$. □

COROLLARY 3.4.3. — *The connection of objects in* $\underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma}$ *is topologically nilpotent.*

Define a map $\alpha_N : \mathcal{E} \to \mathbf{R}$ by

$$\alpha_N(\sum a_n x^n) = \sup_{n \leqslant N} |a_n|$$

for any integer N. Note that (i) $a \in \mathcal{E}^\dagger$ if and only if $\alpha_N(a) \leqslant c\xi^{-N}$ for any integer N for some $c > 0$ and $0 < \xi < 1$ and (ii) if $\alpha_N(a) \leqslant c_a\xi^{-N}$ and $\alpha_N(b) \leqslant c_b\xi^{-N}$, then $\alpha_N(ab) \leqslant c_a c_b \xi^{-N}$

PROPOSITION 3.4.4. — *Any object* M *of* $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}^\dagger,\sigma}$ *satisfies the condition (OC).*

Proof. — Keep the notation as in the proof of (3.4.2). By (3.4.1) we have only to prove the condition (OC) for one basis of M. Choose a positive integer k, a basis $\{e_1, e_2, \cdots, e_r\}$ of M and a real number $0 < \xi < 1$ such that $\alpha_N(w) < p^{-1/(p^k(p-1))} \min\{\xi^{-N}, 1\}$ and $\alpha_N(C) \leqslant p^{-(p^k-1)/(p-1)} \min\{\xi^{-N}, 1\}$ for any integer N. Then one can easily check that $\alpha_N(C^{[n]}) \leqslant p^{-(i+1)(p^k-1)/(p-1)} \min\{\xi^{-N}, 1\}$ for $n = ip^k + j$ $(i \geqslant 0, 0 < j \leqslant p^k)$. By the calculation of valuations as in the proof of (3.4.2) we have $\alpha_N(C^{[n]}w^n/n!) \leqslant \min\{\xi^{-N}, 1\}$. Since $\sum_{n=0}^{\infty} C^{[n]}w^n/n!$ is convergent in $M_r(\mathcal{E})$ by (3.4.2), $\sum_{n=0}^{\infty} C^{[n]}w^n/n!$ is convergent in $M_r(\mathcal{E}^\dagger)$. □

Let σ_1 and σ_2 be Frobenius on R. For an object M of $\underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma_2}$, define an R-linear homomorphism

$$\epsilon_{\sigma_1,\sigma_2} : \sigma_1^* M \to \sigma_2^* M$$

by

$$\epsilon_{\sigma_1,\sigma_2}(a \otimes m) = a \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\sigma_1(x)}{\sigma_2(x)} - 1 \right)^n \otimes \nabla^{[n]}(m).$$

Since one knows the identity

$$\sigma_1(a) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\sigma_1(x)}{\sigma_2(x)} - 1 \right)^n \sigma_2(\delta^{[n]}(a))$$

for any $a \in \mathcal{E}$, the map $\epsilon_{\sigma_1,\sigma_2}$ is well-defined and continuous by (3.4.2) and (resp. (3.4.3)). By easy calculations we have

LEMMA 3.4.5. — *Let σ_1, σ_2 and σ_3 be Frobenius on R. Then*

(i) $\epsilon_{\sigma_1,\sigma_1} = \mathrm{id}$;

(ii) $\epsilon_{\sigma_1,\sigma_3} = \epsilon_{\sigma_1,\sigma_2}\epsilon_{\sigma_2,\sigma_3}$.

Define a functor

$$\tilde{\epsilon}_{\sigma_1,\sigma_2} : \underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma_2} \to \underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma_1}$$

by

$$(M, \varphi, \nabla) \mapsto (M, \varphi_{\sigma_2} \circ \epsilon_{\sigma_1,\sigma_2}|_{1\otimes M}, \nabla).$$

LEMMA 3.4.6. — *Under the notation as above, the triple $(M, \varphi_{\sigma_2} \circ \epsilon_{\sigma_1,\sigma_2}|_{1\otimes M}, \nabla)$ is an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma_1}$.*

Proof. — Put $\varphi_1 = \varphi_{\sigma_2} \circ \epsilon_{\sigma_1,\sigma_2}|_{1\otimes M}$. By (3.4.5) $\epsilon_{\sigma_1,\sigma_2}$ is isomorphic, hence $(\varphi_1)_{\sigma_1}$ is isomorphic. An easy calculation implies the commutative of φ_1 and ∇. □

LEMMA 3.4.7. — *Let σ_1, σ_2 and σ_3 be Frobenius on R. Then*

(i) $\tilde{\epsilon}_{\sigma_1,\sigma_1} = \mathrm{id}$;

(ii) $\tilde{\epsilon}_{\sigma_1,\sigma_3} = \tilde{\epsilon}_{\sigma_1,\sigma_2}\tilde{\epsilon}_{\sigma_2,\sigma_3}$.

LEMMA 3.4.8. — (1) *The functor $\tilde{\epsilon}_{\sigma_1,\sigma_2}$ commutes with tensor products and duals.*

(2) *For a finite separable extension $f : F \to E$ in F^{sep}, the functor $\tilde{\epsilon}_{\sigma_1,\sigma_2}$ commutes with f^* and f_*.*

PROPOSITION 3.4.9. — *Let σ_1 and σ_2 be Frobenius on R and let M be an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma_2}$. Then the slopes of M for Frobenius structures coincide with those of $\tilde{\epsilon}_{\sigma_1,\sigma_2}(M)$. In other words,*

$$\begin{array}{rl} \mathrm{Newton}(\tilde{\epsilon}_{\sigma_1,\sigma_2}(M)) & = \mathit{Newton}(M) \\ (\mathit{resp.}\ \mathrm{Newton}_\eta(\tilde{\epsilon}_{\sigma_1,\sigma_2}(M)) & = \mathrm{Newton}_\eta(M) \\ \mathrm{Newton}_s(\tilde{\epsilon}_{\sigma_1,\sigma_2}(M)) & = \mathrm{Newton}_s(M)) \end{array}$$

if $R = \mathcal{E}$ or $\mathcal{E}^\dagger$ (resp. if $R = S_K$).

Proof. — We have only to prove the assertion in the case where $R = \mathcal{E}$ and M is pure of slopes 0 by (3.1.6). We can choose a suitable basis of M

with $A_{M,e} \in GL_r(O_{\mathcal{E}})$ and $\epsilon_{\sigma_1,\sigma_2}(e_i) \equiv e_i \pmod{m_{\mathcal{E}}}$. Therefore, we have the assertion. □

Now we have obtained

THEOREM 3.4.10. — *The category $\underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma}$ is independent of the choice of Frobenius up to canonical equivalence.*

4. Quasi-unipotent φ-∇-modules.

4.1. Fix a Frobenius φ on $\mathcal{R}$. We define quasi-unipotent φ-∇-modules.

DEFINITION 4.1.1. — (1) A ∇-module M (resp. a φ-∇-module M) over $\mathcal{R}$ is unipotent if and only if M is a successive extension of the unit object $(\mathcal{R}, d)$ (resp. (M, ∇) is a unipotent ∇-module).

(2) A ∇-module M (resp. a φ-∇-module M) over $\mathcal{R}$ is quasi-unipotent if and only if there exists a finite separable extension $f : F \to E$ such that the inverse image f^*M is unipotent.

(3) We denote by $\underline{\mathbf{M}}^{\nabla,qu}_{\mathcal{R}}$ (resp. $\underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma}$) the full subcategory of $\underline{\mathbf{M}}^{\nabla}_{\mathcal{R}}$ (resp. $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$) whose objects consist of quasi-unipotent ∇-modules (resp. φ-∇-modules).

By the standard arguments we have

PROPOSITION 4.1.2. — (1) *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence in $\underline{\mathbf{M}}^{\nabla}_{\mathcal{R}}$ (resp. $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$). M_2 is quasi-unipotent if and only if both M_1 and M_3 are quasi-unipotent.

(2) *The category $\underline{\mathbf{M}}^{\nabla,qu}_{\mathcal{R}}$ (resp. $\underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma}$) is an abelian subcategory of $\underline{\mathbf{M}}^{\nabla}_{\mathcal{R}}$ (resp. $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$) with tensor products and duals.*

PROPOSITION 4.1.3. — *Let $f : F \to E$ be a finite separable extension in F^{sep}.*

(1) *Let M be an object of $\underline{\mathbf{M}}^{\nabla}_{\mathcal{R}}$ (resp. $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$). M is quasi-unipotent if and only if f^*M is quasi-unipotent.*

(2) *Let M be an object of $\underline{\mathbf{M}}^{\nabla}_{\mathcal{R}_E}$ (resp. $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R}_E,\sigma}$). M is quasi-unipotent if and only if f_*M is quasi-unipotent.*

Proof. — The assertion on inverse images is easy. In the case of direct images we may assume that the extension E is Galois over F by (1) and (4.1.2). For $\tau \in \mathrm{Gal}(E/F)$, denote by M_τ the ∇-module (resp. φ-∇-module) whose $\mathcal{R}_E$-action is defined by $(a, m) \mapsto \tau(a)m$ for $a \in \mathcal{R}_E$ and $m \in M$. Then $f^* f_* M \cong \bigoplus_{\tau \in \mathrm{Gal}(E/F)} M_\tau$. The assertion (2) easily follows from the isomorphism. □

Example 4.1.4. — (1) Any φ-∇-module M over $\mathcal{R}$ of rank one is quasi-unipotent. Indeed, if we fix a base e of M, then $A_{M,e} \in \mathcal{R}^\times = (\mathcal{E}^\dagger)^\times$. By the relation (3.2.2) we have $C_{M,e} \in \mathcal{E}^\dagger$. Hence, M has an $\mathcal{E}^\dagger$-lattice and it is quasi-unipotent by [Cr1, 4.11] (or (2) below).

(2) Any φ-∇-module over $\mathcal{R}$ which has an etale $\mathcal{E}^\dagger$-lattice is quasi-unipotent [TN1, 4.2.6]. ("Etale" means that all slopes of Frobenius are 0.)

4.2. We show some properties of unipotent φ-∇-modules.

PROPOSITION 4.2.1. — (1) *An object in* $\underline{\mathbf{M\Phi}}^{\nabla, qu}_{\mathcal{R},\sigma}$ *has an* $\mathcal{E}^\dagger$*-lattice.*

(2) *Assume that* σ *is Frobenius on* S_K. *An object of* $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$ *is unipotent if and only if it has an* S_K*-lattice.*

Remark 4.2.2. — The $\mathcal{E}^\dagger$-lattice (resp. the S_K-lattice) is not unique in Proposition (4.2.1).

Proposition (4.2.1)(1) (resp. (2)) follows from Lemma (4.2.5) (resp. Lemmas (4.2.6) and (4.2.7)) below.

Put $u \in (\mathcal{E}^\dagger)^\times$ to be $\sigma(x) = x^q u$ for the Frobenius σ. Then $|u-1|_G < 1$ and one can define $\log(u)$ in $\mathcal{E}^\dagger$. If σ is a Frobenius on S_K, then $\log(u)$ belongs to S_K. Note that $\mu = \mu(x, \sigma) = \dfrac{\delta_x(\sigma(x))}{\sigma(x)} = q + \dfrac{\delta_x(u)}{u}$ and $\delta_x(\log(u)) = \dfrac{\delta_x(u)}{u}$.

LEMMA 4.2.3. — *Let* $C_1 = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & 0 \end{pmatrix}$ *(resp.* $C_2 = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & 0 \end{pmatrix}$*) be a matrix of degree* r_1 *(resp.* r_2*). A matrix* $Q \in M_{r_1,r_2}(\mathcal{R})$ *(resp.* $Q \in M_{r_1,r_2}(K[[x]])$*) satisfies the relation*

$$\delta_x(Q) + C_1 Q = \mu Q C_2$$

if and only if

$$Q = \begin{cases} \begin{pmatrix} 0 & \cdots & 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{r_1} \\ & & & & \ddots & \ddots & \vdots \\ & & & & & \ddots & q^{r_1-2}\alpha_2 \\ 0 & & & & & & q^{r_1-1}\alpha_1 \end{pmatrix} & \text{if } r_1 \leqslant r_2 \\ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{r_2} \\ 0 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & q^{r_2-2}\alpha_2 \\ & & \ddots & q^{r_2-1}\alpha_1 \\ 0 & & & 0 \end{pmatrix} & \text{if } r_1 \geqslant r_2 \end{cases}$$

with $\alpha_1 = \beta_1, \alpha_2 = \beta_1 \log(u) + \beta_2, \cdots, \alpha_r = \dfrac{\beta_1}{(r-1)!}\log^{r-1}(u) + \dfrac{\beta_2}{(r-2)!}\log^{r-2}(u) + \cdots + \beta_r$ *for some* $\beta_i \in K$.

Proof. — We use Lemma (2.3.1) to show the assertion. Assume that $Q = (q_{i,j})$ is a solution of the differential equation above.

First we prove that $q_{r_1,j} = 0$ $(1 \leqslant j < r_2)$ and q_{r_1,r_2} is contained in K. Since $\delta_x(q_{r_1,1}) = 0$, $q_{r_1,1}$ is contained in K. Then the identity $\delta_x(q_{r_1,2}) = \mu q_{r_1,1}$ implies that $q_{r_1,1} = 0$ and $q_{r_1,2}$ is contained in K. Repeating these, we proved the assertion.

Secondly we prove that $q_{i,1} = 0$ $(2 \leqslant i)$ and $q_{1,1}$ is contained in K. Assume that $q_{i+1,1} = \cdots = q_{r_2,1} = 0$. Since $\delta_x(q_{i,1}) + q_{i+1,1} = 0$, $q_{i,1}$ is contained in K. So the assertion follows from $\delta_x(q_{i-1,1}) + q_{i,1} = 0$.

Thirdly we prove that, if $q_{i,n+i}$ is a linear combination of $1, \log(u)$, $\log^2(u), \cdots$ over K and if $q^{-i+1}q_{i,n+i}$ does not depend on i when n is fixed, then $q_{i,n+1+i}$ is a linear combination of $1, \log(u), \log^2(u), \cdots$ over K and $q^{-i+1}q_{i,n+1+i}$ is independent on i. The former assertion holds by the equation $\delta_x(q_{i,j}) + q_{i+1,j} = \mu q_{i,j-1}$ $(i < r_1, j > 1)$ and $\mu = q + \dfrac{\delta_x(u)}{u}$ and by two assertions above. Moreover $q^{-i+1}q_{i,n+1+i}$ does not depend on i up to constant terms. (When $q_{i,1}$ (resp. $q_{r_1,j}$) appears, $q^{-i+1}q_{i,n+1+i} = 0$ and $q^{i-1}q_{i,n+1+i}$ does not depend on i up to constant terms.) Since

$$\begin{aligned} \delta_x(q_{i,n+1+(i+1)}) &= \mu q_{i,n+1+i} - q_{i+1,n+1+(i+1)} \\ &= \text{constant term } + \frac{\delta_x(u)}{u} q_{i,n+1+i}, \end{aligned}$$

the constant term must vanish. Hence, the later assertion also holds.

Finally we have got the relation $\delta_x(q_{i,r_2}) = \mu q_{i,r_2-1} - q_{i+1,r_2} = \dfrac{\delta_x(u)}{u} q_{i,r_2-1}$. Therefore, Q has a form as in the assertion. The converse can be easily checked. □

Let $f : F \to E$ be a finite separable extension in F^{sep}. Denote by x (resp. y) a lift of uniformizer *of* F (resp. E) in $\mathcal{E}^\dagger = \mathcal{E}_F^\dagger$ (resp. $\mathcal{E}_E^\dagger$)). Using similar arguments as in Lemma (4.2.3) and by Lemma (2.3.1) we obtain

LEMMA 4.2.4. — *Under the notation as above, let* $C_1 = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & 0 \end{pmatrix}$ *(resp.* $C_2 = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & 0 \end{pmatrix}$*) be a matrix of degree* r_1 *(resp.* r_2*). A matrix* $Q \in M_{r_1,r_2}(\mathcal{R}_E)$ *satisfies the differential equation*

$$\delta_x(Q) + C_1 Q = Q C_2$$

for the derivation $\delta_x = x\dfrac{d}{dx}$ *if and only if*

$$Q = \begin{cases} \begin{pmatrix} 0 & \cdots & 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{r_1} \\ & & & & \ddots & \ddots & \vdots \\ & & & & & \ddots & \alpha_2 \\ \mathbf{0} & & & & & & \alpha_1 \end{pmatrix} & \text{if } r_1 \leqslant r_2 \\ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{r_2} \\ 0 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \alpha_2 \\ & & \ddots & \alpha_1 \\ \mathbf{0} & & & 0 \end{pmatrix} & \text{if } r_1 \geqq r_2 \end{cases}$$

for some $\alpha_i \in K_E$.

COROLLARY 4.2.5. — (1) *Under the notation as above, assume furthermore that* M *is a unipotent* ∇*-module over* $\mathcal{R}_E$*. Then there is a basis* $\{e_1, e_2, \cdots, e_r\}$ *of* M *such that, if we define a matrix* $C_{M,e,x} \in M_r(\mathcal{R}_E)$ *by*

$$\nabla(e_1, e_2, \cdots, e_r) = \frac{dx}{x} \otimes (e_1, e_2, \cdots, e_r) C_{M,e,x},$$

$$C_{M,e,x} = \begin{pmatrix} C_1 & & & \mathbf{0} \\ & C_2 & & \\ & & \ddots & \\ \mathbf{0} & & & C_s \end{pmatrix} \quad \textit{with} \quad C_i = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & 0 \end{pmatrix}.$$

Moreover, if M has a σ-linear homomorphism $\varphi : M \to M$ which is compatible with the connection and if L_E is an $\mathcal{E}_E^\dagger$-subspace which is generated by $\{e_1, e_2, \cdots, e_r\}$, then L_E is stable under φ.

(2) *Let M be an object of $M_{\mathcal{R}}^{\nabla, qu}$ and let $f : F \to E$ be a finite separable extension in F^{sep} such that f^*M is unipotent. If $\{e_1, e_2, \cdots, e_r\}$ is a basis of f^*M as in (1) and if we denote by L_E the $\mathcal{E}_E^\dagger$-subspace which is generated by $\{e_1, e_2, \cdots, e_r\}$, then L_E is stable under the action of* $\mathrm{Gal}(E/F)$.

Proof. — (1) We use induction on r. Let $\{e_1, e_2, \cdots, e_{r-1}, e'\}$ be a basis of M such that $C_{M,e',x} = \begin{pmatrix} C_{11} & C_{12} \\ 0 & 0 \end{pmatrix}$ with C_{11} as in the assertion and some $C_{12} \in \mathcal{R}^{r-1}$. Using (2.3.1), one can get a matrix of type $Q = \begin{pmatrix} 1 & Q_{12} \\ 0 & 1 \end{pmatrix}$ with $Q_{12} \in \mathcal{R}^{r-1}$ such that $(e_1, e_2, \cdots, e_{r-1}, e')Q$ is the desired basis. Let $\{e_1, e_2, \cdots, e_r\}$ be a basis as in the former assertion. Then we have $\delta_x(A_{M,e}) + C_{M,e,x}A_{M,e} = \mu(x,\sigma)A_{M,e}C_{M,e,x}$ by the commutativity of Frobenius and connection. By (4.2.3) there is a matrix $A_x \in GL_r(\mathcal{E}^\dagger)$ which satisfies the relation $\delta_x(A_x) + C_{M,e,x}A_x = \mu(x,\sigma)A_xC_{M,e,x}$. Hence we have

$$\delta_x(A_{M,e}{A_x}^{-1}) + C_{M,e,x}A_{M,e}{A_x}^{-1} = A_{M,e}{A_x}^{-1}C_{M,e,x}$$

and $A_{M,e}{A_x}^{-1} \in GL_r(K_E)$ by (4.2.4). The assertion (2) easily follows from the commutativity of the Galois action and the connection and by (4.2.4). □

Let M be an object in $\underline{\mathbf{M}}_{S_K}^{\nabla}$. Put $\overline{M} = M/xM$ (resp. $N_M = \overline{\nabla\left(x\frac{d}{dx}\right)}$ to be the induced K-linear map). By the relation (3.2.2) we have

LEMMA 4.2.6. — *For any object M of $\underline{\mathbf{M\Phi}}_{S_K,\sigma}^{\nabla}$, the K-linear map N_M is nilpotent.*

LEMMA 4.2.7. — *Let M be an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{S_K,\sigma}$ and let $\{e_1, e_2, \cdots, e_r\}$ be a basis of M. Put C_0 to be the representation matrix of the K-linear map N_M for the basis $\{\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_r\}$. Then there exists a solution $Q \in 1_r + xM_r(K[[x]])$ of the system of linear differential equations*

$$\delta_x(Q) + C_{M,e}Q = QC_0$$

such that Q belongs to $GL_r(\mathcal{R})$.

Proof. — Since all proper values of C_0 are 0 (4.2.6), one can uniquely solve the system of differential equation above in $M_r(K[[x]])$ with $Q \pmod{xK[[x]]} = 1_r$. Put $A_0 = Q^{-1}A\sigma(Q)$. Then the pair (A_0, C_0) satisfies the relation (3.2.2.). Hence, A_0 is contained in $GL_r(S_K)$ by (4.2.3). If we denote by γ the radius of convergence of Q, then $0 < \gamma \leqslant 1$ and the radius of convergence of $\sigma(Q)$ is γ^q. By the relation $QA_0 = A\sigma(Q)$ we have

$$\min\{\gamma, 1\} = \min\{\gamma^q, 1\}.$$

Hence, $\gamma = 1$ and Q is contained in $M_r(\mathcal{R})$. Consider the dual object $M^\vee$ of M and the dual basis $\{e^\vee{}_1, e^\vee{}_2, \cdots, e^\vee{}_r\}$. Then there is a matrix $Q^\vee \in M_r(K[[x]]) \bigcap M_r(\mathcal{R})$ with $Q^\vee \pmod{xK[[x]]} = 1_r$ and $\delta_x(Q^\vee) - {}^tC_{M,e}Q^\vee = -Q^{\vee t}C_0$. So we have

$$\delta_x(Q^\vee Q) + C_0 Q^\vee Q = Q^\vee Q C_0.$$

Therefore Q is invertible by (4.2.4). □

4.3. Let K' be an extension of K which is complete under the extension of the valuation of K and put $\mathcal{R}_{K'} = \mathcal{R}_{K',x}$ to be an extension of $\mathcal{R}$. Denote by $g^*_{K'/K} : \underline{\mathbf{M}}^{\nabla}_{\mathcal{R}} \to \underline{\mathbf{M}}^{\nabla}_{\mathcal{R}_{K'}}$ the natural functor which is defined by the scalar extension. If the Frobenius σ on K extends on K', then the Frobenius σ on $\mathcal{R}$ extends on $\mathcal{R}_{K'}$. (The extension of the Frobenius on $\mathcal{R}_{K'}$ is uniquely determined by the extension of the Frobenius on K'.) In this case there is a natural functor $g^*_{K'/K} : \underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R}} \to \underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R}_{K'}}$.

PROPOSITION 4.3.1. — *Under the notation as above, let σ be a Frobenius on $\mathcal{R}$ and let M be an object of $M^{\nabla,qu}_{\mathcal{R}}$. Then there exists a finite extension K' over K and a positive integer d such that the Frobenius σ on K extends on K' and that $g^*_{K'/K}M$ has a Frobenius structure with respect to σ^d. In other words, there exists a σ^d-linear homomorphism $\varphi_d : M \to M$ such that the triple $(\mathcal{R}_{K'} \bigotimes_{\mathcal{R}} M, \varphi_d, \nabla)$ is an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R}_{K'},\sigma^d}$.*

Proof. — Let $f : F \to E$ be a finite Galois extension in F^{sep} such that f^*M is unipotent. Let $\{\rho_\lambda\}$ be the finite set of all irreducible representations of $\mathrm{Gal}(E/F)$ in $\mathbf{Q}_p{}^{\mathrm{alg}}$. Choose a finite extension K' over K and a positive integer d such that (1) K' contains all eigenvalues of ρ_λ, (2) σ extends on K' and (3) $\sigma^d \circ \rho_\lambda = \rho_\lambda$. We can choose such K' and d by (2.4.1). Replacing K, q and σ into K', q^d and σ^d, we may assume that all eigenvalues of ρ_λ are contained in K and $\sigma \circ \rho_\lambda = \rho_\lambda$.

Let $\{e_1, e_2, \cdots, e_r\}$ be a basis of $\mathcal{R}_E \bigotimes_{\mathcal{R}} M$ such that $C_{M,e} \in M_r(K)$ (4.2.5) and denote by L_E (resp. Γ_E) the $\mathcal{E}_E^\dagger$-subspace (resp. the K-subspace) of $\mathcal{R}_E \bigotimes_{\mathcal{R}} M$ which is generated by $\{e_1, e_2, \cdots, e_r\}$. We prove that there exists a Frobenius structure φ on f^*M which commutes with the action of $\mathrm{Gal}(E/F)$. By (4.2.4) Γ_E is stable under the action of $\mathrm{Gal}(E/F)$. By the assumption and Schur's Lemma Γ_E is a direct sum of $\Gamma_{E,\lambda}$ such that the Galois group $\mathrm{Gal}(E/F)$ acts on $\Gamma_{E,\lambda}$ via ρ_λ and that $\nabla\left(x\dfrac{d}{dx}\right)(\Gamma_{E,\lambda}) \subset \Gamma_{E,\lambda}$. So it is enough to prove the existence of Frobenius structure on $\mathcal{R}_E \bigotimes_K \Gamma_{E,\lambda}$ which commutes with the Galois action. Since $C_{f^*M,e}$ is nilpotent and the Galois action commutes with the nilpotent endomorphism $\nabla|_{\Gamma_{E,\lambda}}$, one can choose a basis $\{e_{11}^\lambda, \cdots, e_{1r_\lambda}^\lambda, \cdots, e_{tr_\lambda}^\lambda\}$ of $\Gamma_{E,\lambda}$ such that $\{e_{ij}^\lambda\}_{1 \leqslant j \leqslant r_\lambda}$ is a basis of the irreducible component on which $\mathrm{Gal}(E/F)$ acts via ρ_λ and that the differential structure is given by a direct sum of the type $C_{M,e^\lambda} = \begin{pmatrix} 0_{r_\lambda} & 1_{r_\lambda} & & \mathbf{0} \\ & \ddots & \ddots & \\ & & 0_{r_\lambda} & 1_{r_\lambda} \\ \mathbf{0} & & & 0_{r_\lambda} \end{pmatrix}$ by Schur's Lemma. Here r_λ is the degree of ρ_λ. Hence, there exists a Frobenius structure φ which commutes with the Galois action by (4.2.3) and the condition (3) above in this proof. Of course, L_E is stable under φ. Put $L = L_E^{\mathrm{Gal}(E/F)}$ to be the Galois invariant part. Then $(L, \nabla|_L)$ is an $\mathcal{E}^\dagger$-lattice of M and L is stable under φ. □

From this proposition we know that, if one want to study some properties of quasi-unipotent ∇-modules, then it is enough to work on φ-∇-modules.

4.4. Let σ_1 and σ_2 be Frobenius on $\mathcal{R}$. Define a functor

$$\tilde{\epsilon}^{qu}_{\sigma_1,\sigma_2} : \underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma_2} \to \underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma_1}$$

as follows. For an object M of $\underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma_2}$ and for an $\mathcal{E}^\dagger$-lattice L of M (4.2.1), put

$$\tilde{\epsilon}^{qu}_{\sigma_1,\sigma_2}(M) = \mathcal{R}\bigotimes_{\mathcal{E}^\dagger}\tilde{\epsilon}_{\sigma_1,\sigma_2}(L).$$

(See the definition of $\tilde{\epsilon}_{\sigma_1,\sigma_2}$ in (3.4).)

LEMMA 4.4.1. — *The construction of the functor $\tilde{\epsilon}^{qu}_{\sigma_1,\sigma_2}(M)$ is independent of the choice of $\mathcal{E}^\dagger$-lattices.*

Proof. — Let L^λ (resp. $\{e^\lambda{}_1, e^\lambda{}_2, \cdots, e^\lambda{}_r\}$) be an $\mathcal{E}^\dagger$-lattice of an object M of $\underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma_2}$ (resp. a basis of L^λ) ($\lambda = \alpha, \beta$). Denote by $\epsilon^{\lambda,qu}_{\sigma_1,\sigma_2}$ the map which is defined using L^λ ($\lambda = \alpha, \beta$). Define a matrix $Q \in GL_r(\mathcal{R})$ by $(e^\alpha{}_1, e^\alpha{}_2, \cdots, e^\alpha{}_r) = (e^\beta{}_1, e^\beta{}_2, \cdots, e^\beta{}_r)Q$ and put a matrix Ω^λ to be $\epsilon^{\lambda,qu}_{\sigma_1,\sigma_2}(1 \otimes (e^\lambda{}_1, e^\lambda{}_2, \cdots, e^\lambda{}_r)) = (1 \otimes (e^\lambda{}_1, e^\lambda{}_2, \cdots, e^\lambda{}_r))\Omega_\lambda$. It is enough to prove that the diagram

$$\begin{array}{ccc} \sigma_1^*M & \stackrel{\epsilon^{\alpha,qu}_{\sigma_1,\sigma_2}}{\rightarrow} & \sigma_2^*M \\ \| & & \| \\ \sigma_1^*M & \underset{\epsilon^{\beta,qu}_{\sigma_1,\sigma_2}}{\rightarrow} & \sigma_2^*M \end{array}$$

is commutative. In other words, we have only to prove $\sigma_2(Q)\Omega^\alpha = \Omega^\beta\sigma_1(Q)$.

Assume that $A_{M,e^\lambda,\sigma_i}, C_{M,e^\lambda}$ ($\lambda = \alpha, \beta$ and $i = 1, 2$) and Q are convergent and σ_1 (resp. σ_2) is defined on the annulus $\gamma \leqslant |x| < 1$ for some $\gamma < 1$. Define a K-algebra

$$\mathcal{E}(\gamma) = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \;\middle|\; \begin{array}{l} a_n \in K, |a_n|\gamma^n \text{ is bounded,} \\ |a_n|\gamma^n \to 0 \ (n \to -\infty) \end{array} \right\}.$$

Then $\mathcal{E}(\gamma)$ is complete under the norm $|\sum a_n x^n|_\gamma = \sup_n |a_n|\gamma^n$ and σ_i ($i = 1, 2$) induces a map on $\mathcal{E}(\gamma)$. The pair $(A_{M,e^\lambda,\sigma_i}, C_{M,e^\lambda})$ ($\lambda = \alpha, \beta$ and $i = 1, 2$) define an $\mathcal{E}(\gamma)$ module $L_i^\lambda(\gamma)$ with a connection and a Frobenius structure with respect to σ_i ($i = 1, 2$). Since Q is contained in $GL_n(\mathcal{E}(\gamma))$, $L_i^\alpha(\gamma)$ is isomorphic to $L_i^\beta(\gamma)$ ($i = 1, 2$). By the similar arguments as in (3.4) we can define a similar map of $\epsilon_{\sigma_1,\sigma_2}$ for $\mathcal{E}(\gamma)$ and the matrix Ω_λ is the representative matrix of this map for the basis $\{e^\lambda{}_1, e^\lambda{}_2, \cdots, e^\lambda{}_r\}$. Therefore, we have $\sigma_2(Q)\Omega_\alpha = \Omega_\beta\sigma_1(Q)$. □

LEMMA 4.4.2. — *Let σ_1, σ_2 and σ_3 be Frobenius on $\mathcal{R}$. Then we have*

(i) $\tilde{\epsilon}_{\sigma_1,\sigma_1} = \mathrm{id}$;

(ii) $\tilde{\epsilon}_{\sigma_1,\sigma_3} = \tilde{\epsilon}_{\sigma_1,\sigma_2}\tilde{\epsilon}_{\sigma_2,\sigma_3}$.

THEOREM 4.4.3. — *The category* $\underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma}$ *is independent of the choice of Frobenius on* $\mathcal{R}$ *via the functor* $\tilde{\epsilon}^{qu}_{\sigma_1,\sigma_2}$.

Remark 4.4.4. — The author does not know whether the category $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$ is independent of the choice of Frobenius on $\mathcal{R}$ or not. But it is expected that the natural functor $\underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma} \to \underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$ is an equivalence.

5. Slope filtration for Frobenius structures.

In this section we define a slope filtration for Frobenius structures and prove that a φ-∇-module over $\mathcal{R}$ is quasi-unipotent if and only if it has a slope filtration.

5.1. Fix a Frobenius σ on $\mathcal{R}$.

DEFINITION 5.1.1. — *Let* M *be an object of* $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$. *An increasing filtration* $\{S_\gamma M\}_{\gamma\in\mathbf{Q}}$ *of* M *is a slope filtration for Frobenius structures if and only if it satisfies the condition as follows:*

(i) $S_\gamma M$ *is a sub* φ-∇-*module of* M *over* $\mathcal{R}$;

(ii) $S_\gamma M = 0$ $(\gamma << 0)$ *and* $S_\gamma M = M$ $(\gamma >> 0)$;

(iii) *for a sufficiently small positive rational number* ϵ, *there exists an* $\mathcal{E}^\dagger$-*lattice* L_γ *of* $S_\gamma M/S_{\gamma-\epsilon}M$ *which is pure of slope* γ.

PROPOSITION 5.1.2. — *If* L *is an object of* $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}^\dagger,\sigma}$ *pure of slope* γ, *then there are a finite separable extension* $f : F \to E$ *and a basis* $\{e_1, e_2, \cdots, e_r\}$ *of* f^*M *such that* $C_{f^*M,e} = 0$.

Proof. — Replacing (M, φ, ∇) into $(M, a\varphi^d, \nabla)$ for a suitable positive integer d and $a \in K$, we may assume $\gamma = 0$. The assertion follows [TN2, 4.2.6]. □

PROPOSITION 5.1.3. — *Let* $\eta : M_1 \to M_2$ *be a morphism of* $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$. *Assume that both* M_1 *and* M_2 *have a slope filtration* $S_\gamma M_i$ $(i = 1, 2)$ *for Frobenius structures. Then* η *is strict for filtrations, that is,* $\eta(S_\gamma M_1) = \eta(M_1) \bigcap S_\gamma M_2$ *for any* $\gamma \in \mathbf{Q}$.

Proposition (5.1.3) follows from Lemma (5.1.4) below.

LEMMA 5.1.4. — *Let M_1 (resp. M_2) be an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$ with an $\mathcal{E}^{\dagger}$-lattice L_1 (resp. L_2) pure of slope γ_1 (resp. γ_2).*

(1) *If $\gamma_1 \neq \gamma_2$, then there is no nontrivial morphism from M_1 to M_2.*

(2) *If $\gamma_1 = \gamma_2$, then any morphism $\eta_1 : M_1 \to M_2$ preserves the $\mathcal{E}^{\dagger}$-lattice, that is, $\eta(L_1) = \eta(M_1) \bigcap L_2$.*

Proof. — (1) Since $\mathrm{Hom}_{\mathbf{M\Phi}_{\mathcal{R},\sigma}}(M_1, M_2) \cong \mathrm{Hom}_{\mathbf{M\Phi}_{\mathcal{R},\sigma}}(\mathcal{R}, M_1^{\vee} \otimes M_2)$, we have only to prove the assertion in the case where $M_1 = \mathcal{R}$ and M_2 is an arbitrary M with $\mathcal{E}^{\dagger}$-lattice L pure of slopes γ. There exist a finite separable extension $f : F \to E$ in F^{sep} and an element $A \in GL_r(K)$ such that M is isomorphic to $((\mathcal{R}_E)^r, A\sigma, d)$ by (5.1.2). One can easily see that there is no morphism from the unit object to f^*M if $\gamma \neq 0$.

The assertion (2) follows (2.2.3) and (5.1.2). □

COROLLARY 5.1.5. — *A slope filtration for Frobenius structures of an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$ is unique.*

5.2. We state one of our main local theorems.

THEOREM 5.2.1. — *Let M be an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$. M is quasi-unipotent if and only if M has a slope filtration $\{S_\gamma M\}_{\gamma \in \mathbf{Q}}$ for Frobenius structures.*

Proof. — It is enough to prove the assertion in the case where $\sigma(x) = x^q$ by (3.4.9), (3.4.10) and (4.4.3). Let $f : F \to E$ be a finite separable extension in F^{sep} such that f^*M is unipotent. Then there exists a $\mathrm{Gal}(E/F)$-stable K-lattice Γ_E of f^*M. In fact, choose a basis $\{e_1, e_2, \cdots, e_r\}$ of f^*M as in (4.2.5) and put Γ_E to be a K_E-subspace of f^*M which is generated by $\{e_1, e_2, \cdots, e_r\}$. Here K_E is the finite unramified extension with residue class field k_E. Then Γ_E is stable under the Frobenius structure φ and the action $\mathrm{Gal}(E/F)$ by (4.2.4) and (4.2.5), that is, $\nabla|_{\Gamma_E} \circ \varphi|_{\Gamma_E} = q\varphi|_{\Gamma_E} \circ \nabla|_{\Gamma_E}$. By the theory of φ-spaces with a nilpotent structure over a complete discrete valuation field we have a slope filtration $\{S_\gamma \Gamma_E\}$ for the Frobenius structure $\varphi|_{\Gamma_E}$ of Γ_E which is compatible with the nilpotent operator $\nabla|_{\Gamma_E}$. Moreover the theory of slopes implies that the filtration $\{S_\gamma \Gamma_E\}$ is compatible with the action of $\mathrm{Gal}(E/F)$ since $\varphi|_{\Gamma_E}$ commutes with the action of $\mathrm{Gal}(E/F)$. Define a filtration $\{S_\gamma M\}$ of

M by

$$S_\gamma M = \mathcal{R} \bigotimes_{\mathcal{E}^\dagger} (\mathcal{E}_E^\dagger \bigotimes_{K_E} S_\gamma \Gamma_E)^{\mathrm{Gal}(E/F)}.$$

$\{S_\gamma M\}$ is a slope filtration for Frobenius structures of M by (2.2.4) and (3.3.5). The converse follows from (5.1.2). □

Remark 5.2.2. — In Theorem (5.2.1) the slope filtration $\{S_\gamma M\}$ of M is split as φ-modules (not as ∇-modules) over $\mathcal{R}$ if we choose a Frobenius $\sigma(x) = x^q$, because the filtration $\{S_\gamma \Gamma_E\}$ of Γ_E over K_E is split as φ-Gal(E/F)-modules in the above proof. In general cases the slope filtration is not always split as φ-modules.

6. Quasi-unipotent overconvergent F-isocrystals on a curve.

In this section we give a definition of quasi-unipotent overconvergent F-isocrystals on a curve and apply our local study to them. We use some results on overconvergent F-isocrystals on curves from [Be1], [Be2], [Be3] and [Cr1].

6.1. Let k (resp. K) be a perfect field of positive characteristic p (resp. a complete discrete valuation field with the residue class field k and with a Frobenius σ). Let X be a smooth curve over Spec k which is geometrically connected. For a closed point $s \in X$, denote by $k(s)$ (resp. $K(s)$) the residue class field at s (resp. the finite unramified extension of K with the residue class field $k(s)$).

Let U be a dense open subscheme of X and put $Z = X - U$. Fix a closed point $s \in X$ and denote by $\mathcal{X}$ a formal scheme over Spf O_K which is a lifting of X/Spec k and formally smooth around x. Choose a section $x \in \Gamma(O_{\mathcal{X}})$ which is a lifting of a local parameter of O_X at s. Since $\mathcal{X}$/Spf O_K is formally smooth at s, the completion of $O_{\mathcal{X}}$ at s is isomorphic to $O_{K(s)}[[x]]$. Put $\mathcal{R}_s$ (resp. $\mathcal{E}_s$, resp. $\mathcal{E}_s^\dagger$, resp. $S_{K(s)}$) to be $\mathcal{R}_{x,K(s)}$, (resp. $\mathcal{E}_{x,K(s)}$, resp. $\mathcal{E}^\dagger_{x,K(s)}$, resp. $K \bigotimes_{O_K} O_{K(s)}[[x]]$). Therefore, we have an injective homomorphism

$$i_s : \Gamma(O_{]U[}) \to \mathcal{E}_s \quad (x \mapsto x)$$

of K-algebras. The map i_s is independent of the choice of the lifting of parameter via the natural isomorphism $\mathcal{E}^\dagger_{x,K(s)} \cong \mathcal{E}^\dagger_{x',K(s)}$ for any parameter x'. Especially, if $s \in U$, then $i_s(\Gamma(O_{]U[})) \subset S_{K(s)}$. By [Cr1, 4.7.] we have

LEMMA 6.1.1. — *Assume that X is affine and $U = X - \{s\}$. Under the notation as above, we have*

$$i_s(\Gamma(O_{]X[})) = \mathrm{Im}(i_s) \bigcap S_{K(s)};$$
$$i_s(\Gamma(j^\dagger O_{]X[})) = \mathrm{Im}(i_s) \bigcap \mathcal{E}^\dagger_s,$$

where $j :]U[\to \mathcal{X}^{an}$.

By the construction, $i_s\Big(x\dfrac{d}{dx}(u)\Big) = \delta_x(i_s(u))$ for any section $u \in \Gamma(O_{]U[})$. If $\sigma : O_{]U[} \to O_{]U[}$ is a lifting of q-th power map on O_U ($q = p^a$) which is an extension of the Frobenius σ on K, then σ extends on $\mathcal{E}_s$ (resp. $S_{K(s)}$ if $s \in U$). We call the extension σ a Frobenius on $O_{]U[}$.

Denote by $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) the abelian category of overconvergent isocrystals on U/K around Z (resp. the category of overconvergent F^a-isocrystals on U/K around Z) [Be3, (2.2.10)]. By the natural extension $i_{\mathcal{R}_s} : \Gamma(j^\dagger O_{]X[}) \to \mathcal{R}_s$ of scalar there is a functor

$$i^*_{\mathcal{R}_s} : \underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathbf{M}}^\nabla_{\mathcal{R}_s}$$

which is factored via the natural functor $i^*_{\mathcal{E}^\dagger_s} : \underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathbf{M}}^\nabla_{\mathcal{E}^\dagger_s}$ (resp. $i^*_{S_{K(s)}} : \underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathbf{M}}^\nabla_{S_{K(s)}}$ if $s \in U$). For any Frobenius σ on $O_{]X[}$, we also have a natural functor

$$i^*_{\mathcal{R}_s,\sigma} : F^a\text{-}\underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathbf{M\Phi}}^\nabla_{\mathcal{R}_s,\sigma}$$

which is factored via the natural functor $i^*_{\mathcal{E}^\dagger_s,\sigma} : F^a$-$\underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathbf{M\Phi}}^\nabla_{\mathcal{E}^\dagger_s,\sigma}$ (resp. $i^*_{S_{K(s)},\sigma} : F^a$-$\underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathbf{M\Phi}}^\nabla_{S_{K(s)},\sigma}$ if $s \in U$). One can easily see that the functor $i^*_{\mathcal{R}_s}$ (resp. $i^*_{\mathcal{R}_s,\sigma}$) is independent of all choices up to canonical transformations. One can also see that the functor $i^*_{\mathcal{R}_s,\sigma}$ is independent of the choice of Frobenius σ up to the functor $\tilde{\epsilon}_{\sigma_1,\sigma_2}$ by the definition of F-isocrystals, Proposition (3.4.10) and Lemma (4.3.1).

Now we define a quasi-unipotent overconvergent isocrystal. Our definition differs from that in [Cr2, 10.11], but we will prove that our definition is equivalent to Crew's one in Theorem (6.1.6).

DEFINITION 6.1.2. — (1) *An object $\mathcal{M}$ of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger$*

$(U, X/K)$ is unipotent at a closed point $s \in X$ if and only if $i^*_{\mathcal{R}_s}\mathcal{M}$ is unipotent. An object $\mathcal{M}$ of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) is unipotent if and only if $\mathcal{M}$ is unipotent at any closed point on X.

(2) An object $\mathcal{M}$ of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) is quasi-unipotent at a closed point $s \in X$ if and only if $i^*_{\mathcal{R}_s}\mathcal{M}$ is quasi-unipotent. An object $\mathcal{M}$ of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) is quasi-unipotent if and only if $\mathcal{M}$ is quasi-unipotent at any closed point on X. Denote by $\underline{\mathrm{Isoc}}^\dagger(U, X/K)^{qu}$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)^{qu}$) the full subcategory of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) which consists of quasi-unipotent objects.

PROPOSITION 6.1.3. — *The category $\underline{\mathrm{Isoc}}^\dagger(U, X/K)^{qu}$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)^{qu}$) is an abelian subcategory of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) which is closed under subquotients, tensor products and duals.*

Let $\iota : Y \subset X$ (resp. $V \subset U$) be a non-empty open subscheme and put $Z_Y = Y - V$. Denote by $\iota^\dagger : \underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathrm{Isoc}}^\dagger(V, Y/K)$ (resp. $\iota^\dagger : F^a$-$\underline{\mathrm{Isoc}}^\dagger(U, X/K) \to F^a$-$\underline{\mathrm{Isoc}}^\dagger(V, Y/K)$) the natural inverse image functor which is induced by ι. By the definition we have

PROPOSITION 6.1.4. — *Under the notation as above, let $\mathcal{M}$ be an object of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$). If $\mathcal{M}$ is unipotent (resp. quasi-unipotent), then $\iota^\dagger\mathcal{M}$ is so. Assume furthermore that $Y = X$, then $\mathcal{M}$ is unipotent (resp. quasi-unipotent) if and only if $\iota^\dagger\mathcal{M}$ is so.*

Let $f : Y \to X$ be a finite morphism of smooth curves over Spec k and put $U_Y = Y \times_X U$ and $Z_Y = Y \times_X Z$. Assume that the restriction $f_U : U_Y \to U$ of f is finite and etale. Since one can choose a lifting $\mathcal{Y}$ of Y such that $]U_Y[\to]U[$ is finite etale and $j^\dagger\mathcal{O}_{]Y[}$ is finite of degree $\deg(f)$ over $j^\dagger\mathcal{O}_{]X[}$ locally at s, one can define the inverse image functor (resp. the direct image functor)

$$\begin{aligned} f^* &: \underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathrm{Isoc}}^\dagger(U_Y, Y/K) \\ (\text{resp.}\quad f_* &: \underline{\mathrm{Isoc}}^\dagger(U_Y, Y/K) \to \underline{\mathrm{Isoc}}^\dagger(U, X/K)) \end{aligned}$$

by $f^*\mathcal{M} = j^\dagger\mathcal{O}_{]Y[} \bigotimes_{f^{-1}j^\dagger\mathcal{O}_{]X[}} f^{-1}\mathcal{M}$ (resp. the restriction $j^\dagger\mathcal{O}_{]X[} \to f_*j^\dagger\mathcal{O}_{]Y[}$ of scalar). One can also define the inverse image functor f^* and the direct image functor f_* for F-isocrystals. Let $t \in Y$ be a closed point with

$f(t) = s$. Choose a formally lifting $\mathcal{Y}$ over Spf O_K of $Y/\mathrm{Spec}\ k$ which is formally smooth around t, a lifting $f : \mathcal{Y} \to \mathcal{X}$ over Spf O_K of $f : Y \to X$, a section $y \in \Gamma(O_{\mathcal{Y}})$ which is a lifting of a local parameter at t. Such lifting f always exists locally on $\mathcal{X}$ and our arguments below work well on this situation. Then f induces an injection $f : \mathcal{R}_s \to \mathcal{R}_t$ of K-algebras and we have natural commutative diagrams

$$\begin{array}{ccc} \underline{\mathrm{Isoc}}^\dagger(U, X/K) & \xrightarrow{f^*} & \underline{\mathrm{Isoc}}^\dagger(U_Y, Y/K) \\ i^*_{\mathcal{R}_s} \downarrow & & \downarrow i^*_{\mathcal{R}_t} \\ \underline{\mathbf{M}}^\nabla_{\mathcal{R}_s} & \xrightarrow[f^*]{} & \underline{\mathbf{M}}^\nabla_{\mathcal{R}_t} \end{array}$$

and

$$\begin{array}{ccc} \underline{\mathrm{Isoc}}^\dagger(U_Y, Y/K) & \xrightarrow{f_*} & \underline{\mathrm{Isoc}}^\dagger(U, X/K) \\ i^*_{\mathcal{R}_t} \downarrow & & \downarrow i^*_{\mathcal{R}_s} \\ \underline{\mathbf{M}}^\nabla_{\mathcal{R}_t} & \xrightarrow[f_*]{} & \underline{\mathbf{M}}^\nabla_{\mathcal{R}_s}. \end{array}$$

If σ is a Frobenius on $O_{]U[}$, then σ extends uniquely on $O_{]U_Y[}$ since f_U is etale. We also have commutative diagrams for F-isocrystals as in above diagrams. By Proposition (4.1.3) and (6.1.3) we have

PROPOSITION 6.1.5. — *Under the notation as above,*

(1) *an object $\mathcal{M}$ of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) is quasi-unipotent if and only if $f^*\mathcal{M}$ is quasi-unipotent;*

(2) *an object $\mathcal{M}$ of $\underline{\mathrm{Isoc}}^\dagger(U_Y, Y/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U_Y, Y/K)$) is quasi-unipotent if and only if $f_*\mathcal{M}$ is quasi-unipotent.*

Now we compare Crew's definition to ours.

THEOREM 6.1.6. — *Let $\mathcal{M}$ be an object of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger$ $(U, X/K)$. $\mathcal{M}$ is quasi-unipotent if and only if there is a finite morphism $f : Y \to X$ of smooth curves over* Spec k *and a nonempty open subscheme $\iota : V \to U$ such that $f_V : V_Y \to V$ is etale and that $f_V^*\iota^\dagger\mathcal{M}$ is unipotent.*

Proof. — Assume that $\mathcal{M}$ is quasi-unipotent. Denote by $K(X)$ the field of rational functions of X. Since Z is a finite set, there is a finite separable extension L of $K(X)$ such that, for any point $s \in Z$ and for any place t of L above s, $f^*_{t\mapsto s} i^*_{\mathcal{R}_s}\mathcal{M}$ is unipotent over $\mathcal{R}_t(= \mathcal{R}_{L_t})$. Here $K(X)_s$ (resp. L_t) is completion of $K(X)$ (resp. L) at s (resp. t) and $f_{t\mapsto s} : K(X)_s \to L_t$ is a structure map. Define a smooth curve Y over

k by the normalization of X in L. Since L is separable over $K(X)$, the natural morphism $f : Y \to X$ is generically etale. Therefore we obtain the assertion by (4.1.3). The converse follows from (4.1.3). □

Remark 6.1.7. — Matsuda pointed out that, either if X is affine or if the number of geometric points in $X - U$ is greater than 1, then one can choose a finite covering Y of X such that U_Y is etale over U in Theorem 6.1.6 by [Ka2, 2.1.6].

6.2. We give some examples of quasi-unipotent overconvergent F-isocrystals. By Proposition (4.2.1) we have

PROPOSITION 6.2.1. — *A convergent F-isocrystal on X/K is quasi-unipotent.*

DEFINITION 6.2.2. — *Let $\mathcal{M}$ be an object of $F^a\text{-}\underline{\mathrm{Isoc}}^\dagger(U, X/K)$. An increasing filtration $\{S_\gamma \mathcal{M}\}_{\gamma \in \mathbf{Q}}$ of M is a slope filtration for Frobenius structures if and only if it satisfies the conditions as follows:*

(i) *$S_\gamma \mathcal{M}$ is a subobject of $\mathcal{M}$ in $F^a\text{-}\underline{\mathrm{Isoc}}^\dagger(U, X/K)$;*

(ii) *$S_\gamma \mathcal{M} = 0$ ($\gamma << 0$) and $S_\gamma \mathcal{M} = \mathcal{M}$ ($\gamma >> 0$);*

(iii) *for a Frobenius σ on $j^\dagger O_{]U[}$, $\{i^*_{\mathcal{R}_s} S_\gamma \mathcal{M}\}_\gamma$ is a slope filtration for Frobenius structures of $i^*_{\mathcal{R}_s}\mathcal{M}$ of $\underline{\mathbf{M}\Phi}^\nabla_{\mathcal{R}_s,\sigma}$ at any point $s \in X$.*

The condition (iii) above is independent of the choice of Frobenius by Proposition (3.4.9). By Theorem (5.2.1) we have

PROPOSITION 6.2.3. — *If an object $\mathcal{M}$ of $F^a\text{-}\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ has a slope filtration for Frobenius structures, then $\mathcal{M}$ is quasi-unipotent.*

COROLLARY 6.2.4 ([Cr1, 4.12]). — *An overconvergent F^a-isocrystal on U/K around Z of rank one is quasi-unipotent.*

COROLLARY 6.2.5. — *A unit-root overconvergent F^a-isocrystal on U/K around Z is quasi-unipotent.*

Example 6.2.6. — Let p be an odd prime. Let $k = \mathbf{F}_p$, $K = \mathbf{Q}_p(\pi)$ with $\pi^{p-1} = -p$ and σ be a continuous lifting of p-th power map on K with $\sigma(\pi) = \pi$. Put $X = \mathbb{P}^1_k$ (resp. $U = \mathfrak{G}\mathrm{m}_k$, resp. $Z = \{0, \infty\}$) and $\mathcal{X} = \widehat{\mathbb{P}}^1$ over $\mathrm{Spf}\, O_K$ with a coordinate x. In [Dw] B. Dwork constructed the Bessel overconvergent F-isocrystal $\mathcal{M}$ on U/K around Z. $\mathcal{M}$ is of rank 2 and is defined by the following differential and Frobenius structures:

$$\nabla(e_1, e_2) = dx \otimes (e_1, e_2) \begin{pmatrix} 0 & -x^{-1} \\ -\pi^2 & 0 \end{pmatrix}$$

$$\varphi(e_1, e_2) = (e_1, e_2) \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

on the strict neighbourhood $|x| \leqq \gamma$ for some $\gamma > 1$ of $]U[_{\mathcal{X}}$ with $\begin{pmatrix} a_1(0) & a_2(0) \\ a_3(0) & a_4(0) \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & p \end{pmatrix}$, $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (mod π) and $\det \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = p$.

Claim. — *$\mathcal{M}$ is quasi-unipotent.*

By Proposition (4.2.1) $\mathcal{M}$ is unipotent on any closed point $s \in X - \{\infty\}$. Now we discuss the quasi-unipotency of $\mathcal{M}$ at ∞ following the arguments of [Dw, Section 8]. We change the coordinate x into x^{-1} and denote by $F = k((x))$ the completion of the field of fractions of the local ring $O_{X\infty}$ at the infinity. Define a tamely ramified extension $E = k((y))$ over F with $4y^2 = x$ and choose a lifting y of the parameter of $\mathcal{R}_E$ with $4y^2 = x$. Then the differential structure of $i_\infty^* \mathcal{M}$ over $\mathcal{R}_E$ is given by

$$\nabla(e_1, e_2) = \frac{dy}{y} \otimes (e_1, e_2) \begin{pmatrix} 0 & 2 \\ 2^{-1}\pi^2 y^{-2} & 0 \end{pmatrix}.$$

If $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is a solution of the differential equation $\delta_y \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 2^{-1}\pi^2 y^{-2} & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$, then z_1 satisfies the differential equation $\delta_y^2(z_1) = \pi^2 y^{-2} z_1$. Consider the formal solution $z_1 = y^{\frac{1}{2}} u_\pm(y) \exp(\pm \pi y^{-1})$. Then $u_\pm = u_\pm(y)$ satisfies the differential equation:

$$4y\delta_y^2(u_\pm) + 4(y \mp 2\pi)\delta_y(u_\pm) + xu_\pm = 0.$$

By easy calculations we have

$$u_\pm = 1 + \sum_{n=1}^{\infty} (\pm 1)^n \frac{((2n-1)!!)^2}{(8\pi)^n n!} y^n,$$

where $(2n-1)!! = 1 \times 3 \times \cdots \times (2n-1)$, and $u_\pm$ is convergent on the unit disk $|y| < 1$. Put a matrix

$$Q = \begin{pmatrix} u_+ & u_- \\ \delta_y(u_+) + (\frac{1}{2} - \pi y^{-1})u_+ & \delta_y(u_-) + (\frac{1}{2} + \pi y^{-1})u_- \end{pmatrix}.$$

Since $\delta_y(\det Q) = -\det Q$, we have $\det Q = 2\pi y^{-1}$ and $Q \in GL_2(\mathcal{R}_E)$. Change the basis (e_1, e_2) into $(e_+, e_-) = (e_1, e_2)Q$. By our construction we have

$$\nabla(e_+, e_-) = \frac{dy}{y} \otimes (e_+, e_-)C \quad \text{with } C = \begin{pmatrix} -\frac{1}{2} + \pi y^{-1} & 0 \\ 0 & -\frac{1}{2} - \pi y^{-1} \end{pmatrix}.$$

Put a matrix $A = A_{i^*_\infty \mathcal{M}, e_\pm}$. Note that $\sigma(y) = 2^{p-1} y^p$, and the pair (A, C) satisfies the relation $\delta_y(A) + CA = pA\sigma(C)$. Since $\exp(2\pi y^{-1})$ is not contained in $\mathcal{R}_E$, we have

$$A = \begin{pmatrix} \alpha_+ y^{-\frac{p-1}{2}} \exp(\pi(y^{-1} - \sigma(y^{-1})) & 0 \\ 0 & \alpha_- y^{-\frac{p-1}{2}} \exp(-\pi(y^{-1} - \sigma(y^{-1}))) \end{pmatrix}$$

for some $\alpha_+, \alpha_- \in K^\times$ with $\alpha_+\alpha_- = 2^{1-p}p$. Hence, $\mathcal{M}$ is quasi-unipotent at ∞ by the example (4.1.4). Finally we determine slopes of $\mathcal{M}$ at ∞. Since $\tau(y) = -y$ for the nontrivial element τ in $\mathrm{Gal}(E/F)$, $e_+ + e_-$ and $ye_+ - ye_-$ is a basis of $i^*_\infty \mathcal{M}$ over $\mathcal{R}_F$. By the commutativity between the Galois action and the Frobenius structure we have

$$\varphi(e_+ + e_-) = b_1(e_+ + e_-) + b_2(ye_+ - ye_-) \quad \text{with } b_1, b_2 \in \mathcal{R}_F.$$

On the other hand we have

$$\varphi(e_+ + e_-) = \alpha_+ y^{\frac{-p-1}{2}} \exp(\pi(y^{-1} - \sigma(y^{-1})))e_+ \\ + \alpha_- y^{\frac{-p-1}{2}} \exp(-\pi(y^{-1} - \sigma(y^{-1})))e_-.$$

Comparing both identities, we obtain $v_p(\alpha_+) = v_p(\alpha_-) = \dfrac{1}{2}$ for $\alpha_+\alpha_- = 2^{1-p}p$. Therefore, all slopes of $\mathcal{M}$ at ∞ are $\dfrac{1}{2}$ by Proposition (3.3.5).

BIBLIOGRAPHY

[Be1] P. Berthelot, Cohomologie rigide et théorie de Dwork : Le cas des Sommes Exponentielles, Astérisque, 119-120 (1984), 17-49.

[Be2] P. Berthelot, Géométrie rigide et cohomologie des variétés algébriques de caractéristique p, Soc. Math. de France 2e Série, Mémoire n° 23 (1986), 7-32.

[Be3] P. Berthelot, Cohomologie rigide et cohomologie rigide à supports propres Première Partie, preprint.

[BO] P. Berthelot, A. Ogus, Notes on Crystalline cohomology, Math Notes 21 (1978), Princeton.

[Cr1] R. Crew, F-isocrystals and p-adic representations, Proc. of Symp. in Pure Math., 46 (1987), 111-138.

[Cr2] R. CREW, Finiteness theorem for the cohomology of an overconvergent isocrystal on a curve, preprint.

[CS] B. CHIARELLOTTO, B. LE STUM, F-isocristaux unipotents, preprint.

[De] M. DEMAZURE, Lecture on p-divisible groups, Lecture Notes in Math. 302, Springer-Verlag 1972.

[Dw] B. DWORK, Bessel Functions as p-adic functions of the argument, Duke Math., 41 (1974), 711-738.

[Fo] J.-M. FONTAINE, Représentation p-adiques des corps locaux, Grothendieck Festschrift II, Progress in Math. 87, 249-309 : Birkhäuser 1990.

[Ka1] N. KATZ, Slope filtration of F-crystals, Astérisque, 63 (1979), 113-164.

[Ka2] N. KATZ, Local-to-global extensions of representations of fundamental groups, Ann. Inst. Fourier, Grenoble, 36-4 (1986), 69-106.

[Ma] S. MATSUDA, Local indices of p-adic differential operators corresponding to Artin-Schreier-Witt coverings, Duke Math. J., 77 (1995), 607-625.

[Ro] P. ROBBA, Sur les équations differentielles linéaires p-adiques II, Pacific J. Math., 98 (1982), 393-418.

[Se1] J.-P. SERRE, Corps Locaux, Hermann, Paris, 1962.

[Se2] J.-P. SERRE, Représentations Linéaires des Groupes Finis, Hermann, Paris, 1967.

[TN1] N. TSUZUKI, Finite local monodromy of overconvergent unit-root F-isocrystal on a curve, preprint.

[TN2] N. TSUZUKI, The local index and the Swan conductor, to appear in Compositio Math.

Manuscrit reçu le 5 novembre 1996,
accepté le 8 septembre 1997.

Nobuo TSUZUKI,
Hiroshima University
Faculty of Science
Department of Mathematics
Higashi-Hiroshima, 739, (Japan).
tsuzuki@math.sci.hiroshima-u.ac.jp