ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 48, nº 1 (1998), p. 281-321
<http://www.numdam.org/item?id=AIF_1998__48_1_281_0>

Ann. Inst. Fourier, Grenoble
48, 1 (1998), 281–321

THE COHOMOLOGY RING OF POLYGON SPACES

by J-C. HAUSMANN and A. KNUTSON

Contents.

Introduction.

A "polygon space" $\mathrm{Pol}\,(\alpha_1, \alpha_2, ...\alpha_m), \alpha_i \in \mathbf{R}_+$ can be seen to arise in several ways:

1. the family of piecewise linear paths in $\mathbf{R}^3$, whose ith step (which is of length α_i) can proceed in any direction subject to the polygon ending where it begins, considered up to rotation and translation;

Both authors thank the Fonds National Suisse de la Recherche Scientifique for its support.
Key words: Polygon spaces – Toric manifolds – Groebner bases.
Math. classification: 58Dxx – 14Mxx – 13P10.

2. the "semistable" configurations of m weighted points in $\mathbf{C}P^1$, the ith of weight α_i, where a configuration is considered unstable if more than half the total weight sits in one place, modulo Möbius transformations. This description is of the most classical interest [DM], particularly in the equal-weight case [Po], since the quotient by Sym_m is a compactification of the moduli space of m-pointed genus zero curves;

3. (when the α_i are integral) the geometric invariant theory quotient of the Grassmannian of 2-planes in $\mathbf{C}^n$ by T^n, where the $\{\alpha_i\}$ specify an action of T^n on the canonical bundle.

The connection of the first to the second is made in [Kl] and [KM]; the second to the third in [GM], [GGMS] and the first to the third in [HK]. This paper draws much from the polygonal intuition and will concentrate on the first.

In this paper we compute the integer cohomology rings of these spaces, in the (generic) case that they are smooth. There are partial results in the literature. Klyachko [Kl] showed that the cohomology groups were torsion-free and calculated their rank. Brion [Br] (and later Kirwan [K1]) calculated the rational cohomology ring in the equal-weight case, and also the equal-weight case with an odd number of sides modulo the symmetric group. It seems that a slight refinement of Brion's method would require that one only invert the prime 2.

Our approach is very different, and makes heavy use of toric varieties, whose integer cohomology rings are known by the theorem of Danilov. While a polygon space is not (usually) a toric variety itself, it embeds in one in a very special way: as a transverse self-intersection of a toric subvariety.

This gives a map from the cohomology ring of the ambient toric variety, our *upper path space*, to that of the polygon space itself. We then have four tasks to complete:

1. Compute the cohomology ring of the upper path space, using (a mild extension of) Danilov's result.

2. Show that the restriction map on cohomology is surjective.

3. Show that the kernel of this map is the annihilator of the Poincaré dual of the submanifold.

4. Compute the annihilator.

The first is a very polygon-theoretic argument, and is in Section 5. We prove the second and third as part of a more general study of *even-cohomology spaces*, based on the fact that $H^{\text{odd}} = 0$ for not only the polygon space and upper path space, but also the difference[(1)].

This is in Section 3, where this machinery also provides a simple formula for the Betti numbers of the polygon space (Section 4). The form in which we obtain the cohomology ring of the upper path space gives a simple guess for the annihilator; to show this is the entire annihilator we use a Gröbner basis argument, in Section 6.

The classical interest in these spaces is especially strong in the equal-weight case, in that there is then an action of the symmetric group, and the quotient is then a compactification of the moduli space of m-times-punctured Riemann spheres. Our approach requires us to single out one edge, breaking this symmetry. However, a circle bundle associated to each edge gives a natural list of degree 2 classes, permuted by the action of S_m in the equilateral case. In Section 7 we locate these in our presentation, and show they generate the $\mathbf{Z}[\frac{1}{2}]$ cohomology ring. This gives a manifestly symmetric presentation which is actually simpler.

But breaking the symmetry is unavoidable, in a very precise sense, if one wants to compute the *integer* cohomology ring. While the action of S_m on the second rational cohomology group is the standard one on $\mathbf{Q}^m$ [Kl], it is *not* the standard one on the second integral cohomology group – there is no $\mathbf{Z}$-basis permuted by S_m. In Section 8 we show this, but also show that the action becomes standard if one inverts (the necessarily odd number) m.

The main reason to avoid inverting 2 is to compute the $\mathbf{Z}/2$-cohomology ring of the *planar* polygon space, which we do in Section 9.

Lastly, if the edge chosen is the longest one, our formulae are no worse and frequently much more computationally effective than the symmetric versions with $\mathbf{Z}[\frac{1}{2}]$ coefficients. This and much else can be seen in Section 10 on examples.

Acknowledgements: Both authors are grateful to Shaun Martin for many useful conversations concerning this problem. Susan Tolman made

(1) In fact Claims 2 and 3 hold much more generally, and are the basis of Shaun Martin's unpublished but very influential G-to-T argument [Ma]. Owing to our very special even-cohomology circumstances we can give a self-contained argument.

valuable comments on a preliminary version of this paper. The second author wishes to thank the University of Geneva for its hospitality while this work was being done.

1. The polygon spaces.

Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{R}^m_+$. Let $S^2_{\alpha_i}$ denote the sphere in $\mathbf{R}^3$ with radius α_i. Identifying $\mathbf{R}^3$ with $so(3)^*$, the Lie-Kirillov-Kostant-Souriau symplectic structure gives $S^2_{\alpha_i}$ the symplectic volume$= 2\alpha_i$. Let us consider the manifold

$$\prod_{i=1}^{m-1} S^2_{\alpha_i} \subset (\mathbf{R}^3)^m$$

equipped with the product symplectic structure. We imagine $(\rho_1, \dots, \rho_m) \in (\mathbf{R}^3)^m$ as a path starting from the origin of m successive steps ρ_i and thus call $\prod_{i=1}^{m} S^2_{\alpha_i}$ the *path space* for α. The Hamiltonian actions on $\prod_{i=1}^{m} S^2_{\alpha_i}$ which are relevant to us are

a) the diagonal SO_3-action with moment map $\mu(\rho) := \sum_{i=1}^{m-1} \rho_i$, "end-point"

b) its restriction to SO_2 with moment map $\overline{\mu}(\rho) := \zeta(\sum_{i=1}^{m-1} \rho_i)$, "height of endpoint", where ζ is the projection $\zeta(x, y, z) = z$.

c) the $(SO_2)^m$-action with moment map $\hat{\mu}(\rho) = \big(\zeta(\rho_1), \dots, \zeta(\rho_{m-1})\big)$, "height of each step".

This action makes $\prod_{i=1}^{m-1} S^2_{\alpha_i}$ a toric manifold. The moment polytope (image of $\hat{\mu}$) is the box $\prod_{i=1}^{m-1} [-\alpha_i, \alpha_i]$.

Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{R}^m_+$. We consider the space $\mathrm{Pol}\,(\alpha)$ of configurations in $\mathbf{R}^3$ of a polygon with m edges of length $\alpha_1, \dots, \alpha_m$, modulo rotation, and call it the *polygon space* for α. The precise definition is

$$\mathrm{Pol}\,(\alpha) := \left\{ (\rho_1, \dots, \rho_m) \in (\mathbf{R}^3)^m \,\middle|\, \forall i, |\rho_i| = \alpha_i \text{ and } \sum_{i=1}^{m} \rho_i = 0 \right\} \Big/ SO_3$$

where SO_3 acts on $(\mathbf{R}^3)^m$ diagonally. We say that α is *generic* if the equation $\sum_{i=1}^{m} \varepsilon_i \alpha_i = 0$ has no solution with $\varepsilon_i = \pm 1$ (there are no lined configurations). In this paper, α will always be assumed generic.

The spaces $\mathrm{Pol}(\alpha)$ have been studied for instance in [Kl], [KM] and [HK]. When α is generic it is shown that $\mathrm{Pol}(\alpha)$ is a closed smooth manifold of dimension $2(m-3)$ which naturally carries a symplectic form ω. More precisely, $\mathrm{Pol}(\alpha)$ occurs as symplectic reductions of the path spaces

$$\mathrm{Pol}(\alpha) = \mu^{-1}(0)/SO_3 = \left(\prod_{i=1}^{m} S^2_{\alpha_i}\right) \mathbin{\mathop{/\!\!/}_{0}} SO_3 \cong \left(\prod_{i=1}^{m-1} S^2_{\alpha_i}\right) \mathbin{\mathop{/\!\!/}_{\alpha_m}} SO_3.$$

This last object is "paths of $m-1$ steps of lengths $\alpha_1, \dots, \alpha_{m-1}$, whose endpoint is at distance α_m from the origin, modulo $SO(3)$", and obviously corresponds to the polygons as previously described. It is this last picture, in which the mth edge plays a distinguished role, that will be of most use to us.

The *abelian polygon space* $\mathrm{APol}(\alpha)$ is defined as

$$\mathrm{APol}(\alpha) := \left\{ (\rho_1, \dots, \rho_m) \in (\mathbf{R}^3)^m \;\middle|\; |\rho_i| = \alpha_i \text{ and } \zeta\left(\sum_{i=1}^{m-1} \rho_i\right) = \alpha_m \right\} \Big/ SO_2.$$

The word "abelian" is used because $\mathrm{APol}(\alpha)$ is a symplectic reduction of the path space by the maximal torus SO_2 of SO_3:

$$\mathrm{APol}(\alpha) = \zeta^{-1}(\alpha_m)/SO_2 = \prod_{i=1}^{m-1} S^2_{\alpha_i} \mathbin{\mathop{/\!\!/}_{\alpha_m}} SO_2.$$

The space $\mathrm{APol}(\alpha)$ is visualized as the space of piecewise linear= $(m-1)$-chains (with edge lengths $\alpha_1, \dots, \alpha_{m-1}$) which terminate on the plane $z = \alpha_m$, modulo rotations about the z-axis. The symplectic manifold $\mathrm{APol}(\alpha)$ is of dimension $2(m-2)$ and contains $\mathrm{Pol}(\alpha)$ (those that also end on the z-axis) as a symplectic submanifold of codimension 2.

The $(SO_2)^{m-1}$-action on $\prod_{i=1}^{m-1} S^2_{\alpha_i}$ descends to a Hamiltonian action on $\mathrm{APol}(\alpha)$. It is effective once we divide by the diagonal subgroup SO_2 in $(SO_2)^{m-1}$. With this action, $\mathrm{APol}(\alpha)$ is a toric manifold with moment polytope

$$\Xi_\alpha := \left\{ (x_1, \dots, x_{m-1}) \in \prod_{i=1}^{m-1} [-\alpha_i, \alpha_i] \;\middle|\; \sum_{i=1}^{m-1} x_i = \alpha_m \right\}.$$

The *upper path space* UP (α) is defined as

$$\mathrm{UP}\,(\alpha):=\left\{\rho=(\rho_1,\ldots,\rho_{m-1})\in(\mathbf{R}^3)^{m-1}\,\middle|\,\zeta(\sum_{i=1}^{m-1}\rho_i)\geqslant\alpha_m \text{ and } |\rho_i|=\alpha_i\right\}\Big/\sim$$

where the equivalence relation "$\sim$" is defined as follows: $\rho\sim\rho'$ if $\rho=\rho'$ or if

$$\zeta\left(\sum_{i=1}^{m-1}\rho_i\right)=\alpha_m \quad\text{and } [\rho]=[\rho'] \text{ in APol}\,(\alpha).$$

One can see UP (α) as the result of a symplectic cut (in the sense of [Le]) of the path space $\prod_{i=1}^{m-1} S^2_{\alpha_i}$ at the level α_m of the moment map $\overline{\mu}$. The space UP (α) is thus a closed symplectic manifold of dimension $2(m-1)$ and is a compactification of some open set of $\prod_{i=1}^{m-1} S^2_{\alpha_i}$. It contains APol (α) as a codimension 2 symplectic submanifold (one dimension lost by the height restriction, the other by the circle quotient).

The Hamiltonian action of the torus $(SO_2)^{m-1}$ on the path space descends to an effective action on UP (α). Therefore, UP (α) is a toric manifold with moment polytope

$$\hat{\Xi}_\alpha:=\left\{(x_1,\ldots,x_{m-1})\in\prod_{i=1}^{m-1}[-\alpha_i,\alpha_i]\;\middle|\;\sum_{i=1}^{m-1}x_i\geqslant\alpha_m\right\}.$$

Observe that the codimension 2 submanifold APol $(\alpha)\subset$ UP (α) corresponds to the facet (= codimension 1 face) Ξ_α of the moment polytope $\hat{\Xi}_\alpha$ for UP (α). We shall prove now the important fact that Pol (α) is obtainable as a transverse intersection of APol (α) with itself.

The *vertical path space* is defined by

$$\mathrm{VP}\,(\alpha):=\{\rho\in\mathrm{UP}\,(\alpha)\mid\rho \text{ terminates on the } z\text{-axis}\}\subset\mathrm{UP}\,(\alpha).$$

It is a codimension 2 submanifold of UP (α) which we now show intersects APol (α) transversally in Pol (α). Consider the open subset $W:=\mu^{-1}(\mathbf{R}^3-\{0\})$ of the path space $\prod_{i=1}^{m-1} S^2_{\alpha_i}$.

The map $\rho\mapsto\mu(\rho)/|\mu(\rho)|$ is a fibration $W\to S^2$ (see [Ha, (1.3)]). The fiber V over $(0,0,1)$ is an SO_2-invariant codimension 2 submanifold of W which project onto VP (α). The manifold W intersects $M:=\hat{\mu}^{-1}(\{\alpha_m\})$

transversally in $\mu^{-1}(0,0,\alpha_m)$. The SO_2-action on W induces a SO_2-action on $\mathrm{UP}\,(\alpha)$ for which $\mathrm{APol}\,(\alpha)$ is a set of fixed points. For each $\rho \in \mathrm{Pol}\,(\alpha)$, the tangent space $T_\rho\mathrm{UP}\,(\alpha)$ decomposes

$$T_\rho\mathrm{UP}\,(\alpha) \simeq T_\rho\mathrm{APol}\,(\alpha) \oplus \mathbf{R}\,v \oplus \mathbf{R}\,r(v)$$

where $v \in T_\rho\mathrm{VP}\,(\alpha)$ and $r \in SO_2$ is the rotation of angle $\pi/4$. As $\mathrm{VP}\,(\alpha)$ is SO_2-invariant, the vector $r(v)$ belongs to $T_\rho\mathrm{VP}\,(\alpha)$ and thus $T_\rho\mathrm{UP}\,(\alpha) \simeq T_\rho\mathrm{APol}\,(\alpha) + T_\rho\mathrm{VP}\,(\alpha)$.

PROPOSITION 1.1. — *There is a smooth isotopy* $\varphi_t : \mathrm{VP}\,(\alpha) \to \mathrm{UP}\,(\alpha)$ *such that* $\varphi_0(\rho) = \rho$ *and* $\varphi_1(\mathrm{VP}\,(\alpha)) = \mathrm{APol}\,(\alpha)$.

Proof. — $\varphi_t(\rho)$ is the image of ρ by the rotation about the y-axis of angle

$$t \cos^{-1} \frac{\alpha_m}{\zeta(\rho)} \ .$$

COROLLARY 1.2. — a) $\mathrm{Pol}\,(\alpha)$ *is a transverse intersection of* $\mathrm{APol}\,(\alpha)$ *with itself.*

b) $\mathrm{VP}\,(\alpha)$ *and* $\mathrm{APol}\,(\alpha)$ *are diffeomorphic rel their common* $\mathrm{Pol}\,(\alpha)$.

We note in passing that both $\mathrm{UP}\,(\alpha)$ and $\mathrm{APol}\,(\alpha)$ are symplectomorphic to polygon spaces, though we will not use this fact elsewhere in the paper.

PROPOSITION 1.3. — *For δ big enough, one has symplectomorphisms*

$$\mathrm{UP}\,(\alpha_1,\dots,\alpha_m) \cong \mathrm{APol}\,(\alpha_1,\dots,\alpha_{m-1},\delta-\alpha_m,\delta)$$

$$\mathrm{APol}\,(\alpha_1,\dots,\alpha_m) \cong \mathrm{Pol}\,(\alpha_1,\dots,\alpha_{m-1},\delta+\alpha_m,\delta).$$

Proof. — The symplectomorphisms come from the fact that the above spaces are toric manifolds with isomorphic moment polytopes. For the first one, the moment polytopes are $\Xi_{\alpha_1,\dots,\alpha_{m-1},\delta-\alpha_m,\delta}$ and $\hat{\Xi}_{(\alpha_1,\dots,\alpha_m)}$ and $(x_1,\dots,x_m) \mapsto (x_1,\dots,x_{m-1})$ gives the required isomorphism. The condition on δ is $\delta > \alpha_m$.

For the second case, let $\delta > \sum_{j=1}^{m-1} \alpha_j$. Consider the functions $d_i : \mathrm{Pol}\,(\alpha_1,\dots,\alpha_{m-1},\delta+\alpha_m,\delta) \to \mathbf{R}$ $(i = 0,\dots,m-1)$ given by

$$d_i(\rho) := \Big\| -\rho_m + \sum_{j=1}^{i} \alpha_j \Big\|.$$

Because of the condition on δ, the functions d_i never vanish and are thus smooth. It was shown in [KM] that they Poisson-commute and generate an effective T^{m-2}-action ($d_0 = \delta$ and $d_{m-1} = \delta + \alpha_m$ are constant). This makes $\mathrm{Pol}\,(\alpha_1, \ldots, \alpha_{m-1}, \delta+\alpha_m, \delta)$ a toric manifold with moment polytope Δ the set of $(x_0, \ldots, x_{m-1}) \in \mathbf{R}^m$ satisfying $x_0 = \delta$, $x_{m-1} = \delta + \alpha_m$ and the triangle inequalities:

$$x_{i-1} + \alpha_i \geqslant x_i, \quad x_i + \alpha_i \geqslant x_{i+1}, \quad x_{i-1} + x_i \geqslant \alpha_i$$

for $i = 1, \ldots, m-1$ (see [HK], §5). By the condition on δ, the third inequality is automatically satisfied and the two others amount to $x_i \in [x_{i-1} - \alpha_i, x_{i-1} + \alpha_i]$ ($i = 1, \ldots, m-2$). Changing variables

$$(x_0, \ldots, x_{m-1}) \mapsto \big(x_1 - x_0, x_2 - x_1, \ldots, x_{m-1} - x - m - 2\big)$$

gives an isomorphism between Δ and $\Xi_{(\alpha_1, \ldots, \alpha_{m-1}, \delta+\alpha_m, \delta)}$. □

2. Short and long subsets.

Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbf{R}^m_+$. A subset $J \subset \{1, \ldots, m\}$ is *short* if

$$\sum_{j \in J} \alpha_j \leqslant \sum_{j \notin J} \alpha_j.$$

Equivalently, J is short iff

$$\sum_{j=1}^{m} (-1)^{\chi_J(i)} \alpha_j \geqslant 0$$

where $\chi_S : \mathbf{N} \to \{0, 1\}$ is the characteristic function of S. For example, the empty set is short, and singletons are short iff $\mathrm{Pol}\,(\alpha) \neq \varnothing$. More generally a set S is short exactly if there exist configurations in $\mathrm{Pol}\,(\alpha)$ with all edges in S parallel. Observe that the equalities in the above definitions cannot occur if α is assumed to be generic. Define

$$\mathcal{S} := \mathcal{S}(\alpha) := \big\{J \subset \{1, \ldots, m\} \;\big|\; J \text{ is short}\big\}.$$

The collection $\mathcal{S}$ is partially ordered by inclusion. Every subset of a short subset is short, and thus the poset $\mathcal{S}$ is determined by its maximal elements.

Example 2.1. — For $m = 4$, there are, up to poset isomorphism, two possibilities for $\mathcal{S}$. Listing only the maximal subsets, they are:

a) $\mathcal{S}(\alpha) \supset \{\{1,2\},\{1,3\},\{2,3\},\{4\}\}$; example: $\alpha = (1,1,1,2)$.

b) $\mathcal{S}(\alpha) \supset \{\{1,4\},\{2,4\},\{3,4\}\}$; example: $\alpha = (2,2,2,1)$.

While the length vector $\alpha \in \mathbf{R}^m_+$ defines $\mathrm{Pol}\,(\alpha)$ up to symplectomorphism, we shall see that the diffeomorphism type of $\mathrm{Pol}\,(\alpha)$ is determined by the combinatorial data $\mathcal{S}(\alpha)$.

Define $F_\alpha : \prod\limits_{i=1}^{m} S^2 \longrightarrow \mathbf{R}^3$ by $F_\alpha(z_1,\ldots,z_m) := \sum\limits_{i=1}^{m} \alpha_i z_i$. The map F_α is smooth and SO_3-equivariant for the diagonal action of SO_3 on $\prod\limits_{i=1}^{m} S^2$ and the natural action on $\mathbf{R}^k$. Let $A_\alpha := F_\alpha^{-1}(0)$. By what is in Section 1, the manifold $\mathrm{Pol}\,(\alpha)$ is diffeomorphic to $A(\alpha)/SO_3$ and $A(\alpha)$ is the total space of a principal SO_3-bundle $\xi(\alpha)$.

PROPOSITION 2.2. — *Let α and α' be generic elements in $\mathbf{R}^m_+$. A poset isomorphism $\phi : \mathcal{S}(\alpha) \xrightarrow{\simeq} \mathcal{S}(\alpha')$ determines (up to isotopy) an SO_3-equivariant diffeomorphism between $A(\alpha)$ and $A(\alpha')$.*

Proof. — The poset isomorphism ϕ is first of all a permutation of $\{1,\ldots,m\}$. The correspondence $\rho_i \mapsto \rho_{\phi(i)}$ gives an equivariant diffeomorphism from $A(\alpha_1,\ldots,\alpha_m)$ onto $A(\alpha_{\phi(1)},\ldots,\alpha_{\phi(m)})$. Therefore, one can assume that $\mathcal{S}(\alpha) = \mathcal{S}(\alpha')$ and that $\phi = \mathrm{id}$.

For $t \in [0,1]$, let $\alpha(t) := t\alpha + (1-t)\alpha' \in \mathbf{R}^m_+$. Define $\beta : [0,1] \times \prod\limits_{i=1}^{m} S^2 \to [0,1] \times \mathbf{R}^3$ by $\beta(t,z_1,\ldots,z_m) := (t, F_{\alpha(t)}(z_1,\ldots,z_m))$. The inequalities involved in the definition of $\mathcal{S}(\alpha)$ are all strictly verified for $\alpha(t)$. This shows that $\alpha(t)$ is generic for all t. One deduces that all the points of $[0,1] \times \{0\}$ are regular values of β. The manifold $W := \beta^{-1}([0,1] \times \{0\})$ is then an= SO_3-equivariant cobordism from $M(\alpha)$ to $M(\alpha')$ and $\beta|\, W : W \to [0,1] \times \{0\}$ is an SO_3-invariant map without critical points. Choose an SO_3-invariant Riemannian metric on W. The gradient flow of β on W then produces an SO_3-equivariant diffeomorphism from $A(\alpha)$ to $A(\alpha')$.

□

COROLLARY 2.3. — *If α and α' are generic and $\mathcal{S}(\alpha) \simeq \mathcal{S}(\alpha')$ then there is a diffeomorphism $h : \mathrm{Pol}\,(\alpha) \to \mathrm{Pol}\,(\alpha')$ such that $h^*\xi_{\alpha'} = \xi_\alpha$.*

Remark 2.4. — a) For the examples of 2.1, both $\mathrm{Pol}\,(1,1,1,2)$ and $\mathrm{Pol}\,(2,2,2,1)$ are diffeomorphic to the sphere S^2. But $\xi(1,1,1,2)$ is the non-trivial SO_3-bundle over S^2 whereas $\xi(2,2,2,1)$ is the trivial one (see example 10.3).

b) We have no counterexample to the converse of Corollary 2.3.

c) Proposition 2.2 works for polygons in $\mathbf{R}^k$ (see [Ha]). But, for $k > 3$, even if α is generic, the action of SO_k is not free on $A(\alpha)$ and thus Corollary 2.3 does not make sense.

Let $\alpha \in \mathbf{R}^m_+$ and let $\mathcal{S} := \mathcal{S}(\alpha)$. For $k \in \{1,2,\dots,m\}$, we introduce the subposet $\mathcal{S}_k$ of $\mathcal{S}$:

$$\mathcal{S}_k = \mathcal{S}_k(\alpha) := \{J \subset \{1,\dots,m\} - \{k\} \mid J \cup \{k\} \in \mathcal{S}\}.$$

In the subsequent sections, we give the Poincaré polynomial and presentations of the cohomology ring of Pol$\,(\alpha)$ in terms of $\mathcal{S}_m$. Proposition 2.5 below together with Corollary 2.3 imply that the diffeomorphism type of Pol$\,(\alpha)$ is determined by any of the subposets $\mathcal{S}_k$.

Proposition 2.5. — *Let α and α' be generic elements in $\mathbf{R}^m_+$. Suppose that there are $k, k' \in \{1,2,\dots,m\}$ such that there is a poset isomorphism $\varphi : \mathcal{S}_k(\alpha) \xrightarrow{\simeq} \mathcal{S}'_k(\alpha')$. Then any bijection $\Phi : \{1,2,\dots,m\} \xrightarrow{\simeq} \{1,2,\dots,m\}$ which extends φ and satisfies $\Phi(k) = k'$ is a poset isomorphism from $\mathcal{S}(\alpha)$ onto $\mathcal{S}(\alpha')$.*

Proof. — Let Φ be a permutation of $\{1,2,\dots,m\}$ as in the statement 2.5. By renumbering the components of α using the permutation Φ, one can assume that $k = k'$, $\mathcal{S}_k = \mathcal{S}'_k$ and $\Phi = =$ id. It then suffices to prove that $\mathcal{S}_k = \mathcal{S}'_{k'}$ implies $\mathcal{S} = \mathcal{S}'$.

Let $J \subset \{1,2,\dots,m\}$ and let $\bar{J} := \{1,2,\dots,m\} - J$. If $k \in J$, then $J \in \mathcal{S}$ iff $J - \{k\} \in \mathcal{S}_k$. If $k \notin J$, then $k \in \bar{J}$ and $J \in \mathcal{S}$ iff $\bar{J} - \{k\} \notin \mathcal{S}_k$. This gives a procedure to decide whether or not $J \in \mathcal{S}$ by only knowing $\mathcal{S}_k$. Therefore $\mathcal{S}_k$ determines $\mathcal{S}$. □

A set J which is not short is called *long*. The following notation will be used

$$\mathcal{L} := \big\{J \subset \{1,\dots,m\} \bigm| J \text{ is long}\big\}$$

and

$$\mathcal{L}_m := \big\{J \subset \{1,\dots,m-1\} \bigm| J \cup \{m\} \text{ is long}\big\} \subset \mathcal{L}\,.$$

3. Pairs of even-cohomology manifolds.

We will call a topological space X an *even-cohomology space* if its cohomology groups $H^*(X;\mathbf{Z})$ vanish for $*$ odd. We write $H^*(X)$ for $H^*(X;\mathbf{Z})$.

PROPOSITION 3.1. — *Let M be a closed oriented manifold of dimension n with n even. Let Q be a closed oriented submanifold of M of codimension r. Suppose that Q and $M-Q$ are even-cohomology spaces. Then one has short exact sequences*

$$(1)\qquad 0 \longrightarrow H_{n-*}(M-Q) \longrightarrow H^*(M) \xrightarrow{i^*} H^*(Q) \longrightarrow 0$$

and

$$(2)\qquad 0 \longrightarrow H^{*-r}(Q) \longrightarrow H^*(M) \xrightarrow{j^*} H^*(M-Q) \longrightarrow 0$$

where i^* and j^* are the ring homomorphism induced by the inclusions. In particular, M is an even-cohomology space.

Proof. — Let T be a closed tubular neighborhood of Q in M. Consider the long cohomology exact sequence of the pair (M,T)

$$\cdots \to H^{*-1}(T) \to H^*(M,T) \to H^*(M) \to H^*(T) \to H^{*+1}(M,T) \to \cdots.$$

One has $H^*(T) \cong H^*(Q)$, and excision and Poincaré duality produce the isomorphisms

$$H^*(M,T) \cong H^*(M - \operatorname{int} T, \partial T) \cong H_{n-*}(M - \operatorname{int} T) \cong H_{n-*}(M-Q)$$

which give sequence (1).

Sequence (2) comes from the cohomology exact sequence of the pair $(M, M - \operatorname{int} T)$. Indeed, one has $H^*(M - \operatorname{int} T) \cong H^*(M-Q)$ and the isomorphisms

$$H^*(M, M - \operatorname{int} T) \cong H^*(T, \partial T) \cong H^{*-r}(Q)$$

are given by excision and the Thom isomorphism. □

COROLLARY 3.2. — *Let M and Q be as in 3.1. The Poincaré polynomials of M, Q are calculable from the Poincaré polynomial of $M-Q$ by*

$$(1-t^r)P_Q(t) = P_{M-Q}(t) - t^n P_{M-Q}(1/t)$$

and

$$(1-t^r)P_M(t) = P_{M-Q}(t) - t^{n+r} P_{M-Q}(1/t).$$

Proof. — The two exact sequences of 3.1 give the equations

$$\begin{array}{rclcl} P_M(t) & = & P_Q(t) & + & t^n P_{M-Q}(1/t) \\ P_M(t) & = & t^r P_Q(t) & + & P_{M-Q}(t) \end{array}$$

from which the equations of 3.2 are deduced. □

By 3.1, $H^*(Q)$ is isomorphic to the quotient of $H^*(M)$ by the ideal $\ker i^*$. We shall use the following:

PROPOSITION 3.3. — *Let M and Q be as in 3.1. The kernel of $i^* : H^*(M) \longrightarrow H^*(Q)$ is the annihilator of the cup product by the Poincaré dual class of Q.*

Proof. — That the annihilator contains $\ker i^*$ is a very general fact, as we now show. Let $[M] \in H_n(M)$ and $[Q] \in H_{n-r}(Q)$ be the fundamental classes. The Poincaré dual $q \in H^r(M)$ of Q is determined by the equation $q \cap [M] = i_*([Q])$.

Let $a \in H^*(M)$. By standard properties of cup and cap products (see [Sp], Chapter 5 §6), one has

$$i_*\big(i^*(a) \cap q)\big) = a \cap i_*([Q]) = a \cap (q \cap [M]) = (a \cup q) \cap [M].$$

Therefore, i= $i^*(a) = 0$ then $a \cup q = 0$.

The reverse implication is true if i_* is injective, which will follow from Q and $M - Q$ being even-cohomology spaces. By the universal coefficient theorem, an even-cohomology space is an even-homology space and, as in the proof of 3.1, one gets the short exact sequence

$$0 \longrightarrow H_*(Q) \xrightarrow{i_*} H_*(M) \longrightarrow H^{n-*}(M - Q) \longrightarrow 0.$$

Therefore i_* is injective and $a \cup q = 0$ implies that $i^*(a) = 0$. □

4. Poincaré polynomials of polygon spaces.

Before working out the cohomology rings of the polygon spaces in the next section, we give here their Betti numbers, in the form of the Poincaré polynomial. These are easy to obtain from Corollary 3.2. Different formulae for the Poincaré polynomial of $\mathrm{Pol}\,(\alpha)$ were already obtained in [Kl], §2.2, by different methods, as well as the following lemma ([Kl], Corollary 2.2.2):

LEMMA 4.1. — Pol(α) *is an even-cohomology space.*

Proof. — Let us consider the diagonal-length function $\delta : \mathrm{Pol}(\alpha) \to \mathbf{R}$ given by $\delta(\rho) := |\rho_m - \rho_{m-1}|$. It is smooth if $\alpha_{m-1} \neq \alpha_m$ which can be assumed since by Proposition 2.2 changing the α_i's slightly for a generic α does not change the diffeomorphism class of Pol(α).

By [Ha, Theorem 3.2], δ is a Morse-Bott function. The critical points are of even index and are isolated except possibly for the two extrema. The pre-image $M_{\max}$ of the maximum is either a point or $\mathrm{Pol}(\alpha_1, \ldots, \alpha_{m-2}, \alpha_m + \alpha_{m-1})$. For the pre-image $M_{\min}$ of the minimum, there are three possibilities:

- one point
- $\mathrm{Pol}(\alpha_1, \ldots, \alpha_{m-2}, \alpha_m - \alpha_{m-1})$
- a 2-sphere bundle over $\mathrm{Pol}(\alpha_1, \ldots, \alpha_{m-2}, \alpha_m - \alpha_{m-1})$ (when the minimum is 0).

This enables us to prove Lemma 4.1 by induction on m. A 2-sphere bundle over an even cohomology space is an even cohomology space using the Gysin sequence. Therefore, in all the cases $M_{\min}$ and therefore $\mathrm{Pol}(\alpha) - M_{\max}$ are even cohomology manifolds. If $M_{\max}$ is a point, we are done. Otherwise, we use Proposition 3.1 to deduce that Pol(α) is an even cohomology manifold. □

We now use the inclusion $\mathrm{Pol}(\alpha) \subset \mathrm{APol}(\alpha)$ to obtain the Poincaré polynomials for the various polygon spaces. They are given in terms of the posets $\mathcal{S} := \mathcal{S}(\alpha)$ or $\mathcal{S}_m := \mathcal{S}_m(\alpha)$ introduced in § 2.

PROPOSITION 4.2. — *The open manifolds* $\mathrm{APol}(\alpha) - \mathrm{Pol}(\alpha)$ *and* $\mathrm{UP}(\alpha) - \mathrm{APol}(\alpha)$ *are both even-cohomology spaces with the same Poincaré polynomial,* $\sum_{J \in \mathcal{S}_m} t^{2|J|}$.

Proof. — Using 1.2, one can replace APol(α) by VP(α). Consider the function $d : \mathrm{UP}(\alpha) - \mathrm{APol}(\alpha)$ defined by $d(\rho) := -\zeta\Big(\sum_{i=1}^{m-1} \rho_i\Big)$ and denote by d^v its restriction to $\mathrm{VP}(\alpha) - \mathrm{Pol}(\alpha)$.

By [Ha, Theorem 3.2] the map d^v is a Morse function with a critical point of index $2|J|$ for each $J \in \mathcal{S}_m$ (the critical point is the lined configuration with all the ρ_i pointing upwards if $i \notin S$ and downward otherwise). This proves 4.2 for APol(α).

A small rotation around a horizontal axis will decrease $\zeta\Big(\sum_{i=1}^{m-1}\rho_i\Big)$ and so increase d. The slope is positive away from $\mathrm{VP}(\alpha)$ and thus d has no critical points other than those of d^v. At one of these critical points, the rotation can be used to check the non-degeneracy and show that the index is the same for d as for d^v. □

The above two results, using 3.2, give the following:

COROLLARY 4.3. — *The various polygon spaces are even-cohomology spaces. Their Poincaré polynomials are*

$$P_{\mathrm{Pol}\,(\alpha)} = \frac{1}{1-t^2}\sum_{J\in\mathcal{S}_m}\left(t^{2|J|} - t^{2(m-|J|-2)}\right)$$

$$P_{\mathrm{APol}\,(\alpha)} = \frac{1}{1-t^2}\sum_{J\in\mathcal{S}_m}\left(t^{2|J|} - t^{2(m-|J|-1)}\right)$$

$$P_{\mathrm{UP}\,(\alpha)} = \frac{1}{1-t^2}\sum_{J\in\mathcal{S}_m}\left(t^{2|J|} - t^{2(m-|J|)}\right).$$

Remark 4.4. — The following expression for $P_{\mathrm{Pol}\,(\alpha)}$ was obtained, using another method, by Klyachko [Kl, Theorem 2.2.4]:

$$P_{\mathrm{Pol}\,(\alpha)} = \frac{1}{t^2(t^2-1)}\left((1+t^2)^{m-1} - \sum_{J\in\mathcal{S}} t^{2|J|}\right).$$

This formula gives $P_{\mathrm{Pol}\,(\alpha)}$ in terms of $\mathcal{S}(\alpha)$ whereas those of 4.3 are in terms of $\mathcal{S}_m(\alpha)$. This illustrates that $\mathcal{S}_m$ determines $\mathcal{S}$ (Proposition 2.5). Both expressions have advantages: the one in terms of $\mathcal{S}$ is more symmetric whereas those using $\mathcal{S}_m$ have many fewer monomials and lead to easier computations (see §10).

COROLLARY 4.5. — *If m is odd, so that the dimension of* $\mathrm{Pol}\,(\alpha)$ *is a multiple of 4, the signature of the polygon space* $\mathrm{Pol}\,(\alpha)$ *is* $\sum_{J\in\mathcal{S}_m}(-1)^{|J|}$.

Proof. — In [Kl] it is proven that the Hodge numbers h^{pq} of the Kähler manifold $\mathrm{Pol}\,(\alpha)$ vanish except for the diagonal h^{pp}. Then the Hodge signature theorem [GrHa] implies that the signature is the Poincaré polynomial evaluated at $t = i$. □

This in turn is the Euler characteristic of the associated planar polygon space (discussed further in Section 9), and one plus that of the poset $\mathcal{S}_m - \{\varnothing\}$ which is a simplicial complex.

5. The cohomology of the upper path space.

In this section we give a presentation of the cohomology ring of the upper path space, the toric manifold with moment polytope $\hat{\Xi}_\alpha$. The cohomology ring of a toric manifold is given by Danilov's theorem (see [Fu, Chapter 5], [DJ, Theorem 4.14]) which we recall below in a version useful for us.

Let M^{2n} be a compact symplectic toric manifold (acted on by the standard torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$). Suppose that the moment polytope $\Delta := \mu(M) \subset \mathbf{R}^n$ is given by a family of inequalities indexed by a finite set $\mathcal{J}$:

$$\Delta = \{x \in \mathbf{R}^n \mid \langle x, w_j\rangle \leqslant \lambda_j,\ j \in \mathcal{J}\}$$

where $w_j \in \mathbf{Z}^n$ is primitive and $\lambda_j \in \mathbf{R}$. Let $\mathcal{F}_j$ be the hyperplane $\{x \in \mathbf{R}^n \mid \langle x, w_j\rangle = \lambda_j\}$. We suppose that the $\mathcal{F}_j$'s are distinct. As $\mathcal{J}$ is finite, the facet-hyperplanes of Δ must belong to the family and will be indexed by $\mathcal{J}_0 \subset \mathcal{J}$. Observe that

$$j \in \mathcal{J}_0 \quad \Leftrightarrow \quad \operatorname{codim}\big(\mathcal{F}_j \cap \Delta\big) = 1.$$

Let $\{e_1, \dots, e_n\}$ be a basis of $\mathbf{R}^n$. Danilov's theorem gives a presentation of the ring $H^*(M)$ with a generator $F_j \in H^2(M)$ for each hyperplane $\mathcal{F}_j$:

THEOREM 5.1 (Danilov). — *The cohomology ring $H^*(M)$ is the quotient of the polynomial ring $\mathbf{Z}[F_j;\, j \in \mathcal{J}]$, where each F_j is of degree 2, by the ideal $\mathcal{I}$ generated by the two families of relators:*

$$\sum_{j\in\mathcal{J}} \langle e_i, v_j\rangle F_j \quad i = 1, \dots, n \qquad \textit{(linear relators)}$$

$$\prod_{j\in\mathcal{B}} F_j \qquad \textit{if} \quad \operatorname{codim} \bigcap_{j\in\mathcal{B}} (\mathcal{F}_j \cap \Delta) > |\mathcal{B}| \qquad \textit{(intersection monomials).}$$

Remark 5.2. — 1. The statements of Danilov's theorem in the literature are only for $\mathcal{J} = \mathcal{J}_0$, but any generator F_j for $j \notin \mathcal{J}_0$ is in $\mathcal{I}$ using b) with $\mathcal{B} = \{j\}$.

2. When $j \in \mathcal{J}_0$, the preimage $\mu^{-1}(\mathcal{F}_j)$ is a codimension 2 submanifold representing the Poincaré dual class of F_j.

3. The class $[\omega] \in H^2(M;\mathbf{R})$ of the symplectic form satisfies

$$[\omega] = \sum_{j\in\mathcal{J}} \lambda_j F_j$$

(see [Gu, p. 132]; the different sign comes from the fact that our vectors w_j are pointing out of Δ, contrarily to those in [Gu]).

We now apply 5.1 to $\Delta = \hat{\Xi}_\alpha$. Let $e_1, \dots, e_{m-1}$ be the standard basis of $\mathbf{R}^{m-1}$. The polytope $\hat{\Xi}_\alpha$ is the subset of $\mathbf{R}^{m-1}$ subject to the inequalities

$$\begin{array}{ll} \langle x, e_i\rangle \leqslant \alpha_i & j = 1,\dots m-1 \\ \langle x, -e_i\rangle \leqslant \alpha_i & j = 1,\dots m-1 \\ \langle x, -\sum\limits_{i=1}^{m-1} e_i\rangle \leqslant -\alpha_m. & \end{array}$$

The relevant hyperplanes will be called

$$\mathcal{U}_i := \{\langle x, e_i\rangle = \alpha_i\}\,,\ \mathcal{V}_i := \{\langle x, e_i\rangle = -\alpha_i\} \ \text{ and } \mathcal{R} := \left\{ \langle x, \sum_{i=1}^{m-1} e_i\rangle = \alpha_m = \right\}$$

with corresponding classes $U_i, V_i, R \in H^2(\mathrm{UP}\,(\alpha))$. Set

$$\tilde{\mathcal{U}}_i := \hat{\mu}^{-1}(\mathcal{U}_i)\ ,\ \tilde{\mathcal{V}}_i := \hat{\mu}^{-1}(\mathcal{V}_i)\ ,\ \tilde{\mathcal{R}} := \hat{\mu}^{-1}(\mathcal{R}).$$

The first two are those polygons whose ith step points straight up, or straight down; the third is the abelian polygon space.

If $A \subset \{1,\dots,m-1\}$, define $\tilde{\mathcal{U}}_A := \bigcap\limits_{i\in A} \tilde{\mathcal{U}}_i$ or $\tilde{\mathcal{V}}_A := \bigcap\limits_{i\in A} \tilde{\mathcal{V}}_i$.

LEMMA 5.3. — *Let $A, B \subset \{1,\dots,m-1\}$ such that $A\cap B = \varnothing$. Then the image under $\hat{\mu} : \mathrm{UP}\,(\alpha) \longrightarrow \mathbf{R}$ of $\tilde{\mathcal{U}}_A \cap \tilde{\mathcal{V}}_B$ is the interval*

$$\hat{\mu}(\tilde{\mathcal{U}}_A \cap \tilde{\mathcal{V}}_B) = \left[-\sum_{i=1}^{m-1} (-1)^{\chi_A(i)}\,\alpha_i\,,\ \sum_{i=1}^{m-1} (-1)^{\chi_B(i)}\,\alpha_i \right] \cap [\alpha_m, \infty).$$

Proof. — Recall that $\hat{\mu}(\rho)$ is the height of the endpoint of ρ. The highest it can get is when all edges point straight up (except those in B, required to point down); the lowest is when all edges point straight down (except for those in A) or at $z = \alpha_m$. □

We now work out the presentation of $H^2(\mathrm{UP}(\alpha))$ given by Danilov's theorem with all the generators U_i, V_i and R. Recall that $\mathcal{L}$ is the collection of long subsets of $\{1,\ldots,m\}$ and $\mathcal{L}_m$ the collection of subsets $L \subseteq \{1,\ldots,m-1\}$ such that $L \cup \{m\} \in \mathcal{L}$.

PROPOSITION 5.4. — *The ring $H^*(\mathrm{UP}(\alpha))$ is the quotient of the polynomial ring generated in degree 2 by the classes R, U_i and V_i ($i = 1,\ldots,m-1$), divided by the ideal generated by the following relators:*

$$\begin{array}{lll} \text{(a)} & U_i - V_i - R & i = 1,\ldots,m-1 \\ \text{(b)} & U_i V_i & i = 1,\ldots,m-1 \\ \text{(c)} & \prod_{i\in L} V_i & L \subset \{1,\ldots,m-1\} \text{ and } L \in \mathcal{L}_m \\ \text{(d)} & R\prod_{i\in L} U_i & L \subset \{1,\ldots,m-1\} \text{ and } L \in \mathcal{L}. \end{array}$$

Proof. — The relators (a) are the linear relators of Danilov's theorem. Clearly $\mathcal{U}_i \cap \mathcal{V}_i = \varnothing$ (an edge cannot point both up and down) whence relators (b).

Suppose that $\rho \in \mathcal{V}_L$. If $L \in \mathcal{L}_m$ then $\sum_{i=1}^{m-1} (-1)^{\chi_L(i)} \alpha_i < \alpha_m$. By Lemma 5.3, one has

$$\hat{\mu}(\rho) \leqslant \sum_{i=1}^{m-1} (-1)^{\chi_B(i)} \alpha_i < \alpha_m$$

which contradicts $\rho \in \mathrm{UP}(\alpha)$. Therefore $\mathcal{V}_L = \varnothing$ if $L \in \mathcal{L}_m$ which gives relators (c). In words, a path that steps down too much cannot end above $z \geqslant \alpha_m$.

Similarly, if $L \subset \{1,\ldots,m-1\}$ and $\rho \in \mathcal{U}_A$, then

$$\alpha_m < -\sum_{i=1}^{m-1} (-1)^{\chi_A(i)} \alpha_i \leqslant = \hat{\mu}(\rho)$$

and thus $\mathcal{R} \cap \mathcal{U}_A = \varnothing$ which produces relators (d). In words, a path that steps up too much cannot end at $z = \alpha_m$.

We have thus proved that the families (b)–(d) are indeed intersection monomials. We now prove that any intersection monomial is a multiple of these. By Danilov's theorem, an intersection monomial C is square-free so of the form

$$C = \prod_{i\in A} U_i \prod_{j\in B} V_j \quad \text{or} \quad C = R\prod_{i\in A} U_i \prod_{j\in B} V_j.$$

If $i \in A \cap B$ then C is a multiple of $U_i V_i$. Therefore, one may suppose that $A \cap B = \varnothing$.

If $C = \prod_{i \in A} U_i \prod_{j \in B} V_j$ is an intersection monomial, then $\operatorname{codim}(\mathcal{U}_A \cap \mathcal{V}_B \cap \hat{\Xi}_\alpha) \geqslant |A \cup B|$. So by Lemma 5.3 we know $\sum_{i=1}^{m-1} (-1)^{\chi_B(i)} \alpha_i \leqslant \alpha_m$. But genericity implies that this inequality is strict. Therefore, the inequality $\sum_{i=1}^{m-1} (-1)^{\chi_B(i)} \alpha_i < \alpha_m$ holds, that is $B \in \mathcal{L}_m$ and thus C is a multiple of $\prod_{j \in B} V_j$, an intersection monomial in (c).

Consider now the case $C = R \prod_{i \in A} U_i \prod_{j \in B} V_j$. Thus $\operatorname{codim}(\mathcal{U}_A \cap \mathcal{V}_B \cap \mathcal{R} \cap \hat{\Xi}_\alpha) \geqslant |A \cup B| + 1$. By Lemma 5.3, this would not happen if

$$-\sum_{i=1}^{m-1} (-1)^{\chi_A(i)} \alpha_i < \alpha_m < \sum_{i=1}^{m-1} (-1)^{\chi_B(i)} \alpha_i$$

and equalities never occur. We saw before that the inequality $\sum_{i=1}^{m-1} (-1)^{\chi_B(i)} \alpha_i < \alpha_m$ makes C a multiple of an intersection monomial of (c). The other possibility is $\alpha_m < -\sum_{i=1}^{m-1} (-1)^{\chi_A(i)} \alpha_i$ which is equivalent to $A \in \mathcal{L}$ and makes C a multiple of the intersection monomial $R \prod_{i \in A} U_i$ of (d). □

The presentation of $H^*(\mathrm{UP}(\alpha))$ which will turn out to be useful is the following one:

THEOREM 5.5. — *The ring $H^*(\mathrm{UP}(\alpha))$ is the quotient of the polynomial ring generated in degree 2 by the classes R and V_i for $1 \leqslant i \leqslant m-1$, divided by the ideal $\mathcal{I}$ generated by the following families of relators:*

(R1) $V_i^2 + RV_i$ $\qquad 1 \leqslant i \leqslant m-1$

(R2) $\prod_{i \in L} V_i$ $\qquad L \in \mathcal{L}_m$

(R3) $R^2 \sum_{\substack{S \subset L \\ S \in \mathcal{S}_m}} \left(\prod_{i \in S} V_i \right) R^{|L-S|-1}$ $\qquad L \subset \{1, \ldots, m-1\}$ *and* $L \in \mathcal{L}$.

Proof. — This presentation is obtained by algebraically transforming that of Proposition 5.4. The linear relations $U_i = V_i + R$ of 5.4 are absorbed by reducing the family of generators to R and the V_i's. Replacing U_i by

$V_i + R$ in monomials (b) gives relators (R1). Relators (R2) are just relators (c) (one could of course restrict to minimal sets $L \in \mathcal{L}_m$). Relators (R3) are obtained by expanding monomials (d):

$$R\prod_{i\in L} U_i = R\sum_{S\subset L}\left(\prod_{i\in S} V_i\right) R^{|L-S|} = R\sum_{\substack{S\subset L\\ S\in\mathcal{S}_m}}\left(\prod_{i\in S} V_i\right) R^{|L-S|}$$

(the second equality is obtained thanks to relators (R2) which kill $\Pi_S V_i$ if $S \notin= \mathcal{S}_m$). As $L \in \mathcal{L}$ and $S \in \mathcal{S}_m$, one has $S \neq L$. Therefore $|L - S| \geqslant 1$ and one can pull out one more R to get relators (R3). □

6. The cohomology rings of APol (α) and Pol (α).

THEOREM 6.1. — *The cohomology rings of the abelian polygon space and actual polygon space can be obtained from that of the upper path space as follows:*

$$H^*(\mathrm{APol}\,(\alpha)) \cong H^*(\mathrm{UPol}\,(\alpha))/\mathrm{Ann}(R)$$
$$H^*(\mathrm{Pol}\,(\alpha)) \cong H^*(\mathrm{UPol}\,(\alpha))/\mathrm{Ann}(R^2)$$

where $\mathrm{Ann}(x) := \{y \in H^*(\mathrm{UPol}(\alpha)) : yx = 0\}$.

Proof. — By construction, R is Poincaré dual in UP (α) to APol (α). By Proposition 1.2, the Poincaré dual to Pol (α) in UP (α) is R^2. All the spaces under consideration are even-cohomology spaces by 4.1 and 4.2. Therefore the result follows from Proposition 3.3. □

It remains to calculate these annihilators, or equivalently the "ideal quotients"

$$\mathcal{I} : R^k := \{y \in \mathbf{Z}[V_i, R] : R^k y \in \mathcal{I}\}, \qquad k = 1, 2$$

where $\mathcal{I}$ is the ideal found in Theorem 5.5 defining $H^*(\mathrm{UPol}\,(\alpha))$. Manifestly these contain the families (1), (2), and $R^{-k}(3)$ (recall that R^2 divides all the relators in the third family). We will show that these do in fact generate the ideal quotients.

If (1)-(3) were a *Gröbner basis* for the ideal, this would be straightforward (see the lemma below); it is not in general, but we will show that it is close enough.

We take the computational viewpoint of Gröbner bases, that they provide a recognition algorithm for elements of an ideal – a polynomial is an element of $\mathcal{I}$ if the reduction algorithm (defined below) can reduce it to zero. Conveniently, any list of generators of an ideal can be finitely extended to a Gröbner basis by adding S-polynomials (also defined below); if all S-polynomials reduce to 0, the basis is Gröbner. While all necessary definitions are given here, our reference for these theorems is [Ei].

Given a polynomial p we wish to check for $\mathcal{I}$-membership, a well-ordering of all monomials respecting multiplication ($a < b$ implies $ac < bc$ for all a, b, c), and a list $\{r_i\}$ of generators of the ideal, the reduction algorithm is defined as follows. Within each r_i is an initial monomial m_i (with respect to the well-ordering). If one of those m_i divides a monomial $m_i l$ of p, "reduce" p to $p - r_i l$ (which is in $\mathcal{I}$ exactly if p itself was). This kills the $m_i l$ in p. This algorithm terminates; $\{r_i\}$ is called a Gröbner basis if $p \in \mathcal{I}$ implies that it terminates at zero, *no matter what order the reductions take place.* This powerful independence makes it very easy to prove things about Gröbner bases.

Given two relations r_1, r_2 with initial monomials m_1, m_2, there are two ways to reduce the monomial $lcm(m_1, m_2)$: to $\dfrac{m_2}{gcd(m_1, m_2)}(m_1 - r_1)$ and to $\dfrac{m_1}{gcd(m_1, m_2)}(m_2 - r_2)$. Their difference

$$S(r_1, r_2) := \frac{m_2}{gcd(m_1, m_2)}(r_1 - m_1) - \frac{m_1}{gcd(m_1, m_2)}(r_2 - m_2)$$

is called the *S-polynomial* of r_1 and r_2, and is manifestly in the ideal; if the list $\{r_i\}$ cannot reduce these to 0, it certainly isn't Gröbner. There are two convenient converses to this fact [Ei]:

(1) if all the S-polynomials do reduce to 0, the list $\{r_i\}$ is a Gröbner basis;

(2) if not, one can add those S-polynomials as new elements of the list, a process that eventually terminates at a Gröbner basis.

The following lemma points out the relevance of Gröbner bases to calculating $\mathcal{I} : R^k$. In what follows we use a *reverse lexicographic order* for R; this means that monomials are ordered first by their power of R (with low powers earlier in the order), and only then by other criteria (which we leave unspecified).

LEMMA 6.2. — *Let $\mathcal{I} \leqslant \mathbf{Z}[x_i, R]$ be a homogeneous ideal in a polynomial ring, and $\{r_i\}$ be a Gröbner basis of $\mathcal{I}$, with respect to a "revlex" order for R. Then $\{r_i/gcd(r_i, R^k)\}$ is a Gröbner basis for $\mathcal{I} : R^k$.*

Proof. — First we show that each $r_i/gcd(r_i, R^k)$ is in fact in $\mathcal{I} : R^k$:

$$R^k r_i/gcd(r_i, R^k) = \frac{R^k}{gcd(r_i, R^k)} r_i \in \mathcal{I}.$$

Second, that this list $\{r_i/gcd(r_i, R^k)\}$ is powerful enough to reduce any element p of $\mathcal{I} : R^k$ to zero. To see this, follow the reductions of $R^k p$, an element of $\mathcal{I}$, by the (assumed) Gröbner basis $\{r_i\}$. The possible reductions of $R^k p$ using $\{r_i\}$ correspond exactly to possible reductions of p using $\{r_i/gcd(r_i, R^k)\}$, because reducing $R^k p$ using r_i necessarily uses a multiple of $lcm(R^k, r_i)$, which we divide by R^k to get the corresponding reduction of p using $lcm(R^k, r_i)/R^k = r_i/gcd(r_i, R^k)$. □

This technique of linking one reduction algorithm to another will be used again in what follows; we will say that the reductions are *parallel* in $(p, \{r_i\})$ and $(p', \{r_i'\})$ given a correspondence between possible reductions of p using $\{x_i\}$ and possible reductions of p' using $\{x_i'\}$.

Unfortunately, the list of relations $(R1) - (R3)$ in Theorem 5.5 is not generally a Gröbner basis, and extending it to one seems difficult – in particular, defining the problem would require more precise specification of the monomial order, such as an ordering on the edges. Luckily, this list is close enough to being Gröbner to calculate the annihilators we need.

THEOREM 6.3. — *Let $\mathcal{I} \leqslant \mathbf{Z}[\{x_i\}, R]$ be a homogeneous ideal in a polynomial ring, and $\{r_i\}$ generate $\mathcal{I}$ as an ideal. Assume that all S-polynomials of pairs r_i, r_j, such that neither is a multiple of R^k, reduce to zero with respect to an elimination order for R. Then $\{r_i/gcd(r_i, R^k)\}$ generates $\mathcal{I} : R^k$ as an ideal.*

This is a weaker requirement than in Lemma 6.2, which required that *all* S-polynomials reduce to zero; this will let us ignore the third family of relators in 5.5.

Proof. — The argument is this: we complete $\{r_i\}$ to a Gröbner basis for $\mathcal{I}$, and show that the parallel completion of $\{r_i/gcd(r_i, R^k)\}$ is to a Gröbner basis of $\mathcal{I} : R^k$. Therefore $\{r_i/gcd(r_i, R^k)\}$ generates $\mathcal{I} : R^k$.

Let r_1, r_2 be generators such that r_2 is divisible by R^k. Consider the S-polynomial $s := S(r_1, r_2)$. We claim that the reductions are parallel for $(s, \{r_i\})$ and $(s/R^k, \{r_i/gcd(r_i, R^k)\})$.

For this to make sense, we first must establish that $R^k | s$. Let R^j be the highest power of R dividing r_1. Then by our assumption on the order, $R^j | m_1$ and $R^k | m_2$. In our formula for

$$s := \frac{m_2}{gcd(m_1, m_2)}(m_1 - r_1) - \frac{m_1}{gcd(m_1, m_2)}(m_2 - r_2)$$

we can then see that $R^{k-j} | \frac{m_2}{gcd(m_1, m_2)}$, $R^j | (m_1 - r_1)$, and $R^k | (m_2 - r_2)$, so $R^k | S$.

Second, we must establish a correspondence between the possible reductions. This is as before: reducing s by adding a multiple of r_i necessarily adds a multiple of $lcm(r_i, R^k)$, which corresponds to reducing s/R^k by adding a multiple of $lcm(r_i, R^k)/R^k = r_i/gcd(r_i, R^k)$.

Now consider the process of extending $\{r_i\}$ to a Gröbner basis by tossing in a S-polynomial which cannot reduce to zero. By the assumption, it must be of the above type (one of the relations is divisible by R^k), at which point it parallels an S-polynomial in $\{r_i/gcd(r_i, R^k)\}$. What this establishes is that the $\{r_i/gcd(r_i, R^k)\}$ can generate a Gröbner basis of $\mathcal{I} : R^k$. In particular they generate $\mathcal{I} : R^k$. □

In the case at hand, with the three families of relations

(R1) $V_i^2 + RV_i$ $\qquad 1 \leqslant i \leqslant m$

(R2) $\prod_{\substack{i \in L \\ \{i\} \in \mathcal{S}_m}} V_i$ $\qquad L \in \mathcal{L}_m$

(R3) $R^2 \sum_{\substack{S \subset L \\ S \in \mathcal{S}_m}} \left(\prod_{i \in S} V_i \right) R^{|L-S|-1}$ $\qquad L \subset \{1, \dots, m-1\}$ *and* $L \in \mathcal{L}$

we have to check the (R1)-(R1), (R1)-(R2), and (R2)-(R2) S-polynomials – the lemma lets us ignore the S-polynomials with (R3), since those are all divisible by R^2.

One standard observation [Ei] is that the S-polynomial of two elements (r_1, r_2) with relatively prime initial terms m_1, m_2 is necessarily trivial. Here the initial terms are

(R1) V_i^2 for (R1) and $\prod_{j\in L} V_j$ for (R2).

(R1)–(R1): Each pair of initial terms is relatively prime.

(R1)–(R2): V_i^2 and $\prod_{j\in L} V_j$ have greatest common divisor V_i if $i \in L$, and are otherwise relatively prime.

$$S(V_i^2 + RV_i, \prod_{j\in L} V_j) = \left(\prod_{j\in L-\{i\}} V_j\right)(RV_i) - 0 = R\prod_{j\in L} V_j \equiv 0.$$

(R2)–(R2): The S-polynomial of two monomial relations is automatically zero, no reduction necessary.

Using 5.5 and 6.1, we have just proved

THEOREM 6.4. — 1) *The cohomology ring of the abelian polygon space* APol(α) *is*

$$H^*(\mathrm{APol}(\alpha)) = \mathbf{Z}[R, V_1, \ldots, V_{m-1}]/\mathcal{I}_{\mathrm{APol}}$$

where R and V_i are of degree 2 and $\mathcal{I}_{\mathrm{APol}}$ is the ideal generated by the three families

$$\begin{array}{lll} \text{(R1)} & V_i^2 + RV_i & i = 1, \ldots, m-1 \\ \text{(R2)} & \prod_{i\in L} V_i & L \in \mathcal{L}_m \\ \text{(R3)} & R\sum_{\substack{S\subset L\\ S\in\mathcal{S}_m}}\left(\prod_{i\in S} V_i\right)R^{|L-S|-1} & L \subset \{1,\ldots,m-1\} \textit{ and } L \in \mathcal{L}. \end{array}$$

2) *The cohomology ring of the actual polygon space* Pol(α) *is*

$$H^*(\mathrm{Pol}) = \mathbf{Z}[R, V_1, \ldots, V_{m-1}]/\mathcal{I}_{\mathrm{Pol}}$$

where R and V_i are of degree 2 and $\mathcal{I}_{\mathrm{Pol}}$ is generated by the three families

$$\begin{array}{lll} \text{(R1)} & V_i^2 + RV_i & i = 1, \ldots, m-1 \\ \text{(R2)} & \prod_{i\in L} V_i & L \in \mathcal{L}_m \\ \text{(R3)} & \sum_{\substack{S\subset L\\ S\in\mathcal{S}_m}} = \left(\prod_{i\in S} V_i\right)R^{|L-S|-1} & L \subset \{1,\ldots,m-1\} \textit{ and } L \in \mathcal{L}. \end{array}$$

As an example, we give the expression of the class $[\omega] \in H^2(\mathrm{Pol}\,(\alpha)); \mathbf{R})$ of the symplectic form in terms of the generators R and V_i:

PROPOSITION 6.5. — *The class* $[\omega] \in H^2(\mathrm{Pol}\,(\alpha)); \mathbf{R})$ *is given by*

$$[\omega] \;=\; \Big(- \alpha_m + \sum_{j=1}^{m-1} \alpha_j \Big) R + \;= 2 \sum_{\{j\} \in \mathcal{S}_m} \alpha_j V_j .$$

Proof. — From Remark 3 of 5.2, one gets

$$[\omega] \;=\; -\alpha_m R + \sum_{i=1}^{m-1} \alpha_i \big(U_i - V_i\big)$$

which is put in the required form by using the relations $U_i = V_i + R$ of 5.4. Observe that the formula of 6.5 is actually also valid in $H^2(\mathrm{UP}\,(\alpha); \mathbf{R})$ and in $H^2(\mathrm{APol}\,(\alpha); \mathbf{R})$. □

As a consequence, one has a sufficient condition for the class $[\omega] \in H^2(\mathrm{Pol}\,(\alpha)); \mathbf{R})$ to be integral:

COROLLARY 6.6. — *If* $\alpha \in \mathbf{Z}^m$ *then* $[\omega] \in H^2(\mathrm{Pol}\,(\alpha); \mathbf{Z})$.

7. Natural bundles over polygon spaces.

Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbf{R}^m_+$ be generic. For $j \in \{1, \ldots, m\}$ we define $A_j := A_j(\alpha) \subset (\mathbf{R}^3)^m$ by

$$A_j := \left\{ \rho = (\rho_1, \ldots, \rho_m) \in (\mathbf{R}^3)^m \,\middle|\, |\rho_i| = \alpha_i \text{ and } \sum_{i=1}^{m} \rho_i = 0 \text{ and } \rho_j = (0, 0, \alpha_j) \right\}.$$

As α is generic, the diagonal SO_2-action on $(\mathbf{R}^3)^m$ is free on A_j and one has $\mathrm{Pol}\,(\alpha) = A_j/SO_2$. Therefore, $A_j \to \mathrm{Pol}\,(\alpha)$ is a principal SO_2-bundle ξ_j determined by its Chern (or Euler) class $c_j := c_1(\xi_j) \in H^2(\mathrm{Pol}\,(\alpha); \mathbf{Z})$.

As in Section 2, let us consider

$$A := A(\alpha) := \left\{ (\rho_1, \ldots, \rho_m) \in (\mathbf{R}^3)^m \,\middle|\, \sum_{i=1}^{m} \rho_i = 0 \text{ and } |\rho_i| = \alpha_i \right\} \subset (\mathbf{R}^3)^m.$$

As α is generic, the quotient map $A \longrightarrow \mathrm{Pol}\,(\alpha)$ is a principal SO_3-bundle denoted by $\xi := \xi(\alpha)$ (write the elements of $\mathbf{R}^3$ as row vectors, so that SO_3 acts on the right on them). The bundle ξ is determined by its Stiefel-Whitney class $w_2(\xi) \in H^2(\mathrm{Pol}\,(\alpha); \mathbf{Z}_2)$ and its Pontrjagin class $p := p_1(\xi) \in H^4(\mathrm{Pol}\,(\alpha); \mathbf{Z})$.

PROPOSITION 7.1. — *For each $j \in \{1, \dots, m\}$ the bundle ξ_j is a SO_2-reduction of ξ, that is, A is SO_3-equivariantly diffeomorphic to $A_j \times_{SO_2} SO_3$.*

Proof. — The SO_3-equivariant diffeomorphism from $A_j \times_{SO_2} SO_3$ onto A is given by $(\rho, \beta) \mapsto (\rho)\beta$. □

COROLLARY 7.2. — *For each $j \in \{1, \dots, m\}$ one has $c_j^2 = p$ and $c_j = w_2(\xi) \bmod 2$.*

Proof. — The first equation is shown in [MS], Corollary 15.8, p 179. The second is classical between Euler and Stiefel-Whitney classes ([MS], Property 9.5, p. 99). □

By 6.4 the Chern classes c_j are expressible in terms of the classes R and V_j. The formulae are:

PROPOSITION 7.3. — *In $H^2(\mathrm{Pol}\,(\alpha); \mathbf{Z})$, one has*

$$c_i = \begin{cases} R + 2V_i & \text{for } i = 1, \dots, m-1 \\ -R & \text{for } i = m. \end{cases}$$

(In particular $c_i = R$ if $\{i\} \in \mathcal{L}_m$.)

Proof. — Define $B \subset (\mathbf{R}^3)^{m-1}$ by

$$B := \left\{ (\rho_1, \dots, \rho_{m-1}) \in (\mathbf{R}^3)^m \;\middle|\; |\rho_i| = \alpha_i \text{ and } \zeta\left(\sum_{i=1}^m \rho_i\right) = \alpha_m \right\}.$$

As α is generic, SO_2 acts freely on B making it a principal SO_2-bundle ψ over $\mathrm{APol}\,(\alpha)$. One has a commutative diagram

$$\begin{array}{ccc} A_m & \xrightarrow{\tilde{i}} & B \\ \downarrow & & \downarrow \\ \mathrm{Pol}\,(\alpha) & \xrightarrow{i} & \mathrm{APol}\,(\alpha) \end{array}$$

where the inclusion $\tilde{i} : A_m \hookrightarrow B$ is anti-equivariant: $j(\rho \cdot \beta) = \rho \cdot \beta^{-1}$ (since, for the identification of $\mathrm{Pol}\,(\alpha)$ as a subspace of $\mathrm{APol}\,(\alpha)$, the vector ρ_m must point downwards). Therefore $c_m = -i^*(c_1(\psi))$ and the equality $c_m = -R$ in $H^2(\mathrm{Pol}\,(\alpha)))$ is equivalent to $c_1(\psi) = R$ in $H^2(\mathrm{APol}\,(\alpha))$

(recall that R denotes a class in $H^2(\mathrm{UP}\,(\alpha))$ as well as its its images in $H^2(\mathrm{APol}\,(\alpha))$ and $H^2(\mathrm{Pol}\,(\alpha))$).

Let T be a tubular neighbourhood of $\tilde{\mathcal{R}} = \mathrm{APol}\,(\alpha)$ in $\mathrm{UP}\,(\alpha)$. The retraction $T \to \tilde{\mathcal{R}}$ is the disc bundle associated to ψ. The class $R \in H^*(\mathrm{UP}\,(\alpha))$ being the Poincaré dual of $\tilde{\mathcal{R}}$, it is the image of the Thom class, $\mathrm{Thom}(\psi) \in H^2(T, \partial T)$, under the homomorphism

$$H^2(T, \partial T) \xrightarrow{\cong} H^2(\mathrm{UP}\,(\alpha), \mathrm{UP}\,(\alpha) - \mathrm{int}\, T) \longrightarrow H^2(\mathrm{UP}\,(\alpha)).$$

Therefore $R \in H^2(\mathrm{APol}\,(\alpha))$ is the image of $\mathrm{Thom}(\psi)$ under the homomorphism $H^2(T, \partial T) \to H^2(T) \cong H^2(\mathrm{UP}\,(\alpha))$ which, by one of the definitions of the Euler class ([Hu], §16.7), is equal to $c_1(\psi)$. Thus, we have proven that $c_m = -R$.

By the Duistermaat-Heckmann theorem [Gu, Theorem 2.7], one has

$$-c_1(\psi) = \frac{\partial}{\partial \alpha_m}[\omega]$$

in $H^2(\mathrm{APol}\,(\alpha); \mathbf{R})$ and thus

$$c_m = \frac{\partial}{\partial \alpha_m}[\omega] \tag{1}$$

in $H^2(\mathrm{Pol}\,(\alpha); \mathbf{R})$. Finally, by symmetry (any edge can be the "last" one):

$$c_j = \frac{\partial}{\partial \alpha_j}[\omega].$$

Applying this formula to the expression of $[\omega]$ given in Proposition 6.5 gives the equations of 7.3.

COROLLARY 7.4. — *The classes c_i generate $H^*(\mathrm{Pol}\,(\alpha); \mathbf{Z}[1/2])$.*

Remark 7.5. — a) By 7.2 the classes c_i generate a 1-dimensional space in $H^2(\mathrm{Pol}\,(\alpha); \mathbf{Z}_2)$. As $\mathrm{Pol}\,(\alpha)$ has an even-dimensional cell decomposition, $H^2(\mathrm{Pol}\,(\alpha); \mathbf{Z})$ is free abelian with rank equal of the dimension of $H^2(\mathrm{Pol}\,(\alpha); \mathbf{Z}_2)$. Therefore, the classes c_i do not generate $H^2(\mathrm{Pol}\,(\alpha); \mathbf{Z})$ unless $\mathcal{S}_m = \{\varnothing\}$.

b) In the proof of Proposition 7.3 the formula $c_m = -R$ could have been obtained directly from equation (1) and Proposition 9.3. The advantage of the previous argument is to be applicable to planar polygon spaces (see Proposition 9.3).

c) By [Fu, p. 109], the total Chern class of the (tangent bundle of the) upper path space is given by

$$c(\mathrm{UP}\,(\alpha)) \;=\; (1+R) = \prod_{i=1}^{m}(1+U_i)\prod_{j=1}^{m-1}(1+V_j).$$

Using the relation $U_i = V_i + R$ and $(1+R)^2 c(\mathrm{Pol}\,(\alpha =)) = c(\mathrm{UP}\,(\alpha))$ gives

$$c_1(\mathrm{Pol}\,(\alpha)) \;=\; (m-2)R + 2\sum_{i\in S_m} V_i \;=\; \sum_{i=1}^{m} c_i.$$

d) Using 6.5 and 7.4, one gets the nicest expression for the cohomology class $[\omega] \in H^2(\mathrm{Pol}\,(\alpha)); \mathbf{R})$ of the symplectic form:

$$[\omega] \;=\; \sum_{i=1}^{m} \alpha_i\, c_i.$$

This is no surprise since it is essentially how we calculated the c_i in 7.3.

The great advantage of the $\{c_i\}$ over the generators $\{R, V_i\}$ is that they are manifestly natural under permutation of the edges. Given a permutation $\pi \in \mathrm{Sym}_m$, there is an isomorphism of $\mathrm{Pol}\,(\alpha)$ and $\mathrm{Pol}\,(\pi\alpha)$ given by reordering the steps. (This is a little confusing in the polygon description, since one naturally thinks of keeping adjacent edges adjacent; instead one should simply think of a list of m vectors whose sum is zero, modulo rotation.) From the geometric construction of the c_i, one sees that under

$$\mathrm{Pol}\,(\alpha) \to \mathrm{Pol}\,(\pi\alpha)$$

giving

$$H^2(\mathrm{Pol}\,(\pi\alpha)) \to H^2(\mathrm{Pol}\,(\alpha))$$

we have

$$c_i \mapsto c_{\pi(i)}.$$

PROPOSITION 7.6. — *The $\{c_i\}$ and Pontrjagin class p are the generators in a manifestly S_m-invariant presentation of the cohomology ring with coefficients in $\mathbf{Z}[1/2]$:*

$$H^*(\mathrm{Pol}\,(\alpha); \mathbf{Z}[1/2]) = \mathbf{Z}[1/2][c_1, \ldots, c_m, p]/\mathcal{I}_c$$

where c_i is of degree 2 and p of degree 4 and $\mathcal{I}_c$ is the ideal generated by the two families

$$\text{(R1)}\quad c_i^2 - p \qquad\qquad i = 1, \ldots, m-1$$

$$\text{(R2)}\quad \sum_{\substack{M \subset L \\ |M| \not\equiv |L| \bmod 2}} \left(\prod_{i \in M} c_i \right) p^{(|L-M|-1)/2} \qquad L \in \mathcal{L}.$$

Proof. — It is easiest to see this by returning to the original presentation in 5.4. There were two steps necessary in Theorem 5.5 to turn this presentation for the upper path space into one for the polygon space; (1) for each $L \subseteq \{1, \ldots, m-1\}, L \in \mathcal{L}_m$ giving a relator (d), subtract the corresponding relator (c) associated to $L \cup \{m\}$, then (2) divide the difference by R^2 (now c_m^2).

Rewritten in terms of $c_i = R + 2V_i = 2U_i - R$ and p, safely ignoring factors of 2, and performing the above two steps on (d), the relations (b)-(d) become

$$\text{(b)}\quad (c_i + c_m)(c_i - c_m) \qquad\qquad i = 1, \ldots, m-1$$

$$\text{(c)}\quad \prod_{i \in L} (c_i + c_m) \qquad\qquad L \subset \{1, \ldots, m-1\} \text{ and } L \in \mathcal{L}_m$$

$$\text{(d')}\quad c_m^{-1} \Big(\prod_{i \in L} (c_i - c_m) - \prod_{i \in L} (c_i + c_m) \Big)$$

$$L \subset \{1, \ldots, m-1\} \text{ and } L \in \mathcal{L}.$$

The relations (b) say that all the c_i^2 are equal, which we knew in our ring, since that's the Pontrjagin class p. Expand (c), pulling out factors of p where possible:

$$\begin{aligned} \prod_{i \in L} (c_i + c_m) &= \sum_{M \leqslant L} c_m^{|L-M|} \prod_{i \in M} c_i \\ &= \sum_{\substack{M \leqslant L \\ |M| \equiv |L|}} p^{|L-M|/2} \prod_{i \in M} c_i + \sum_{\substack{M \leqslant L \\ |M| \not\equiv |L|}} p^{(|L-M|-1)/2} \prod_{i \in M \cup \{m\}} c_i \end{aligned}$$

(where the congruences are mod 2). Note that the products in both terms of this last expression are over subsets of $L \cup \{m\}$ with odd complement. And in fact, every such subset appears this way exactly once; this is relation (R2) in the case that the long subset of $\{1, \ldots, m\}$ contains m.

A similar analysis of (d') gives the relations (R2) for the case that the long subset does not contain m; the negative terms in the first expression cancel those subsets with even complement. □

This presentation is of most use in the case that $\pi\alpha = \alpha$, and the induced isomorphism of polygon spaces is an automorphism. In this case π preserves the collection of long subsets, and so permutes the relations given. We emphasize that the generator p is not necessary; its virtue is in giving a much more efficient presentation.

8. Equilateral polygon spaces.

In this section we study the equilateral case, i.e. $\alpha_i = 1$ for all i. As usual, we require α to be generic, which in this case means exactly that m is odd. For the rest of this section we will use the notation Pol_m for the equilateral case with m sides.

This space carries an action of S_m. It is the one most commonly studied in algebraic geometry, because the quotient Pol_m/S_m is a compactification of the moduli space of m *unordered* points in CP^1 – in turn, the moduli space of m-times-punctured genus zero algebraic curves. Computing the cohomology ring of this space is a classical problem, first solved by Brion [Br].

Since this space is an orbifold, it is most natural to consider its rational[(2)] cohomology, particularly since one has a way to compute it:

$$H^*(\mathrm{Pol}_m/\mathrm{Sym}_m; \mathbf{Q}) \cong H^*(\mathrm{Pol}_m; \mathbf{Q})^{\mathrm{Sym}_m}.$$

This requires one to understand the action of Sym_m on $H^*(\mathrm{Pol}_m; \mathbf{Q})$ (first computed in [Kl]); for our purposes, since we know it is generated in degree 2, we need only understand the action on H^2. And this is easy, since the $\{c_i\}$ provide a basis demonstrating that the representation is the usual one of Sym_m on $\mathbf{Q}^m$ by permutation matrices.

In this section we show that the action of Sym_m on the *integral* cohomology group $H^2(\mathrm{Pol}_m)$ is *not* the standard one. And while the set $\{c_i\}$ shows that it suffices to invert the prime 2, it is not necessary; in particular it suffices to invert the primes dividing m (which, recall, is odd). This will find application in Section 9, Theorem 9.4.

(2) In fact one only has to invert primes up to $(m-1)/2$. The maximal stabilizers of Sym_m come on triangles, and the very largest one is $\mathrm{Sym}_{(m-1)/2} \times \mathrm{Sym}_{(m-1)/2} \times \mathrm{Sym}_1$.

We then finish the section by calculating the rational cohomology ring of $\mathrm{Pol}_m/\mathrm{Sym}_m$, by a different method than those in [Br] [KiPoly].

Let n be the product of the primes we are willing to invert. To standardize the action on $H^2(\mathrm{Pol}_m; \mathbf{Z}[1/n])$, we need to find a set $\{b_1, \dots, b_m\} \in H^2(\mathrm{Pol}_m; \mathbf{Z})$ satisfying two criteria:

(C1) $\pi^*(b_i) = b_{\pi(i)}$ for all $i \in 1, \dots, m$ and $\pi \in= \mathrm{Sym}_m$

(C2) $\{b_1, \dots, b_m\}$ is a basis of $H^2(\mathrm{Pol}_m; \mathbf{Z}[1/n])$.

(It suffices to take the b_i in $H^2(\mathrm{Pol}_m; \mathbf{Z})$, since one can simply multiply them all by n until this is true.) To determine when $\{b_i\}$ is a basis, we make a matrix converting from the known basis $\{R, V_i\}$ to $\{b_i\}$ and see if its determinant is a unit in $\mathbf{Z}[1/n]$.

We first find all solutions to criterion (C1). We look first for a vector b_m invariant under permutations of the first $m-1$ edges, and then construct the other $m-1$ vectors from it by applying the cycle $(123\dots m)$. Rationally it is easy to find all vectors invariant under permutations of the first $m-1$ edges; take as the rational basis c_m and $\sum_{i=1}^{m} c_i$, where $\{c_i\}$ are the Chern classes from Section 7.

In our basis these are $-R$ and $(m-2)R + 2\sum_{i=1}^{m-1} V_i$. These two vectors do not span the intersection of the $\mathbf{Q}$-space they generate with the lattice $H^2(\mathrm{Pol}_m; \mathbf{Z})$. To get all those vectors, we must take combinations of the form

$$b_m := \frac{x}{2}R + \frac{y}{2}\left((m-2)R + 2\sum_{i=1}^{m-1} V_i\right)$$

where $x \equiv y \bmod 2$ so that R has an integer coefficient.

This basis was chosen to have easy transformation properties under $(123\dots m)$. The first cycles through the other $c_i = R + 2V_i$, whereas the second is fixed. For convenience of notation set $Z := m/2 - 1$. The $\{b_i\} \to \{R, V_i\}$ transformation matrix is then

$$x/2 \begin{pmatrix} -1 & & & & \\ 1 & 2 & & & \\ 1 & & 2 & & \\ 1 & & & 2 & \\ \vdots & & & & \ddots \end{pmatrix} + y \begin{pmatrix} Z & 1 & 1 & \cdots \\ Z & 1 & 1 & \cdots \\ Z & 1 & 1 & \cdots \\ Z & 1 & 1 & \cdots \\ & \vdots & & \end{pmatrix}$$

We now compute the determinant. Subtract rows 2 through m from their previous row, from the top down (nothing propagates).

$$x/2\begin{pmatrix} -2 & -2 & & & \\ & 2 & -2 & & \\ & & 2 & -2 & \\ & & & \ddots & \\ 1 & & & & 2 \end{pmatrix} + y\begin{pmatrix} & \vdots & & \\ & 0 & & \\ & 0 & & \\ & 0 & & \\ Z & 1 & 1 & \cdots \end{pmatrix}.$$

Subtract the first column from the second.

$$x/2\begin{pmatrix} -2 & & & & \\ & 2 & -2 & & \\ & & 2 & -2 & \\ & & & \ddots & \\ 1 & -1 & & & 2 \end{pmatrix} + y\begin{pmatrix} & \vdots & & & \\ & 0 & & & \\ & 0 & & & \\ & 0 & & & \\ Z & 1-Z & 1 & 1 & \cdots \end{pmatrix}.$$

Now add the second to the third, the third to the fourth, and so on (this time things propagate).

$$x/2\begin{pmatrix} -2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & \ddots & \\ 1 & -1 & -1 & \cdots & 1 \end{pmatrix} + y\begin{pmatrix} & \vdots & & & \\ & 0 & & & \\ & 0 & & & \\ & 0 & & & \\ Z & 1-Z & 2-Z & \cdots & (m-1-Z) \end{pmatrix}.$$

This is a lower triangular matrix whose determinant is

$$-x^{m-1}(x/2 + y(m-1-(m/2-1))) = -x^{m-1}(x+ym)/2$$

(recall that m is odd, and $x \equiv y \bmod 2$, so this is actually an integer).

THEOREM 8.1. — *The action of* Sym_m *on the integral cohomology group* $H^2(\mathrm{Pol}_m)$ *is never the standard permutation representation. To standardize it it suffices to invert* 2 *or* m.

Proof. — At this point we are asking if $-x^{m-1}(x+ym)/2$ can ever be a unit in $\mathbf{Z}$, that is to say, ± 1. The first factor forces us to take $x = \pm 1$. Then y cannot be zero, since x and y have the same parity, so

$$|ym + x| \geqslant |ym| - |x| = |y|m - 1 \geqslant m - 1 \geqslant 4$$

so its quotient by 2 cannot be as small as ± 1. (Note that the case $m = 3$ fails for a simpler reason.)

To get the two possibilities advertised, take $x = 2, y = 0$ for $\mathbf{Z}[1/2]$ (this is just the c_i basis) and $x = m, y = 1$ for $\mathbf{Z}[1/m]$. □

It is not too hard to find the exact conditions on n making the representation standard. The reader may find it amusing to show that for $m = 5$, it is necessary and sufficient that n be divisible by a prime congruent to $0, 2, 3 \bmod 5$.

Hereafter in this section we work with rational coefficients, and the $\{c_i\}$ basis of $H^2(\operatorname{Pol}_m; \mathbf{Q})$. In the equilateral case, the presentation 7.6 of the cohomology ring is particularly simple. The minimal long subsets are exactly those with $(m+1)/2$ edges, giving relators in degree $m-1$.

Focus first on the relators $(R1)\, c_i^2 - p$, where p was the extra "generator" in degree 4. These generate a sub-ideal agreeing with the whole ideal up to degree $m-1$, and are easily seen to be a Gröbner basis for this sub-ideal.

LEMMA 8.2. — *Let σ_i be the ith symmetric polynomial in the $\{c_j\}$. Then the subspace of $\mathbf{Q}[p, \{c_i\}]$ generated by p and the $\{\sigma_i\}$ maps onto the Sym_m-invariant part of $H^*(\operatorname{Pol}_m; \mathbf{Q})$. For $* \leqslant m-3$ this map is an isomorphism.*

Proof. — Consider the series of maps

$$\{\{c_i^2 - p\}\text{-reduced polynomials}\} \to \mathbf{Q}[p, \{c_i\}] \\ \to \mathbf{Q}[p, \{c_i\}]/(c_i^2 - p) \to H^*(\operatorname{Pol}_m; \mathbf{Q}).$$

The relations $\{c_i^2 - p\}$ are easily seen to be a Gröbner basis for the ideal they generate, with respect to a reverse lexicographic order making powers of p late in the order. Consequently the subspace of $\mathbf{Q}[p, \{c_i\}]$ of reduced polynomials maps isomorphically onto the quotient $\mathbf{Q}[p, \{c_i\}]/(c_i^2 - p)$ [Ei]. So the composition of the first two maps above is an isomorphism, and therefore the composition of all three is an epimorphism. In degrees below the omitted relations, of degree $m-1$, the last map is an isomorphism.

The $\{c_i^2 - p\}$-reduced polynomials are exactly combinations of p and the $\{c_i\}$ that are square-free in the $\{c_i\}$. The condition of being square-free is preserved by the action of Sym_m permuting the $\{c_i\}$. So this composite map is actually an Sym_m-epimorphism, and we can find the invariants up in our subspace rather than looking in the quotient. These are exactly polynomials in p and $\{c_i\}$ symmetric in the $\{c_i\}$, which are generated by p and the elementary symmetric polynomials in $\{c_i\}$. □

COROLLARY 8.3. — *The* Sym_m*-invariant part of* $H^*(\mathrm{Pol}\,_m;\mathbf{Q})$ *is generated by* σ_1 *of degree 2 and* p *of degree 4, with no relations up to degree* $m-3$.

Proof. — The σ_i are generated by σ_1 and p:

$$\sigma_1\sigma_i = (i+1)\sigma_{i+1} + (m-(i-1))p\sigma_{i-1}$$

(where $\sigma_0 := 1$, $\sigma_{-1} = 0$). To see this, imagine multiplying a product $\prod_{i\in S} c_i$ by c_j. Either $j \notin S$, in which case the product becomes one longer, or $j \in S$, in which case two c_j's cancel to become a p. The coefficients arise this way: in a product of $i+1$ things, any of them may be the new one, whereas in a product of $i-1$, any one of the missing ones may be the one that just cancelled. □

Since we now know the Betti numbers up to the middle dimension $m-3$, by Poincaré duality we know all of them, and as in [Br] the Poincaré polynomial is quickly determined to be

$$P_{\mathrm{Pol}\,_m/\mathrm{Sym}_m} = \frac{(1-t^{m-1})(1-t^{m+1})}{(1-t^2)(1-t^4)}.$$

In particular, since we know that there are only two generators (in degrees 2 and 4), we know there are only two relations (in degrees $m-1$ and $m+1$).

The relation in degree $m-1$ is not unexpected; it is the symmetric combination of all the relations of degree $m-1$ in $H^*(\mathrm{Pol}\,_m)$. To find the one of degree $m+1$, we form for each $i = 1,\dots,m$ and $L =\ni i$, $|L| = (m+1)/2$ the S-polynomial of the corresponding relators in (R1), (R2). Our relation is the symmetric combination of those (one must check that it is not σ_1 times the previous relator). We omit the computations as the result is known [Br].

THEOREM 8.4. — *The rational cohomology ring of the equilateral polygon space mod permutations is*

$$H^*(\mathrm{Pol}\,_m/\mathrm{Sym}_m;\mathbf{Q}) = \mathbf{Q}[p,\{\sigma_i\}]/\mathcal{I}$$

where p *is of degree 4,* σ_i *of degree* $2i$ *for=* $i = 0,\dots,(m-1)/2$ *and* $\mathcal{I}$ *is generated by the family*

(R1) $\quad \sigma_1\sigma_i = (i+1)\sigma_{i+1} + (m-(i-1))p = \sigma_{i-1} \qquad i = 0,\dots,(m-3)/2$

and the two relators

$$\sum_{i\equiv(m-1)/2 \bmod 2} \binom{m-i}{(m+1)/2-i} p^{(\frac{m-1}{2}-i)/2}\sigma_i$$

and

$$\sum_{i\equiv(m+1)/2} \binom{\frac{m+1}{2}}{i} \Big/ \binom{m}{i} p^{\frac{1}{2}(\frac{m+1}{2}-i)}\sigma_i.$$

It is worth explaining here exactly what problem Brion addresses, since it is not obviously the one above. In both cases one is studying the action of $SO(3)\times \mathrm{Sym}_m$ on $\prod_{i=1}^{m} S_1^2$. In the approach above, we first perform the symplectic reduction by $SO(3)$, producing the equilateral polygon space, and then take the quotient by Sym_m.

One can perform these tasks in the opposite order. Regarding the S^2's as CP^1's, one has available the celebrated homeomorphism[(3)] of $(\prod_{i=1}^{m} CP^1)/\mathrm{Sym}_m$ with CP^m, taking the m numbers to their elementary symmetric combinations. This latter space is in turn the projectivization of the $m+1$-dimensional irreducible representation of $SU(2)$ – on the projective space, the action factors through $SO(3)$. It is this question – the cohomology of $CP^m//SO(3)$ – that Brion addresses and solves.

9. Planar polygon spaces.

In this section, we study the *planar polygon space*:

$$\mathrm{Pol}_{\mathbf{R}}(\alpha) := \left\{(\rho_1,\dots,\rho_m)\in(\mathbf{R}^2)^m \,\middle|\, |\rho_i| = \alpha_i \text{ and } \sum_{i=1}^{m}\rho_i = 0\right\}\Big/ O_2$$

where O_2 acts on $(\mathbf{R}^2)^m$ diagonally. The more classical quotient by SO_2, denoted by $\mathrm{Pol}\,(\alpha;\mathbf{R}^2)$, will also be considered. We assume α generic, so the actions are free. The space $\mathrm{Pol}_{\mathbf{R}}(\alpha)$ is then a smooth manifold of dimension $m-3$ and $\mathrm{Pol}\,(\alpha;\mathbf{R}^2)\to\mathrm{Pol}_{\mathbf{R}}(\alpha)$ is a 2-fold cover.

(3) The orbifold structure is different, but this is not relevant for the rational cohomology.

The $O(2)$-quotient $\mathrm{Pol}_{\mathbf{R}}(\alpha)$ is more natural for us because it is a submanifold of $\mathrm{Pol}(\alpha)$. It can be interpreted as a "real part" of the Kähler manifold $\mathrm{Pol}(\alpha)$: it is the fixed point set of the antiholomorphic involution $\rho \mapsto r \circ \rho$ where r is the reflection $r(x,y,z) = (x,-y,z)$. More about that is to be found in [HK, §3 and 4]. The planar upper path space $\mathrm{UP}_{\mathbf{R}}(\alpha)$ and abelian polygon space $\mathrm{APol}_{\mathbf{R}}(\alpha)$ are defined accordingly and can be seen as real parts of $\mathrm{UP}(\alpha)$ and $\mathrm{APol}(\alpha)$.

We shall prove that a well-known phenomenon for Grassmannians, toric manifolds, etc., also holds true for polygon spaces:

THEOREM 9.1. — *Let P (respectively: $P_{\mathbf{R}}$) stand for* $\mathrm{UP}(\alpha)$, $\mathrm{APol}(\alpha)$ *or* $\mathrm{Pol}(\alpha)$ *(respectively:* $\mathrm{UP}_{\mathbf{R}}(\alpha)$, $\mathrm{APol}_{\mathbf{R}}(\alpha)$ *or* $\mathrm{Pol}_{\mathbf{R}}(\alpha)$*). Then there is is a ring isomorphism*

$$H^{2*}(P;\mathbf{Z}_2) \xrightarrow{\simeq} H^*(P_{\mathbf{R}};\mathbf{Z}_2)$$

sending elements of degree $2d$ to elements of degree d.

For instance, for $P = \mathrm{Pol}_r(\alpha)$, one gets:

COROLLARY 9.2. — *The cohomology ring of the planar polygon space* $\mathrm{Pol}_{\mathbf{R}}(\alpha)$ *with $\mathbf{Z}_2$ as coefficient is*

$$H^*(\mathrm{Pol}_{\mathbf{R}}(\alpha);\mathbf{Z}_2) = Z_2[R, V_1, \ldots, V_{m-1}]/\mathcal{I}_{\mathrm{Pol}}$$

where R and V_i are of degree 1 and $\mathcal{I}_{\mathrm{Pol}}$ is generated by the three families

$$\text{(R1)}\quad V_i^2 + RV_i \qquad\qquad i = 1,\ldots,m-1$$

$$\text{(R2)}\quad \prod_{i\in L} V_i \qquad\qquad L \in \mathcal{L}_m$$

$$\text{(R3)}\quad \sum_{\substack{S\subset L\\ S\in\mathcal{S}_m}} = \Big(\prod_{i\in S} V_i\Big) R^{|L-S|-1} \qquad L \subset \{1,\ldots,m-1\} \textit{ and } L \in \mathcal{L}.$$

Proof of 9.1. — As seen in Section 1, the manifolds $\mathrm{UP}(\alpha)$ and $\mathrm{APol}(\alpha)$ are toric manifolds. Therefore, Theorem 9.1 is true by [DJ, Theorem 4.14] and a proof is only required for $\mathrm{Pol}(\alpha)$.

We first establish that for each $k \in \mathbf{N}$:

$$\text{(1)}\qquad \dim H^k(\mathrm{Pol}_{\mathbf{R}}(\alpha);\mathbf{Z}_2) \leqslant \dim H^{2k}(\mathrm{Pol}(\alpha);\mathbf{Z}_2)$$

where dim means the dimension as a vector space over the field $\mathbf{Z}_2$. This is done by induction on the number m of edges. The statement is trivial for $m = 3$ where $\mathrm{Pol}_{\mathbf{R}}(\alpha) = \mathrm{Pol}(\alpha) =$ one point. It is also obviously true for $m = 4, 5$ by the list of all polygon spaces (see [HK, Section 6]).

We use the notations of the proof of Lemma 4.1. By [Ha, Theorem 3.2], the diagonal-length function $\delta : \mathrm{Pol}(\alpha) \longrightarrow \mathbf{R}$ given by $\delta(\rho) := |\rho_m - \rho_{m-1}|$ is a Morse-Bott function on $\mathrm{Pol}_{\mathbf{R}}(\alpha)$. The critical points are the same as those for $\mathrm{Pol}(\alpha)$ but, for each of them, the index is divided by 2. They are isolated except possibly for the two extrema. The pre-image $M_{\max}$ of the maximum is either a point or $\mathrm{Pol}_{\mathbf{R}}(\alpha_1, \dots, \alpha_{m-2}, \alpha_m + \alpha_{m-1})$. For the pre-image $M_{\min}$ of the minimum, there are three possibilities:

- one point
- $\mathrm{Pol}_{\mathbf{R}}(\alpha_1, \dots, \alpha_{m-2}, \alpha_m - \alpha_{m-1})$
- a circle bundle over $\mathrm{Pol}_{\mathbf{R}}(\alpha_1, \dots, \alpha_{m-2}, \alpha_m - \alpha_{m-1})$ (when the minimum is 0).

By induction on m, inequality (1) holds for $M_{\min}$, $M_{\max}$ and $\mathrm{Pol}_{\mathbf{R}}(\alpha) - M_{\max}$. As in Proposition 3.1 one gets an exact sequence

$$\begin{aligned} &\longrightarrow H^{*-1}(M_{\max}) \longrightarrow H_{n-*}(\mathrm{Pol}_{\mathbf{R}}(\alpha) - M_{\max}) \longrightarrow \\ &\qquad\longrightarrow H^*(\mathrm{Pol}_{\mathbf{R}}(\alpha)) \xrightarrow{i^*} H^*(M_{\max}) \longrightarrow . \end{aligned} \tag{2}$$

For $\mathrm{Pol}(\alpha)$ this exact sequence is cut into short ones by Proposition 3.1. This enables us to propagate inequality (1) to $\mathrm{Pol}_{\mathbf{R}}(\alpha)$.

As in the proof of Theorem 6.1, the class $R^2 \in H^*(\mathrm{UP}_{\mathbf{R}}(\alpha); \mathbf{Z}_2)$ is Poincaré dual to $\mathrm{Pol}_{\mathbf{R}}(\alpha)$. By the proof of Proposition 3.3, the annihilator $\mathrm{Ann}(R^2)$ in $H^*(\mathrm{UP}_{\mathbf{R}}(\alpha); \mathbf{Z}_2)$ of the cup product with R^2 contains the kernel $\ker i^*$ of $i^* : H^*(\mathrm{UP}_{\mathbf{R}}(\alpha); \mathbf{Z}_2) \to H^*(\mathrm{Pol}_{\mathbf{R}}(\alpha); \mathbf{Z}_2)$. Combining with inequality (1) gives the following sequence of inequalities:

$$\begin{array}{rcl} \dim\left(H^*(\mathrm{UP}_{\mathbf{R}}(\alpha); \mathbf{Z}_2)/\mathrm{Ann}(R^2)\right) & \leqslant & \\ \dim\left(H^*(\mathrm{UP}_{\mathbf{R}}(\alpha); \mathbf{Z}_2)/\ker i^*\right) & = & \dim = \mathrm{Image}\, i^* \leqslant \\ \dim H^*(\mathrm{Pol}_{\mathbf{R}}(\alpha); \mathbf{Z}_2) & \leqslant & \\ \dim H^*(\mathrm{Pol}(\alpha); \mathbf{Z}_2) & = & \dim\left(H^*(\mathrm{UP}_{\mathbf{R}}(\alpha); \mathbf{Z}_2)/\mathrm{Ann}(R^2)\right), \end{array}$$

the last equation being Theorem 6.4. The two ends being equal, all the above inequalities are equalities. Therefore $\mathrm{Ann}(R^2) = \ker i^*$, i^* is surjective, $\dim H^k(\mathrm{Pol}_{\mathbf{R}}(\alpha); \mathbf{Z}_2) = \dim H^{2k}(\mathrm{Pol}(\alpha); \mathbf{Z}_2)$ and one has an isomorphism

$$H^*(\mathrm{Pol}_{\mathbf{R}}(\alpha); \mathbf{Z}_2) \simeq H^*(\mathrm{UP}_{\mathbf{R}}(\alpha); \mathbf{Z}_2))/\mathrm{Ann}(R^2).$$

This proves Theorem 9.1. □

We now turn our attention to the 2-fold cover $\kappa : \mathrm{Pol}\,(\alpha; \mathbf{R}^2) \to \mathrm{Pol}_{\mathbf{R}}\,(\alpha)$. Seen as a principal O_1 cover, it is determined by its Stiefel-Whitney class $w_1(\kappa) \in H^1(\mathrm{Pol}_{\mathbf{R}}\,(\alpha); \mathbf{Z}_2)$.

PROPOSITION 9.3. — $w_1(\kappa) = R$.

Proof. — The O_1-bundle κ is the planar analogue of U_1-bundle ξ_m introduced in Section 7. The proof that $c_1(\xi_m) = -R$ (proof of Proposition 7.3) then works mod2 to give $w_1(\kappa) = R$. □

Lastly, we discuss equilateral planar polygons. We cannot say much about the quotient by the symmetric group since those calculations involve inverting the prime 2. There is something to say about the action:

PROPOSITION 9.4. — *The action of S_m on $H^2(\mathrm{Pol}_{\mathbf{R}\,m}; \mathbf{Z}_2)$ is the standard one on $\mathbf{Z}_2^m$.*

Proof. — Take the $\mathbf{Z}[1/m]$-basis of $H^2(\mathrm{Pol}_m)$ from Theorem 8.1. This becomes a basis once m is invertible, which it is over $\mathbf{Z}_2$. □

10. Examples.

10.1. Suppose $\mathcal{S}_m(\alpha) = \{\varnothing\}$, for example if $\alpha = (1, \ldots, 1, m-2)$. It follows from [Ha, Proposition (4.1)] that $\mathrm{Pol}\,(\alpha)$ is diffeomorphic to the complex projective space $\mathbf{C}P^{m-3}$. Knowing this, we can test our different results for the homology or cohomology of $\mathrm{Pol}\,(\alpha)$.

As $\mathcal{S}_m(\alpha) = \{\varnothing\}$, the expression of the Poincaré polynomial $P_{\mathrm{Pol}\,(\alpha)}$ given in Theorem 4.3 is a 1-term sum:

$$P_{\mathrm{Pol}\,(\alpha)} = \frac{1 - t^{2(m-2)}}{1 - t^2} = 1 + t^2 + \cdots + t^{2(m-3)}$$

which is indeed the Poincaré polynomial of $\mathbf{C}P^{m-3}$. Observe that the formula for $P_{\mathrm{Pol}\,(\alpha)}$ in terms of $\mathcal{S}$ given in Remark 4.4 would have $2^{m-1} - 1$ terms!

For the cohomology ring $H^*(\mathrm{Pol}\,(\alpha)))$, Theorem 6.4 asserts that it is the quotient of $\mathbf{Z}[R, V_1, \ldots, V_{m-1}]$ by an ideal $\mathcal{I}$ generated by the families of relators (R1), (R2) and (R3). As $\{i\} \in \mathcal{L}_m$ for all $1 \leqslant i \leqslant= m-1$, all the V_i's are killed by (R2) and (R1) becomes empty. Family (R3) contains one element, for $L = \{1, \ldots, m-1\}$. As $\mathcal{S}_m(\alpha =) = \{\varnothing\}$, this relator is R^{m-2}. Thus $H^*(\mathrm{Pol}\,(\alpha)) = \mathbf{Z} = [R]/(R^{m-2})$, the cohomology ring of $\mathbf{C}P^{m-3}$.

In the planar case, one has $\mathrm{Pol}_{\mathbf{R}}\,(\alpha) \simeq \mathbf{R}P^{m-3}$ and $\mathrm{Pol}\,(\alpha; \mathbf{R}^2) \simeq S^{m-3}$.

10.2. Consider the case where $\mathcal{S}_m(\alpha)$ contains $\{1,\ldots,m-3\}$ (for instance: $\alpha = (\varepsilon,\ldots,\varepsilon,1,1,1)$ with $(m-3)\varepsilon < 1$). Then, any $r \in \mathrm{Pol}\,(\alpha)$ has a unique representative ρ with $\rho_m = (1,0,0)$ and $\rho(m-1) = (x,y,0)$ with $y > 0$. The class r is then determined by $\rho(1),\ldots,\rho(m-3)$ and there is no constraint on these vectors. Therefore, $\mathrm{Pol}\,(\alpha)$ is symplectomorphic to $\prod_{i=1}^{m-3} S^2_{\alpha_i}$. In the planar case, $\mathrm{Pol}_{\mathbf{R}}\,(\alpha)$ is diffeomorphic to $\prod_{i=1}^{m-3} S^1$. The space $\mathrm{Pol}\,(\alpha;\mathbf{R}^2)$ is not connected: $\mathrm{Pol}\,(\alpha;\mathbf{R}^2) \simeq S^0 \times \prod_{i=1}^{m-3} S^1$.

Let us compute the cohomology ring

$$H^*(\mathrm{Pol}\,(\alpha)) = \mathbf{Z}[R, V_1,\ldots,\ V_{m-1}]/\mathcal{I}.$$

One has $\mathcal{S}_m = \Delta^{m-4}$ and the minimal elements of $\mathcal{L}_m$ are the singletons $\{m-2\}$ and $\{m-1\}$. Therefore, relators (R2) reduce to V_{m-2} and V_{m-1}. The minimal $L \subset \{1,\ldots,m-1\}$ in $\mathcal{L}$ is $L = \{m-2, m-1\}$. For this L, relator (R3) is R. The other relators of the family (R3) all have R as a factor. Finally, using (R1), one finds

$$H^*(\mathrm{Pol}\,(\alpha)) = \mathbf{Z}[V_1,\ldots,V_{m-3}]/(V_1^2,\ldots,V_{m-3}^2)$$

as expected.

In the particular case $m = 3$, the cohomology ring reduces to the degree 0 part (no wonder since a triangle space is just a point).

10.3. Consider the two cases of quadrilaterals mentioned in Section 2: $\alpha = (1,1,1,2)$ and $\alpha' = (1,2,2,2)$. As $\mathcal{S}_4(\alpha) = \{\varnothing\}$, we are in case 10.1 and $H^*(\mathrm{Pol}\,(\alpha)) = \mathbf{Z}[R]/(R^2)$. The case α' is like example 10.2 and $H^*(\mathrm{Pol}\,(\alpha')) = \mathbf{Z}[V_1]/(V_1^2)$ (in particular, $R = 0$). Therefore, $\xi(\alpha)$ is the non-trivial SO_3-bundle over S^2 whereas $\xi(\alpha')$ is the trivial one. In the same way, $\mathrm{Pol}\,(\alpha;\mathbf{R}^2) \to \mathrm{Pol}_{\mathbf{R}}\,(\alpha)$ is the connected 2-fold cover of S^1 whereas $\mathrm{Pol}\,(\alpha';\mathbf{R}^2) \to \mathrm{Pol}_{\mathbf{R}}\,(\alpha')$ is the trivial cover.

10.4. *The regular pentagon:* $\alpha = (1,1,1,1,1)$. The space $\mathrm{Pol}\,(\alpha)$ is a smooth manifold diffeomorphic to $(S^2 \times S^2)\sharp 3\overline{CP}^2 \simeq \mathbf{C}P^2\sharp 4\overline{CP}^2$ [Kl] [HK, (6.3)].

One has $\mathcal{S}_5 = \{\{1\},\{2\},\{3\},\{4\}\}$ and therefore $H^*(\mathrm{Pol}\,(\alpha)))$ is generated by $R, V_1,\ldots,V_4 \in H^2(\mathrm{Pol}\,(\alpha))$. The minimal elements of $\mathcal{L}_5$ are the doubletons $\{i,j\}$ for $i,j = 1,2,3,4$; hence family (R2) is generated by relators V_iV_j. The subsets of $\{1,2,3,4\}$ which are elements of $\mathcal{L}$ are

$L_j := \{1,2,3,4\} - \{j\}$ and $L := \{1,2,3,4\}$. This gives rise to five relators in family (R3):

$$\begin{array}{llcl} (1) & L=\{1,2,3\} & : & R^2+RV_1+RV_2+RV_3 \\ (2) & L=\{1,2,4\} & : & R^2+RV_1+RV_2+RV_4 \\ (3) & L=\{1,3,4\} & : & R^2+RV_1+RV_3+RV_4 \\ (4) & L=\{2,3,4\} & : & R^2+RV_2+RV_3+RV_4 \\ (5) & L=\{1,2,3,4\} & : & R^3+R^2V_1+R^2V_2+R^2V_3+R^2V_4. \end{array}$$

One deduces that $RV_1 = RV_2 = RV_3 = RV_4$, and $R^2 = -3RV_1$. One also has relators (R1) : $V_i^2 + RV_i$. One then checks that everything in degree 3 vanishes. Let us take $T = R + V_1 + V_2 + V_3 + V_4$ and the V_i's as a basis for $H^2(\mathrm{Pol}\,(\alpha))$ and RV_1 as a basis of $H^4(\mathrm{Pol}\,(\alpha))$. With these bases, the cup product $H^2(\mathrm{Pol}\,(\alpha)) \times H^2(\mathrm{Pol}\,(\alpha)) \to H^4(\mathrm{Pol}\,(\alpha))$ is given by the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

which is indeed the intersection form of $\mathbf{C}P^2 \sharp 4\overline{\mathbf{C}P}^2$.

By Proposition 6.5, the class $[\omega] \in H^2(\mathrm{Pol}\,(\alpha))$ of the symplectic form ω is

$$[\omega] = 3R + 2V_1 + 2V_2 + 2V_3 + 2V_4$$

and therefore $[\omega]^2 = 5RV_1$. The Liouville volume $\int_{\mathrm{Pol}\,(\alpha)} \omega^2/2$ is then $5/2$. We get exactly the area of the "moment polytope" of [HK, Figure 3] (we put "moment polytope" between quotes since the regular pentagon space is only a limit case of toric manifold; see [HK, (6.3)]). This illustrates the Duistermaat-Heckmann theorem.

10.5. The pentagon spaces for generic α's are all classified [HK, (6.2)]. They are toric manifolds and thus classified by their moment polytope. The reader can check, as for the regular pentagon space, that one gets the correct intersection forms for these 4-manifolds and that the Liouville volume is the area of the moment polytope.

10.6. Consider the hexagon spaces $\mathrm{Pol}\,(\alpha)$ and $\mathrm{Pol}\,(\alpha')$ for

$$\alpha := (2,2,3,5,5,10) \text{ and } \alpha' :== (2,2,3,5,5,8).$$

One has $\mathcal{S}_6(\alpha) = \{\varnothing, \{1\}, \{2\}, \{3\}\}$ and $\mathcal{S}_6(\alpha') = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}\}$. Using 4.3, one sees that these polygon spaces cannot be distinguished by their Poincaré polynomial:

$$P_{\mathrm{Pol}\,(\alpha)} = P_{\mathrm{Pol}\,(\alpha')} = 1 + 4t^2 + 4t^4 + t^6.$$

As for the cohomology rings, we deduce from Theorem 6.4 that $H^*(\mathrm{Pol}\,(\alpha))$ has the following description:

	generators:	further relations:
$H^2(\mathrm{Pol}\,(\alpha))$	R, V_1, V_2, V_3	
$H^4(\mathrm{Pol}\,(\alpha)))$	R^2, V_1^2, V_2^2, V_3^2	$V_1V_2 = V_1V_3 = V_2V_3 = 0\,,\; V_i^2 = -RV_i$
$H^6(\mathrm{Pol}\,(\alpha))$	$V_1^3 = V_2^3 = V_3^3$	$2V_1^3 = -R^3$

whereas, for $H^*(\mathrm{Pol}\,(\alpha'))$, one has

	generators:	further relations:
$H^2(\mathrm{Pol}\,(\alpha'))$	R, V_1, V_2, V_3	
$H^4(\mathrm{Pol}\,(\alpha'))$	$R^2, V_1^2, V_2^2, V_1V_2$	$V_1V_3 = V_2V_3 = 0\,,\; V_i^2 = -RV_i\,,\; V_3^2 = -R^2$
$H^6(\mathrm{Pol}\,(\alpha'))$	$V_3R^2 = R^3$	$V_1^3 = V_2^3 = 0,\; RV_1V_2 = -R^3.$

These two rings are nonisomorphic, even over $\mathbf{Z}_4$. It is a computer algebra exercise to show that in $H^*(\mathrm{Pol}\,(\alpha)) \otimes \mathbf{Z}_4$ there are 72 elements x with $x^3 = 0$ whereas this number is 80 for $H^*(\mathrm{Pol}\,(\alpha')) \otimes \mathbf{Z}_4$. One can check by hand the more relevant fact that there is no isomorphism between $H^*(\mathrm{Pol}\,(\alpha))$ and $H^*(\mathrm{Pol}\,(\alpha))$ preserving the classes R's. Indeed, R^3 mod 2 generates $H^6(\mathrm{Pol}\,(\alpha'); \mathbf{Z}_2)$ whereas $R^3 = 0$ in $H^6(\mathrm{Pol}\,(\alpha); \mathbf{Z}_2)$. But, by 7.2 and 7.3, R mod 2 is the second Stiefel-Whitney class of the SO_3-bundle $\xi(\alpha)$ defined in Section 7. In particular, $A(\alpha)$ and $A(\alpha')$ are not SO_3-equivariantly diffeomorphic.

BIBLIOGRAPHY

[Br] M. Brion, Cohomologie équivariante des points semi-stables, J. Reine Angew. Math., 421 (1991), 125–140.

[DM] P. Deligne & G. Mostow, Monodromy of hypergeometric functions and non-lattice integral monodromy, Publication de l'IHES, 43 (1986), 5–90

[DJ] M. Davis & T. Januszkiewicz Convex polytopes, Coxeter orbifolds and torus actions, Duke Math. J., 62 (1991), 417–451

[Ei] D. Eisenbud, Commutative algebra, Springer-Verlag, 1995.

[Fu] W. Fulton, Introduction to toric varieties, Princeton Univ. Press, 1993.

[GGMS] I. Gel'fand & M.Goresky & R. MacPherson & V. Serganova, Combinatorial geometries, convex polyedra and Schubert cells, Adv. Math., 63 (1987), 301–316.

[GM] I. Gel'fand & R. MacPherson Geometry in Grassmannians and a generalization of the dilogarithm, Adv. Math., 44 (1982), 279–312.

[GH] P. Griffiths & J.Harris, Principles of algebraic geometry, Wiley Classics Library, 1978.

[GS] V. Guillemin, & S. Sternberg, The coefficients of the Duistermaat-Heckman polynomial and the cohomology ring of reduced spaces. Geometry, topology, and physics, Conf. Proc. Lecture Notes Geom. Topology, VI International Press, 1995.

[Gu] V. Guillemin, Moment maps and combinatorial invariants of Hamiltonian T^n-spaces, Birkhäuser, 1994.

[Ha] J-C. Hausmann, Sur la topologie des bras articulés. In "Algebraic Topology, Poznan", Springer Lectures Notes, 1474 (1989), 146–159.

[HK] J-C. Hausmann & A. Knutson, Polygon spaces and Grassmannians, L'Enseignement Mathématique, 43 (1997), 173–198.

[Hu] D. Husemoller, Fibre bundles (2nd ed.), Springer Verlag, 1975.

[KM] M. Kapovich, & J.Millson, The symplectic geometry of polygons in Euclidean space, J. of Diff. Geometry, 44 (1996), 479–513.

[K1] F. Kirwan, The cohomology rings of moduli spaces of vector bundles over Riemann surfaces, J. Amer. Math. Soc., 5 (1992), 902 and 904.

[K2] F. Kirwan, Cohomology of quotients in symplectic and algebraic geometry, Princeton University Press, 1984.

[Kl] A. Klyachko, Spatial polygons and stable configurations of points in the projective line. in: Algebraic geometry and its applications (Yaroslavl, 1992), Aspects Math., Vieweg, Braunschweig (1994) 67–84.

[Le] E. Lerman, Symplectic cuts, Math. Res. Lett., 2 (1995), 247–258.

[Ma] S. Martin, Doctoral thesis (in preparation).

[MS] J. Milnor & J. Stasheff, Characteristic Classes, Princeton Univ. Press, 1974.

[Popp] H. Popp, Moduli theory and classification of algebraic varieties, Springer-Verlag, 1977.

[Sp] E. Spanier, Algebraic topology, McGraw-Hill, 1966.

Manuscrit reçu le 27 juin 1997,
accepté le 21 octobre 1997.

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