Annales de l'institut Fourier

Bernard Le Stum
Adolfo Quirós

Transversal crystals of finite level

Annales de l'institut Fourier, tome 47, nº 1 (1997), p. 69-100
<http://www.numdam.org/item?id=AIF_1997__47_1_69_0>

Ann. Inst. Fourier, Grenoble
47, 1 (1997), 69–100

TRANSVERSAL CRYSTALS OF FINITE LEVEL

by B. LE STUM and A. QUIRÓS (*)

CONTENTS

INTRODUCTION

In [Q] the second author studies families of strongly divisible filtered F-crystals in relation with Griffiths transversality. In his book [O2] Ogus

(*) Partially supported by the DGICYT project n° PB91-0370-C02-02.
Key words: Griffiths transversality – Isogeny – Divided power of finite level – Differential operator – Frobenius structure.
Math. classification: 14F30.

introduces the notion of T-crystal (T for *transversal*), which provides an excellent context to study this kind of questions. He uses it to prove a version of Mazur's theorem on the relation between the action of Frobenius and the Hodge filtration on crystalline cohomology which is valid for cohomology with coefficients in an F-crystal. As applications, he gets results about Newton and Hodge polygons (Katz conjecture) and degeneration of the Hodge spectral sequence. One of his key results shows that there is an equivalence between F-spans and T-crystals, provided we restrict to objects of width less than p.

In his letter to Illusie [B3], Berthelot developes the theory of crystals of level m. We use this new theory to extend Ogus' theorem to objects of width less than p^{m+1}: after defining T-m-crystals and F-m-spans, we show that one can identify T-m-crystals of width less than p^{m+1} with a full subcategory of F-m-spans.

More precisely: let S be a torsion free p-adic formal scheme, S_0 its reduction mod p and X a smooth S_0-scheme. A T-m-crystal on X/S is a crystal E of level m with a filtration Fil by submodules which after saturation (see Definition 1.1.6), behaves like a filtration by subcrystals. If $F: X \to X'$ is the relative Frobenius of X/S_0, an F-m-span is a p-isogeny $\Phi: F^{m+1^*}E \to E'$ of p-torsion free m-crystals. We prove (Theorem 4.3.6) that if (E, Fil) is a p-torsion free T-m-crystal on X/S such that $\mathrm{Fil}^{p^{m+1}} \subset pE$, then there exists a unique F-m-span $\Phi: F^{m+1^*}E \to E'$ such that, up to saturation, $F^{m+1^*}\mathrm{Fil}$ coincides with the filtration M defined by $M^k: = \Phi^{-1}(p^k E')$. This construction is functorial in (E, Fil) and the functor is fully faithful.

In order to prove this theorem, we consider a lifted situation: X is a smooth formal S-scheme, F_0 is the relative Frobenius of X_0 over S_0, $F: X \to X'$ is a lifting of F_0 and we assume that there are coordinates $t_1, \ldots, t_d$ on X and X' such that $F(t_i) = t_i^p$. Then T-m-crystals correspond to Griffiths transversal $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules that are also transversal to the m-PD-ideal (p) and F-m-spans correspond to p-isogenies of $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules. We prove the theorem in this local situation (Theorem 2.3.3 and Corollary 3.3.5).

Let us briefly describe the structure of this paper: in the first part, we recall Ogus' notion of transversality and Berthelot's notion of partial divided power structures as well as some properties of p-isogenies in this context. In the second part, we first recall Berthelot's theory of differential operators of finite level, we define Griffiths transversality for $\mathcal{D}^{(m)}$-modules and we build the local version of our functor. In the third part, we define and study p-m-curvature for $\mathcal{D}^{(m)}$-modules in characteristic p and we use this notion to prove the fullfaithfulness of our functor in a local situation. In the fourth part, we recall Berthelot's theory of m-crystals, we define T-m-crystals and F-m-spans and we deduce our main theorem from its local version. In the fifth and last part, we study the behavior of T-m-crystals and F-m-spans when m varies and use it to show that our results provide some improvement on Ogus' theory.

Acknowledgments. — The first (resp. second) author wishes to thank the Universidad Autónoma de Madrid (resp. the Université de Rennes I) for its hospitality. We both want to express our gratitude to Pierre Berthelot and Arthur Ogus who allowed us to see preliminary versions of their articles. We also thank the referee for his very careful reading of the manuscript. Many of his comments helped us a lot in polishing this article.

Conventions. — We let p be a non zero prime and $m \in \mathbb{N}$. All formal schemes are p-adic formal schemes. All schemes are locally killed by some power of p and might hence be considered as formal schemes. Also, all PD-structures are compatible with p. We will use the subindex 0 to indicate reduction mod p. We will adopt the standard multiindex notation, and if $\underline{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$, we will write $|\underline{k}| = k_1 + \cdots + k_d$.

1. PRELIMINARIES

1.1. Transversal filtrations.

We briefly recall the notion of a transversal module from [O2]. We call transversal what Ogus calls G-transversal and almost transversal what he calls G'-transversal. Let us first fix some terminology and notations:

1.1.1. DEFINITION. — *Let A be a ring (in a topos). A module filtration* Fil *on an A-module M is a decreasing filtration by submodules* Fil^k *such that there exists an integer a such that* $\mathrm{Fil}^a = M$*. It is called effective if we can take $a = 0$. In general, if we set* $\mathrm{Fil}[r]^k := \mathrm{Fil}^{k+r}$*, we see that* $\mathrm{Fil}[a]$ *is an effective filtration on M. If $\varphi : (\mathcal{T}, A') \to (\mathcal{T}, A)$ is a morphism of ringed sites, (M, Fil) is a filtered A-module and* Fil^k_φ *denotes the image of* $\varphi^* \mathrm{Fil}^k$ *in* $\varphi^* M$*, then* $\varphi^*(M, \mathrm{Fil}) := (\varphi^* M, \mathrm{Fil}\,\varphi)$ *is called the inverse image of* (M, Fil).

In this article, in order to simplify the notations, we will only consider effective filtrations.

1.1.2. DEFINITION. — *A ring filtration on a ring A is a module filtration $I^{(*)}$ such that $I^{(k)}I^{(\ell)} \subset I^{(k+\ell)}$. If $(A, I^{(*)})$ is a filtered ring, we set $I := I^{(1)}$ and we say that a filtered module (M, Fil) has width at most w (with respect to I) if there exists an integer a such that $\mathrm{Fil}^a = M$ and $\mathrm{Fil}^{a+w+1} \subset IM$. A filtered ringed site $(\mathcal{T}, A, I^{(*)})$ is a site endowed with a filtered ring. A morphism of filtered ringed sites*

$$\varphi : (\mathcal{T}', A', I'^{(*)}) \longrightarrow (\mathcal{T}, A, I^{(*)})$$

is a morphism of ringed sites such that $\varphi^* I^{(k)}$ maps into $I'^{(k)}$ for all k.

1.1.3. DEFINITION. — *A filtered module (M, Fil) in a filtered ringed site $(\mathcal{T}, A, I^{(*)})$ is transversal (a T-module for short) if it satisfies*

$$IM \cap \mathrm{Fil}^k = I \,\mathrm{Fil}^{k-1} + I^{(2)} \,\mathrm{Fil}^{k-2} + I^{(3)} \,\mathrm{Fil}^{k-3} + \cdots$$

for all k. It is almost transversal if

$$IM \cap \mathrm{Fil}^k \subset I \,\mathrm{Fil}^{k-1} + I^{(2)} \,\mathrm{Fil}^{k-2} + I^{(3)} \,\mathrm{Fil}^{k-3} + \cdots$$

for all k and saturated if $I^{(k)} \,\mathrm{Fil}^{\ell} \subset \mathrm{Fil}^{\ell+k}$ for all k, ℓ.

Since there will sometimes be several ring filtrations involved, we will, if necessary, say (almost) transversal to $I^{(*)}$ and saturated with respect to $I^{(*)}$. If $I^{(k)} = I^k$ for all k, we will just say (almost) transversal to I and saturated with respect to I.

1.1.4. *Example.* — A filtered module (M, Fil) in a ringed site $(\mathcal{T}, A)$ is transversal to an ideal I of A if and only if it satisfies $IM \cap \mathrm{Fil}^k = I \,\mathrm{Fil}^{k-1}$ for all k.

1.1.5. *Remark.* — A filtered module is transversal if and only if it is almost transversal and saturated.

Starting from any almost transversal filtration, there exists a natural process that turns it into a transversal one:

1.1.6. DEFINITION. — *If (M, Fil) is a filtered module on a filtered ringed site $(\mathcal{T}, A, I^{(*)})$, we set*

$$\overline{\mathrm{Fil}}^k = \mathrm{Fil}^k + I \,\mathrm{Fil}^{k-1} + I^{(2)} \,\mathrm{Fil}^{k-2} + I^{(3)} \,\mathrm{Fil}^{k-3} + \cdots.$$

We call $(M, \overline{\mathrm{Fil}})$ the saturation of (M, Fil).

1.1.7. PROPOSITION (see [O2], 2.3.1).

(i) *The filtration* $\overline{\mathrm{Fil}}$ *is the finest filtration on* M *that is saturated and coarser than the given one.*

(ii) *If* (M, Fil) *is almost transversal, then its saturation is transversal.*

This saturation process is specially useful in view of the following result:

1.1.8. PROPOSITION (see [O2], 2.2.1). — *Let*

$$\varphi : (\mathcal{T}', A', I'^{(*)}) \longrightarrow (\mathcal{T}, A, I^{(*)})$$

be a morphism of filtered ringed sites such that the natural map $\varphi^{-1}A/I \to A'/I'$ *is flat. If* (M, Fil) *is an almost transversal module, then so is* $\varphi^*(M, \mathrm{Fil})$.

1.2. p-isogenies.

We introduce the m-PD-filtration $(p, \{\ \})$ and we describe transversality with respect to this filtration in terms of p-isogenies.

1.2.1. DEFINITION. — *If* A *is a* $\mathbb{Z}_{(p)}$*-algebra and* M, M' *two* p*-torsion free* A*- modules, a* p*-isogeny* $\Phi : M \to M'$ *of width at most* w *is an injective homomorphism* $\Phi : M \to M' \otimes \mathbb{Q}$ *of* A*-modules such that there exists an integer* a *such that* $p^{a+w+1}M' \subset \Phi(M) \subset p^a M'$. *It is called effective if one can take* $a = 0$. *In general, if we set* $\Phi[r] = p^{-r}\Phi$, *we see that* $\Phi[a]$ *is effective.*

As we do for filtrations, we will only consider effective p-isogenies.

Transversality with respect to p, meaning to the ideal (p), has a very nice interpretation in terms of p-isogenies:

1.2.2. PROPOSITION (see [O2], 5.1.2). — *The functor* $\Phi \mapsto (M, \mathrm{Fil})$, *where* $\mathrm{Fil}^k = \Phi^{-1}(p^k M')$, *is an equivalence from the category of* p*-isogenies of width at most* w *onto the category of filtered modules transversal to* p *of width at most* w.

Actually, the filtration that will naturally appear in the sequel is not $(p)^k$ but the m-PD-filtration defined below (and generalized in Definition 1.3.4).

1.2.3. DEFINITION. — *For* $k = qp^m + r$ *with* $0 \leq r < p^m$, *we let* $p^{\{k\}} := p^k/q!$. *The m-PD-filtration* $(p)^{\{k\}}$ *on a* $\mathbb{Z}_p$*-algebra* A *is the finest* *ring filtration such that* $p^{\{k\}} \in (p)^{\{k\}}$. *We will also write* $(p, \{\ \})$ *for this filtration.*

In the sequel, we will also need the notion of modified binomial coefficients. Let us recall what they are:

1.2.4. DEFINITION. — *If* $\underline{k}'$ *and* $\underline{k}'' \in \mathbb{N}^d$, *and*

$$\begin{aligned} \underline{k}' &= \underline{q}' pm + \underline{r}', & 0 \le \underline{r}' < pm, \\ \underline{k}'' &= \underline{q}'' pm + \underline{r}'', & 0 \le \underline{r}'' < pm, \\ \underline{k} &= \underline{k}' + \underline{k}'' = \underline{q}\, pm + \underline{r}, & 0 \le \underline{r} < pm, \end{aligned}$$

one sets:

$$\left\{ \begin{matrix} \underline{k} \\ \underline{k}' \end{matrix} \right\} := \frac{\underline{q}\,!}{\underline{q}'\,!\,\underline{q}''\,!} \in \mathbb{N} \quad \text{and} \quad \left\langle \begin{matrix} \underline{k} \\ \underline{k}' \end{matrix} \right\rangle := \binom{\underline{k}}{\underline{k}'} \left\{ \begin{matrix} \underline{k} \\ \underline{k}' \end{matrix} \right\}^{-1} \in \mathbb{Z}_p.$$

Proposition 1.2.2 is still valid for the m-PD-filtration under some assumptions on the width:

1.2.5. PROPOSITION (see [O2], 2.3.5). — *The functor « saturation with respect to* $(p, \{\ \})$ *» from the category of filtered modules transversal to* p *to the category of filtered modules transversal to* $(p, \{\ \})$ *is an equivalence of categories when restricted to objects of width less than* p^{m+1}.

1.2.6. COROLLARY. — *The functor* $\Phi \mapsto (M, \mathrm{Fil})$ *where* Fil^k *is the saturation of* $\Phi^{-1}(p^k M')$ *with respect to* $(p, \{\ \})$ *is an equivalence from the category of* p*-isogenies of width less than* p^{m+1} *onto the category of filtered modules transversal to* $(p, \{\ \})$ *of width less than* p^{m+1}.

1.3. m-PD-structures.

We recall Berthelot's theory of partial divided powers from [B4] which generalizes the usual divided power structures in [B1].

1.3.1. DEFINITION. — *Let* Y *be a formal scheme. An* m*-PD-structure on a coherent ideal* I *in* $\mathcal{O}_Y$ *is the data of a PD-ideal* $(J, [\ ])$ *in* I *such that* $I^{(p^m)} + pI \subset J$ *(where* $I^{(p^m)}$ *is the ideal locally generated by* f^{p^m} *with* $f \in I$*). We say that* I *is an* m*-PD-ideal or that* (Y, I, J) *is a formal* m*-PD-scheme. We will drop* J*, or even* I*, from the notations when no confusion should arise. If* $f \in I$ *and* $k = q\,p^m + r$ *with* $0 \le r < p^m$, we write

$$f^{\{k\}} := f^r \left(f^{p^m}\right)^{[q]}.$$

1.3.2. Definition. — *Let $(S, \mathfrak{a}, \mathfrak{b})$ be a formal m-PD-scheme. The m-PD-structure on $\mathfrak{a}$ extends to a formal S-scheme X if the PD-structure on $\mathfrak{b}$ extends to a PD-structure on X (compatible with p). An m-PD-structure (I, J) on a formal S-scheme Y is said to be compatible with $(S, \mathfrak{a}, \mathfrak{b})$ if the m-PD-structure on $\mathfrak{a}$ extends to Y, the PD-structure on $J + (p)$ is compatible with the PD-structure on $\mathfrak{b} + (p)$ and $I \cap (\mathfrak{b}\mathcal{O}_Y + (p))$ is a sub PD-ideal of $\mathfrak{b}\mathcal{O}_Y + (p)$. We then say that (Y, I, J) is a formal m-PD-S-scheme.*

1.3.3. Definition. — *Let $(S, \mathfrak{a}, \mathfrak{b})$ be a formal m-PD-scheme. A morphism of formal m-PD-S-schemes is a morphism of formal schemes $\varphi : Y' \to Y$ such that $\varphi^{-1}(I) \subset I'$ and $(Y', J') \to (Y, J)$ is a morphism of formal PD-schemes. If (Y, I, J) is a formal m-PD-S-scheme and X is the closed formal subscheme of Y defined by I, we say that $X \hookrightarrow Y$ is an m-PD-immersion.*

The following generalizes Definition 1.2.3 and agrees with Berthelot's new definition that replaces [B4] 1.3.8 and 1.3.7.

1.3.4. Proposition and Definition (see [B5]). — *If (Y, I, J) is a formal m-PD-S-scheme, then there exists a finest ring filtration $(I, \{\ \}) := I^{\{*\}}$ on $\mathcal{O}_Y$ such that*

(i) $I^{\{1\}} = I$,

(ii) $I^{\{n\}} \cap (J + \mathfrak{b}\mathcal{O}_Y + p\mathcal{O}_Y)$ *is a sub PD-ideal of* $J + \mathfrak{b}\mathcal{O}_Y + p\mathcal{O}_Y$,

(iii) $x^{\{h\}} \in I^{\{nh\}}$ *whenever* $x \in I^{\{n\}}$.

It is called the m-PD-filtration on $\mathcal{O}_Y$ with respect to (I, J). Then $(Y, \mathcal{O}_Y, I^{\{n\}})$ is a filtered ringed site. Moreover, any morphism of formal m-PD-S-schemes induces a morphism of the corresponding filtered ringed sites.

Universal m-PD-immersions do exist:

1.3.5. Proposition and Definition (see [B4], 2.1.1). — *Let S be a formal m-PD-scheme, X a formal S-scheme to which the m-PD-structure of S extends and $i : X \hookrightarrow Y$ an immersion into a formal S-scheme. Then i factors as an m-PD-S-immersion $X \hookrightarrow P^n_{X/S(m)}(Y)$ followed by a morphism $\varphi : P^n_{X/S(m)}(Y) \to Y$ having the following universal property: any morphism $Y' \to Y$ inducing $X' \to X$, where $X' \hookrightarrow Y'$ is an m-PD-S-immersion whose ideal satisfies $I^{\{n+1\}} = 0$, factors uniquely through φ.*

We say that $P^n_{X/S(m)}(Y)$ is the n-th m-PD-neighborhood of X in Y and we write $\mathcal{P}^n_{X/S(m)}(Y)$ for its structural sheaf.

1.3.6. *Remark.* — If $X \hookrightarrow Y$ is an immersion of schemes (locally killed by a power of p) then there exists an m-PD-S-immersion $X \hookrightarrow P_{X/S(m)}(Y)$ with the same universal property but without nilpotency condition on I. We call $P_{X/S(m)}(Y)$ the *m-PD-neighborhood of X in Y*, and write $\mathcal{P}^n_{X/S(m)}(Y)$ for its structural sheaf.

1.3.7. Definition. — *If i is the diagonal immersion*

$$X \hookrightarrow Y := X \times_S X,$$

then we drop Y from the notations in 1.3.5 *and* 1.3.6 *and we call $\mathcal{P}^n_{X/S(m)}$ the sheaf of m-th principal parts of order at most n.*

2. DIFFERENTIAL OPERATORS OF LEVEL m AND GRIFFITHS TRANSVERSALITY

2.1. Differential operators of level m.

We will now recall from [B4] Berthelot's theory of differential operators of finite level.

Let $(S, \mathfrak{a}, \mathfrak{b})$ be a formal m-PD-scheme and X a smooth formal S-scheme to which the m-PD-structure of S extends. We consider $\mathcal{P}^n_{X/S(m)}$ as an $\mathcal{O}_X$-module using the first projection $X \times_S X \to X$ and we note $\theta : \mathcal{O}_X \to \mathcal{P}^n_{X/S(m)}$ the map induced by the second projection. We first recall the definition of differential operators of level m:

2.1.1. Definition. — *The $\mathcal{O}_X$-dual $\mathcal{D}^{(m)}_{X/S\,n}$ to $\mathcal{P}^n_{X/S(m)}$ is called the sheaf of differential operators of level m and order at most n. The natural maps $\mathcal{P}^{n'}_{X/S(m)} \to \mathcal{P}^n_{X/S(m)}$ for $n \leq n'$ induce injections $\mathcal{D}^{(m)}_{X/S\,n} \hookrightarrow \mathcal{D}^{(m)}_{X/S\,n'}$ and we set*

$$\mathcal{D}^{(m)}_{X/S} = \bigcup_n \mathcal{D}^{(m)}_{X/S\,n}.$$

Moreover, the natural maps

$$\mathcal{P}^{n+n'}_{X/S(m)} \longrightarrow \mathcal{P}^n_{X/S(m)} \otimes \mathcal{P}^{n'}_{X/S(m)}$$

induce bilinear maps

$$\mathcal{D}^{(m)}_{X/S\,n} \times \mathcal{D}^{(m)}_{X/S\,n'} \longrightarrow \mathcal{D}^{(m)}_{X/S\,n+n'}$$

which make $\mathcal{D}^{(m)}_{X/S}$ into a ring called the ring of differential operators of level m. Its p-adic completion will be denoted by $\widehat{\mathcal{D}}^{(m)}_{X/S}$.

2.1.2. *Remark.* — If $t_1, \dots, t_d$ are local coordinates on X and

$$\tau_i : = \theta(t_i) - t_i \quad \text{for all } i,$$

then $\mathcal{P}^n_{X/S(m)}$ is a free $\mathcal{O}_X$-module on the $\underline{\tau}^{\{\underline{k}\}}$ with $|\underline{k}| \leq n$.

We let $\{\underline{\partial}^{\langle \underline{k} \rangle}\}$ be the dual basis to $\{\underline{\tau}^{\{\underline{k}\}}\}$ in $\mathcal{D}^{(m)}_{X/S\,n}$.

If $\underline{k} = \underline{q}p^m + \underline{r} < p^{m+1}$, we set

$$\underline{\partial}^{[\underline{k}]} : = \underline{\partial}^{\langle \underline{k} \rangle}/q!.$$

If $n < p^{m+1}$, then the $\underline{\tau}^{\underline{k}}$ with $|\underline{k}| \leq n$ form a basis for $\mathcal{P}^n_{X/S(m)}$ and the $\underline{\partial}^{[\underline{k}]}$ form the dual basis in $\mathcal{D}^{(m)}_{X/S\,n}$. Note that $\mathcal{D}^{(m)}_{X/S}$ is generated as an $\mathcal{O}_X$-algebra by the $\partial_i^{[p^j]} = \partial_i^{\langle p^j \rangle}$ for $j \leq m$.

2.1.3. *Remark.* — If $\varphi : Y \to X$ is a morphism of smooth formal S-schemes and $\mathcal{F}$ is a $\mathcal{D}^{(m)}_{X/S}$-module then $\varphi^*\mathcal{F}$ has a natural structure of $\mathcal{D}^{(m)}_{Y/S}$-module that can be described locally as follows. Let $t_1, \dots, t_d$ be local coordinates on X, $t'_1, \dots, t'_d$ be local coordinates on Y and $\{\tau_i\}$ and $\{\tau'_k\}$ be the corresponding sections of $\mathcal{P}^n_{X/S(m)}$ and $\mathcal{P}^n_{Y/S(m)}$. If $\varphi^*(\tau_i^{\{j\}}) = \sum f^{i,j}_{k,\ell} \tau_k'^{\{\ell\}}$ and s is a section of $\mathcal{F}$, we have

$$\partial_i^{\langle j \rangle}(\varphi^*(s)) = \sum f^{k,\ell}_{i,j}\, \varphi^*(\partial_k^{\langle \ell \rangle}(s)).$$

As in the classical case, $\mathcal{D}^{(m)}$-modules have an interpretation in terms of stratifications:

2.1.4. Proposition (see [B4], 2.3.2). — *If $\mathcal{F}$ is an $\mathcal{O}_X$-module, it is equivalent to give it a structure of $\mathcal{D}^{(m)}_{X/S}$-module or an m-PD-stratification (defined in the obvious way).*

2.1.5. Definition. — *A $\mathcal{D}^{(m)}_{X/S}$-module (or $\widehat{\mathcal{D}}^{(m)}_{X/S}$-module) is locally (topologically) quasi-nilpotent if locally, given any section s, we have $\partial_i^{\langle N \rangle}(s) \to 0$ as $N \to \infty$ for any index i.*

It follows from Proposition 4.1.7 and Proposition 4.1.8 below that this definition does not depend on the choice of the local coordinate system.

2.1.6. Proposition (generalization of [B1], II. 4.1.3). — *If X is a smooth S-scheme (with p locally nilpotent) and $\mathcal{F}$ is an $\mathcal{O}_X$-module, it is equivalent to give it a structure of locally quasi-nilpotent $\mathcal{D}^{(m)}_{X/S}$-module or an m-HPD-stratification (defined in the obvious way).*

We will also have to consider formal S-schemes that are not necessarily smooth. In order to deal with this situation we need to introduce the following terminology (see also [B4], 2.3.4 and 2.3.5):

2.1.7. Definition. — *Let X be an S-scheme and $X \hookrightarrow Y$ a closed immersion into a smooth formal S-scheme. It follows from Proposition 4.1.5 below that $\mathcal{P}_{X/S(m)}(Y)$ has a natural structure of $\mathcal{D}^{(m)}_{Y/S}$-module. A $\mathcal{P}_{X/S(m)}(Y)$-$\mathcal{D}^{(m)}_{Y/S}$-module is a $\mathcal{D}^{(m)}_{Y/S}$-module $\mathcal{F}$ with a structure of $\mathcal{P}_{X/S(m)}(Y)$-module such that, locally, given any sections f of $\mathcal{P}_{X/S(m)}(Y)$ and s of $\mathcal{F}$, we have*

$$\underline{\partial}^{\langle \underline{k} \rangle}(fs) = \sum \left\{ \begin{matrix} \underline{k} \\ \underline{j} \end{matrix} \right\} \underline{\partial}^{\langle \underline{j} \rangle}(f)\, \underline{\partial}^{\langle \underline{k}-\underline{j} \rangle}(s).$$

It follows from Proposition 4.1.7 and Proposition 4.1.8 below that this definition does not depend on the choice of the local coordinate system.

2.2. Griffiths transversality for $\mathcal{D}^{(m)}$-modules.

We define Griffiths transversality for $\mathcal{D}^{(m)}$-modules and interpret it in terms of stratifications.

Let S be a formal m-PD-scheme and X a smooth formal S-scheme. The following generalizes the usual notion of Griffiths transversality:

2.2.1. Definition. — *A filtered $\mathcal{D}^{(m)}_{X/S}$-module $(\mathcal{F}, \mathrm{Fil})$ is a $\mathcal{D}^{(m)}_{X/S}$-module $\mathcal{F}$ together with a filtration by sub $\mathcal{O}_X$-modules. We say that $(\mathcal{F}, \mathrm{Fil})$ is Griffiths transversal if whenever $P \in \mathcal{D}^{(m)}_{X/S\,n}$, we have $P(\mathrm{Fil}^k) \subset \mathrm{Fil}^{k-n}$ and that it is horizontal if the Fil^k are $\mathcal{D}^{(m)}_{X/S}$-submodules. A filtered $\widehat{\mathcal{D}}^{(m)}_{X/S}$-module $(\mathcal{F}, \mathrm{Fil})$ is a complete $\widehat{\mathcal{D}}^{(m)}_{X/S}$-module $\mathcal{F}$ together with a filtration by complete sub $\mathcal{O}_X$-modules. We say that it is Griffiths transversal or horizontal if it is so $\mathrm{mod}\ p^n$ for all n.*

2.2.2. *Remarks.*

(i) What we call Griffiths transversal corresponds to what is simply called a filtration on a $\mathcal{D}$-module in the classical situation.

(ii) Assume we have local coordinates $t_1, \ldots, t_d$. In order to show that $(\mathcal{F}, \mathrm{Fil})$ is Griffiths transversal it is sufficient to check that $\partial_i^{[p^j]} \mathrm{Fil}^k \subset \mathrm{Fil}^{k-p^j}$ for $j \leq m$ and all i.

Here is the interpretation of Griffiths transversality in terms of stratifications:

2.2.3. DEFINITION. — *Let $(\mathcal{F}, \mathrm{Fil})$ be a filtered $\mathcal{O}_X$-module with an m-PD-stratification $\{\varepsilon_n : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}\}$. We call the stratification transversal if ε_n induces an isomorphism between $\overline{\mathrm{Fil}}^k_{p_2}$ and $\overline{\mathrm{Fil}}^k_{p_1}$ for all n.*

2.2.4. PROPOSITION. — *Let $\mathcal{F}$ be a $\mathcal{D}^{(m)}_{X/S}$-module and Fil^k a filtration on $\mathcal{F}$ by sub $\mathcal{O}_X$-modules. Then $\mathcal{F}$ is Griffiths transversal if and only if the corresponding m-PD-stratification is transversal.*

Proof. — Let $\mathcal{I}$ be the ideal of X in $P^n_{X/S(m)}$, $p_1, p_2 : P^n_{X/S(m)} \to X$ the projections, $\varepsilon : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}$ the n-th Taylor isomorphism of $\mathcal{F}$ and

$$\begin{aligned} \theta : \mathcal{F} &\longrightarrow p_1^*\mathcal{F}, \\ e &\longmapsto \varepsilon(1 \otimes e) \end{aligned}$$

the n-th Taylor map. Assume first the m-PD-stratification to be transversal. Since ε induces an isomorphism between $\overline{\mathrm{Fil}}^k_{p_2}$ and $\overline{\mathrm{Fil}}^k_{p_1}$, then

$$\begin{aligned} \theta\,\mathrm{Fil}^k \subset \overline{\mathrm{Fil}}^k_{p_1} &= \mathrm{Fil}^k_{p_1} + \mathcal{I}\,\mathrm{Fil}^{k-1}_{p_1} + \mathcal{I}^{\{2\}}\,\mathrm{Fil}^{k-2}_{p_1} + \cdots + \mathcal{I}^{\{n\}}\,\mathrm{Fil}^{k-n}_{p_1} \\ &\subset \mathrm{Fil}^{k-n}_{p_1}. \end{aligned}$$

If $P : \mathcal{P}^n_{X/S(m)} \to \mathcal{O}_X$ is a differential operator of level m and order less than n, then P acts on $\mathcal{F}$ as the composite of θ and $p_1^*(P)$ (*i.e.* $P(e) = (P \otimes \mathrm{Id})(\theta(e))$ so that $P\,\mathrm{Fil}^k \subset \mathrm{Fil}^{k-n}$. Thus, we see that $\mathcal{F}$ is Griffiths transversal. Conversely, assume that $\mathcal{F}$ is Griffiths transversal. We want to check that ε induces an isomorphism between $\overline{\mathrm{Fil}}^k_{p_2}$ and $\overline{\mathrm{Fil}}^k_{p_1}$ and we may assume that we have local coordinates $t_1, \dots, t_d$ on X. Thanks to the cocycle condition, it is sufficient to show that $\theta(\mathrm{Fil}^k) \subset \overline{\mathrm{Fil}}^k_{p_1}$. But if $e \in \mathrm{Fil}^k$ then

$$\theta(e) = \sum \partial^{\langle j \rangle}(e)\tau^{\{j\}} \in \sum \mathcal{I}^{\{j\}}\,\mathrm{Fil}^{k-j}_{p_1} = \overline{\mathrm{Fil}}^k_{p_1}. \qquad \square$$

The same is true for hyperstratifications. Let S be an m-PD-scheme and X a smooth S-scheme.

2.2.5. DEFINITION. — *If $(\mathcal{F}, \mathrm{Fil})$ is a filtered $\mathcal{O}_X$-module, we call an m-HPD-stratification $\varepsilon : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}$ on $\mathcal{F}$ transversal if ε induces an isomorphism between $\overline{\mathrm{Fil}}^k_{p_2}$ and $\overline{\mathrm{Fil}}^k_{p_1}$.*

2.2.6. PROPOSITION. — *An m-HPD-stratification $\varepsilon : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}$ on a filtered $\mathcal{O}_X$-module $(\mathcal{F}, \mathrm{Fil})$ is transversal if and only if $(\mathcal{F}, \mathrm{Fil})$ is Griffiths transversal.*

Proof. — Same as Proposition 2.2.4. □

2.3. Griffiths transversality and p-isogenies.

We are going to build the local version of the functor of our main theorem.

Let S be a formal m-PD-scheme, X a formal S-scheme, F_0 the relative Frobenius of X_0 over S_0 and $F: X \to X'$ a lifting of F_0. We assume that there are local coordinates $t_1, \dots, t_d$ on X and X' such that $F^*(t_i) = t_i^p$.

We will write $X_0^{(m+1)}$ for the pull back of X_0 by the $m+1$ iterate of F_0, and, with the usual slight abuse of notation, we will call

$$F_0^{m+1}: X_0 \longrightarrow X_0^{(m+1)}$$

this $m+1$ iterate of F_0 and $F^{m+1}: X \to X^{(m+1)}$ a lifting obtained by iterating the above process.

2.3.1. Lemma. — *If s is a section of a $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$-module $\mathcal{E}$, then for $\underline{k} < p^{m+1}$, we have, with $a_{\underline{j},\underline{k}} \in \mathbb{Z}$,*

$$\underline{\partial}^{[\underline{k}]}\left(F^{m+1^*}(s)\right) = \sum p^{\underline{j}} a_{\underline{j},\underline{k}}\, t^{\underline{j}p^{m+1}-\underline{k}}\, F^{m+1^*}\left(\underline{\partial}^{[\underline{j}]}(s)\right).$$

Proof. — For $n = p^{m+1} - 1$, we have in $\mathcal{P}^n_{X^{(m+1)}/S(m)}$

$$\begin{aligned} F^{m+1^*}(\tau_i) &= (t_i + \tau_i)^{p^{m+1}} - t_i^{p^{m+1}} \\ &= \sum_{k=1}^{p^{m+1}} \binom{p^{m+1}}{k} t_i^{p^{m+1}-k} \tau_i^k \\ &= \sum_{k=1}^{p^{m+1}-1} p\, c_{i,k}\, t_i^{p^{m+1}-k} \tau_i^k \end{aligned}$$

with $c_{i,k} \in \mathbb{Z}$. Thus we can write

$$F^{m+1^*}(\underline{\tau}^{\underline{j}}) = \sum p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, t^{\underline{j}p^{m+1}-\underline{k}}\, \underline{\tau}^{\underline{k}}$$

with $a_{\underline{j},\underline{k}} \in \mathbb{Z}$. Therefore, if s is a section of $\mathcal{E}$, we have

$$\underline{\partial}^{[\underline{k}]}\left(F^{m+1^*}(s)\right) = \sum p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, t^{\underline{j}p^{m+1}-\underline{k}} F^{m+1^*}\left(\underline{\partial}^{[\underline{j}]}(s)\right). \qquad \square$$

This lemma allows us to show that Frobenius pulls back transversal modules to horizontal modules:

2.3.2. PROPOSITION. — *If* $(\mathcal{E}, \mathrm{Fil})$ *is a Griffiths transversal* $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$*-module (or* $\widehat{\mathcal{D}}^{(m)}_{X^{(m+1)}/S}$*-module) on* $X^{(m+1)}$ *which is saturated with respect to* $(p, \{\ \})$*, then* $F^{m+1^*}(\mathcal{E}, \mathrm{Fil})$ *is horizontal.*

Proof. — We have seen that if s is a section of $\mathcal{E}$, then for $\underline{k} < p^{m+1}$, we have

$$\underline{\partial}^{[\underline{k}]}\big(F^{m+1^*}(s)\big) = \sum p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, t^{\underline{j}p^{m+1}-\underline{k}} F^{m+1^*}\big(\partial^{[\underline{j}]}(s)\big).$$

Since $(\mathcal{E}, \mathrm{Fil})$ is Griffiths transversal, we know that if $s \in \mathrm{Fil}^{\ell}$, we have $(\underline{\partial}^{[\underline{j}]}(s)) \in \mathrm{Fil}^{\ell-[\underline{j}]}$. It follows that $F^{m+1^*}(\partial^{[\underline{j}]}(s)) \in \mathrm{Fil}^{\ell-[\underline{j}]}$ so that

$$p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, t^{\underline{j}p^{m+1}-\underline{k}} F^{m+1^*}\big(\underline{\partial}^{[\underline{j}]}(s)\big) \in p^{\underline{j}}\, \mathrm{Fil}^{\ell-[\underline{j}]}.$$

Since $(\mathcal{E}, \mathrm{Fil})$ is saturated with respect to $(p, \{\ \})$, so is $F^{m+1^*}(\mathcal{E}, \mathrm{Fil})$ and therefore

$$\begin{aligned}\underline{\partial}^{[\underline{j}]}\big(F^{m+1^*}(s)\big) &= \sum p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, t^{\underline{j}p^{m+1}-\underline{k}} F^{m+1^*}\big(\underline{\partial}^{[\underline{j}]}(s)\big)\\ &\in \sum p^{\underline{j}}\, \mathrm{Fil}^{\ell-[\underline{j}]} = \sum p^{\{\underline{j}\}}\, \mathrm{Fil}^{\ell-[\underline{j}]} \subset \mathrm{Fil}^{\ell}.\end{aligned}$$ □

2.3.3. THEOREM. — *Assume* S *has no* p*-torsion. Let* $(\mathcal{E}, \mathrm{Fil})$ *be a* p*-torsion free Griffiths transversal* $\widehat{\mathcal{D}}^{(m)}_{X^{(m+1)}/S}$*-module of width less than* p^{m+1} *which is transversal to* $(p, \{\ \})$*. Then there exists a unique* p*-isogeny* $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ *of* $\widehat{\mathcal{D}}^{(m)}_{X/S}$*-modules such that* $F^{m+1^*}\mathrm{Fil}^k$ *is the saturation of* $\Phi^{-1}(p^k\mathcal{F})$ *with respect to* $(p, \{\ \})$.

Proof. — Follows from Corollary 1.2.6 and Proposition 2.3.2. □

2.3.4. DEFINITION. — *Given any lifting* $F : X \to X'$ *of the relative Frobenius of* X_0 *over* S_0*, an* F^{m+1}*-*p*-isogeny on* X/S *will be a* p*-isogeny of the form* $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ *where* $\mathcal{E}$ *is a* $\widehat{\mathcal{D}}^{(m)}_{X^{(m+1)}/S}$*-module and* $\mathcal{F}$ *is a* $\widehat{\mathcal{D}}^{(m)}_{X/S}$*-module.*

2.3.5. — Theorem 2.3.3 gives a functor μ from the category of p-torsion free Griffiths transversal $\widehat{\mathcal{D}}^{(m)}_{X^{(m+1)}/S}$-module of width less than p^{m+1} that are transversal to $(p, \{\ \})$ to the category of F^{m+1}-p-isogenies of width less than p^{m+1} on X/S. We will show in section 3.3 that this functor is fully faithful.

3. $\mathcal{D}^{(m)}$-MODULES IN CHARACTERISTIC p AND GRIFFITHS TRANSVERSALITY

3.1. p-m-curvature of a $\mathcal{D}^{(m)}$-module.

We define p-m-curvature for $\mathcal{D}^{(m)}$-modules in characteristic p and study the relation between it being zero and horizontal sections.

Let S be a scheme of characteristic p and X a smooth S-scheme. We let

- $\mathcal{D}^{(m)+}_{X/S}$ be the kernel of the canonical map $\mathcal{D}^{(m)}_{X/S} \to \mathcal{O}_X$;
- $\mathcal{K}^{(m)}_{X/S}$ be the kernel of the canonical map $\mathcal{D}^{(m)}_{X/S} \to \mathcal{E}nd(\mathcal{O}_X)$.

3.1.1. DEFINITIONS. — *Let $\mathcal{F}$ be a $\mathcal{D}^{(m)}_{X/S}$-module. The sheaf $\mathcal{F}^{\nabla}$ of horizontal sections of $\mathcal{F}$ is the part of $\mathcal{F}$ on which $\mathcal{D}^{(m)+}_{X/S}$ acts as zero. The p-m-curvature of $\mathcal{F}$ is the restriction to $\mathcal{K}^{(m)}_{X/S}$ of the canonical map $\mathcal{D}^{(m)}_{X/S} \to \mathcal{E}nd(\mathcal{F})$.*

3.1.2. *Remark.* — Let $\mathcal{F}$ be a $\mathcal{D}^{(m)}_{X/S}$-module. Then it follows from [B4], 2.2.6, that $\mathcal{F}$ has zero p-m-curvature if, locally on X, we have for all i, $\partial_i^{\langle p^{m+1}\rangle}(s) = 0$ for any $s \in \mathcal{F}$. In particular, in case $m = 0$, zero p-m-curvature is the same as zero p-curvature.

Let $F: X \to X'$ be the relative Frobenius of X over S.

3.1.3. LEMMA. — *If $\mathcal{E}$ is a $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$-module, then $\mathcal{D}^{(m)+}_{X/S}$ acts as zero on sections of the form $F^{m+1^*}(s)$ with $s \in \mathcal{E}$.*

Proof. — This is a local question. We have

$$\begin{aligned} F^{m+1^*}(\tau_i) &= (t_i + \tau_i)^{p^{m+1}} - t_i^{p^{m+1}} \\ &= \sum_{k=1}^{p^{m+1}} \binom{p^{m+1}}{k} t_i^{p^{m+1}-k} \tau_i^k \\ &= \tau_i^{p^{m+1}} = p!\, \tau_i^{\{p^{m+1}\}} = 0. \end{aligned}$$

It follows that, if $0 < j < p^{m+1}$, then $F^{m+1^*}(\tau^j) = 0$, so that, for any section s of $\mathcal{E}$, we have $\partial^{[j]}(F^{m+1^*}(s)) = 0$. □

3.1.4. PROPOSITION. — *The trivial $\mathcal{D}^{(m)}_{X/S}$-module $\mathcal{O}_X$ has zero p-m-curvature and the canonical map $\mathcal{O}_{X^{(m+1)}} \to F^{m+1}_* \mathcal{O}_X^{\nabla}$ is bijective.*

Proof. — The first assertion is an obvious consequence of the definition. The second one is local and we may therefore choose local coordinates $t_1, \dots, t_d$. These coordinates define an étale map from X to $\mathbb{A}^d_S$. The relative Frobenius being cartesian with respect to étale morphisms and to base change, this map provides us with an isomorphism

$$F_*^{m+1}\mathcal{O}_X \cong \mathcal{O}_{X^{(m+1)}} \otimes_{\mathbb{F}_p[t_1,\dots,t_d]} \mathbb{F}_p[t_1, \dots, t_d]^{(m+1)}$$

where $\mathbb{F}_p[t_1, \dots, t_d]^{(m+1)}$ is $\mathbb{F}_p[t_1, \dots, t_d]$ seen as a module over itself via the $(m+1)$-st power of Frobenius. If $\mathbb{F}_p[t_1, \dots, t_d]_{<p^{(m+1)}}$ denotes the space of polynomials of degree stricly less than p^{m+1} in each variable, the canonical map

$$\mathbb{F}_p[t_1, \dots, t_d] \otimes_{\mathbb{F}_p} \mathbb{F}_p[t_1, \dots, t_d]^{(m+1)}_{<p^{(m+1)}} \longrightarrow \mathbb{F}_p[t_1, \dots, t_d]^{(m+1)}$$

is bijective and therefore

$$F_*^{m+1}\mathcal{O}_X \cong \mathcal{O}_{X^{(m+1)}} \otimes_{\mathbb{F}_p} \mathbb{F}_p[t_1, \dots, t_d]_{<p^{(m+1)}}.$$

Since $F_*^{m+1}\mathcal{D}^{(m)+}_{X/S}$ acts as zero on $\mathcal{O}_{X^{(m+1)}}$, we are reduced to showing that if $f \in \mathbb{F}_p[t_1, \dots, t_d]_{<p^{(m+1)}}$ and $\mathcal{D}^{(m)+}_{X/S}$ acts as zero on f, then $f \in \mathbb{F}_p$. One may first prove that if A is an $\mathbb{F}_p$-algebra and $f \in A[t^{p^j}]$ is such that $\partial^{\langle p^j \rangle}(f) = 0$, then $f \in A[t^{p^{j+1}}]$ and then use induction on d. The details are left to the reader. □

3.1.5. Proposition

(i) *If $\mathcal{F}$ is a $\mathcal{D}^{(m)}_{X/S}$-module then $F_*^{m+1}\mathcal{F}^\nabla$ is a sub $\mathcal{O}_{X^{(m+1)}}$-module of $F_*^{m+1}\mathcal{F}$.*

(ii) *If $\mathcal{E}$ is a $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$-module then $F^{m+1^*}\mathcal{E}$ has zero p-m-curvature.*

Proof. — Again, these are local questions. For the first assertion, we have to show that if s is a section of $\mathcal{F}^\nabla$ and f is a section of $\mathcal{O}_{X^{(m+1)}}$ then $\underline{\partial}^{\langle \underline{k} \rangle}((F^{m+1^*}(f)s) = 0$ for $\underline{k} \neq 0$. For the second one, we have to show that if s is a section of $\mathcal{F}$ and f is a section of $\mathcal{O}_X$, then $\partial_i^{\langle p^{m+1} \rangle}(fF^{m+1^*}(s)) = 0$. Using the formula

$$\underline{\partial}^{\langle \underline{k} \rangle}(fs) = \sum \left\{ \begin{matrix} \underline{k} \\ \underline{j} \end{matrix} \right\} \underline{\partial}^{\langle \underline{j} \rangle}(f)\, \underline{\partial}^{\langle \underline{k}-\underline{j} \rangle}(s),$$

both statements are easy consequences of Lemma 3.1.3 and Proposition 3.1.4. □

3.2. Cartier's theorem for $\mathcal{D}^{(m)}$-modules.

We generalize Cartier's theorem (see [K], 5.1) to $\mathcal{D}^{(m)}_{X/S}$-modules. We let S, X and $F : X \to X'$ be as in section 3.1.

3.2.1. Lemma. — *Let $t_1, \dots, t_d$ be local coordinates on X and*

$$P := \sum_{\underline{k}<p^{m+1}} (-\underline{t})^{\underline{k}} \underline{\partial}^{[\underline{k}]}.$$

If $\mathcal{F}$ is a $\mathcal{D}^{(m)}_{X/S}$-module with zero p-m-curvature, then P is a projector from $\mathcal{F}$ onto $\mathcal{F}^{\nabla}$.

Proof. — We follow the first part of the proof of Proposition 5.1 in [K]. Since $\mathcal{F}$ has zero p-m-curvature, we have $\partial_i^{\langle j\rangle}(s) = 0$ for $j \geq p^{m+1}$. There should therefore be no confusion if we write $\underline{\partial}^{[\underline{j}]}(s) = 0$ for $\underline{j}$ such that $\max(j_i) \geq p^{m+1}$. If $s \in \mathcal{F}$, we have

$$\begin{aligned}
\underline{\partial}^{[\underline{j}]}(P(s)) &= \underline{\partial}^{[\underline{j}]}\Big(\sum (-\underline{t})^{\underline{k}}\, \underline{\partial}^{[\underline{k}]}(s)\Big)\\
&= \sum\sum \underline{\partial}^{[\underline{i}]}((-\underline{t})^{\underline{k}})\,(\underline{\partial}^{[\underline{j}-\underline{i}]}\,\underline{\partial}^{[\underline{k}]})(s)\\
&= \sum\sum (-1)^{\underline{i}}\binom{\underline{k}}{\underline{i}}(-\underline{t})^{\underline{k}-\underline{i}}\binom{\underline{k}+\underline{j}-\underline{i}}{\underline{k}}\,\underline{\partial}^{[\underline{k}+\underline{j}-\underline{i}]}(s)\\
&= \sum\sum (-1)^{\underline{i}}\binom{\underline{\ell}+\underline{i}}{\underline{i}}(-\underline{t})^{\underline{\ell}}\binom{\underline{\ell}+\underline{j}}{\underline{\ell}+\underline{i}}\,\underline{\partial}^{[\underline{\ell}+\underline{j}]}(s)\\
&= \sum\Big(\sum (-1)^{\underline{i}}\binom{\underline{\ell}+\underline{i}}{\underline{i}}\binom{\underline{\ell}+\underline{j}}{\underline{\ell}+\underline{i}}\Big)(-\underline{t})^{\underline{\ell}}\,\underline{\partial}^{[\underline{\ell}+\underline{j}]}(s)
\end{aligned}$$

and, if $\underline{j} \neq 0$, we have

$$\sum (-1)^{\underline{i}}\binom{\underline{\ell}+\underline{i}}{\underline{i}}\binom{\underline{\ell}+\underline{j}}{\underline{\ell}+\underline{i}} = \binom{\underline{\ell}+\underline{j}}{\underline{\ell}}\sum(-1)^{\underline{i}}\binom{\underline{j}}{\underline{i}} = 0.$$

Thus we see that P maps f into $\mathcal{F}^{\nabla}$. Since P restricts to the identity on $\mathcal{F}^{\nabla}$, it is a projector from $\mathcal{F}$ onto $\mathcal{F}^{\nabla}$. □

3.2.2. Proposition. — *Let $\mathcal{F}$ be a $\mathcal{D}^{(m)}_{X/S}$-module with zero p-m-curvature. Then the canonical map $F^{m+1^*} F^{m+1}_{*}\mathcal{F}^{\nabla} \to \mathcal{F}$ is an isomorphism.*

Proof. — We follow the end of the proof of Proposition 5.1 in [K]. The question is local on X and we may therefore assume that we have local coordinates $t_1, \dots, t_d$. We have seen in Lemma 3.2.1 that P is a projector from $\mathcal{F}$ onto $\mathcal{F}^{\nabla}$. It follows that the map

$$T : \mathcal{F} \longrightarrow F^{m+1^*} F_*^{m+1} \mathcal{F}^{\nabla}, \qquad s \longmapsto \sum_{\underline{k} < p^{m+1}} \underline{t}^{\underline{k}} \otimes P \underline{\partial}^{[\underline{k}]}(s)$$

is well defined. Let us show that T is a right inverse to the canonical map $U : F^{m+1^*} F_*^{m+1} \mathcal{F}^{\nabla} \to \mathcal{F}$. If $s \in \mathcal{F}$, then

$$\begin{aligned}
(U \circ T)(s) &= \sum \underline{t}^{\underline{k}} P \underline{\partial}^{[\underline{k}]}(s) \\
&= \sum \underline{t}^{\underline{k}} \sum (-\underline{t})^{\underline{\ell}} \, \underline{\partial}^{[\underline{\ell}]} \, \underline{\partial}^{[\underline{k}]}(s) \\
&= \sum \sum (-1)^{\underline{\ell}} \underline{t}^{\underline{k}+\underline{\ell}} \binom{\underline{k}+\underline{\ell}}{\underline{\ell}} \underline{\partial}^{[\underline{k}+\underline{\ell}]}(s) \\
&= \sum \Big(\sum (-1)^{\underline{\ell}} \binom{\underline{j}}{\underline{\ell}} \Big) \underline{t}^{\underline{j}} \underline{\partial}^{[\underline{j}]}(s) = s.
\end{aligned}$$

We have seen that $F_*^{m+1} \mathcal{O}_X^{\nabla} = \mathcal{O}_{X^{(m+1)}}$ and it follows that U is a bijection in the case $\mathcal{F} = \mathcal{O}_X$. Hence, T is also a left inverse to U in this case, which implies that for any $f \in \mathcal{O}_X$, we have $T(f) = f \otimes 1$. In general, we have for $f \in \mathcal{O}_X$ and $s \in \mathcal{F}^{\nabla}$,

$$\begin{aligned}
(T \circ U)(f \otimes s) = T(fs) &= \sum \underline{t}^{\underline{k}} \otimes P \underline{\partial}^{[\underline{k}]}(fs) \\
&= \sum \underline{t}^{\underline{k}} \otimes P \underline{\partial}^{[\underline{k}]}(f) s \\
&= \Big(\sum \underline{t}^{\underline{k}} \otimes P \underline{\partial}^{[\underline{k}]}(f) \Big)(1 \otimes s) \\
&= T(f)(1 \otimes s) = (f \otimes 1)(1 \otimes s) = f \otimes s. \qquad \square
\end{aligned}$$

3.2.3. Proposition. — *Let $\mathcal{E}$ be a $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$-module, $\mathcal{F} = F^{m+1^*} \mathcal{E}$ (as $\mathcal{D}^{(m)}_{X/S}$-module) and $\eta : \mathcal{E} \to F_*^{m+1} \mathcal{F}$ be the adjunction map. Then*

(i) *The map η induces a natural isomorphism $\mathcal{E} \cong F_*^{m+1} \mathcal{F}^{\nabla}$ of $\mathcal{O}_{X^{(m+1)}}$-modules.*

(ii) *In the situation of Lemma 3.2.1, the action of P on $F_*^{m+1} \mathcal{F}$ factors through η.*

(iii) *If $\mathcal{F}'$ is a sub-$\mathcal{D}^{(m)}_{X/S}$-module of $\mathcal{F}$, then the natural map $F^{m+1^*} F_*^{m+1} \mathcal{F} \to \mathcal{F}$ induces an isomorphism $F^{m+1^*}(\eta^{-1}(F_*^{m+1} \mathcal{F}')) \cong \mathcal{F}'$.*

Proof. — We know from Proposition 3.1.5 (ii) that $\mathcal{F}$ has zero p-m-curvature. It follows from Proposition 3.2.2 that

$$F^{m+1^*}\mathcal{E} \cong F^{m+1^*}F_*^{m+1}\mathcal{F}^\nabla$$

and we use the faithful flatness of F to obtain assertion (i).

In order to prove assertion (ii), we recall from Lemma 3.2.1 that the image of P acting on $\mathcal{F}$ is (contained in) $\mathcal{F}^\nabla$. It therefore follows from (i) that the action of P on $F_*^{m+1}\mathcal{F}$ factors through

$$\eta : \mathcal{E} \cong F_*^{m+1}\mathcal{F}^\nabla \longrightarrow F_*^{m+1}\mathcal{F}.$$

Finally, for (iii), since $\mathcal{F}$ has zero p-m-curvature, so does $\mathcal{F}'$. The map η being functorial, it follows from (i) that it induces $\mathcal{F}'^\nabla \cong F_*^{m+1}\mathcal{F}'$ so that

$$\mathcal{F}' \cong F^{m+1^*}\mathcal{F}'^\nabla \cong F^{m+1^*}\big(\eta^{-1}(F_*^{m+1}\mathcal{F}')\big). \qquad \square$$

3.2.4. Corollary (Cartier's theorem). — *The functors $\mathcal{E} \mapsto F^{m+1^*}\mathcal{E}$ and $\mathcal{F} \mapsto F_*^{m+1}\mathcal{F}^\nabla$ give an equivalence between the category of $\mathcal{O}_{X^{(m+1)}}$-modules and the category of $\mathcal{D}^{(m)}_{X/S}$-modules with zero p-m-curvature.* $\square$

3.3. F^{m+1}-p-isogenies and Griffiths transversality.

We have built in section 2.3 a functor μ that associates F^{m+1}-p-isogenies to some filtered $\widehat{\mathcal{D}}^{(m)}$-modules. We are now going to define a functor α from F^{m+1}-p-isogenies to filtered $\widehat{\mathcal{D}}^{(m)}$-modules that will allow us to prove that μ is fully faithful.

The setting is as in section 2.3: S is a p-torsion free formal scheme, X is a smooth formal S-scheme, F_0 is the relative Frobenius of X_0 over S_0 and $F : X \to X'$ is a lifting of F_0. We also assume that there are local coordinates $t_1, \ldots, t_d$ on X and X' such that $F^*(t_i) = t_i^p$.

If $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ is an F^{m+1}-p-isogeny on X/S, we consider the filtration M on $F^{m+1^*}\mathcal{E}$ given by

$$M^k : = \Phi^{-1}(p^k\mathcal{F})$$

and the filtration Fil on $\mathcal{E}$ given by

$$\mathrm{Fil}^k : = \eta^{-1}(F_*^{m+1}M^k),$$

where $\eta : \mathcal{E} \to F_*^{m+1}F^{m+1^*}\mathcal{E}$ is the adjunction map. We will write $\overline{\mathrm{Fil}}$ for the saturation of Fil with respect to $(p, \{\ \})$. This way, we get a functor

$$\alpha : (\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}) \longmapsto (\mathcal{E}, \overline{\mathrm{Fil}})$$

with values in the category of filtered $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules transversal to $(p, \{\ \})$.

3.3.1. Lemma. — *If*

$$P := \sum_{\underline{k}<p^{m+1}} (-\underline{t})^{\underline{k}} \underline{\partial}^{[\underline{k}]},$$

then there exists Q, *reducing to* 1 mod p, *such that*

$$P\big(F^{m+1^*}(s)\big) = F^{m+1^*}\big(Q(s)\big)$$

for any section s *of a* $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$*-module* $\mathcal{E}$.

Proof. — From Lemma 2.3.1, we deduce that

$$\underline{t}^{\underline{k}} \underline{\partial}^{[\underline{k}]}\big(F^{m+1^*}(s)\big) = \sum p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, \underline{t}^{\underline{j}p^{m+1}} F^{m+1^*}\big(\underline{\partial}^{[\underline{j}]}(s)\big) = F^{m+1^*}\big(Q_{\underline{k}}(s)\big)$$

where $Q_{\underline{k}} : = \sum p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, \underline{t}^{\underline{j}} \underline{\partial}^{[\underline{j}]}$ and we let

$$Q = \sum_{\underline{k}<p^{m+1}} (-1)^{\underline{k}} Q_{\underline{k}}.$$

□

The following result is of technical nature and is needed in the next proposition:

3.3.2. Lemma. — *Let* $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ *be an* F^{m+1}*-p-isogeny on* X/S *and* M, Fil *and* η *as above. Then* $\eta_0 : \mathcal{E}_0 \to F^{m+1}_{0*} F^{m+1^*}_0 \mathcal{E}_0$ *is strictly compatible with the induced filtrations (i.e. we have* $\mathrm{Fil}^k_0 = \eta_0^{-1}(F^{m+1}_{0*} M^k_0)$*).*

Proof. — We follow the proof of Theorem 2.2 of [O1]. The map is clearly compatible with the induced filtrations and we are left with proving the strictness. Let $s_0 \in \mathcal{E}_0$ be such that $\eta_0(s_0) \in F^{m+1}_{0*} M^k_0$. We want to prove that there exists a lifting $s \in \mathcal{E}$ of s_0 such that $\Phi(\eta(s)) = p^k s'$. It is clearly sufficient to show that for any i there exists a lifting $s \in \mathcal{E}$ of s_0, and u such that $\Phi(\eta(s) + p^i u) = p^k s'$ and then take $i = k$. We prove this by induction on i, the case $i = 1$ being just our assumption.

So, let us assume that $s \in \mathcal{E}$ is a lifting of s_0 such that

$$\Phi(\eta(s) + p^i u) = p^k s'.$$

Since Φ is a morphism of $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules, it commutes with the operator P of the lemma. Using Lemma 3.3.1, we have

$$\begin{aligned} p^k P(s') &= P(p^k s') = P\big(\Phi(\eta(s) + p^i u)\big) \\ &= \Phi\big(P(\eta(s)) + P(p^i u)\big) = \Phi\big(\eta(Q(s)) + p^i P(u)\big). \end{aligned}$$

We have seen in Proposition 3.2.3 (ii) that the action of P on $F_0^{m+1^*}\mathcal{E}_0$ factors through $\eta_0 : \mathcal{E}_0 \to F_{0*}^{m+1}F_0^{m+1^*}\mathcal{E}_0$. We can therefore write

$$P(u) = \eta(v) + pw.$$

It follows that

$$p^k P(s') = \Phi\big(\eta(Q(s)) + p^i\eta(v) + p^{i+1}w\big) = \Phi\big(\eta(Q(s) + p^i v) + p^{i+1}w\big).$$

It just remains to observe that $Q(s) + p^i v$ is a lifting of s_0 since Q is the identity mod p. □

3.3.3. Proposition. — *Let $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ be an F^{m+1}-p-isogeny on X/S, and M and* Fil *as above. Then we have $F^{m+1^*}\,\mathrm{Fil}^k = M^k$.*

Proof. — We follow the proof of Lemma 5.2.11 in [O2]. The modules $\mathcal{E}$ and $\mathcal{F}$ are p-torsion free and the filtrations Fil^k and M^k are transversal to p. From this, we deduce that the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & F^{m+1^*}\,\mathrm{Fil}^{k-1} & \xrightarrow{p} & F^{m+1^*}\,\mathrm{Fil}^k & \longrightarrow & F_0^{m+1^*}\,\mathrm{Fil}_0^k & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M^{k-1} & \xrightarrow{p} & M^k & \longrightarrow & M_0^k & \longrightarrow & 0 \end{array}$$

has exact rows. Hence, by induction, it is sufficient to prove that $F_0^{m+1^*}\,\mathrm{Fil}_0^k = M_0^k$ But we have seen in Proposition 3.2.3 (iii) that

$$F_0^{m+1^*}\big(\eta_0^{-1}(F_{0*}^{m+1}M_0^k)\big) = M_0^k$$

and we know from Lemma 3.3.2 that $\eta_0^{-1}(F_{0*}^{m+1}M_0^k) = \mathrm{Fil}_0^k$. □

We will show in Proposition 5.2.5 that the filtration $\overline{\mathrm{Fil}}$ in the definition of α is not always Griffiths transversal when $m > 0$. Nevertheless, for the functor μ of 2.3.5, we have the following:

3.3.4. Theorem. — *When restricted to the essential image of μ, the functor α is a quasi-inverse to μ.*

Proof. — Follows from Proposition 3.3.3. □

3.3.5. Corollary. — *The functor μ is fully faithful.* □

4. TRANSVERSAL m-CRYSTALS

4.1. m-crystals.

We recall Berthelot's theory of m-crystals from [B3].

Let $(S, \mathfrak{a}, \mathfrak{b})$ be a formal m-PD-scheme. If X is an S-scheme, we will always assume that the m-PD-structure of S extends to X.

4.1.1. Definition. — *If $X \hookrightarrow Y$ is an m-PD-S-immersion of S-schemes, we say that Y is an m-PD-S-thickening of X.*

4.1.2. Definition. — *Let X be an S-scheme. The m-th crystalline site of X/S is the category $\mathrm{Cris}^{(m)}(X/S)$ of m-PD-S-thickenings $U \hookrightarrow Y$ with U open in X, endowed with a suitable topology. As in the classical case, the site $\mathrm{Cris}^{(m)}(X/S)$ is functorial in X/S.*

4.1.3. *Remark.* — There exists a unique sheaf $\mathcal{I}^{\{n\}}_{X/S}$ on $\mathrm{Cris}^{(m)}(X/S)$ whose value on (Y, I, J) is $I^{\{n\}}$. We will write

$$\mathcal{O}_{X/S} := \mathcal{I}^{\{0\}}_{X/S} \quad \text{and} \quad \mathcal{I}_{X/S} := \mathcal{I}^{\{1\}}_{X/S}.$$

It is clear that $(\mathrm{Cris}^{(m)}(X/S), \mathcal{O}_{X/S}, \mathcal{I}^{\{n\}}_{X/S})$ is a filtered ringed site.

4.1.4. Definition. — *Let X be an S-scheme. To any sheaf E on $\mathrm{Cris}^{(m)}(X/S)$ and any object Y of $\mathrm{Cris}^{(m)}(X/S)$, one associates in the obvious way a sheaf E_Y on Y. If E is an $\mathcal{O}_{X/S}$-module, any morphism $\varphi : Y' \to Y$ of m-PD-thickenings gives a natural morphism $\varphi^* E_Y \to E_{Y'}$. We call E an m-crystal if these maps are all bijective.*

The proofs of the following statements are straightforward generalizations of those of the analogous results from [B1]. They should appear in a forthcoming article of Berthelot as announced in [B4].

4.1.5. Proposition. — *If $X \hookrightarrow Y$ is a closed immersion of S-schemes and E is an m-crystal on X, then $i_* E$ is an m-crystal on Y.*

4.1.6. Corollary. — *If $\overline{S} = \mathrm{Spec}\, \mathcal{O}_S/\mathfrak{a}$ and $\overline{X} = X \times_S \overline{S}$, then the restriction functor $\mathrm{Cris}^{(m)}(X/S) \to \mathrm{Cris}^{(m)}(\overline{X}/S)$ induces an equivalence between the categories of m-crystals on X/S and on $\overline{X}/S$.*

4.1.7. Proposition. — *Let $i : X \hookrightarrow Y$ be a closed immersion of S-schemes with Y smooth. Then the functor $E \mapsto E_Y := (i_* E)_Y$ is an equivalence of categories between m-crystals on X and locally quasi-nilpotent $\mathcal{P}_{X/S(m)}(Y)$-$\mathcal{D}^{(m)}_{Y/S}$-modules.*

4.1.8. Proposition. — *Let X be a smooth formal S-scheme and let X_n denote its reduction* mod p^{n+1}. *The functor*

$$E \longmapsto E_X := \varprojlim E_{X_n}$$

is an equivalence of categories between m-crystals on X_0 and locally topologically quasi-nilpotent complete $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules.

4.2. T-m-Crystals.

We define T-m-crystals and relate them to differential modules. Note that we call T-m-crystals what Ogus would call proto-T-m-crystals.

Let S be a formal m-PD-scheme.

4.2.1. Proposition and Definition. — *Let $f : (U', Y') \to (U, Y)$ be a morphism of m-PD-S-thickenings such that $U' \to U$ is flat and $(\mathcal{F}, \mathrm{Fil})$ a T-module on $(Y, \mathcal{O}_Y, \mathcal{I}^{\{n\}})$. Then $Tf^*(\mathcal{F}, \mathrm{Fil}) := (f^*\mathcal{F}, \overline{\mathrm{Fil}}^k_f)$ is a T-module called the T-inverse image of $(\mathcal{F}, \mathrm{Fil})$.*

Proof. — This follows from Proposition 1.1.7 (ii) and Proposition 1.1.8. □

4.2.2. Definition. — *Let X be an S-scheme. If E is any T-module on* $\mathrm{Cris}(X/S)^{(m)}$ *and Y any object of* $\mathrm{Cris}(X/S)^{(m)}$, *then E_Y is in a natural way a T-module. If $f : Y' \to Y$ is a morphism in* $\mathrm{Cris}(X/S)^{(m)}$, *then there is a natural morphism of filtered modules $Tf^*E_Y \to E_{Y'}$. We call E a T-m-crystal if these maps are all isomorphisms of filtered modules (i.e. such that $\overline{\mathrm{Fil}}^k_f = \mathrm{Fil}^k$).*

The category of T-m-crystals is functorial with respect to flat morphisms: if $\varphi : X' \to X$ is a flat morphism and E a T-m-crystal on X/S, then

$$T\varphi^*(E, \mathrm{Fil}) : = (\varphi^*E, \overline{\mathrm{Fil}}^k_\varphi)$$

is a T-m-crystal.

4.2.3. *Example.* — The *trivial* T-m-crystal is $(\mathcal{O}_{X/S}, \mathcal{I}^{\{k\}}_{X/S})$ whose value at X is the trivial filtered module $\mathcal{O}_X = \mathrm{Fil}^0 \supset \mathrm{Fil}^1 = 0$.

The following generalize Proposition 3.2.2 and Theorem 3.2.3 of [O2]:

4.2.4. Proposition. — *If $i : X \hookrightarrow Y$ is a closed immersion into a smooth S-scheme and E a T-m-crystal on X/S, then*

$$i_*(E, \mathrm{Fil}) := (i_*E, i_* \mathrm{Fil})$$

is a T-m-crystal which is transversal to $(i_*\mathcal{I}_{X/S}, \{\ \})$.

Proof. — Same proof as [O2], 3.2.2. □

4.2.5. Proposition. — *Let $i : X \hookrightarrow Y$ be a closed S-immersion into a smooth S-scheme. Then the functor $E \mapsto E_Y$ is an equivalence of categories between T-m-crystals on X and Griffiths transversal locally quasi-nilpotent $\mathcal{P}_{X/S(m)}(Y)$-$\mathcal{D}^{(m)}_{Y/S}$-modules which are transversal to the m-PD-filtration of $\mathcal{P}_{X/S(m)}(Y)$.*

Proof. — Let $p_1, p_2 : P_X(Y^2) \to P_X(Y)$ be the projections. If E is a T-m-crystal, we have an isomorphism of filtered modules

$$\varepsilon : Tp_2^* E_Y \xrightarrow{\sim} E_{Y^2} \xleftarrow{\sim} Tp_1^* E_Y ,$$

which means that the HPD-stratification $\varepsilon : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}$ is transversal and therefore, by Proposition 2.2.6, that E_Y is Griffiths transversal.

Conversely, let $\mathcal{F}$ be a Griffiths transversal locally quasi-nilpotent $\mathcal{P}_{X/S(m)}(Y)$-$\mathcal{D}^{(m)}_{Y/S}$-module which is transversal to the m-PD-filtration of $\mathcal{P}_{X/S(m)}(Y)$. There exists, by Proposition 4.1.7, a unique m-crystal E such that $E_Y = \mathcal{F}$. Let $X \hookrightarrow T$ be an m-PD-thickening. Since Y is smooth, i extends locally on T to a map $f : T \to Y$ which in turn extends to an m-PD-morphism $g : T \to P_X(Y)$. We then set

$$\mathrm{Fil}^k E_T = \overline{\mathrm{Fil}}^k_g ,$$

so that $(E_T, \mathrm{Fil}) = Tg^*(\mathcal{F}, \mathrm{Fil})$. If this is well defined, it is clear that we obtain a quasi-inverse to our functor. It is actually sufficient to check that the HPD-stratification $\varepsilon : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}$ is transversal. But this follows again from Proposition 2.2.6. □

4.2.6. Corollary. — *Let X be a smooth formal S-scheme. Then the functor $E \mapsto E_X$ is an equivalence of categories between T-m-crystals on X_0/S and locally topologically quasi-nilpotent Griffiths transversal complete $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules transversal to $(p, \{\ \})$.* □

4.3. T-m-crystals and F-m-spans.

We define F-m-spans and use them to describe T-m-crystals.

Let S be a formal m-PD-scheme, X a smooth S_0-scheme, and $F:X \to X'$ the relative Frobenius of X over S_0.

4.3.1. Definition. — *If (E, Fil) is a filtered m-crystal where the* Fil^k *are not merely sub modules but sub m-crystals, then we say that (E, Fil) is horizontal.*

Note that a horizontal filtered m-crystal is not a T-m-crystal. Let us describe the saturation process:

4.3.2. Proposition

(i) *Any horizontal filtered m-crystal (E, Fil) on X/S that is almost transversal to $(p, \{\ \})$ is almost transversal to $(\mathcal{I}_{X/S}, \{\ \})$. In particular, $(E, \overline{\mathrm{Fil}})$ is a T-m-crystal.*

(ii) *The functor $(E, \mathrm{Fil}) \to (E, \overline{\mathrm{Fil}})$ from the category of horizontal filtered m-crystals on X/S that are transversal to $(p, \{\ \})$, to the category of T-m-crystals is fully faithful.*

Proof.

(i) Let $X \hookrightarrow T$ be an m-PD-immersion and I the ideal of X in T. We have to show that (E_T, Fil) is almost transversal to $(I, \{\ \})$. This question is local on T. The scheme X being smooth over S_0, it locally lifts to a smooth formal scheme Y over S. Since Y is smooth and $X \hookrightarrow T$ is nilpotent, there exists, locally on T, a map $\varphi: T \to Y$ that induces the identity on X. The m-PD-structure on T is compatible with $(p, \{\ \})$, so that the map φ is an m-PD-morphism. Since (E_Y, Fil) is almost transversal to $(p, \{\ \})$, it follows from Proposition 1.1.8 that (E_T, Fil) is almost transversal to $(I, \{\ \})$. Applying Proposition 1.1.7 (ii), we get the last assertion.

(ii) We have to show that $\overline{\mathrm{Fil}}^k$ determines Fil^k. This is a local question on X. The scheme X being smooth over S_0, it locally lifts to a smooth formal scheme Y over S. Since (E_Y, Fil) is saturated with respect to $(p, \{\ \})$, we have $\overline{\mathrm{Fil}}^k E_Y = \mathrm{Fil}^k E_Y$. It follows from Corollary 4.2.6 that $\mathrm{Fil}^k E$ is determined by $\mathrm{Fil}^k E_Y$ and hence by $\overline{\mathrm{Fil}}^k E$. □

4.3.3. Definition. — *If (E, Fil) is in the image of this last functor, we call it a horizontal T-m-crystal.*

We are now able to globalize the local results of parts 2 and 3:

4.3.4. PROPOSITION. — *If* (E, Fil) *is a* T*-*m*-crystal on* $X^{(m+1)}/S$*, then* $TF^{m+1^*}(E, \mathrm{Fil})$ *is a horizontal* T*-*m*-crystal.*

Proof. — This follows from Proposition 2.3.2 and Proposition 4.3.2 (i). □

4.3.5. DEFINITION. — *An* F*-*m*-span is a* p*-isogeny* $\Phi : F^{m+1^*}E \longrightarrow E'$ *of* m*-crystals.*

4.3.6. THEOREM. — *Assume* S *has no* p*-torsion. Let* (E, Fil) *be a* p*-torsion free* T*-*m*-crystal on* $X^{(m+1)}/S$ *of width less than* p^{m+1}*. Then there exists a unique* F*-*m*-span* $\Phi : F^{m+1^*}E \rightarrow E'$ *of width less than* p^{m+1} *such that the saturations of* $F^{m+1^*}\mathrm{Fil}^k$ *and* $\Phi^{-1}(p^kE')$ *with respect to* $(\mathcal{I}_{X/S}, \{\ \})$ *coincide. This construction is functorial in* (E, Fil) *and the functor is fully faithful.*

Proof. — Follows from Theorem 2.3.3, Proposition 4.3.2 (ii) and Corollary 3.3.5. □

5. COMPARISON OF TRANSVERSALITY PROPERTIES FOR VARIOUS LEVELS

From now on, m' will be an integer larger than m and $\{\ \}'$ will denote divided powers of level m'. We will also write $d := m' - m$.

5.1. Changing level and Griffiths transversality.

After recalling how to obtain a $\mathcal{D}^{(m)}$-module from a $\mathcal{D}^{(m')}$-module, we show that, for filtered $\mathcal{D}^{(m')}$-modules transversal to p of width at most p^{m+1}, Griffiths transversality can be checked on the corresponding filtered $\mathcal{D}^{(m)}$-module. We give a counterexample for higher width.

5.1.1. — We recall some results from [B4].

(i) If Y is a formal scheme and I is a coherent ideal in $\mathcal{O}_Y$, then any m-PD-structure $(J, [\ ])$ on I is also an m'-PD-structure on I. If $(S, \mathfrak{a}, \mathfrak{b})$ is a formal m-PD-scheme and (Y, I, J) is a formal m-PD-S-scheme, then it is also a formal m'-PD-S-scheme. We should also remark that the m'-PD-filtration is finer than the m-PD-filtration.

(ii) Let S be a formal m-PD-scheme, X a formal S-scheme to which the m-PD-structure of S extends and $i : X \hookrightarrow Y$ an immersion into a formal

S-scheme, then there are canonical maps $P^n_{X/S(m')}(Y) \to P^n_{X/S(m)}(Y)$. They are bijective for $n < p^{m+1}$.

(iii) Assume now that X is smooth over S. Then we get canonical maps

$$\mathcal{D}^{(m)}_{X/S\,n} \longrightarrow \mathcal{D}^{(m')}_{X/S\,n}$$

that are bijective for $n < p^{m+1}$. They glue to give canonical maps $\mathcal{D}^{(m)}_{X/S} \to \mathcal{D}^{(m')}_{X/S}$ and, after completion, $\widehat{\mathcal{D}}^{(m)}_{X/S} \to \widehat{\mathcal{D}}^{(m')}_{X/S}$. We can therefore consider any $\mathcal{D}^{(m')}_{X/S}$-module (resp. $\widehat{\mathcal{D}}^{(m')}_{X/S}$-module) as a $\mathcal{D}^{(m)}_{X/S}$-module (resp. $\widehat{\mathcal{D}}^{(m)}_{X/S}$-module).

(iv) Assume moreover that S has no p-torsion. Then one easily checks that the obvious functor from $\widehat{\mathcal{D}}^{(m')}_{X/S}$-modules to $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules is faithful. It is even fully faithful when restricted to p-torsion free objects.

Let S be a formal m-PD-scheme and X a smooth formal S-scheme to which the m-PD-structure of S extends. If $(\mathcal{F}, \mathrm{Fil})$ is a Griffiths transversal $\mathcal{D}^{(m')}_{X/S}$-module (resp. $\widehat{\mathcal{D}}^{(m')}_{X/S}$-module), then it is also Griffiths transversal as a $\widehat{\mathcal{D}}^{(m)}_{X/S}$-module (resp. $\widehat{\mathcal{D}}^{(m)}_{X/S}$-module).

The converse is true under some additional hypothesis:

5.1.2. PROPOSITION. — *Let $(\mathcal{F}, \mathrm{Fil})$ be a filtered $\mathcal{D}^{(m')}_{X/S}$-module of width at most p^{m+1} that is Griffiths transversal as a $\mathcal{D}^{(m)}_{X/S}$-module and transversal to p. Then it is also Griffiths transversal as a $\mathcal{D}^{(m')}_{X/S}$-module.*

Proof. — We have to show that, if $P \in \mathcal{D}^{(m')}_{X/S}$ is an m'-PD-differential operator of order at most n, then $P(\mathrm{Fil}^k) \subset \mathrm{Fil}^{k-n}$. Thanks to 5.1.1 (iii), we may assume that $n \geq p^{m+1}$. We proceed by induction on k.

- If $k \leq p^{m+1}$, then $\mathrm{Fil}^{k-n} = \mathcal{F}$ and our assertion is trivial.
- If $k > p^{m+1}$, transversality to p and the condition on the width give us that $\mathrm{Fil}^k = p\,\mathrm{Fil}^{k-1}$. It follows that

$$P(\mathrm{Fil}^k) = pP(\mathrm{Fil}^{k-1}) \subset p\,\mathrm{Fil}^{k-1-n} \subset \mathrm{Fil}^{k-n}. \qquad \square$$

The bound on the width is sharp as the following shows:

5.1.3. *Example.* — We take X to be the affine line over S and we consider $(\mathcal{F}, \mathrm{Fil})$ where $\mathcal{F} = \mathcal{O}_X$ and Fil is defined as follows:

- for $0 \leq k \leq p^{m+1}$, Fil^k is the ideal generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq k$;

- for $k > p^{m+1}$, Fil^k is the ideal generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq p^{m+1} - 1$, together with $p^{k-p^{m+1}-1}t^{p^{m+1}}$.

It is clear that $(\mathcal{F}, \mathrm{Fil})$ is a filtered $\mathcal{D}^{(m')}_{X/S}$-module of width $p^{m+1} + 1$. It is transversal to p because, for $k \leq p^{m+1}$, both $(p) \cap \mathrm{Fil}^k$ and $p\,\mathrm{Fil}^{k-1}$ are generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq k-1$, together with $p\,t^k$, while $(p) \cap \mathrm{Fil}^{p^{m+1}+1}$ and $p\,\mathrm{Fil}^{p^{m+1}}$ are generated by the elements $p^{p^{m+1}+1-i}t^i$ for $0 \leq i \leq p^{m+1} - 1$, together with $p\,t^{p^{m+1}}$.

To show that $(\mathcal{F}, \mathrm{Fil})$ is Griffiths transversal as a $\mathcal{D}^{(m)}_{X/S}$-module, let us remark that

$$\partial^{[r]}(p^{k-i}t^i) = \begin{cases} \binom{i}{r}\, p^{k-i}t^{i-r} & \text{if } r \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\partial^{[r]}(\mathrm{Fil}^k) \subset \mathrm{Fil}^{k-r}$ when $0 \leq k \leq p^{m+1}$. Moreover, when $r \leq p^m$, we have $\binom{p^{m+1}}{r} \in (p)$ and therefore

$$\partial^{[r]}(\mathrm{Fil}^{p^{m+1}+1}) \subset p\,\mathrm{Fil}^{p^{m+1}-r} \subset \mathrm{Fil}^{p^{m+1}+1-r}.$$

Nevertheless, $(\mathcal{F}, \mathrm{Fil})$ is not Griffiths transversal as a $\mathcal{D}^{(m')}_{X/S}$-module because $t^{p^{m+1}} \in \mathrm{Fil}^{p^{m+1}+1}$ but $\partial^{[p^{m+1}]}(t^{p^{m+1}}) = 1 \notin \mathrm{Fil}^1$.

5.2. Frobenius descent and F^{m+1}-p-isogenies.

We are going to apply Berthelot's theory of Frobenius descent to F^{m+1}-p-isogenies and use it to study the question of the surjectivity of the functor μ of 2.3.5.

Let S be a formal m-PD-scheme and X a smooth formal S-scheme to which the m-PD-structure of S extends. Let F_0 be the relative Frobenius of X_0 over S_0 and $F : X \to X'$ a lifting of F_0. We briefly recall Berthelot's unpublished theory of Frobenius descent.

5.2.1. Proposition (see [B5]). — *The morphism*

$$F^d \times_S F^d : X \times_S X \longrightarrow X^{(d)} \times_S X^{(d)}$$

induces for all n, *a unique morphism*

$$F^d : P^n_{X/S(m')} \longrightarrow P^n_{X^{(d)}/S(m)}$$

compatible with the PD-structures (taking into account the PD-ideal of S*). It is also compatible with the partial divided power filtrations.*

It follows that, if $\mathcal{E}$ is a $\mathcal{D}^{(m)}_{X^{(d)}/S}$-module, then $F^{d^*}(\mathcal{E})$ has a natural structure of $\mathcal{D}^{(m')}_{X/S}$-module.

5.2.2. THEOREM (see [B5]). — *If S is a scheme, the functor $\mathcal{E} \mapsto F^{d^*}(\mathcal{E})$ induces an equivalence between the categories of $\mathcal{D}^{(m)}_{X^{(d)}/S}$-modules and $\mathcal{D}^{(m')}_{X/S}$-modules.*

It follows that the functor $\mathcal{E} \mapsto F^{d^*}(\mathcal{E})$ induces an equivalence between the category of complete $\widehat{\mathcal{D}}^{(m)}_{X^{(d)}/S}$-modules and the category of complete $\widehat{\mathcal{D}}^{(m')}_{X/S}$-modules. From Proposition 1.2.2, we get an equivalence between the category of p-isogenies of complete $\widehat{\mathcal{D}}^{(m)}_{X^{(d)}/S}$-modules and the category of p-isogenies of complete $\widehat{\mathcal{D}}^{(m')}_{X/S}$-modules. Thus, we get:

5.2.3. COROLLARY. — *The functor F^{d^*} makes the full subcategory of F^{m+1}-p-isogenies on $X^{(d)}/S$ consisting of those $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ where $\mathcal{E}$ is a $\widehat{\mathcal{D}}^{(m')}_{X^{(m'+1)}/S}$-module equivalent to the category of $F^{m'+1}$-p-isogenies on X/S.*

5.2.4. LEMMA. — *Let $(\mathcal{F}, \mathrm{Fil})$ be a filtered $\mathcal{D}^{(m)}_{X/S}$-module of width less than p^{m+1} that is transversal to p and $\overline{\mathrm{Fil}}$ the saturation of Fil with respect to $(p, \{\ \})$. Then $(\mathcal{F}, \mathrm{Fil})$ is Griffiths transversal if and only if $(\mathcal{F}, \overline{\mathrm{Fil}})$ is Griffiths transversal.*

Proof. — The filtrations are identical up to order $(p^{m+1} - 1)$ and, for any $k \geq 0$, we have

$$\mathrm{Fil}^{p^{m+1}-1+k} = p^k\, \mathrm{Fil}^{p^{m+1}-1} \quad \text{and} \quad \overline{\mathrm{Fil}}^{p^{m+1}-1+k} = (p)^{\{k\}}\overline{\mathrm{Fil}}^{p^{m+1}-1}. \qquad \square$$

Assume now that S is a p-torsion free formal PD-scheme and that there are local coordinates $t_1, \ldots, t_d$ on X and X' such that $F^*(t_i) = t_i^p$.

5.2.5. PROPOSITION. — *The functor μ of 2.3.5 is not in general an equivalence of categories for $m > 0$. However, it becomes an equivalence when restricted to objects of width at most p.*

Proof. — Let $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ be an F^{m+1}-p-isogeny on X/S of width less than p^{m+1}. By Corollary 5.2.3, it corresponds to a unique F-p-isogeny $\Phi^0 : F^*\mathcal{E} \to \mathcal{F}'$ on $X^{(m)}/S$. We have shown in section 3.3 how to associate to Φ^0 a filtration Fil on $\mathcal{E}$ that is transversal to p. Thanks to

Proposition 3.3.3 and [O2], 5.2.12, the filtered module $(\mathcal{E}, \mathrm{Fil})$ is Griffiths transversal as a $\widehat{\mathcal{D}}^{(0)}_{X^{(m+1)}/S}$-module. It follows from Lemma 5.2.4 and Proposition 3.3.3 that Φ will be in the essential image of μ if and only if $(\mathcal{E}, \mathrm{Fil})$ is Griffiths transversal as a $\widehat{\mathcal{D}}^{(m)}_{X^{(m+1)}/S}$-module. If the width is at most p this is always the case by Proposition 5.1.2, while Example 5.1.3 shows that this needs not happen for higher width. □

5.2.6. *Example.* — For $m > 0$, we can give an explicit F^{m+1}-p-isogeny of width less than p^{m+1} on the formal affine line X which is not in the essential image of μ. We take $\mathcal{E} = \mathcal{O}_{X^{(m+1)}}$ and we let $\mathcal{F}$ be the ideal of $\mathcal{O}_X$ generated by the elements $p^{p+1-i}t^{ip^{m+1}}$ for $0 \leq i \leq p-1$, together with $t^{p^{m+2}}$. It is a sub $\widehat{\mathcal{D}}^{(m)}$-module of $\mathcal{O}_X$ and we let the p-isogeny $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ be multiplication by p^{p+1}. If we apply the functor α to this F^{m+1}-p-isogeny, we get the saturation of the following filtration:

- for $0 \leq k \leq p$, Fil^k is the ideal generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq k$;
- for $k > p$, Fil^k is the ideal generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq p-1$, together with $p^{k-p-1}t^p$.

It is not Griffiths transversal because $t^p \in \mathrm{Fil}^{p+1}$ but $\partial^{[p]}(t^p) = 1$ is not in Fil^1 and we can use Lemma 5.2.4. □

5.2.7. *Remark.* — Let $\Phi : F^*\mathcal{E} \to \mathcal{F}$ and $\Phi' : F^*\mathcal{F} \to \mathcal{G}$ be two F-p-isogenies of width less than p. From [O2], 5.2.13, or Proposition 5.2.5, they are in the essential image of the fuctor μ for level 0. Assume that $\mathcal{E}$ and $\mathcal{G}$ are $\widehat{\mathcal{D}}^{(1)}$-modules and that $\Phi' \circ F^*(\Phi) : F^{2^*}\mathcal{E} \to \mathcal{G}$ is a morphism of $\widehat{\mathcal{D}}^{(1)}$-modules. Then it is an F^2-p-isogeny of width less than $(2p-1)$, and one may wonder if it is in the essential image of μ. One can show that this is true if $p = 2$, but if $p > 2$ the answer is no in general as the following example on the formal affine line shows:

We take $\mathcal{E} = \mathcal{O}$, we let $\mathcal{F}$ be the ideal of $\mathcal{O}$ generated by p^2, pt^p and t^{2p}, and $\mathcal{G}$ be the ideal of $\mathcal{O}$ generated by the elements $p^{p+1-i}t^{ip^2}$ with $0 \leq i \leq p-1$, together with t^{p^3}. The p-isogenies Φ and Φ' are multiplication by p^2 and p^{p-1}, respectively. The composition of $F^*(\Phi)$ and Φ is Example 5.2.6 in the case $m = 1$. □

5.3. Changing level for T-m-crystals and F-m-spans.

We study the behavior of the functors relating T-m-crystals and F-m-spans when the level changes and derive some consequences.

5.3.1. LEMMA. — *The functor «saturation with respect to $(p, \{\ \})$» from the category of filtered modules transversal to $(p, \{\ \}')$ to the category of filtered modules transversal to $(p, \{\ \})$ gives an equivalence of categories when restricted to objects of width less than p^{m+1}.*

Proof. — This is an immediate consequence of Proposition 1.2.5. □

Let $(S, \mathfrak{a}, \mathfrak{b})$ be a formal m-PD-scheme. If X is an S-scheme, it follows from 5.1.1 (i) that $\mathrm{Cris}^{(m)}(X/S)$ is a subsite of $\mathrm{Cris}^{(m')}(X/S)$. By restriction, any sheaf on $\mathrm{Cris}^{(m')}(X/S)$ defines a sheaf on $\mathrm{Cris}^{(m)}(X/S)$. The m'-PD-filtration restricts to a filtration on the structural sheaf $\mathcal{O}_{X/S}^{(m)}$ of $\mathrm{Cris}^{(m)}(X/S)$ that is finer than the m-PD-filtration.

Using restriction and then saturation with respect to the m-PD-filtration, any T-module E on $\mathrm{Cris}(X/S)^{(m')}$ defines a T-module on $\mathrm{Cris}(X/S)^{(m)}$. It is clear that this process is functorial and that, when applied to T-m'-crystals, it produces T-m-crystals.

Assume from now on that S has no p-torsion and that X is a smooth S_0-scheme.

5.3.2. PROPOSITION. — *Consider the functor that associates a T-m-crystal to a T-m'-crystal. Restricted to p-torsion free T-m'-crystals of width less than p^{m+1}, it is fully faithful and its essential image is the full subcategory of p-torsion free T-m-crystals of width less than p^{m+1} whose underlying crystal is the restriction of an m'-crystal.*

Proof. — This is a local question and all our constructions are functorial. Using Corollary 4.2.6 and Lemma 5.3.1, the first assertion is a consequence of 5.1.1 (iv) and the second follows from Proposition 5.1.2. □

Let $F: X \to X'$ be the relative Frobenius of X over S_0. We will write $(X/S)_{\mathrm{cris}}^{(m)}$ for the crystalline topos of level m. In [B3] Berthelot shows that the morphism of crystalline topoi of level m induced by F^d factors canonically through the restriction map $(X/S)_{\mathrm{cris}}^{(m)} \to (X/S)_{\mathrm{cris}}^{(m')}$ to give a morphism

$$F^d : (X/S)_{\mathrm{cris}}^{(m')} \longrightarrow (X^{(d)}/S)_{\mathrm{cris}}^{(m)}.$$

Under the equivalence of Corollary 4.1.8, this construction is compatible with that of Proposition 5.2.1.

5.3.3. Proposition. — *The functor F^{d^*} makes the full subcategory of F-m-spans on $X^{(d)}/S$ consisting of those $\Phi : F^{m+1^*}E \to E'$ where E is an m'-crystal on $X^{(m'+1)}/S$ equivalent to the category of F-m'-spans on X/S.*

Proof. — This is a again a local question. Using Corollary 4.2.6, the assertion reduces to Proposition 5.2.3. □

5.3.4. *Remark.* — When restricted to objects of width less than p^{m+1}, we have commutative diagrams:

$$\begin{array}{ccc}
\begin{array}{l}\textit{p-torsion free}\\ T\text{-}m'\text{-}\textit{crystals}\\ \textit{on } X^{(m'+1)}/S\end{array} & \hookrightarrow & F\text{-}m'\text{-}\textit{spans on } X/S \\
\big\downarrow & & \big\downarrow \\
\begin{array}{l}\textit{p-torsion free}\\ T\text{-}m\text{-}\textit{crystals}\\ \textit{on } X^{(m'+1)}/S\end{array} & \hookrightarrow & F\text{-}m\text{-}\textit{spans on } X^{(d)}/S
\end{array}$$

where the horizontal arrows come from Theorem 4.3.6 and the vertical ones from Proposition 5.3.2 and Proposition 5.3.3; and, when S is a PD-scheme:

$$\begin{array}{ccc}
\begin{array}{l}\textit{p-torsion free}\\ T\text{-}m\text{-}\textit{crystals}\\ \textit{on } X^{(m+1)}/S\end{array} & \hookrightarrow & F\text{-}m\text{-}\textit{spans on } X/S \\
\big\downarrow & & \big\downarrow \\
\begin{array}{l}\textit{p-torsion free}\\ T\text{-}0\text{-}\textit{crystals}\\ \textit{on } X^{(m+1)}/S\end{array} & \hookrightarrow & F\text{-}0\text{-}\textit{spans on } X^{(m)}/S
\end{array}$$

where the top arrow comes from Theorem 4.3.6, the bottom one from Theorem 5.2.13 of [O2] and the vertical ones from Proposition 5.3.2 and Proposition 5.3.3.

5.3.5. Proposition. — *The construction of* 4.3.6 *does not give an equivalence of categories in general. However, if S is a PD-scheme, it becomes an equivalence when restricted to objects of width at most p.*

Proof. — Follows from Corollary 4.2.6 and Proposition 5.2.5. □

BIBLIOGRAPHY

[B1] P. BERTHELOT, Cohomologie cristalline des schémas de caractéristique $p > 0$, Lecture Notes in Math., 407, Springer Verlag (1974).

[B2] P. BERTHELOT, Cohomologie rigide et théorie des $\mathcal{D}$-modules, Proc. Conf. p-adic Analysis (Trento 1989), Lecture notes in Math., 1454, Springer-Verlag (1990).

[B3] P. BERTHELOT, Letter to Luc Illusie, 1990.

[B4] P. BERTHELOT, $\mathcal{D}$-modules arithmétiques I: Opérateurs différentiels de niveau fini, Ann. Scient. Éc. Norm. Sup., 29, 4e série (1996), 185–272.

[B5] P. BERTHELOT, $\mathcal{D}$-modules arithmétiques II: Descente par Frobenius, work in progress.

[BO] P. BERTHELOT, A. OGUS, Notes on crystalline cohomology, Princeton University Press, 1978.

[K] N. KATZ, Nilpotent connexions and the monodromy theorem: applications of a result of Turritin, Publ. Math. I.H.E.S, 39 (1970), 355–412.

[O1] A. OGUS, F-Crystals and Griffiths Transversality, Proc. Int. Symp. on Algebraic Geometry, Kyoto 1977 (1977), 15–44.

[O2] A. OGUS, F-Crystals, Griffiths Transversality, and the Hodge decomposition, Astérisque, 221 (1994).

[Q] A. QUIRÓS, On the divisibility preperties of families of filtered F-crystals, Math. Ann., 288 (1990), 389–412.

Manuscrit reçu le 15 septembre 1995,
révisé le 27 juin 1996,
accepté le 9 septembre 1996.

B. LE STUM,
IRMAR
Université de Rennes I
Campus de Beaulieu
35042 Rennes cedex (France).
lestum@univ-rennes1.fr

A. QUIRÓS,
Departamento de Matemáticas
Universidad Autónoma de Madrid
Ciudad Universitaria de Cantoblanco
E–28049 Madrid (España).
adolfo.quiros@uam.es