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SHARP L^p-L^q ESTIMATES FOR A CLASS OF AVERAGING OPERATORS

by A. IOSEVICH and E. SAWYER(*)

1. INTRODUCTION

Let S be a smooth hypersurface in R^n. Let

$$Tf(x) = \int_S f(x-y)\, d\sigma(y), \tag{1}$$

where $d\sigma$ is a smooth compactly supported measure on S. We consider the problem of determining the optimal range of exponents (p,q), such that

$$\|Tf\|_{L^q(R^n)} \leqslant C_{p,q} \|f\|_{L^p(R^n)}, \quad f \in \mathcal{S}(R^n). \tag{2}$$

It is known that if S has everywhere non-vanishing Gaussian curvature, then (2) holds if and only if $\left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{T}_n$, where $\mathcal{T}_n$ is the triangle with vertices $(0,0)$, $(1,1)$, and $\left(\frac{n}{n+1}, \frac{1}{n+1}\right)$ (see [Litt], [Str] and [St1]). If the Gaussian curvature on S is allowed to vanish, sharp (L^p, L^q) bounds are in general very difficult to obtain.

Let

$$F[d\sigma](\xi) = \int_S e^{-ix\cdot\xi} d\sigma(x),$$

where $d\sigma(x)$ is defined above. It is known that if $|F[d\sigma](\xi)| \leqslant C(1+|\xi|)^{-\rho}$, $\rho > 0$, (which holds for some $\rho > 0$ if the Gaussian curvature vanishes of

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at most finite order), then (2) holds whenever

$$1 \leqslant p \leqslant 2, \quad \frac{1}{p} - \frac{1}{2} \leqslant \frac{1}{2}\left(\frac{\rho}{\rho+1}\right), \tag{3}$$

and $q = p'$, the conjugate exponent (see [Litt] and [Str]). However, if the Gaussian curvature is allowed to vanish, this result is not sharp in the sense that even with the optimal decay ρ for $F[d\sigma]$, (3) does not cover the full range of exponents (p, q) such that estimate (2) holds, even after we interpolate with the trivial $L^1 \to L^1$ and $L^\infty \to L^\infty$ estimates.

The purpose of this paper is to determine the optimal range of exponents (p, q) such that (2) holds in the case when S is a graph of a homogeneous function Φ of degree $m \geqslant 2$. We begin with a lemma that shows that we can never have $\left(\frac{1}{p}, \frac{1}{q}\right)$ outside the trapezoid with vertices

$$(0,0),\ (1,1),\ \left(\frac{n}{n+m-1}, \frac{1}{n+m-1}\right),$$

and

$$\left(1 - \frac{1}{n+m-1}, 1 - \frac{n}{n+m-1}\right).$$

DEFINITION 1. — *Denote by $\mathcal{Q}(N, \rho)$ the trapezoid with vertices*

$$(0,0),\ (1,1),\ \left(\frac{N\rho}{(N-1)(\rho+1)}, \frac{\rho}{(N-1)(\rho+1)}\right),$$

$$\textit{and } \left(1 - \frac{\rho}{(N-1)(\rho+1)}, 1 - \frac{N\rho}{(N-1)(\rho+1)}\right).$$

Thus $\mathcal{Q}(N, \rho)$ is the intersection of the triangle $\mathcal{T}_N$ and the half-plane lying on and above the line $\frac{1}{p} - \frac{1}{q} = \frac{\rho}{\rho+1}$.

Note that N plays the role of the effective dimension, while ρ plays the role of the decay of the Fourier transform of the surface carried measure (which coincides with the ρ in the L^ρ condition on Φ below - see [IoSa]).

LEMMA 2. — *Suppose S is the graph of a homogeneous function of degree m. If T is defined as in (1) and (2) holds, then $\left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{Q}\left(n, \frac{n-1}{m}\right)$.*

Proof. — We first test T against $f = \chi_B$, where B is a small ball of radius $\delta > 0$. Then $T\chi_B$ is at least $c\delta^{n-1}$ in absolute value on a set of measure $C\delta$. Thus (2) yields

$$c\delta^{n-1+\frac{1}{q}} \leqslant \|T\chi_B\|_{L^q} \leqslant C\|\chi_B\|_{L^p} = C\delta^{\frac{n}{p}},$$

for δ small, which implies that $n-1+\frac{1}{q} \geqslant \frac{n}{p}$. By duality we also obtain $n-1+\frac{1}{p'} \geqslant \frac{n}{q'}$ or $\frac{n}{q} \geqslant \frac{1}{p}$. These two inequalities show that $\left(\frac{1}{p}, \frac{1}{q}\right)$ must lie in the triangle $\mathcal{T}_n$ with vertices $(0,0)$, $(1,1)$ and $\left(\frac{n}{n+1}, \frac{1}{n+1}\right)$.

Next, a simple homogeneity argument shows that $\left(\frac{1}{p}, \frac{1}{q}\right)$ must lie on or below the line $\frac{1}{p} - \frac{1}{q} = \frac{n-1}{m+n-1}$. Indeed, $T_0 = 2^{(n-1)k} \tau_{-k} T_k \tau_k$ where

$$T_k f(x) = \chi_B\left(2^{k+1}x'\right) \int_{R^{n-1}} f\left(x'-y', x_n - \Phi\left(y'\right)\right) \chi_B\left(2^k y'\right) dy'$$

and $\tau_k f\left(x', x_n\right) = f\left(2^k x', 2^{km} x_n\right)$. Also $T_k \tau_k f = T\tau_k f$ for f supported in $\frac{1}{2}B$. Assuming (2) holds, we thus have

$$\begin{aligned} \left\|T_0 f\right\|_{L^q} &= 2^{(n-1)k} \left\|\tau_{-k} T_k \tau_k f\right\|_{L^q} \\ &= 2^{k\left[(n-1)+\frac{1}{q}(n-1+m)\right]} \left\|T_k \tau_k f\right\|_{L^q} = 2^{k\left[(n-1)+\frac{1}{q}(n-1+m)\right]} \left\|T \tau_k f\right\|_{L^q} \\ &\leqslant C 2^{k\left[(n-1)+\frac{1}{q}(n-1+m)\right]} \left\|\tau_k f\right\|_{L^p} = C 2^{k\left[(n-1)+\left(\frac{1}{q}-\frac{1}{p}\right)(n-1+m)\right]} \|f\|_{L^p}. \end{aligned}$$

Since $\left\|T_0 f\right\|_{L^q} > 0$, we must have $(n-1) + \left(\frac{1}{q} - \frac{1}{p}\right)(n-1+m) \geqslant 0$ as required. Combining the facts that $\left(\frac{1}{p}, \frac{1}{q}\right)$ lies inside the triangle $\mathcal{T}_n$ and on or below the above line, we obtain that $\left(\frac{1}{p}, \frac{1}{q}\right)$ must lie in the trapezoid $\mathcal{Q}\left(n, \frac{n-1}{m}\right)$.

We shall also apply our methods to the question of regularity of the following initial value problem:

$$\text{(4)} \qquad \begin{cases} P(D)u = i\dfrac{\partial u}{\partial t} \\ u(x,0) = f(x), \end{cases}$$

where $(x,t) \in R^n \times R_+$, and $P(D)$ is a differential operator with symbol $P(\xi)$. Kenig, Ponce, and Vega [KPV] showed that if $P(D)$ is an elliptic operator, i.e the principal homogeneous part does not vanish on the unit sphere, and if the determinant of the Hessian matrix of P satisfies

$$|HP(\xi)| \geqslant C|\xi|^{n(m-2)}, \tag{5}$$

then, roughly speaking, the solution $u(\cdot,t)$ has $\frac{n}{2}(m-2)$ derivatives in L^∞ if the initial data f is in L^1.

However, the result in [KPV] does not apply, for example, to the operator with symbol

$$P(\xi) = \xi_1^4 + \xi_2^4 - \xi_1^2\xi_2^2$$

since the determinant of the Hessian matrix of this polynomial,

$$HP(\xi) = -24\left(\xi_1^4 + \xi_2^4 - 6\xi_1^2\xi_2^2\right),$$

vanishes along a line, and consequently the estimate (5) does not hold. In this paper we shall drop the curvature assumption (5), and weaken the ellipticity hypothesis on Φ, for homogeneous operators in dimension $n = 2$. See Theorem 16 below.

2. MINIMAL CURVATURE ASSUMPTIONS - THE L^ρ CONDITION

Let

$$Tf(x) = \int_{R^{n-1}} f(x' - y', x_n - \Phi(y'))\,\psi(y')\,dy', \tag{6}$$

where ψ is a smooth cutoff function and $\Phi \in C^\infty\left(R^{n-1}\setminus\{0\}\right)$ is homogeneous of degree $m \geqslant 2$. In this section we obtain the optimal mapping properties for T in the case when no additional curvature assumptions are placed on the level set $\{\Phi = 1\}$. The mapping properties are then determined by the L^ρ condition introduced in [IoSa], namely that

$$\Phi(\omega)^{-1} \in L^\rho\left(S^{n-2}\right). \tag{7}$$

The main result of this section is the following.

THEOREM 3. — *Let T be defined as in (6) above. Suppose that $\Phi(\omega)^{-1} \in L^\rho\left(S^{n-1}\right)$ with $0 < \rho < \min\left\{\frac{n-1}{m}, \frac{1}{2}\right\}$. Then the norm inequality (2) holds if $\left(\frac{1}{p}, \frac{1}{q}\right)$ lies in the trapezoid $\mathcal{Q}(2,\rho)$. Conversely,* *for each* $\rho < \frac{n-1}{m}$*, there is a* Φ *satisfying the above hypotheses such that (2) fails for* $\left(\frac{1}{p}, \frac{1}{q}\right)$ *outside the trapezoid* $\mathcal{Q}(2, \rho)$.

Remark 1. — The restriction $\rho < \min\left\{\frac{n-1}{m}, \frac{1}{2}\right\}$ is natural here. Indeed, $\rho \leqslant \frac{1}{2}$ follows from the fact that we are only assuming curvature in one (the radial) direction, while $\rho < \frac{n-1}{m}$ is implied by the finiteness of the integral

$$\int_{|x|\leqslant 1} |\Phi|^{-\rho}\, dx = C \int_0^1 r^{-m\rho} r^{n-2} dr.$$

Proof. — Similar ideas to those below appear in [Str] and [RiSt]. Let

$$(8) \quad T^\alpha f(x) = \int_{R^{n-1}} f(x' - y', x_n - \Phi(y'))\, |\Phi(y')|^{\rho\left(\frac{\rho+1}{\rho}\alpha - 1\right)} \psi(y')\, dy',$$

and note that $T^{\frac{\rho}{\rho+1}} f(x) = Tf(x)$. We begin by showing that

$$(9) \quad \begin{aligned} &T^\alpha : L^\infty(R^n) \longrightarrow L^\infty(R^n),\ \mathrm{Re}(\alpha) = 0,\\ &T^\alpha : L^{\frac{3}{2}}(R^n) \longrightarrow L^3(R^n),\ \mathrm{Re}(\alpha) = \frac{1}{3}, \end{aligned}$$

with operator bounds depending only polynomially on $\mathrm{Im}(\alpha)$. Now when $\mathrm{Re}(\alpha) = 0$, then the power of $|\Phi(y')|$ appearing in (8) has real part $-\rho$, and so by the L^ρ condition (7), T^α is convolution with respect to a finite measure. Thus T^α is bounded on $L^\infty(R^n)$ with norm independent of $\mathrm{Im}(\alpha)$.

To obtain the second mapping property in (9), we must exploit the two facts that (i) we have L^2 boundedness for $T^{\frac{1}{3}}$ composed with fractional integration in the x' directions and differentiation in the x_n direction and (ii) we have a bounded kernel for $T^{\frac{1}{3}}$ composed with fractional differentiation in the x' directions and integration in the x_n direction. More precisely, let

$$\begin{aligned}(10)\ U^z f(x) &= f * K^{\frac{1}{3} - \left(\frac{1}{3} - \frac{\rho}{\rho+1}\right)z} * I^z_{x_n}\\ &= \int\!\!\int f(x' - y', x_n - u_n - \Phi(y'))\\ &\qquad \times |\Phi(y')|^{\rho\left(\frac{\rho+1}{\rho}\left[\frac{1}{3} - \left(\frac{1}{3} - \frac{\rho}{\rho+1}\right)z\right] - 1\right)} \psi(y') \frac{|u_n|^{z-1}}{\Gamma(z)} dy' du_n,\end{aligned}$$

where K^α is the kernel of T^α, $I^z_{x_n} = \delta_0(x') \otimes I_z(x_n)$, and $I_z(t) = \dfrac{|t|^{z-1}}{\Gamma(z)}$. If $\operatorname{Re}(z) = 1$, then the kernel of U^z is clearly bounded and so we have

$$U^{1+i\tau} : L^1(R^n) \to L^\infty(R^n) \tag{11}$$

with norm independent of τ. On the other hand, when $z = -\frac{1}{2} + i\tau$, we compute that the power of $|\Phi(y')|$ in (10) is

$$\rho\left(\frac{\rho+1}{\rho}\left[\frac{1}{3} - \left(\frac{1}{3} - \frac{\rho}{\rho+1}\right)\left(-\frac{1}{2} + i\tau\right)\right] - 1\right) = \frac{1}{2} - \rho + ic_\rho\tau$$

and thus that

$$\left(K^{\frac{1}{3}-\left(\frac{1}{3}-\frac{\rho}{\rho+1}\right)z}\right)^\wedge(\xi', \xi_n) = \int e^{-i(y'\cdot\xi' + \xi_n\Phi(y'))}|\Phi(y')|^{\frac{1}{2}-\rho+ic_\rho\tau}\psi(y')\,dy'.$$

In the case $\tau = 0$, we proved in Theorem 4(A) of [IoSa] that the latter integral decays like $C|\xi|^{-\frac{1}{2}}$, and the same proof shows in general that the integral decays like $C(1+|\tau|)|\xi|^{-\frac{1}{2}}$. For the sake of completenesss we give the proof. Begin by letting $\alpha = \frac{1}{2} - \rho + ic_\rho\tau$, $\Omega = \left\{\omega \in S^{n-2} : |\Phi(\omega)| > \frac{1}{|\xi_n|}\right\}$ and writing

$$\begin{aligned} F(\xi', \xi_n) &= \int e^{i(x\cdot\xi' + \xi_n\Phi(x))}|\Phi(x)|^\alpha \psi(x)dx \\ &= \int_{\frac{x}{|x|}\in S^{n-2}\setminus\Omega} + \int_{\frac{x}{|x|}\in\Omega} = F^{S^{n-2}\setminus\Omega} + F^\Omega. \end{aligned}$$

Now if we take absolute values inside the first integral, we see immediately that

$$\begin{aligned} \left|F^{S^{n-2}\setminus\Omega}(\xi', \xi_n)\right| &\leqslant C\int_{\left\{\omega\in S^{n-2}:|\Phi(\omega)|\leqslant\frac{1}{\lambda}\right\}} |\Phi(\omega)|^{\frac{1}{2}-\rho}\,d\omega \\ &\leqslant C|\xi_n|^{-\frac{1}{2}+\rho}\left|\left\{\omega \in S^{n-2} : |\Phi(\omega)|^{-1} > |\xi_n|\right\}\right| \\ &\leqslant C|\xi_n|^{-\frac{1}{2}}, \end{aligned}$$

since $\Phi(\omega)^{-1} \in$ *weak* $L^\rho\left(S^{n-2}\right)$.

To handle the second integral, let φ be supported in the annulus

$$\mathcal{A} = \left\{x \in R^{n-1} : \frac{1}{2} \leqslant |x| \leqslant 2\right\}$$

so that

$$\sum_{k=0}^{\infty} \varphi\left(2^k x\right) = 1, \qquad |x| \leqslant 1.$$

Setting

$$F_k^{\Omega}\left(\xi', \xi_n\right) = \int_{\frac{x}{|x|} \in \Omega} e^{i(x\cdot\xi' + \xi_n \Phi(x))} \varphi\left(2^k x\right) \psi(x) \left|\Phi(x)\right|^{\alpha} dx,$$

we obtain

$$\begin{aligned} F^{\Omega}(\xi, \lambda) &= \sum_{k=0}^{\infty} F_k^{\Omega}\left(\xi', \xi_n\right) = \sum_{k=0}^{\infty} 2^{-k(n-1)} 2^{-km\alpha} \widetilde{F}_k^{\Omega}\left(2^{-k}\xi', 2^{-km}\xi_n\right), \\ &= \left(\sum_{2^{km} \geqslant |\xi_n \Phi(\omega)|} + \sum_{2^{km} < |\xi_n \Phi(\omega)|}\right) 2^{-k(n-1)} 2^{-km\alpha} \widetilde{F}_k^{\Omega}\left(2^{-k}\xi, 2^{-km}\lambda\right) \\ &= I + II, \end{aligned}$$

where

$$\begin{aligned} \left|\widetilde{F}_k^{\Omega}\left(2^{-k}\xi', 2^{-km}\xi_n\right)\right| &= \Bigg| \int_{\frac{x}{|x|} \in \Omega} e^{i(x\cdot 2^{-k}\xi' - 2^{-km}\xi_n \Phi(x))} \varphi(x) \\ &\qquad\qquad \psi(2^{-k}x) \left|\Phi(x)\right|^{\alpha} dx \Bigg| \end{aligned}$$

$$\begin{aligned} &\leqslant \int_{\Omega} \left| \int e^{i\left\{r\left(2^{-k}\omega\cdot\xi'\right) - r^m\left(2^{-km}\xi_n \Phi(\omega)\right)\right\}} \varphi(r) \psi(2^{-k}r) r^{m\alpha+n-2} dr \right| \\ &\qquad\qquad \left|\Phi(\omega)\right|^{\frac{1}{2}-\rho} d\omega \\ &\leqslant C(1+|\tau|) \int_{\Omega} \left(1 + 2^{-km}\left|\xi_n \Phi(\omega)\right|\right)^{-\frac{1}{2}} \left|\Phi(\omega)\right|^{-\frac{1}{2}-\rho} d\omega \end{aligned}$$

since the curve (r, r^m) is nondegenerate for $r \in \left(\frac{1}{2}, 2\right)$ and $\left|\frac{d}{dr}\left\{\varphi(r)\psi(2^{-k}r) r^{m\alpha+n-2}\right\}\right| \leqslant C(1+|\tau|)$.

Thus we have

$$|I| \leqslant C \int_{\Omega} \left(\sum_{2^{km} \geqslant |\xi_n \Phi(\omega)|} 2^{-k(n-1)} 2^{-km\left(\frac{1}{2}-\rho\right)} \right) |\Phi(\omega)|^{\frac{1}{2}-\rho} d\omega$$

$$\leqslant C \int_{\Omega} \left(|\xi_n \Phi(\omega)|\right)^{-\frac{n-1}{m}-\left(\frac{1}{2}-\rho\right)} |\Phi(\omega)|^{\frac{1}{2}-\rho} d\omega$$

$$\leqslant C |\xi_n|^{-\left(\frac{1}{2}-\rho\right)-\frac{n-1}{m}} \int_{\left\{\omega \in S^{n-2} : |\Phi(\omega)| > \frac{1}{|\xi_n|}\right\}} |\Phi(\omega)|^{-\frac{n-1}{m}} d\omega$$

$$= C |\xi_n|^{-\left(\frac{1}{2}-\rho\right)-\frac{n-1}{m}} \int_0^{\lambda} \frac{n-1}{m} t^{\frac{n-1}{m}-1} \left|\left\{\omega \in S^{n-2} : t < |\Phi(\omega)|^{-1} < |\xi_n|\right\}\right| dt$$

$$\leqslant C |\xi_n|^{-\left(\frac{1}{2}-\rho\right)-\frac{n-1}{m}} \int_0^{\lambda} \frac{n-1}{m} t^{\frac{n-1}{m}-1} C (1+t)^{-\rho} dt \leqslant C |\xi_n|^{-\frac{1}{2}},$$

since $\Phi^{-1} \in weak\ L^{\rho}\left(S^{n-2}\right)$. Also,

$$|II| \leqslant C(1+|\tau|) \int_{\Omega} \sum_{2^{km} < |\xi_n \Phi(\omega)|} 2^{-k(n-1)} 2^{-km\left(\frac{1}{2}-\rho\right)} \left(2^{-km} |\xi_n \Phi(\omega)|\right)^{-\frac{1}{2}} |\Phi(\omega)|^{\frac{1}{2}-\rho} d\omega$$

$$\leqslant C(1+|\tau|) |\xi_n|^{-\frac{1}{2}} \int_{\Omega} \left(\sum_{2^{km} < |\xi_n \Phi(\omega)|} \left(2^{km}\right)^{\rho-\frac{n-1}{m}} \right) |\Phi(\omega)|^{-\rho} d\omega$$

$$\leqslant C(1+|\tau|) |\xi_n|^{-\frac{1}{2}} \int_{\Omega} |\Phi(\omega)|^{-\rho} d\omega$$

$$\leqslant C(1+|\tau|) |\xi_n|^{-\frac{1}{2}},$$

since $\rho < \dfrac{n-1}{m}$ and $\Phi^{-1} \in L^{\rho}\left(S^{n-2}\right)$. Altogether this shows

$$\left|\left(K^{\frac{1}{3}-\left(\frac{1}{3}-\frac{\rho}{\rho+1}\right)z}\right)^{\wedge}(\xi', \xi_n)\right| \leqslant C(1+|\tau|) |\xi|^{-\frac{1}{2}}$$

as required since if $|\xi'| \geqslant C|\xi_n|$, then integration by parts yields rapid decay (see Proposition 4, p. 341 in [St2]).

Since

$$\left|\left(I_{x_n}^{z}\right)^{\wedge}(\xi_n)\right| = \left|c|\xi_n|^{-z}\right| = c|\xi_n|^{\frac{1}{2}},$$

Plancherel's theorem together with (10) now shows that

$$U^{-\frac{1}{2}+i\tau} : L^2(R^n) \to L^2(R^n) \tag{12}$$

with norm at most $C(1+|\tau|)$. By analytic interpolation, the maps (11) and (12) yield

$$U^{i\tau} : L^{\frac{3}{2}}(R^n) \to L^3(R^n)$$

with norm at most $C(1+|\tau|)^\beta$. This completes the proof of (9).

Now by analytic interpolation of the maps in (9), we obtain that $T = T^{\frac{\rho}{\rho+1}}$ maps $L^{\frac{\rho+1}{2\rho}}$ to $L^{\frac{\rho+1}{\rho}}$. Thus the mapping set (of points $\left(\frac{1}{p}, \frac{1}{q}\right)$ such that (2) holds) contains the point $\left(\frac{2\rho}{\rho+1}, \frac{\rho}{\rho+1}\right)$, and by duality the point $\left(1-\frac{\rho}{\rho+1}, 1-\frac{2\rho}{\rho+1}\right)$. Using the obvious facts that T is bounded on L^1 and L^∞, we see that the mapping set also contains the points $(1,1)$ and $(0,0)$, and the first part of Theorem 3 now follows from interpolation.

We now turn to the converse assertion in Theorem 3. The following example shows that in general we cannot do better than the trapezoid $\mathcal{Q}\left(2, \frac{n-1}{m}\right)$ without additional curvature on the level set $\{\Phi = 1\}$.

Example 4. — Let Φ be homogeneous of degree m such that the level set $\{\Phi = 1\}$ contains a copy of the cube I^{n-2}. Set $f_\delta = \chi_{B_2(\delta)\times I^{n-2}}$ where the two cubes I^{n-2} are parallel and $B_2(\delta)$ is a disk of radius $\delta > 0$. With T as in (6), we note that $\|f_\delta\|_p = c\delta^{\frac{2}{p}}$ and $\|Tf_\delta\|_q \geqslant c\delta \cdot \delta^{\frac{1}{q}}$ since Tf_δ is at least $c\delta$ on a set of measure $c\delta$. We thus obtain from (2) that

$$c\delta^{1+\frac{1}{q}} \leqslant \|Tf_\delta\|_q \leqslant C\|f_\delta\|_p = Cc\delta^{\frac{2}{p}}$$

for δ small, and so $1+\frac{1}{q} \geqslant \frac{2}{p}$. This is precisely the estimate obtained in Lemma 2 in the case $n=2$, and if we now use duality and argue as we did there, we obtain that $\left(\frac{1}{p}, \frac{1}{q}\right)$ must lie in the triangle with vertices $(0,0)$, $(1,1)$ and $\left(\frac{2}{3}, \frac{1}{3}\right)$. Since $\left(\frac{1}{p}, \frac{1}{q}\right)$ must also lie on or below the line $\frac{1}{p} - \frac{1}{q} = \frac{n-1}{m+n-1}$ (see Lemma 2 again), we see that $\left(\frac{1}{p}, \frac{1}{q}\right)$ lies in the trapezoid $\mathcal{Q}\left(2, \frac{n-1}{m}\right)$.

The next example shows that in Theorem 3, we cannot do better than $\mathcal{Q}(n,\rho)$ without improving the L^ρ condition.

Example 5. — Let $\Phi(x) = x_1^{a_1} \times \ldots x_{n-1}^{a_{n-1}}$ where $a_1 \geqslant a_2 \geqslant \ldots a_{n-1}$. Then $\Phi^{-1} \in L^{\frac{1}{a_1}}\left(S^{n-2}\right)$, so $\rho = \frac{1}{a_1}$. If we apply the scaling argument of Lemma 2 with $n = 2$ and $m = a_1$ to the variable x_1 in Φ, we obtain the restriction $\frac{1}{p} - \frac{1}{q} \leqslant \frac{1}{a_1+1} = \frac{\rho}{\rho+1}$. Since we always have $\left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{T}_n$, we see that T is not bounded for $\left(\frac{1}{p}, \frac{1}{q}\right)$ outside the trapezoid $\mathcal{Q}(n, \rho)$.

Finally, by combining the above two examples, we can construct a Φ satisfying the hypotheses of Theorem 3 such that T is not bounded for $\left(\frac{1}{p}, \frac{1}{q}\right)$ outside the trapezoid $\mathcal{Q}(2, \rho)$. Indeed, choose Φ to be homogeneous of degree $m = a_1 + a_2 + \ldots + a_{n-1}$ where $\frac{1}{\rho} = a_1 \geqslant a_2 \geqslant \ldots a_{n-1}$ such that Φ looks like the function in Example 5 for x near the coordinate planes, and such that the level set $\{\Phi = 1\}$ contains a copy of the cube I^{n-2} away from the coordinate planes as in Example 4. Then just as in Example 5, $\left(\frac{1}{p}, \frac{1}{q}\right)$ must lie on or below the line $\frac{1}{p} - \frac{1}{q} = \frac{\rho}{\rho+1}$, and just as in Example 4, $\left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{T}_2$. Thus T is not bounded for $\left(\frac{1}{p}, \frac{1}{q}\right)$ outside the trapezoid $\mathcal{Q}(2, \rho)$, and this completes the proof of Theorem 3.

3. ADDITIONAL CURVATURE ON THE LEVEL SET

Our first result shows what type of mapping property can be obtained from a given decay on the Fourier transform of a surface carried measure weighted by powers of $|\Phi|$, and the proof is a generalization of that of Theorem 3. More precisely, we consider

$$F_\Phi^z(\xi) = \int_{R^{n-1}} e^{i\left(x'\cdot\xi' + \Phi(x')\xi_n\right)} \left|\Phi(x')\right|^z \psi(x')\, dx' \tag{13}$$

where $\psi \in C_c^\infty\left(R^{n-1}\right)$.

THEOREM 6. — *Let T be defined as in (6) above. Suppose that $\Phi(\omega)^{-1} \in L^\rho\left(S^{n-1}\right)$ with $0 < \rho < \min\left\{\frac{n-1}{m}, \frac{1}{2}\right\}$. In addition, we assume that there is $0 \leqslant \eta \leqslant \frac{n-2}{2}$ such that*

$$\left|F_\Phi^z(\xi)\right| \leqslant C_{\mathrm{Im}(z)}\left(1 + |\xi|\right)^{-\frac{1}{2}-\eta} \tag{14}$$

when $\mathrm{Re}(z) + \rho = \dfrac{1}{2} + \eta$, *and that* $C_{Im(z)}$ *grows at most exponentially in* $\mathrm{Im}(z)$. *Then the norm inequality (2) holds if* $\left(\dfrac{1}{p}, \dfrac{1}{q}\right) \in \mathcal{Q}(2(\eta+1), \rho)$.

Proof. — Let

(15)
$$T^{\alpha} f(x) = \int_{R^{n-1}} f(x' - y', x_n - \Phi(y')) \left|\Phi(y')\right|^{\rho\left(\frac{(N-1)(\rho+1)}{\rho}\alpha - 1\right)} \psi(y')\, dy',$$

where $N = 2(\eta+1) \leqslant n$ is the effective dimension, and note that $T^{\frac{\rho}{(N-1)(\rho+1)}} f(x) = Tf(x)$. We begin by showing that

$$T^{\alpha} : L^{\infty}(R^n) \longrightarrow L^{\infty}(R^n),\ \mathrm{Re}(\alpha) = 0, \tag{16}$$
$$T^{\alpha} : L^{\frac{N+1}{N}}(R^n) \longrightarrow L^{N+1}(R^n),\ \mathrm{Re}(\alpha) = \frac{1}{N+1},$$

with operator bounds depending only polynomially on $\mathrm{Im}(\alpha)$. The first estimate follows as in (9). To obtain the second mapping property, let

$$\begin{aligned} U^{z} f(x) &= f * K^{\frac{1}{N+1} - \left(\frac{1}{N+1} - \frac{\rho}{(N-1)(\rho+1)}\right) z} * I^{z}_{x_n} \\ &= \int\!\!\int f(x' - y', x_n - u_n - \Phi(y')) \\ &\quad \times \left|\Phi(y')\right|^{\rho\left((N-1)\frac{\rho+1}{\rho}\left[\frac{1}{N+1} - \left(\frac{1}{N+1} - \frac{\rho}{\rho+1}\right) z\right] - 1\right)} \\ &\qquad\qquad \psi(y') \frac{|u_n|^{z-1}}{\Gamma(z)} dy'\, du_n, \end{aligned} \tag{17}$$

where K^{α} is the kernel of T^{α}, $I^{z}_{x_n} = \delta_0(x') \otimes I_z(x_n)$, and $I_z(t) = \dfrac{|t|^{z-1}}{\Gamma(z)}$. If $\mathrm{Re}(z) = 1$, then the kernel of U^z is clearly bounded and so we have

$$U^{1+i\tau} : L^{1}(R^n) \to L^{\infty}(R^n) \tag{18}$$

with norm independent of τ. On the other hand, when $z = -\dfrac{N-1}{2} + i\tau$, we compute that the power of $|\Phi(y')|$ in (17) is

$$\begin{aligned} &\rho\left((N-1)\frac{\rho+1}{\rho}\left[\frac{1}{N+1} - \left(\frac{1}{N+1} - \frac{\rho}{(N-1)(\rho+1)}\right)\left(-\frac{N-1}{2} + i\tau\right)\right] - 1\right) \\ &\qquad = \frac{N-1}{2} - \rho + ic_{\rho,N}\tau \end{aligned}$$

and thus that

$$\left(K^{\frac{1}{N+1}-\left(\frac{1}{N+1}-\frac{\rho}{(N-1)(\rho+1)}\right)z}\right)^{\wedge}(\xi',\xi_n)$$

$$= \int e^{-i(y'\cdot\xi'+\xi_n\Phi(y'))}\left|\Phi(y')\right|^{\frac{N-1}{2}-\rho+ic_\rho\tau}\psi(y')\,dy'.$$

By assumption this integral is bounded by $C_\tau(1+|\xi|)^{-\frac{N-1}{2}}$. Since $\left|\left(I^z_{x_n}\right)^{\wedge}(\xi_n)\right| = \left|c\,|\xi_n|^{-z}\right| = c\,|\xi_n|^{\frac{N-1}{2}}$, Plancherel's theorem together with (17) shows that

$$U^{-\frac{N-1}{2}+i\tau} : L^2(R^n) \to L^2(R^n). \tag{19}$$

By analytic interpolation of the maps (18) and (19), we obtain that

$$U^{i\tau} : L^{\frac{N+1}{N}}(R^n) \to L^{N+1}(R^n),$$

and this completes the proof of (16).

Now by analytic interpolation again, we obtain that $T = T^{\frac{\rho}{(N-1)(\rho+1)}}$ maps $L^{\frac{(2\epsilon+1)(\rho+1)}{2(\epsilon+1)\rho}}$ to $L^{\frac{(2\epsilon+1)(\rho+1)}{\rho}}$. The conclusions of the theorem now follow from duality and a further interpolation using the obvious facts that T is bounded on L^1 and L^∞.

Remark 2. — For a homogeneous polynomial of degree m, Theorem 6 does not yield the sharp $L^p - L^q$ mapping properties. Indeed, the weight $|\Phi|^z$ is effective in improving the gain from the radial direction up to $\frac{1}{2}$, but the problem is that the gain from the tangential direction, which could be as little as $\frac{1}{m}$, is not improved at all. This problem is overcome in three dimensions in the next section by introducing an appropriate tangential weight that allows us to gain up to $\frac{1}{2}$ in the tangential direction also. This yields a total gain of $\frac{n-1}{2} = 1$ which produces the sharp mapping properties when $n = 3$.

3.1. An application.

We will use a refinement of Theorem 6 in proving our sharp 3-dimensional results in the next section, but first we give an application to a special class of surfaces in any dimension. We start by giving conditions under which estimate (14) holds.

THEOREM 7. — *Let Φ be homogeneous of degree m with level set $\Sigma = \{\Phi = 1\}$ containing a copy of $\Sigma_k = S^k \times I^{n-2-k}$, and such that the Gaussian curvature of Σ is non-zero outside this copy of Σ_k. Then with $F_\Phi^z(\xi)$ as in (13), we have*

$$|F_\Phi^z(\xi)| \leqslant C_{\mathrm{Im}(z)} (1 + |\xi|)^{-\frac{1}{2}-\frac{k}{2}}$$

for $\mathrm{Re}(z) + \dfrac{n-1}{m} = \dfrac{1}{2} + \dfrac{k}{2}$.

In light of Theorem 6, we now have the following corollary.

COROLLARY 8. — *Let S be the graph of Φ where Φ is as in Theorem 7 above. If T is as in (6), then the norm inequality (2) holds for $\left(\dfrac{1}{p}, \dfrac{1}{q}\right)$ in the trapezoid $Q\left(k+2, \dfrac{n-1}{m}\right)$.*

Proof (of Theorem 7). — We compute

$$\begin{aligned} F_\Phi^z(\xi) &= \int_{R^{n-1}} e^{i(x'\cdot\xi' + \Phi(x')\xi_n)} |\Phi(x')|^z \psi(x')\, dx' \\ &= \int_0^\infty \int_{\Sigma_k} e^{ir\eta\cdot\xi} e^{i\lambda r^m} \frac{d\sigma(\eta)}{|\nabla\Phi(\eta)|} r^{n-2} r^{mz} \psi(r)\, dr, \end{aligned}$$

where $d\sigma$ is surface measure, upon using polar coordinates, making the change of variables sending $\omega \in S^{n-2}$ to $\dfrac{\omega}{\Phi(\omega)^{\frac{1}{m}}}$, observing that $\Phi\left(\dfrac{\omega}{\Phi(\omega)^{\frac{1}{m}}}\right) = 1$, and computing the corresponding Jacobian (see [IoSa]). Note that since Φ is elliptic, Euler homogeneity relations imply that $|\nabla\Phi|$ is bounded from below. Thus $\dfrac{d\sigma(\eta)}{|\nabla\Phi(\eta)|} = d\widetilde{\sigma}(\eta)$ is a smooth measure on Σ_k. We first integrate in the angular variables using stationary phase to see that

$$\int_{\Sigma_k} e^{ir\eta\cdot\xi} \frac{d\sigma(\eta)}{|\nabla\Phi(\eta)|} = e^{irq(\xi)} b(r\xi),$$

where $q(\xi) \approx \sqrt{\xi_1^2 + ... + \xi_k^2}$ and b is a symbol of order $-\frac{k}{2}$ in the variables $\xi_1, ..., \xi_k$, i.e.

$$\left|\partial^{\alpha}_{(\xi_1,...,\xi_k)} b(\xi)\right| \leqslant C(1+q(\xi))^{-\frac{k}{2}-|\alpha|}.$$

It remains to consider the radial integral:

$$\int_0^{\infty} e^{i(rq(\xi)+\lambda r^m)} r^{n-2} r^{mz} b(r\xi)\psi(r)\,dr \tag{20}$$

$$= \left(\int_0^{\lambda^{-\frac{1}{m}}} + \int_{\lambda^{-\frac{1}{m}}}^{\infty}\right) = I + II. \tag{21}$$

Note that

$$|I| \leqslant C\left(\lambda^{-\frac{1}{m}}\right)^{n-1+\mathrm{Re}z}, \qquad for\ \mathrm{Re}z \geqslant 1-n.$$

For II we change variables $r \longrightarrow r\lambda^{-\frac{1}{m}}$ to get

$$II = \lambda^{-\frac{n-1}{m}-z}\int_1^{\infty} e^{i\left(rq(\xi)\lambda^{-\frac{1}{m}}+r^m\right)} r^{n-2} r^{mz} b\left(r\lambda^{-\frac{1}{m}}\xi\right)\psi\left(r\lambda^{-\frac{1}{m}}\right)dr.$$

The phase function $\phi(r) = rq(\xi)\lambda^{-\frac{1}{m}} + r^m$ has the critical point $r_0 = \left(\frac{q(\xi)}{m\lambda^{\frac{1}{m}}}\right)^{\frac{1}{m-1}}$. We first consider the case $r_0 > 2$, set $A = \frac{1}{m}q(\xi)\lambda^{-\frac{1}{m}}$ and decompose the integral into three pieces as follows:

$$\begin{aligned} II &= \lambda^{-\frac{n-1}{m}-z}\left(\int_1^{\frac{1}{2}A^{\frac{1}{m-1}}} + \int_{\frac{1}{2}A^{\frac{1}{m-1}}}^{2A^{\frac{1}{m-1}}} + \int_{2A^{\frac{1}{m-1}}}^{\infty}\right) \\ &= \lambda^{-\frac{n-1}{m}-z}(III + IV + V). \end{aligned}$$

Terms III and V are easily handled by integrations by part. We illustrate with term V. We have

$$V = \int_{2A^{\frac{1}{m-1}}}^{\infty}\left(\left[\frac{1}{\phi'(r)}\frac{d}{dr}\right]^N e^{i\phi(r)}\right) r^{n-2} r^{mz} b\left(r\lambda^{-\frac{1}{m}}\xi\right)\psi\left(r\lambda^{-\frac{1}{m}}\right)dr$$

where $|\phi'(r)| \geqslant cr^{m-1}$ and $\left|b\left(r\lambda^{-\frac{1}{m}}\xi\right)\right| \leqslant C\left(1+r\lambda^{-\frac{1}{m}}q(\xi)\right)^{-\frac{k}{2}} \leqslant C(1+rA)^{-\frac{k}{2}}$. Integrating by parts N times, we thus get

$$V = \int_{2A^{\frac{1}{m-1}}}^{\infty} e^{i\phi(r)}\left[\frac{d}{dr}\frac{1}{\phi'(r)}\right]^N \left(r^{n-2} r^{mz} b\left(r\lambda^{-\frac{1}{m}}\xi\right)\psi\left(r\lambda^{-\frac{1}{m}}\right)\right)dr \tag{22}$$

plus N boundary terms of which the largest is the first:

$$\left| e^{i\phi(r)} \frac{r^{n-2} r^{mz} b\left(r\lambda^{-\frac{1}{m}}\xi\right) \psi\left(r\lambda^{-\frac{1}{m}}\right)}{\phi'(r)} \Bigg|_{r=2A^{\frac{1}{m-1}}}^{r=\infty} \right|$$

$$\leqslant C \frac{\left(A^{\frac{1}{m-1}}\right)^{n-2+m\mathrm{Re}z} \left(A^{\frac{m}{m-1}}\right)^{-\frac{k}{2}}}{\left(A^{\frac{1}{m-1}}\right)^{m-1}} \leqslant C$$

provided $n-2+m\mathrm{Re}(z) - m\frac{k}{2} - (m-1) \leqslant 0$, i.e.

$$\mathrm{Re}(z) \leqslant \frac{k}{2} + 1 - \frac{n-1}{m}. \tag{23}$$

The integral in (22) is dominated by

$$\left| \int_{2A^{\frac{1}{m-1}}}^{\infty} e^{i\phi(r)} \left[\frac{d}{dr}\frac{1}{\phi'(r)}\right]^N \left(r^{n-2} r^{mz} b\left(r\lambda^{-\frac{1}{m}}\xi\right) \psi\left(r\lambda^{-\frac{1}{m}}\right)\right) dr \right|$$

$$\leqslant \int_{2A^{\frac{1}{m-1}}}^{\infty} r^{n-2} r^{mz} (rA)^{-\frac{k}{2}} r^{-mN} dr \leqslant C_N$$

for N large enough since

$$\left| \left[\frac{d}{dr}\right]^j b\left(r\lambda^{-\frac{1}{m}}\xi\right) \right| \leqslant C r^{-\frac{k}{2}-j} A^{-\frac{k}{2}}$$

$$\left| \left[\frac{d}{dr}\right]^j \psi\left(r\lambda^{-\frac{1}{m}}\right) \right| \leqslant C\left(\lambda^{-\frac{1}{m}}\right)^{-j} \leqslant C r^{-j} \quad \text{on supp } \psi^{(j)}$$

$$\left| \left[\frac{d}{dr}\right]^j \frac{1}{\phi'(r)} \right| \leqslant C r^{-m+1-j}$$

for $j \geqslant 1$. Altogether this shows that term V is bounded provided (23) holds.

The main contribution comes from term IV. Recall the critical point of the phase function $\phi(r) = rq(\xi)\lambda^{-\frac{1}{m}} + r^m$ is $r_0 = \left(\frac{q(\xi)}{m\lambda^{\frac{1}{m}}}\right)^{\frac{1}{m-1}} = A^{\frac{1}{m-1}}$. On the support of integration, $\phi''(r) \approx A^{\frac{m-2}{m-1}}$ and so by the van der

Corput lemma, we have

$$\begin{aligned}|IV| &= \left| \int_{\frac{1}{2}A^{\frac{1}{m-1}}}^{2A^{\frac{1}{m-1}}} e^{i\phi(r)} r^{n-2} r^{mz} b\left(r\lambda^{-\frac{1}{m}}\xi\right) \psi\left(r\lambda^{-\frac{1}{m}}\right) dr \right| \\ &\leqslant C \left|\phi''(r_0)\right|^{-\frac{1}{2}} r_0^{n-2+m\mathrm{Re}(z)} (r_0 A)^{-\frac{k}{2}} \\ &\leqslant C A^{-\frac{m-2}{2(m-1)} + \frac{n-2}{m-1} + \frac{m\mathrm{Re}(z)}{m-1} - \frac{mk}{2(m-1)}}.\end{aligned}$$

This latter expression is bounded provided

$$\mathrm{Re}\,(z) \leqslant \frac{k+1}{2} - \frac{n-1}{m}.$$

Combined with (23), this shows that $|II|$ is dominated by $\lambda^{-\frac{n-1}{m}-\mathrm{Re}(z)}$ for $\mathrm{Re}\,(z) \leqslant \dfrac{k+1}{2} - \dfrac{n-1}{m}$ as required in the case $r_0 > 2$. If $r_0 < 2$, we estimate II directly by integration by parts just as in our estimate for term V. This completes the proof of Theorem 7.

4. THREE DIMENSIONS

In this final section, we obtain sharp weighted decay estimates for surface carried measures that will enable us to obtain almost the full decay of $\dfrac{n-1}{2} = 1$ with appropriate radial and tangential weights (see Theorem 11 below). This then allows us to give sharp $L^p - L^q$ estimates for our averaging operators with nondegenerate Φ in $n = 3$ dimensions (see Theorem 12 below).

DEFINITION 9. — *Φ is said to be nondegenerate if $\nabla\Phi(x) \neq 0$ for $x \neq 0$.*

An important part of our argument is the following geometric observation.

LEMMA 10. — *Let $Z_0 = \{(x_1,x_2) : \Phi(x_1,x_2) = 0\}$. Let $Z_1 = \{(x_1,x_2) : \nabla\Phi(x_1,x_2) = (0,0)\}$. Let $Z_2 = \{(x_1,x_2) : H\Phi(x_1,x_2) = 0\}$, where $H\Phi(x_1,x_2)$ denotes the determinant of the Hessian matrix of Φ. Then for each $j = 0,1,2$, $Z_j = \{(0,0)\} \bigcup \bigcup_{k=1}^{N_j} L_k^j$, where each L_k^j is a line through the origin, and $N_j < \infty$. Moreover, $Z_1 = Z_0 \bigcap Z_2$.*

Proof. — Let Φ_j denote the partial derivative of Φ with respect to x_j. Since Φ is homogeneous of degree m, Φ_j is homogeneous of degree $m-1$, and $H\Phi(x_1,x_2)$ is homogeneous of degree $2(m-2)$. By homogeneity, if Z_j contains a point (x_1,x_2), it also contains a line through the origin containing that point. Since Φ is a polynomial, there can be at most a finite number of such lines. This proves the first assertion of the lemma.

By the Euler homogeneity relations,

$$\begin{aligned} m\Phi(x_1,x_2) &= x_1\Phi_1(x_1,x_2) + x_2\Phi_2(x_1,x_2), \\ (m-1)\Phi_1(x_1,x_2) &= x_1\Phi_{11}(x_1,x_2) + x_2\Phi_{12}(x_1,x_2), \\ (m-1)\Phi_2(x_1,x_2) &= x_1\Phi_{21}(x_1,x_2) + x_2\Phi_{22}(x_1,x_2), \end{aligned}$$

where $\{\Phi_{jk}\}$ denotes the second partial derivatives. Hence, $Z_0 \supset Z_1$. If we write the equations for Φ_1 and Φ_2 in matrix form we see that $(m-1)\nabla\Phi(x_1,x_2)$ is obtained by applying the Hessian matrix of Φ to the vector (x_1,x_2). Hence, $Z_2 \supset Z_1$. Putting these observations together we see that $Z_0 \bigcap Z_2 \supset Z_1$.

Conversely, suppose that both Φ and $H\Phi$ vanish along a line through the origin, which without loss of generality we take to be the x_1 -axis. Then since $m\Phi(x_1,0) = x_1\Phi_1(x_1,0)$, we conclude that $\Phi_1(x_1,0) = 0$. Also, $(m-1)\Phi_1(x_1,0) = x_1\Phi_{11}(x_1,0)$, and this then implies that $\Phi_{11}(x_1,0) = 0$. By assumption,

$$H\Phi(x_1,0) = \Phi_{11}(x_1,0)\Phi_{22}(x_1,0) - \Phi_{12}^2(x_1,0) = 0.$$

Since $\Phi_{11}(x_1,0) = 0$, we conclude that $\Phi_{12}(x_1,0) = 0$, which implies that $\Phi_2(x_1,0) = 0$ since $\Phi_2(x_1,0) = x_1\Phi_{21}(x_1,0)$. This proves that $\nabla\Phi(x_1,0) = (0,0)$ and hence that $Z_1 \supset Z_0 \bigcap Z_2$. This completes the proof of the lemma.

Since we are assuming that Φ is nondegenerate, i.e. $\nabla\Phi(x_1,x_2) = 0$ iff $(x_1,x_2) = (0,0)$, Lemma 3 implies that $Z_0 \bigcap Z_2 = (0,0)$. By Lemma 3, $Z_2 = \bigcup\limits_{j=1}^{N} L_j$, where $N < \infty$ and each L_j is a straight line through the origin. Let S_j be a sector which contains L_j. More precisely, let $L_j = \{(r\cos\theta_j, r\sin\theta_j) : r > 0\}$ and let

$$S_j = \{(r\cos\theta, r\sin\theta) : r > 0, \theta_j - \epsilon \leqslant \theta \leqslant \theta_j + \epsilon\},$$

where ϵ is chosen to be small enough so that $S_j \bigcap S_{j'} = \{(0,0)\}$ if $j \neq j'$, and $S_j \bigcap Z_0 = \{(0,0)\}$.

In order to obtain a sharp version of Theorem 6, we will need decay estimates for the Fourier transform of surface carried measures weighted not only by $|\Phi(x')|^{\alpha}$, but also by what is essentially an appropriate local power of $|H\Phi(x')|$. Let

$$A(x) = \prod_{k=1}^{N} (\theta - \theta_k), \qquad \mathit{for}\ x = (r\cos\theta, r\sin\theta).$$

Then we have

THEOREM 11. — *Suppose that Φ is a nondegenerate homogeneous polynomial of degree m, and let*

$$F_{\alpha}(\xi, \lambda) = \int e^{i(x_1\xi_1 + x_2\xi_2 + \Phi(x_1,x_2)\lambda)} |\Phi(x_1, x_2)|^{\alpha} |A(x)|^{\frac{m}{2}\alpha} \psi(x_1, x_2)\, dx_1 dx_2.$$

Then

$$|F_{\alpha}(\xi, \lambda)| \leqslant C(1 + |\lambda| + |\xi|)^{-\frac{2}{m} - \alpha},$$

when $0 \leqslant \alpha < 1 - \dfrac{2}{m}$.

Note that for nondegenerate homogeneous Φ, the zeroes of $\Phi\,|_{S^{n-1}}$ are of first order since $0 = m\Phi(x) = x \cdot \nabla\Phi(x)$ implies that $\nabla\Phi(x) \neq 0$ is tangent to S^{n-1}. Consequently, $\Phi^{-1} \in L^{\rho}\left(S^{n-1}\right)$ for all $0 \leqslant \rho < 1$. Thus the next result can be viewed as extending the conclusion of Theorem 3 from $\mathcal{Q}\left(2, \dfrac{2}{m}\right)$ to $\mathcal{Q}\left(3, \dfrac{2}{m}\right)$ in dimension $n = 3$ for such Φ.

THEOREM 12. — *Suppose that Φ is a nondegenerate homogeneous polynomial of degree m, and let $Tf(x)$ be defined as in (6) with $n = 3$. Then estimate (2) holds if $\left(\dfrac{1}{p}, \dfrac{1}{q}\right)$ is contained in the interior of the trapezoid $\mathcal{Q}\left(3, \dfrac{2}{m}\right)$. The vertices $\left(\dfrac{3}{m+2}, \dfrac{1}{m+2}\right)$ and $\left(\dfrac{m+1}{m+2}, \dfrac{m-1}{m+2}\right)$ (and the segment connecting them) are not included.*

The scaling argument in the introduction shows that this last result is sharp. The result itself follows from the decay estimate in Theorem 11 in the same way that Corollary 8 follows from Theorem 7 above. More precisely, we need the analogue of Theorem 6 with the weight $A(x)$ thrown in. This in turn requires only two superficial changes. First, the L^{∞} bound in (16) follows from the integrability of $|\Phi(x)|^{z} A(x)^{\frac{m}{2}z}$ for $\mathrm{Re}(z) > -\dfrac{2}{m}$.

Second, the number N in the second estimate in (16) should be set to 3. The proof then proceeds as before but using Theorem 11 in the appropriate place.

Proof (of Theorem 11). — Our plan is as follows. Outside the sectors S_j, the Gaussian curvature vanishes only at the origin. In that region we shall obtain the desired result by using a scaling argument which relies on a dyadic decomposition away from the origin. Inside each S_j we shall separate variables in polar coordinates, taking advantage of the fact that $\Phi|_{S_j}$ does not vanish except at the origin.

Let ρ be a smooth cutoff function supported in the intersection of $\mathbb{R}^2\backslash\bigcup_j S_j$ and the annulus $\{(x_1,x_2) : 1 \leqslant \sqrt{x_1^2+x_2^2} \leqslant 4\}$ such that $\sum \rho(2^k x_1, 2^k x_2) \equiv 1$. Let χ denote the characteristic function of $\left(\bigcup_j S_j\right)^c$. Then $F_\alpha(\xi,\lambda) = F_{1,\alpha}(\xi,\lambda) + F_{2,\alpha}(\xi,\lambda)$, where in local coordinates

$$F_{1,\alpha}(\xi,\lambda) = \int e^{i(x_1\xi_1+x_2\xi_2+\Phi(x_1,x_2)\lambda)}\Phi^\alpha(x_1,x_2)\psi(x_1,x_2)\chi(x_1,x_2)dx_1dx_2.$$

Let

$$F_{1,\alpha}^k(\xi,\lambda) = \int e^{i(x_1\xi_1+x_2\xi_2+\Phi(x_1,x_2)\lambda)} \rho(2^k x_1, 2^k x_2)\Phi^\alpha(x_1,x_2)\psi(x_1,x_2)dx_1dx_2.$$

Note that since each S_j is invariant under isotropic dilations and ψ is radial, $\psi(x_1,x_2)\ \times\rho(2^k x_1, 2^k x_2)$ is supported in the intersection of $\left(\bigcup_j S_j\right)^c$ and the annulus

$$\left\{(x_1,x_2) : 2^{-2k} \leqslant x_1^2+x_2^2 \leqslant 2^{-2(k-1)}\right\}.$$

Also, $\sum_k F_{1,\alpha}^k(\xi,\lambda) = F_{1,\alpha}(\xi,\lambda)$.

A change of variables and homogeneity show that

$$F_{1,\alpha}^k(\xi,\lambda) = 2^{-2k}2^{-mk\alpha}F_{1,\alpha}^0(2^{-k}\xi_1, 2^{-k}\xi_2, 2^{-mk}\lambda).$$

Since $F_{1,\alpha}^0(\xi,\lambda)$ is defined over a smooth piece of the hypersurface where the Gaussian curvature does not vanish, it is known that $|F_{1,\alpha}^0(\xi,\lambda)| \leqslant C|\lambda|^{-1}$ (See e.g. [St2], p. 348). We shall also use the fact that $|F_{1,\alpha}^0(\xi,\lambda)| \leqslant C$. We must estimate $\sum_k 2^{-2k}2^{-mk\alpha}F^0_{1,\alpha}(2^{-k}\xi_1, 2^{-k}\xi_2, 2^{-mk}\lambda)$. Splitting up the sum, we get

$$\sum_k 2^{-2k}2^{-mk\alpha}F^0_{1,\alpha}(2^{-k}\xi_1, 2^{-k}\xi_2, 2^{-mk}\lambda)$$

$$= \sum_{|\lambda|\geqslant 2^{mk}} 2^{-2k}2^{-mk\alpha}() + \sum_{|\lambda|\leqslant 2^{mk}} 2^{-2k}2^{-mk\alpha}().$$

To estimate the first term we use the fact that $|F^0_{1,\alpha}(\xi,\lambda)| \leqslant C|\lambda|^{-1}$, whereas to control the second term we just use the fact that $F^0_{1,\alpha}(\xi)$ is bounded. We then see that the absolute values of the first and the second terms are both bounded by $C|\lambda|^{-\frac{2}{m}-\alpha}$, provided that $\alpha < 1-\dfrac{2}{m}$. This completes the argument if $|\lambda| \geqslant C(|\xi_1|+|\xi_2|)$. However, if $|\lambda| \leqslant C(|\xi_1+\xi_2|)$ for a large enough constant C, the gradient of the phase function of $F_{1,\alpha}(\xi)$ never vanishes, and so an integration by parts argument shows that the integral has rapid decay in (ξ_1,ξ_2) (see e.g [St2], p. 341).

Remark 3. — Note that if there were no cutoff function ψ, we would be forced to sum over negative $k's$ also. However, these are easily handled as follows. Ignoring the weights $|\Phi|^{\alpha}|A|^{\frac{m}{2}\alpha}$, which play no role in estimating $F_{1,\alpha}$ anyway, we have

$$\left|F^k_{1,\alpha}(\xi,\lambda)\right| = 2^{2k}\left|F^0_{1,\alpha}\left(2^k\xi, 2^{mk}\lambda\right)\right| \leqslant C\left(2^{-(m-2)k}\lambda\right)^{-1},$$

which sums up over negative $k's$ to λ^{-1}. We make no assertion regarding decay in ξ in this case.

Hence, $|F_{1,\alpha}(\xi,\lambda)| \leqslant C(1+|\xi|)^{-\frac{2}{m}-\alpha}$, if $\alpha < 1-\dfrac{2}{m}$. Note that if $H\Phi$ only vanishes at the origin, $F_{1,\alpha}(\xi,\lambda) = F_{\alpha}(\xi,\lambda)$ and the argument above finishes the proof of the theorem. In the general case however, we must estimate $F_{2,\alpha}(\xi,\lambda)$.

Let $F^j_{2,\alpha}(\xi,\lambda)$ denote the localization of $F_{2,\alpha}$ to the sector S_j. In local polar coordinates,

$$F^j_{2,\alpha}(\xi,\lambda) = \iint_{\theta_j-\epsilon}^{\theta_j+\epsilon} e^{i(r\xi_1 cos\theta + r\xi_2 sin\theta + \lambda r^m\Phi(\omega))} r r^{m\alpha}\Phi^{\alpha}(\omega)\psi(r)A(r\omega)^{\frac{m}{2}\alpha}d\theta dr$$

$$= \int \int_{\theta_j-\epsilon}^{\theta_j+\epsilon} e^{-i(r\xi_1 \frac{\cos\theta}{\Phi^{\frac{1}{m}}(\omega)} + r\xi_2 \frac{\sin\theta}{\Phi^{\frac{1}{m}}(\omega)} + \lambda r^m))} r r^{m\alpha} \psi\left(\frac{r}{\Phi^{\frac{1}{m}}(\omega)}\right)$$

$$\Phi^{-\frac{2}{m}}(\omega) A\left(\frac{r\omega}{\Phi^{\frac{1}{m}}(\omega)}\right)^{\frac{m}{2}\alpha} d\theta dr$$

$$= \int \int_{\Sigma_j} e^{-i(r\xi_1\mu_1 + r\xi_2\mu_2 + \lambda r^m)} r r^{m\alpha} \psi(r|\mu|) \frac{d\sigma(\mu)}{|\nabla \Phi(\mu)|} A(\mu)^{\frac{m}{2}\alpha} dr,$$

where $\omega = (\cos\theta, \sin\theta)$, $\Sigma_j = \{\mu \in S_j : \Phi(\mu) = 1\}$, and $d\sigma(\mu)$ denotes the Lebesgue measure on Σ_j.

In the second line above we made a change of variables sending $r \to r\Phi^{-\frac{1}{m}}(\omega)$. In the third line we made a change of variables $\mu = \dfrac{(\cos\theta, \sin\theta)}{\Phi^{\frac{1}{m}}(\omega)}$, observed that by homogeneity $\Phi(\mu) = 1$, and then computed the corresponding Jacobian. We also used the fact that $A(r\mu) = A(\mu)$.

Note that since inside S_j, Φ does not vanish except at the origin, Σ_j is a compact smooth curve in the plane. The smoothness follows from the fact the the restriction of the gradient does not vanish, and from the implicit function theorem. Also note that our assumption that the gradient of Φ does not vanish away from the origin guarantees that $d\sigma'(\mu) = \dfrac{d\sigma(\mu)}{\nabla\Phi(\mu)}$ is a smooth measure on Σ_j. Since each S_j contains only one line where $H\Phi$ vanishes, Σ_j has exactly one point where the curvature vanishes. Since Φ is a polynomial, the curvature at that point cannot vanish of infinite order. Hence, at that point Σ_j is a curve of finite type M, $2 \leqslant M < \infty$.

We shall first analyze the integral over Σ_j. Our observations above show that after perhaps applying a linear transformation,

$$\Sigma_j = \left\{(s, \phi(s)) : \phi^{(k)}(0) = 0, k < M, \quad \phi^{(M)}(0) = 1, \ |s| < c\right\}.$$

Applying Taylor's theorem we see that $\phi(s) = g(s)s^M$, where g is smooth and $g(0) = 1$. Hence, the integral over Σ_j can be written in the form

$$I_\alpha(\xi_1, \xi_2) = \int e^{-i(rs\xi_1 + rs^M g(s)\xi_2)} s^{\frac{m}{2}\alpha} \rho(s) ds,$$

where without loss of generality ρ is a smooth cutoff function supported in the interval $(-2, 2)$ and $\rho = 1$ near the origin. As before we may assume that $|\xi_1| \leqslant C|\xi_2|$ since otherwise the phase function in the expression above has no critical points and so an integration by parts argument shows that the integral has rapid decay in $|\xi_1|$.

Let ρ_0 be supported in the interval $(1,2)\bigcup(-1,-2)$, such that $\sum\limits_k \rho_0(2^k s) \equiv 1$. Let

$$I_{k,\alpha}(\xi_1,\xi_2) = \int e^{-i(rs\xi_1+g(s)rs^M\xi_2)} s^{\frac{m}{2}\alpha}\rho_0(2^k s)ds.$$

Each $I_{k,\alpha}$ is defined over a a dyadic piece of Σ_j where the curvature does not vanish. To take advantage of this fact we shall need the following stationary phase result (see e.g. [So], p. 48).

LEMMA 13. — *Let S be a smooth hypersurface in R^n with non-vanishing Gaussian curvature and $d\mu$ a smooth compactly supported measure on S. Then*

$$|\widehat{d\mu}(\xi)| \leqslant \text{const.}(1+|\xi|)^{-\frac{n-1}{2}}.$$

Moreover, suppose that $\Gamma \subset R^n\setminus\{0\}$ is the cone consisting of all ξ which are normal to some point $x \in S$ belonging to a fixed relatively compact neighborhood $\mathcal{N}$ of supp $d\mu$. *Then*

$$\left(\frac{\partial}{\partial\xi}\right)^\alpha \widehat{d\mu}(\xi) = O(1+|\xi|)^{-N} \quad \forall N, \text{ if } \xi \notin \Gamma,$$

$$\widehat{d\mu}(\xi) = \sum e^{-i\langle x_j,\xi\rangle} a_j(\xi), \quad \text{if } \xi \in \Gamma,$$

where the finite sum is taken over all $x_j \in \mathcal{N}$ having ξ as the normal and

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha a_j(\xi)\right| \leqslant C_\alpha(1+|\xi|)^{-\frac{n-1}{2}-|\alpha|}.$$

If we make a change of variables sending $s \to 2^{-k}s$, we get

$$I_{k,\alpha}(\xi_1,\xi_2) = 2^{-k}2^{-\frac{m}{2}\alpha k}\int e^{-i(2^{-k}rs\xi_1+2^{-Mk}rs^M g(2^{-k}s)\xi_2)} s^{\frac{m}{2}\alpha}\rho_0(s)ds.$$

The idea here is that as $k \to \infty$, $g(2^{-k}s) \to g(0) = 1$ and hence $I_{k,\alpha}(\xi_1,\xi_2) \approx 2^{-k}2^{-\frac{m}{2}\alpha k} I_{0,\alpha}(2^{-k}\xi_1, 2^{-Mk}\xi_2)$. Using Lemma 4 we write

$$I_{k,\alpha}(\xi_1,\xi_2) = 2^{-k}2^{-\frac{m}{2}\alpha} e^{irq_k(2^{-k}\xi_1,2^{-Mk}\xi_2)} b_{k,\alpha}(2^{-k}\xi_1, 2^{-Mk}\xi_2),$$

where each b_k is a symbol of order $-\dfrac{1}{2}$, q_k is a homogeneous symbol of order 1, and the Hessian of q_k has rank one everywhere.

Note that since the Gauss map is smooth, for k large,

$$q_k(2^{-k}\xi_1, 2^{-Mk}\xi_2) \approx q(2^{-k}\xi_1, 2^{-Mk}\xi_2),$$

where q is the phase function given by Lemma 4 corresponding to the curve $(s, g(0)s^m)$. One can check by a direct computation (see e.g. [Io1], proof of Theorem 1.1) that $q(2^{-k}\xi_1, 2^{-Mk}\xi_2) = q(\xi_1, \xi_2)$. In fact,

$$q(\xi_1, \xi_2) = c\frac{\xi_1^{\frac{M}{M-1}}}{\xi_2^{\frac{1}{M-1}}}.$$

It also follows that there exists a uniform constant $C > 0$ such that

$$C^{-1}|q(\xi_1, \xi_2)| \leqslant |q_k(2^{-k}\xi_1, 2^{-Mk}\xi_2)| \leqslant C|q(\xi_1, \xi_2)|.$$

Similarly, $\{b_k(\xi_1, \xi_2)\}_k$ is contained in the bounded subset of symbols of order $-\frac{1}{2}$ i.e,

$$|D^{\beta}_{(\xi_1,\xi_2)} b_{k,\alpha}(\xi_1, \xi_2)| \leqslant C(|\xi_1| + |\xi_2|)^{-\frac{1}{2}-|\beta|}, \tag{24}$$

where C is a uniform constant.

We must estimate

$$\int_0^{\infty} \sum_k 2^{-k} 2^{-\frac{m}{2}\alpha k} e^{-i(rq_k(2^{-k}\xi_1, 2^{-Mk}\xi_2)+\lambda r^m)} b_{k,\alpha}(r\xi_1 2^{-k}, r\xi_2 2^{-Mk}) r r^{m\alpha} dr.$$

After interchanging the sum and the integral and making a change of variables sending $r \to r\lambda^{-\frac{1}{m}}$, we get $\lambda^{-\frac{2}{m}} \times$

$$\sum_k 2^{-k} 2^{-\frac{m}{2}\alpha} \int_0^{\infty} e^{-i(rq_k(2^{-k}\xi_1, 2^{-Mk}\xi_2)\lambda^{-\frac{1}{m}}+r^m)} b_{k,\alpha}(r\xi_1\lambda^{-\frac{1}{m}}2^{-k}, r\xi_2\lambda^{-\frac{1}{m}}2^{-Mk}) r r^{m\alpha} dr. \tag{25}$$

We estimate the integral first. Let $\Psi(r) = rA_k + r^m$, where

$$A_k = q_k(2^{-k}\xi_1, 2^{-Mk}\xi_2)\lambda^{-\frac{1}{m}} \approx q(\xi_1, \xi_2)\lambda^{-\frac{1}{m}} = B.$$

Differentiating we see that the critical point is at $r_0 = CA_k^{\frac{1}{m-1}}$, whereas $\Psi''(r) = Cr^{m-2}$. We split up the integral in (25) into three parts: $\int_0^{\infty} = \int_0^{\frac{r_0}{2}} + \int_{\frac{r_0}{2}}^{2r_0} + \int_{2r_0}^{\infty}$. The first and third integrals are handled by integration by parts just as in the proof of Theorem 7. The main contribution comes from the second integral $\int_{\frac{r_0}{2}}^{2r_0}$. Applying the van der Corput Lemma (see e.g. [St2], p.332) we see that the expression in (25) is bounded by

$$(26)\quad |\lambda|^{-\frac{2}{m}-\alpha}|B|^{-\frac{m-2}{2(m-1)}+\frac{1}{m-1}+\frac{m\alpha}{m-1}}\sum_k 2^{-k}2^{-\frac{m}{2}\alpha k} \\ |b(r_0\xi_1\lambda^{-\frac{1}{m}}2^{-k}, r_0\xi_2\lambda^{-\frac{1}{m}}2^{-lk})|.$$

We may assume that $|A_k|$ is large because if not, an integration by parts as in the proof of Theorem 7 yields the correct decay. We may assume that $|\xi_1|+|\xi_2|\leqslant C|\lambda|$ for a sufficiently large C, since otherwise an integration by parts argument shows that the integral in question has rapid decay in $|\xi_1|+|\xi_2|$.

We are left to estimate the sum in (26). Recalling the definition of A_k and using estimate (24), we get

$$\sum_k 2^{-k}2^{-\frac{m}{2}\alpha k}|b_k(r_0\xi_1\lambda^{-\frac{1}{m}}2^{-k}, r_0\xi_2\lambda^{-\frac{1}{m}}2^{-Mk})|$$

$$\leqslant C|B|^{-\frac{m}{2(m-1)}}\sum_{B\geqslant 2^{(M-1)k}} 2^{-k}2^{\frac{Mk}{2}}2^{-\frac{m}{2}\alpha k}+C\sum_{B\leqslant 2^{(M-1)k}} 2^{-k}2^{-\frac{m}{2}\alpha k}.$$

Since $m\geqslant M$, both terms are dominated by $C|B|^{-\frac{1}{m-1}-\frac{m\alpha}{2(m-1)}}$. Hence, we see that the expression (26) is dominated by

$$(27)\qquad |\lambda|^{-\frac{2}{m}-\alpha}|B|^{-\frac{m-2}{2(m-1)}+\frac{m\alpha}{2(m-1)}}.$$

Using the fact that $\alpha\leqslant 1-\dfrac{2}{m}$ and that $|B|$ is large, we see that (27) is dominated by $C|\lambda|^{-\frac{2}{m}-\alpha}$, which completes the proof.

Remark 4. — Again, as in Remark 3, it is not hard to see that this estimate for $F_{2,\alpha}$ remains in effect when there is no cutoff function ψ. Again, we make no assertion regarding decay in ξ in this case.

4.1. An application to a class of dispersive equations.

LEMMA 14. — *Let $m>1$ and $\dfrac{1}{2}<\mathrm{Re}(\sigma)<m$. Then for $s\in R$,*

$$\left|\int_0^\infty e^{i(t^m-mst)}t^{\sigma-1}dt\right|\leqslant C_{m,\sigma}\,(1+|s|)^{\frac{-\frac{m}{2}+Re(\sigma)}{m-1}},$$

where, for m and $\mathrm{Re}(\sigma)$ *fixed, $C_{m,\sigma}$ is polynomially increasing with* $\mathrm{Im}\,(\sigma)$.

The next result follows from Lemma 14 and the latter part of Theorem 11, including Remarks 3 and 4.

THEOREM 15. — *Let*

$$F^z(\xi,\lambda) = \int e^{i(x\cdot\xi+\lambda\Phi(x))}\Phi^z(x)A^{\frac{mz}{2}}(x)dx,$$

where Φ and A are defined as in Theorem 11 above. (Note that the integral is over the whole space, i.e. there is no cutoff function.) Then

$$|F^z(\xi,\lambda)| \leqslant C\,|\lambda|^{-\frac{2}{m}-\mathrm{Re}(z)}, \qquad \mathrm{Re}(z) < 1-\frac{2}{m}.$$

THEOREM 16. — *Let*

$$u_\theta(x,t) = \int e^{i(x\cdot\xi+tP(\xi))}\left|P(\xi)^\theta A(\xi)^{\frac{m\theta}{2}}\right|^{(1-\frac{2}{m})}\widehat{f}(\xi)d\xi,$$

where P is a nondegenerate homogeneous polynomial in two variables of degree $m \geqslant 2$, and A is defined as above. Then for any $\epsilon > 0$,

$$\|u_\theta\|_{\frac{2}{1-\mathrm{Re}(\theta)}} \leqslant C_\epsilon |t|^{-\theta(1-\epsilon)}\,\|f\|_{\frac{2}{1+\mathrm{Re}(\theta)}}\,. \tag{28}$$

Proof. — When $\mathrm{Re}(\theta) = 0$, inequality (28) holds by Plancherel's theorem. When $\mathrm{Re}(\theta) = 1$, inequality (28) follows from Young's inequality and Theorem 15 above with (ξ,λ) replaced by $(x-y,t)$, and upon writing $\widehat{p}(\xi)$ as an integral in y. An application of Stein's analytic interpolation theorem then yields the result.

Remark 5. — Note that if $\theta = 0$, $u_\theta = u_0$ is the solution for the initial value problem (4) in the introduction. If we ignore the zeroes of P and A, and observe that P is homogeneous of degree m and that A is homogeneous of degree 0, then (28) with $p = \dfrac{2}{1+\theta}$, says that $u_0(\cdot,t)$ has essentially $(m-2)\left(\dfrac{2}{p}-1\right)$ derivatives in $L^{p'}$ when $f \in L^p$, $1 \leqslant p \leqslant 2$.

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