ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 46, nº 4 (1996), p. 1095-1124

<http://www.numdam.org/item?id=AIF_1996__46_4_1095_0>

Ann. Inst. Fourier, Grenoble
46, 4 (1996), 1095–1124

IDEMPOTENTS IN QUOTIENTS AND RESTRICTIONS OF BANACH ALGEBRAS OF FUNCTIONS

by Thomas Vils PEDERSEN

We say that a commutative Banach algebra $\mathcal{B}$ is *generated by its idempotents* if the algebra of all linear combinations of idempotents in $\mathcal{B}$ is dense in $\mathcal{B}$. For a compact Hausdorff space X, it is easily proved, using the Stone-Weierstrass theorem, that $C(X)$ is generated by its idempotents if and only if X is totally disconnected. In this paper we discuss conditions under which quotient and restriction algebras of certain Banach algebras of functions on the unit circle $\mathbb{T}$ are generated by their idempotents.

For the algebra $\mathcal{A}$ of absolutely convergent Fourier series on $\mathbb{T}$, Kahane ([7], pp. 39–43) has proved that the restriction algebra $\mathcal{A}(E)$ is generated by its idempotents whenever E is a closed set of measure zero, and that there also exists a totally disconnected, closed set of positive measure for which $\mathcal{A}(E)$ is generated by its idempotents. In the other direction, it is known that there exists a totally disconnected set E (necessarily of positive measure) for which $\mathcal{A}(E)$ is not generated by its idempotents.

The algebras that we discuss in this paper are the Beurling and Lipschitz algebras and the algebra of absolutely continuous functions on $\mathbb{T}$.

For the Beurling algebras $\mathcal{A}_\beta$, Zouakia has proved that $\mathcal{A}_\beta(E)$ is generated by its idempotents whenever E is of measure zero and $\beta < \frac{1}{2}$ (thus generalizing the result of Kahane mentioned above). We provide a proof of this result in Section 2. In the other direction, we prove that, for $\beta > \frac{1}{2}$, there exists a closed set $E \subseteq \mathbb{T}$ of measure zero such that $\mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$ is not generated by its idempotents.

Key words: Banach algebras of functions – Idempotents.
Math. classification: 46J10 – 26A16 – 26A45 – 42A16.

In Section 3, we prove that a certain condition on a closed set $E \subseteq \mathbb{T}$ is equivalent to the Lipschitz algebra $\lambda_\gamma(E)$ being generated by its idempotents. This condition is shown to hold for every closed set of measure zero, and we obtain examples of perfect symmetric sets of positive measure for which it holds and of such sets for which the condition does not hold.

Finally, for the algebra $\mathcal{AC}$ of absolutely continuous functions on $\mathbb{T}$, we show that $\mathcal{AC}(E)$ is generated by its idempotents if and only if E is of measure zero (Section 4).

1. INTRODUCTION

By a *Banach function algebra* on a compact Hausdorff space X, we mean a unital, commutative, semisimple Banach algebra $\mathcal{B}$ with character space X. We shall regard $\mathcal{B}$ as an algebra of functions on X. In this paper we shall study only Banach function algebras on the unit circle $\mathbb{T}$ and on closed subsets of $\mathbb{T}$. We often identify $\mathbb{T}$ with $[-\pi, \pi]$ or $[0, 2\pi]$.

Let $\beta \geq 0$ and define the *Beurling algebra* $\mathcal{A}_\beta$ as the algebra of continuous functions on $\mathbb{T}$ whose Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-\mathrm{int}}\, dt, \qquad n \in \mathbb{Z}$$

satisfy

$$\|f\|_{\mathcal{A}_\beta} = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|\,(1+|n|)^\beta < \infty.$$

With the norm $\|\cdot\|_{\mathcal{A}_\beta}$, it is easily seen that $\mathcal{A}_\beta$ is a Banach algebra. Since $((1+|n|)^\beta)^{1/|n|} \to 1$ as $|n| \to \infty$, it follows from [4], pp. 118–120 that the character space of $\mathcal{A}_\beta$ is $\mathbb{T}$, so that $\mathcal{A}_\beta$ is a Banach function algebra on $\mathbb{T}$.

The second class of Banach function algebras on $\mathbb{T}$ that we consider is the class of Lipschitz algebras. For $f \in C(\mathbb{T})$, let

$$\omega_f(h) = \sup\{|f(t) - f(s)| : t, s \in \mathbb{T} \text{ with } |t-s| \leq h\} \quad (h > 0)$$

be the modulus of continuity of f. For $0 < \gamma \leq 1$, define the *Lipschitz algebra* Λ_γ to be the algebra of continuous functions on $\mathbb{T}$ satisfying $\omega_f(h) = O(h^\gamma)$ as $h \to 0$. Hence $f \in \Lambda_\gamma$ if and only if

$$p_\gamma(f) = \sup\left\{\frac{|f(t) - f(s)|}{|t-s|^\gamma} : t, s \in \mathbb{T},\, t \neq s\right\} < \infty.$$

With the norm

$$\|f\|_{\Lambda_\gamma} = \|f\|_\infty + p_\gamma(f)$$
$$= \|f\|_\infty + \sup_{h>0} \frac{\omega_f(h)}{h^\gamma}, \qquad f \in \Lambda_\gamma$$

(where $\|\cdot\|_\infty$ is the uniform norm), Λ_γ becomes a Banach algebra. This algebra was studied by Sherbert, who noted that the character space of Λ_γ is $\mathbb{T}$ ([16], Proposition 2.1). Hence Λ_γ is a Banach function algebra on $\mathbb{T}$.

For $0 < \gamma < 1$, let λ_γ be the subalgebra of Λ_γ of functions satisfying

$$\omega_f(h) = o(h^\gamma) \qquad \text{as } h \to 0.$$

(If we extend this definition to $\gamma = 1$, we simply obtain $\lambda_1 = \mathbb{C}1$.) In Section 3 it becomes apparent that λ_γ rather than Λ_γ provides the right frame of reference for discussing idempotents in restrictions of Lipschitz algebras. We mention in passing that from a Banach algebra point of view, it is often the case that λ_γ is more interesting than Λ_γ. For example, for $s \in \mathbb{T}$, let the translation operator L_s on Λ_γ be defined by $(L_s f)(t) = f(t-s)$ for $t \in \mathbb{T}$ and $f \in \Lambda_\gamma$. It was proved by Mirkil ([12]) that $L_s f \to f$ in Λ_γ as $s \to 0$ if and only if $f \in \lambda_\gamma$. Hence λ_γ is homogeneous in the sense of Shilov, and a remarkably general result of Shilov ([17], Theorem 5.2 or [11], Proposition 20.1) thus implies that λ_γ is the closed subalgebra of Λ_γ generated by the trigonometric polynomials.

Finally, we are also interested in the algebra of continuous functions of bounded variation and the subalgebra of absolutely continuous functions. Recall that a function f on $\mathbb{T}$ is of *bounded variation* if

$$\mathrm{Var}(f) = \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| : 0 = x_0 < x_1 < \ldots < x_n = 2\pi\right\} < \infty.$$

Let $\mathcal{BVC}$ be the algebra of continuous functions of bounded variation on $\mathbb{T}$. Equipped with the norm

$$\|f\|_{\mathcal{BVC}} = \|f\|_\infty + \mathrm{Var}(f), \quad f \in \mathcal{BVC},$$

it is easily seen that $\mathcal{BVC}$ becomes a Banach algebra. Also, $1/f \in \mathcal{BVC}$ whenever $f \in \mathcal{BVC}$ does not have any zeros on $\mathbb{T}$, so it follows from [9], Theorem, p. 204, that the character space of $\mathcal{BVC}$ is $\mathbb{T}$. Hence $\mathcal{BVC}$ is a Banach function algebra on $\mathbb{T}$.

Recall that a function f on $\mathbb{T}$ is said to be *absolutely continuous* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$ whenever $(a_1, b_1), \ldots, (a_n, b_n)$ are pairwise disjoint intervals in $\mathbb{T}$ satisfying

$\sum_{k=1}^{n}(b_k - a_k) < \delta$. Let $\mathcal{AC}$ be the algebra of absolutely continuous functions on $\mathbb{T}$. It is well known (for this and other basic facts about $\mathcal{AC}$, see, for example, [6], Section 18), that $f \in \mathcal{AC}$ if and only if f is differentiable a.e. with $f' \in L^1(\mathbb{T})$ and

$$f(x) - f(y) = \int_y^x f'(t)\,dt \qquad \text{for } x, y \in \mathbb{T}.$$

Obviously a function defined on $\mathbb{T}$ is absolutely continuous if and only if both the real and the imaginary part are, and Banach (see, for example, [6], Theorem 18.25) proved that a real-valued continuous function on $\mathbb{T}$ is absolutely continuous if and only if it is of bounded variation and maps sets of measure zero to sets of measure zero. Furthermore,

$$\mathrm{Var}(f) = \int_{\mathbb{T}} |f'(t)|\,dt \qquad \text{for } f \in \mathcal{AC}.$$

It thus follows that $\mathcal{AC}$ is a closed subalgebra of $\mathcal{BVC}$, and it is easily seen that $\mathcal{AC}$ is a Banach function algebra on $\mathbb{T}$. (Also, the result of Shilov mentioned above implies that, for $f \in \mathcal{BVC}$, we have $L_s f \to f$ in $\mathcal{BVC}$ as $s \to 0$ if and only if $f \in \mathcal{AC}$.)

Recall that a Banach function algebra $\mathcal{B}$ on X is called *regular* if, for every closed set $E \subseteq X$ and $x \in X \backslash E$, there exists $f \in \mathcal{B}$ such that $f(x) = 1$ and $f = 0$ on E. Note that all the algebras $\mathcal{A}_\beta$ $(\beta \geq 0)$, λ_γ $(0 < \gamma < 1)$ and $\mathcal{AC}$ contain $C^\infty(\mathbb{T})$ and thus are regular.

1.1. Ideal structures.

Let $\mathcal{B}$ be a regular Banach function algebra on $\mathbb{T}$. For a closed set $E \subseteq \mathbb{T}$, consider the ideals

$$I_{\mathcal{B}}(E) = \{f \in \mathcal{B} : f = 0 \text{ on } E\},$$
$$J_{\mathcal{B}}(E) = \{f \in \mathcal{B} : f = 0 \text{ on a neighbourhood of } E\}.$$

For $f \in C(\mathbb{T})$, let $Z(f) = \{t \in \mathbb{T} : f(t) = 0\}$, and define the *hull* of a closed ideal I in $\mathcal{B}$ to be

$$h(I) = \bigcap_{f \in I} Z(f).$$

It is well known (see, for example, [9], Corollary 5.7, p. 224) that, if I is a closed ideal in $\mathcal{B}$ with $h(I) = E$, then

$$\overline{J_{\mathcal{B}}(E)} \subseteq I \subseteq I_{\mathcal{B}}(E).$$

Hence, if $J_{\mathcal{B}}(E)$ is dense in $I_{\mathcal{B}}(E)$, then $I_{\mathcal{B}}(E)$ is the only closed ideal in $\mathcal{B}$ with E as hull. In this case, we say that E is of *synthesis* for $\mathcal{B}$. If every closed set $E \subseteq \mathbb{T}$ is of synthesis for $\mathcal{B}$, we say that synthesis holds for $\mathcal{B}$; otherwise that synthesis fails.

Malliavin's famous result ([10]) states that synthesis fails for $\mathcal{A}$, and it is not hard to see that an elaboration due to Kahane ([7], pp. 64–65) actually shows that synthesis fails for $\mathcal{A}_\beta$ for $\beta < \frac{1}{4}$. For $\beta \geq 1$, we have $\mathcal{A}_\beta \subseteq C^1(\mathbb{T})$, which implies that not even singletons are of synthesis for $\mathcal{A}_\beta$. This makes it seem very likely that synthesis fails for $\mathcal{A}_\beta$ for all $\beta \geq 0$, but to our knowledge this is still an open problem.

For the Lipschitz algebras λ_γ $(0 < \gamma < 1)$, Sherbert ([16], Theorem 4.2) proved that synthesis holds. On the other hand, for $0 < \gamma \leq 1$, we can define $f \in \Lambda_\gamma$ by $f(t) = |t|^\gamma$ for $|t| \leq \pi$. For $g \in J_{\Lambda_\gamma}(\{0\})$, we then have $\|f - g\|_{\Lambda_\gamma} \geq 1$, which proves that points are not of synthesis for Λ_γ.

Finally, Shilov (see, for example, [14], A.2.5, pp. 302-303) proved that synthesis holds for $\mathcal{BVC}$ as well as for $\mathcal{AC}$.

1.2. Algebras generated by their idempotents.

Let $\mathcal{B}$ be a regular Banach function algebra on $\mathbb{T}$. When I is a closed ideal in $\mathcal{B}$ with $h(I) = E$, then the quotient algebra $\mathcal{B}/I$ is a regular Banach algebra with character space E and radical $I_{\mathcal{B}}(E)/I$. For a closed set $E \subseteq \mathbb{T}$, the semisimple algebra $\mathcal{B}/I_{\mathcal{B}}(E)$ is a Banach function algebra on E which is easily seen to be isometrically isomorphic to the restriction algebra

$$\mathcal{B}(E) = \{f \in C(E) : \text{ there exists } g \in \mathcal{B} \text{ such that } g|_E = f\},$$

with the norm $\|f\|_{\mathcal{B}(E)} = \inf\{\|g\|_{\mathcal{B}} : g \in \mathcal{B} \text{ and } g|_E = f\}$ for $f \in \mathcal{B}(E)$. When synthesis holds for $\mathcal{B}$, these are the only quotient algebras, but when synthesis fails, we can also consider the non-semisimple quotients $\mathcal{B}/I$, where $\overline{J_{\mathcal{B}}(E)} \subseteq I \subset I_{\mathcal{B}}(E)$ for closed sets $E \subseteq \mathbb{T}$ which are not of synthesis for $\mathcal{B}$. Note that, if I is a closed ideal in $\mathcal{B}$ with $h(I) = E$, then

$$\mathcal{B}/I \simeq \left(\mathcal{B}/\overline{J_{\mathcal{B}}(E)}\right) / \left(I/\overline{J_{\mathcal{B}}(E)}\right), \tag{1}$$

$$\mathcal{B}(E) \simeq (\mathcal{B}/I) \,/\, (I_{\mathcal{B}}(E)/I) \tag{2}$$

(where $\simeq$ indicates an isometric isomorphism). Hence $\mathcal{B}/I$ is generated by its idempotents if $\mathcal{B}/\overline{J_{\mathcal{B}}(E)}$ is, and $\mathcal{B}(E)$ is generated by its idempotents if $\mathcal{B}/I$ is.

We shall first see that the only closed sets $E \subseteq \mathbb{T}$ for which $\mathcal{B}(E)$ can be generated by its idempotents are the totally disconnected sets.

PROPOSITION 1.1. — *Let $\mathcal{B}$ be a regular Banach function algebra on $\mathbb{T}$. Let I be a closed ideal in $\mathcal{B}$ and suppose that $\mathcal{B}/I$ is generated by its idempotents. Then $h(I)$ is totally disconnected.*

Proof. — Suppose that $E = h(I)$ is not totally disconnected. Then E contains a non-empty interval U, and every idempotent in $\mathcal{B}(E)$ is constant on U. On the other hand, $\mathcal{B}(E)$ separates the points of E, so $\mathcal{B}(E)$ is not generated by its idempotents, and the result thus follows from (2). □

Let $E \subseteq \mathbb{T}$ be a closed set and suppose that $F \subseteq E$ is both open and closed in E. It follows directly from the regularity of $\mathcal{B}$ that there exists an idempotent $e \in \mathcal{B}(E)$ such that $e = 1$ on F and $e = 0$ on $E \setminus F$. Also, if I is any closed ideal in $\mathcal{B}$ with $h(I) = E$, then it follows from Shilov's idempotent theorem (see [3], Theorem 5, p. 109) that there exists an idempotent $e \in \mathcal{B}/I$ such that $\widehat{e} = 1$ on F and $\widehat{e} = 0$ on $E \setminus F$, where $\widehat{e}$ is the Gelfand transform of e. If E is totally disconnected, then the subsets of E which are both open and closed form a base for the topology ([15], Corollary, p. 371), and we thus deduce that $\mathcal{B}/I$ contains many idempotents. In the following sections we study whether there are enough to generate $\mathcal{B}/I$.

We shall now give a simple characterization of idempotents in restriction algebras, and show that, in some sense, the linear span of idempotents in $\mathcal{B}(E)$ does not depend on the algebra $\mathcal{B}$. Hence the problem of determining whether $\mathcal{B}(E)$ is generated by its idempotents becomes a problem about approximation in the norm on $\mathcal{B}$.

LEMMA 1.2. — *Let $\mathcal{B}$ be a Banach function algebra on $\mathbb{T}$ and suppose that $C^\infty(\mathbb{T}) \subseteq \mathcal{B}$. Let $E \subseteq \mathbb{T}$ be a closed set and let $g \in \mathcal{B}$. Then $g|_E$ belongs to the linear span of idempotents in $\mathcal{B}(E)$ if and only if $g(E)$ is finite. In the case where this condition is satisfied, there exists $f \in C^\infty(\mathbb{T})$ such that $g|_E = f|_E$.*

Proof. — If $g|_E = \sum_{n=1}^{N} c_n e_n$, where c_n is a constant and $e_n \in \mathcal{B}(E)$ is an idempotent for $n = 1, \dots, N$, then $g(E) \subseteq \left\{ \sum_{n=1}^{N} c_n \varepsilon_n, \text{ where } \varepsilon_n = 0 \text{ or } 1 \right\}$, so $g(E)$ is finite. Conversely, suppose that $g(E)$ is finite, say

$g(E) = \{y_1, \ldots, y_N\}$. Let $E_n = g^{-1}(y_n) \cap E$ and choose $e_n \in C^\infty(\mathbb{T})$ such that $e_n = 1$ on E_n and $e_n = 0$ on $E \setminus E_n$. Then $e_n|_E$ is an idempotent for $n = 1, \ldots, N$, and $g = \sum_{n=1}^{N} y_n e_n$, which finishes the proof. □

If $\mathcal{B} \subseteq C^1(\mathbb{T})$, then

$$\overline{J_{\mathcal{B}}(E)} \subseteq \{f \in \mathcal{B} : f = f' = 0 \text{ on } E\}$$

for every closed set $E \subseteq \mathbb{T}$. Hence, if $e \in \mathcal{B}$ is such that $e + \overline{J_{\mathcal{B}}(E)}$ is an idempotent in $\mathcal{B}/\overline{J_{\mathcal{B}}(E)}$, then $e(E) \subseteq \{0, 1\}$ and $(2e-1)e' = (e^2 - e)' = 0$ on E. Thus $e' = 0$ on E, so we deduce that $\mathcal{B}/\overline{J_{\mathcal{B}}(E)}$ is not generated by its idempotents; even when E is finite. The algebra $\mathcal{B}(E)$ is obviously generated by its idempotents when E is finite. If, however, E is infinite and x is an accumulation point of E, then $e'(x) = 0$ for all idempotents in $\mathcal{B}(E)$, so $\mathcal{B}(E)$ is not generated by its idempotents. For these reasons, we restrict ourselves to algebras $\mathcal{B}$ such that $\mathcal{B} \not\subseteq C^1(\mathbb{T})$. In particular, we shall discuss only the Beurling algebras $\mathcal{A}_\beta$ with $0 \le \beta < 1$.

2. IDEMPOTENTS IN QUOTIENTS OF BEURLING ALGEBRAS

The main result in this section is that, for $\beta > \frac{1}{2}$, there exists a closed set E of measure zero such that $\mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$ is not generated by its idempotents. In contrast, Zouakia ([18], Corollaire 5.13) has proved that $\mathcal{A}_\beta(E)$ and $\mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$ are generated by their idempotents whenever E is of measure zero and $\beta < \frac{1}{2}$. Since this source is rather inaccesible, we include a proof of the result. We complement these results by showing that, for $\beta < \frac{1}{2}$, there exists a closed set $E \subseteq \mathbb{T}$ of positive measure such that $\mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$ is generated by its idempotents, and that there exists a totally disconnected, closed set $E \subseteq \mathbb{T}$ (necessarily of positive measure) such that $\mathcal{A}_\beta(E)$ is not generated by its idempotents for $0 \le \beta < 1$.

The Beurling algebras are defined in terms of their Fourier coefficients and not directly in terms of properties of the functions involved. This often complicates matters, but it does, on the other hand, allow a simple description of their dual spaces. Let $\beta \ge 0$ and write $\mathcal{PM}_\beta$ (pseudomeasures with weight $(1+|n|)^\beta$) for the dual space of $\mathcal{A}_\beta$. It is easily seen that the map

$$T \mapsto \left(\widehat{T}(n)\right),$$

where $\widehat{T}(n) = \langle e^{-\mathrm{int}}, T\rangle$ for $T \in \mathcal{PM}_\beta$ and $n \in \mathbb{Z}$, identifies $\mathcal{PM}_\beta$ with the set of all sequences $(\widehat{T}(n))$ for which

$$\|T\|_{\mathcal{PM}_\beta} = \sup_{n\in\mathbb{Z}} \frac{|\widehat{T}(n)|}{(1+|n|)^\beta} < \infty.$$

Also,

$$\langle f, T\rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\widehat{T}(-n)$$

for $f \in \mathcal{A}_\beta$ and $T \in \mathcal{PM}_\beta$.

We define the support of a pseudomeasure $T \in \mathcal{PM}_\beta$ to be the support of the corresponding distribution T on $\mathbb{T}$, that is, the complement of the largest open set $U \subseteq \mathbb{T}$ for which $\langle f, T\rangle = 0$ for all $f \in C^\infty(\mathbb{T})$ with $\operatorname{supp} f \subseteq U$. We denote the support of T by $\operatorname{supp} T$ and, for a closed set $E \subseteq \mathbb{T}$, let $\mathcal{PM}_\beta(E) = \{T \in \mathcal{PM}_\beta : \operatorname{supp} T \subseteq E\}$. A proof almost identical to [7], p. 29, shows that, if $f \in \mathcal{A}_\beta$ and $T \in \mathcal{PM}_\beta$ with $\operatorname{supp} f \cap \operatorname{supp} T = \emptyset$, then $\langle f, T\rangle = 0$ (so we could as well have defined the support of $T \in \mathcal{PM}_\beta$ by means of all $\mathcal{A}_\beta$ functions). Hence it follows that, for a closed set $E \subseteq \mathbb{T}$, the dual space of the quotient algebra $\mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$ is $\mathcal{PM}_\beta(E)$.

2.1. Quotient algebras generated by their idempotents.

Let $0 \le \beta < \frac{1}{2}$. To $T \in \mathcal{PM}_\beta$, associate $\theta_T \in L^2(\mathbb{T})$ defined by

$$\theta_T(t) = \widehat{T}(0)t + \sum_{n\neq 0} \frac{\widehat{T}(n)}{in} e^{\mathrm{int}} \quad \text{for } -\pi < t \le \pi$$

(convergence in $L^2(\mathbb{T})$), that is, the formal integral of T. Since $t = \sum_{n\neq 0} (i(-1)^n/n)e^{\mathrm{int}}$, we have

$$\theta_T(t) = \sum_{n\neq 0} \frac{\widehat{T}(n) - (-1)^n\widehat{T}(0)}{in} e^{\mathrm{int}} \quad \text{for } -\pi < t \le \pi. \tag{3}$$

We need the following simple partial integration result.

LEMMA 2.1. — *Let $0 \le \beta < \frac{1}{2}$. For $f \in C^\infty(\mathbb{T})$ and $T \in \mathcal{PM}_\beta$, we have*

$$\frac{1}{2\pi}\int_{\mathbb{T}} f'(t)\theta_T(t)\,dt = -\langle f, T\rangle + f(\pi)\widehat{T}(0).$$

Proof. — Since f', $\theta_T \in L^2(\mathbb{T})$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{T}} f'(t)\theta_T(t)\,dt &= \sum_{n=-\infty}^{\infty} \widehat{f'}(n)\widehat{\theta_T}(-n)\\ &= \sum_{n\neq 0} in\widehat{f}(n)\frac{\widehat{T}(-n)-(-1)^n\widehat{T}(0)}{-in}\\ &= -\sum_{n=-\infty}^{\infty}\widehat{f}(n)\widehat{T}(-n)+\sum_{n=-\infty}^{\infty}\widehat{f}(n)(-1)^n\widehat{T}(0)\\ &= -\langle f,T\rangle + f(\pi)\widehat{T}(0),\end{aligned}$$

as required. □

The proof of the following result is omitted, since it is basically the same as that of [2], Proposition 3.2.5.b).

LEMMA 2.2. — *Let $0 \le \beta < \frac{1}{2}$ and let $T \in \mathcal{PM}_\beta$ with $\widehat{T}(0) = 0$. Suppose that $T \in \mathcal{PM}_\beta(\mathbb{T}\setminus V)$ for some open interval $V \subseteq \mathbb{T}$. Then θ_T is constant on V.*

The following proof of Zouakia's result is very similar to his own proof, although we use a different representation of $\mathcal{PM}_\beta$.

THEOREM 2.3 (Zouakia). — *Let $0 \le \beta < \frac{1}{2}$ and let $E \subseteq \mathbb{T}$ be a closed set of measure zero. Then $\mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$ is generated by its idempotents.*

Proof. — We may assume that $\pi \in E$. Let $T \in \mathcal{PM}_\beta(E)$ and suppose that $\langle e,T\rangle = 0$ for every idempotent $e \in \mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$. In particular, $\widehat{T}(0) = \langle 1,T\rangle = 0$. Let $a,b \in \mathbb{T}\setminus E$ with $-\pi < a < b < \pi$ and choose $\varepsilon > 0$ such that $[a-\varepsilon,a]$, $[b,b+\varepsilon] \subseteq (-\pi,\pi)\setminus E$. Choose $f \in C^\infty(\mathbb{T})$ satisfying $f = 1$ on a neighbourhood of $[a,b]$ and supp $f \subseteq (a-\varepsilon,b+\varepsilon)$. Then $f - f^2 \in J_{\mathcal{A}_\beta}(E)$, so $f + \overline{J_{\mathcal{A}_\beta}(E)}$ is an idempotent in $\mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$. Also, $f' = 0$ in a neighbourhood of E, so it follows from the two previous lemmas that

$$\begin{aligned}0 &= \langle f + \overline{J_{\mathcal{A}_\beta}(E)},T\rangle = \langle f,T\rangle = -\frac{1}{2\pi}\int_{\mathbb{T}} f'(t)\theta_T(t)\,dt\\ &= -\frac{1}{2\pi}\int_{a-\varepsilon}^{a} f'(t)\,dt\cdot\theta_T(a) - \frac{1}{2\pi}\int_{b}^{b+\varepsilon} f'(t)\,dt\cdot\theta_T(b)\\ &= \frac{1}{2\pi}\left(\theta_T(b)-\theta_T(a)\right).\end{aligned}$$

Consequently θ_T is constant on $\mathbb{T} \setminus E$. Since E is of measure zero, we deduce from (3) that $\widehat{T}(n) = (-1)^n \widehat{T}(0) = 0$ for $n \in \mathbb{Z}$, and thus $T = 0$. Since $\mathcal{PM}_\beta(E)$ is the dual space of $\mathcal{A}_\beta / \overline{J_{\mathcal{A}_\beta}(E)}$, the result follows from the Hahn-Banach theorem. □

We briefly digress to mention a related result by Bade and Dales. Let $0 \leq \beta < \frac{1}{2}$ and suppose that $E \subseteq \mathbb{T}$ is a closed set which is of measure zero and not of synthesis for $\mathcal{A}_\beta$. It follows from Theorem 2.3 and [1], Lemma 3.3 that the non-semisimple algebra $\mathcal{A}_\beta / \overline{J_{\mathcal{A}_\beta}(E)}$ does not have a strong Wedderburn decomposition. This, however, is just a special case of [1], Theorem 4.3, where the result is shown for all closed sets which are not of synthesis for $\mathcal{A}_\beta$. In view of our results for $\beta > \frac{1}{2}$ (Corollary 2.8), it is nevertheless interesting to note that the results in [1] are only proved for $\beta < \frac{1}{2}$, and that it seems unknown whether they hold for $\beta \geq \frac{1}{2}$.

The following is an immediate consequence of the previous theorem and (2).

COROLLARY 2.4. — *Let $0 \leq \beta < \frac{1}{2}$ and let $E \subseteq \mathbb{T}$ be a closed set of measure zero. Then $\mathcal{A}_\beta(E)$ is generated by its idempotents.*

We also have the following generalization of the result mentioned in [7], p. 43.

PROPOSITION 2.5. — *Let $0 \leq \beta < \frac{1}{2}$. Then there exists a closed set $E \subseteq \mathbb{T}$ of positive measure such that $\mathcal{A}_\beta / \overline{J_{\mathcal{A}_\beta}(E)}$ and $\mathcal{A}_\beta(E)$ are generated by their idempotents.*

Proof. — Let $\varepsilon_n = (n+1)^{\beta - 1}$ for $n \in \mathbb{N}_0$. Following [19], Theorem IX.6.21, we choose a $U(\varepsilon)$-set E of positive measure. If $T \in \mathcal{PM}_\beta(E)$ and $\langle e, T \rangle = 0$ for all idempotents $e \in \mathcal{A}_\beta / \overline{J_{\mathcal{A}_\beta}(E)}$, then it follows as in the proof of Theorem 2.3 that $\theta_T = c$ on $\mathbb{T} \setminus E$ for some constant c. Also, $\widehat{\theta_T}(n) = O(\varepsilon_{|n|})$ as $|n| \to \infty$, so it follows from the proof of [19], Theorem IX.6.21 that $\theta_T = c$ on $\mathbb{T}$. (For $\beta = 0$, it follows from [19], Theorem III.3.8 that the Fourier series of $\theta_T - c$ converges to 0 on $\mathbb{T} \setminus E$. For $\beta > 0$, this need not be true anymore (see [19], Theorem VIII.2.5), but the proof of [19], Theorem IX.6.21 works for $\theta_T - c \in L^2$ with $\theta_T - c = 0$ on $\mathbb{T} \setminus E$.) Hence $c = 0$ and thus $T = 0$, so we deduce that $\mathcal{A}_\beta / \overline{J_{\mathcal{A}_\beta}(E)}$ and thus $\mathcal{A}_\beta(E)$ are generated by their idempotents. □

2.2. Quotient algebras not generated by their idempotents.

We start with the following generalization of [7], p. 42.

PROPOSITION 2.6. — *There exists a totally disconnected, closed set* $E \subseteq \mathbb{T}$ *(necessarily of positive measure) which is of synthesis for* $\mathcal{A}_\beta$ *such that* $\mathcal{A}_\beta(E)$ *is not generated by its idempotents for* $0 \leq \beta < 1$.

Proof. — It follows from [7], p. 42 that there exists a Herz set (and thus a set of synthesis for $\mathcal{A}$) $E \subseteq \mathbb{T}$ such that $\mathcal{A}(E)$ is not generated by its idempotents. Now let $0 \leq \beta < 1$. The injection $\iota : \mathcal{A}_\beta(E) \hookrightarrow \mathcal{A}(E)$ is continuous with dense range, so it follows that $\mathcal{A}_\beta(E)$ is not generated by its idempotents. Furthermore, it can be shown that Herz sets are of synthesis for $\mathcal{A}_\beta$ (see [13], Theorem 2.2.5). □

We now wish to prove that, for $\beta > \frac{1}{2}$, there exists a closed set $E \subseteq \mathbb{T}$ of measure zero such that the quotient algebra $\mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$ is not generated by its idempotents.

PROPOSITION 2.7. — *Let* $E \subseteq \mathbb{T}$ *be a closed set and let* $\beta \geq 0$. *Suppose that there exists a non-zero measure* μ *with support contained in* E *such that* $\widehat{\mu}(n) = O(|n|^{\beta-1})$ *as* $|n| \to \infty$. *Then* $\mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$ *is not generated by its idempotents.*

Proof. — Defining μ' in the sense of distributions (that is, $\langle f, \mu' \rangle = -\langle f', \mu \rangle$ for $f \in C^\infty(\mathbb{T})$), we have $\widehat{\mu'}(n) = in\widehat{\mu}(n)$ for $n \in \mathbb{Z}$, so $\mu' \in \mathcal{PM}_\beta$. It follows from the definition of the support that supp $\mu' \subseteq E$ and thus $\mu' \in \mathcal{PM}_\beta(E)$. Also, we may assume that $E \neq \mathbb{T}$, so that μ is not a constant function and thus $\mu' \neq 0$.

For $g \in \mathcal{A}_\beta$, let $\dot{g}$ (resp. $\ddot{g}$) denote the corresponding element in the quotient algebra $\mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$ (resp. in $\mathcal{A}_\beta(E) \simeq \mathcal{A}_\beta/I_{\mathcal{A}_\beta}(E)$). Let $\dot{e} \in \mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$ be an idempotent (with $e \in \mathcal{A}_\beta$). It follows from (2) that $\ddot{e}$ is an idempotent. Hence, with $E_j = \{t \in E : \ddot{e}(t) = j\}$ for $j = 0, 1$, we see that E_0, E_1 are disjoint, compact sets with $E = E_0 \cup E_1$. Choose $f \in C^\infty(\mathbb{T})$ such that $f = j$ in a neighbourhood of E_j for $j = 0, 1$. Then $\ddot{e} = \ddot{f}$, so

$$\dot{e} - \dot{f} \in I_{\mathcal{A}_\beta}(E)/\overline{J_{\mathcal{A}_\beta}(E)} = \mathrm{rad}(\mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)})$$

(the radical of $\mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$). But $\dot{e}$ and $\dot{f}$ are idempotents, so we deduce that $\dot{e} = \dot{f}$. Since $f' = 0$ on E, we thus have

$$\langle \dot{e}, \mu' \rangle = \langle f, \mu' \rangle = -\langle f', \mu \rangle = 0.$$

From the Hahn-Banach theorem, we thus conclude that $\mathcal{A}_\beta/\overline{J_{\mathcal{A}^\beta}(E)}$ is not generated by its idempotents. □

For $\gamma > -\frac{1}{2}$, Salem ([8], p. 110) proved that, there exists a perfect, closed set $E \subseteq \mathbb{T}$ of measure zero and a non-zero measure μ with support contained in E such that $\widehat{\mu}(n) = O(|n|^\gamma)$ as $|n| \to \infty$. Combining this with the previous proposition, we obtain the following.

COROLLARY 2.8. — *For $\beta > \frac{1}{2}$, there exists a perfect, closed set $E \subseteq \mathbb{T}$ of measure zero such that $\mathcal{A}_\beta/\overline{J_{\mathcal{A}_\beta}(E)}$ is not generated by its idempotents.*

We would also like to prove that the same conclusion holds for the restriction algebras $\mathcal{A}_\beta(E)$, or more precisely that, for $\beta > \frac{1}{2}$, there exists a closed set of measure zero such that $\mathcal{A}_\beta(E)$ is not generated by its idempotents. If the set E in the previous corollary is of synthesis for $\mathcal{A}_\beta$, then this is certainly the case, but we do not even know whether E is of synthesis for $\mathcal{A}$.

For $f \in C^\infty(\mathbb{T})$ with $f = 0$ on E, we have $f^{(n)} = 0$ on E for $n \in \mathbb{N}$, because E is perfect. Hence $\langle f, \mu' \rangle = -\langle f', \mu \rangle = 0$. If $I_{\mathcal{A}_\beta}(E) \cap C^\infty(\mathbb{T})$ is dense in $I_{\mathcal{A}_\beta}(E)$, we thus have $\mu' \in I_{\mathcal{A}_\beta}(E)^\perp$, which is the dual space of $\mathcal{A}_\beta(E)$. The proof of Proposition 2.7 would thus show that $\mathcal{A}_\beta(E)$ is not generated by its idempotents. However, for $f \in I_{\mathcal{A}_\beta}(E) \cap C^\infty(\mathbb{T})$, we have $\sup\{|f(t)| : d(t, Z(f)) \leq \varepsilon\} = O(\varepsilon^n)$ as $\varepsilon \to 0$ for $n \in \mathbb{N}_0$, and it can be shown that this implies that $f \in \overline{J_{\mathcal{A}_\beta}(Z(f))} \subseteq \overline{J_{\mathcal{A}_\beta}(E)}$. Consequently $I_{\mathcal{A}_\beta}(E) \cap C^\infty(\mathbb{T})$ is dense in $I_{\mathcal{A}_\beta}(E)$ if and only if E is of synthesis for $\mathcal{A}_\beta$, so this idea cannot be used to decide whether $\mathcal{A}_\beta(E)$ is generated by its idempotents.

3. IDEMPOTENTS IN RESTRICTIONS OF LIPSCHITZ ALGEBRAS

We begin this section with the following simple result.

PROPOSITION 3.1. — *Let $0 < \gamma \leq 1$ and let $E \subseteq \mathbb{T}$ be an infinite, closed set. Then $\Lambda_\gamma(E)$ is not generated by its idempotents.*

Proof. — We may assume that 0 is an accumulation point of E. Define $f \in \Lambda_\gamma$ by $f(t) = |t|^\gamma$ for $|t| \leq \pi$. Let $g \in \Lambda_\gamma$ with $g(E)$ finite.

Then g is constant on $E \cap [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$, so $\|f - g\|_{\Lambda_\gamma} \geq 1$. It thus follows from Lemma 1.2 that $f|_E$ does not belong to the closed linear span of idempotents in $\Lambda_\gamma(E)$. In particular, $\Lambda_\gamma(E)$ is not generated by its idempotents. □

(For $0 < \gamma < 1$, we could as well have argued as follows. The closed linear span of idempotents in $\Lambda_\gamma(E)$ is contained in $\lambda_\gamma(E)$ by Lemma 1.2. Also, $\lambda_\gamma(E) \subset \Lambda_\gamma(E)$ since $f|_E \notin \lambda_\gamma(E)$, with f as in the proof.)

Because of this result, we shall focus on the algebras $\lambda_\gamma (0 < \gamma < 1)$. In this section we obtain a characterization of those closed sets $E \subseteq \mathbb{T}$ for which $\lambda_\gamma(E)$ is generated by its idempotents. It turns out that we can avoid some technical difficulties by working with Lipschitz algebras on the unit interval rather than the unit circle. We define the Lipschitz algebra $\widetilde{\lambda}_\gamma$ on the unit interval as we defined λ_γ, except that we do not require the functions to be periodic. For a closed set $E \subset [0,1]$, it is easily seen that $\widetilde{\lambda}_\gamma(E) = \{f|_E : f \in \widetilde{\lambda}_\gamma\}$ is isomorphic (but not isometrically isomorphic) to $\lambda_\gamma(e^{i2\pi E})$. In particular, $\widetilde{\lambda}_\gamma(E)$ is generated by its idempotents if and only if $\lambda_\gamma(e^{i2\pi E})$ is. From the point of view of this paper there is thus no difference between λ_γ and $\widetilde{\lambda}_\gamma$. For notational convenience, we write $\widetilde{\lambda}_\gamma$ as λ_γ. Also, we extend the standard terminology and refer to intervals in $[0,1]$ of the form $[0,a)$ or $(a,1]$ for $0 < a < 1$ as open intervals.

The following set function plays a central part. Let $0 < \gamma < 1$ and let $E \subseteq [0,1]$ be a closed set. For a closed interval $F = [x,y] \subseteq [0,1]$, define

$$\rho_{E,\gamma}(F) = \sup\{g(y) - g(x) : g \in \lambda_\gamma \text{ is real-valued, } p_\gamma(g|_F) \leq 1 \text{ and } g(E \cap F) \text{ is finite}\}.$$

Denoting the interior of a set F by F° and the Lebesgue measure on $[0,1]$ by m, we have the following.

Lemma 3.2. — *Let $0 < \gamma < 1$, let $E \subseteq [0,1]$ be a closed set and let $F \subseteq [0,1]$ be a closed interval. Then*

(i) $\rho_{E,\gamma}(F) \leq m(F)^\gamma$.

(ii) *With $F^\circ \setminus E = \bigcup_{n=1}^{\infty} V_n$, where (V_n) is a sequence of pairwise disjoint, open intervals, we have $\rho_{E,\gamma}(F) \leq \sum_{n=1}^{\infty} m(V_n)^\gamma$.*

(iii) *For $F \neq \emptyset$, we have $\rho_{E,\gamma}(F) \geq m(F)^{\gamma-1} m(F \setminus E)$. In particular, $\rho_{E,\gamma}(F) = m(F)^\gamma$ if $m(E \cap F) = 0$.*

Proof. — (i) and (ii) follow directly from the definition. For (iii), let $\varepsilon > 0$ and choose pairwise disjoint, open intervals $V_1, \ldots, V_N \subseteq F^\circ \setminus E$ such that $\sum_{n=1}^{N} m(V_n) \geq m(F \setminus E) - \varepsilon$. Let $g \in \lambda_\gamma$ be a real-valued function which is linearly increasing on each of the sets $V_1, \ldots, V_N$ with slope $m(F)^{\gamma-1}$ and is constant on each of the contiguous intervals. Then $p_\gamma(g|_F) \leq 1$ and

$$g(y) - g(x) = \sum_{n=1}^{N} m(V_n) m(F)^{\gamma-1}$$
$$\geq (m(F \setminus E) - \varepsilon) m(F)^{\gamma-1},$$

so the conclusions follow. □

We shall now obtain a characterization of the closed sets $E \subseteq [0,1]$ for which $\lambda_\gamma(E)$ is generated by its idempotents. In concrete cases it is, however, difficult to decide whether the condition is satisfied, but we shall see that it can be done in certain cases.

THEOREM 3.3. — *Let $0 < \gamma < 1$ and let $E \subseteq [0,1]$ be a closed set. Then $\lambda_\gamma(E)$ is generated by its idempotents if and only if*

$$\rho_{E,\gamma}(F) = m(F)^\gamma$$

for every closed interval $F \subseteq [0,1]$.

Proof. — First, suppose that $\lambda_\gamma(E)$ is generated by its idempotents. Let $F = [x,y] \subseteq [0,1]$ and let $\varepsilon > 0$. Define $f \in \lambda_\gamma$ by $f(t) = m(F)^{\gamma-1} t$ for $t \in [0,1]$. Then $p_\gamma(f|_F) = 1$ and $f(y) - f(x) = m(F)^\gamma$. By Lemma 1.2, there exists $g \in \lambda_\gamma$ real-valued with $g(E \cap F)$ finite and $\|f - g\|_{\lambda_\gamma} \leq \varepsilon$. In particular,

$$g(y) - g(x) \geq m(F)^\gamma - 2\varepsilon$$

and

$$p_\gamma(g|_F) \leq 1 + \varepsilon.$$

Hence $\rho_{E,\gamma}(F) = m(F)^\gamma$.

Conversely, suppose that $\rho_{E,\gamma}(F) = m(F)^\gamma$ for every closed interval $F \subseteq [0,1]$. Since $\rho_{E,\gamma}(\overline{V}) = 0$ for every open interval $V \subseteq E$ by Lemma 3.2 (ii), we deduce that E does not contain any open intervals. Hence E is totally disconnected. Write $[0,1] \setminus E = \bigcup_{n=1}^{\infty} V_n$, where (V_n) is a sequence of pairwise disjoint, open intervals. Let $f \in \lambda_\gamma$ be real-valued (it is sufficient to prove that we can approximate real-valued functions in $\lambda_\gamma(E)$ with linear combinations of idempotents) and let $\varepsilon > 0$. Choose $h_0 > 0$ such that

$$\omega_f(h) \le \varepsilon h^\gamma \qquad \text{for } h \le h_0,$$

and choose $N \in \mathbb{N}$ such that $m(V_n) \le h_0$ for $n > N$. In particular,

$$p_\gamma(f|_{V_n}) \le \varepsilon \qquad \text{for } n > N.$$

Let $F = [x,y]$ be one of the closed intervals constituting $[0,1] \setminus \bigcup_{n=1}^{N} V_n$ (there are $N-1$, N or $N+1$ such intervals), and suppose that F is not a singleton. Choose $K \in \mathbb{N}$ such that $K \ge 2(y-x)/h_0$. For $k = 1,\dots,K$, choose $z_k \in (x+(k-1)(y-x)/K, x+k(y-x)/K) \setminus E$ and remove the open interval V_{n_k} containing z_k. Then $F \setminus \bigcup_{k=1}^{K} V_{n_k}$ is a finite union of closed intervals each of measure at most h_0. If we do this for each of the closed intervals constituting $[0,1] \setminus \bigcup_{n=1}^{N} V_n$ (except for the singletons), we see that there exists a finite number of closed intervals $F_1,\dots,F_M$ such that $m(F_m) \le h_0$ and thus $p_\gamma(f|_{F_m}) \le \varepsilon$ for $m = 1,\dots,M$, and such that $[0,1] \setminus \bigcup_{m=1}^{M} F_m$ is a finite union of intervals V_n including $V_1,\dots,V_N$. Let $m \in \{1,\dots,M\}$ and write $F_m = [x_m, y_m]$. There exists $g_m \in \lambda_\gamma$ real-valued with $g_m(x_m) = f(x_m)$, $g_m(y_m) = f(y_m)$, $g_m(E \cap F_m)$ finite and

$$p_\gamma(g_m|_{F_m}) \le \frac{|f(y_m) - f(x_m)|}{m(F_m)^\gamma} \le p_\gamma(f|_{F_m}) \le \varepsilon.$$

(If F_m is a singleton, then $p_\gamma(g_m|_{F_m}) = 0$.) Define $g \in \lambda_\gamma$ by

$$g = \begin{cases} f & \text{on } [0,1] \setminus \bigcup_{m=1}^{M} F_m \\ g_m & \text{on } F_m \quad \text{for } m = 1,\dots,M. \end{cases}$$

Then $g(E)$ is finite. Let $h = f - g$, and let $x \in [0,1] \setminus \bigcup_{m=1}^{M} F_m$ and $y \in F_m$ with $1 \le m \le M$. We may assume that $x < y$, so we have

$$|h(y) - h(x)| = |h(y)| \le p_\gamma(h|_{F_m})(y - x_m)^\gamma \le p_\gamma(h|_{F_m})(y-x)^\gamma.$$

Also, for $x \in F_{m_1}$ and $y \in F_{m_2}$ with $1 \le m_1, m_2 \le M$ and $x \le y$, we have

$$\begin{aligned} |h(y) - h(x)| &\le \sup_{1 \le m \le M} p_\gamma(h|_{F_m}) \left((y - x_{m_2})^\gamma + (y_{m_1} - x)^\gamma\right) \\ &\le 2 \sup_{1 \le m \le M} p_\gamma(h|_{F_m})(y-x)^\gamma. \end{aligned}$$

Hence we deduce that

$$p_\gamma(h) \le 2 \sup_{1 \le m \le M} p_\gamma(h|_{F_m}) \le 2 \sup_{1 \le m \le M} \left(p_\gamma(f|_{F_m}) + p_\gamma(g|_{F_m})\right) \le 4\varepsilon,$$

and thus $\|h\|_\infty \leq 4h_0^\gamma \varepsilon$. Consequently $\lambda_\gamma(E)$ is generated by its idempotents. □

Hedberg ([5]) has given the following characterization of the closed sets $E \subseteq [0,1]$ for which $\lambda_\gamma(E)$ is generated by its idempotents. For a union $\bigcup_{n=1}^{\infty} V_n$ of pairwise disjoint, open intervals, let $M_\gamma\Big(\bigcup_{n=1}^{\infty} V_n\Big) = \sum_{n=1}^{\infty} m(V_n)^\gamma$. Then $\lambda_\gamma(E)$ is generated by its idempotents if and only if, for every $a \in E$, we have

$$\liminf_{\delta\to 0} \frac{M_\gamma((a-\delta, a+\delta)\setminus E)}{\delta^\gamma} > 0.$$

We find it quite interesting to compare Hedberg's "local" condition to our more "global" version.

The following important corollary follows from Lemma 3.2 (iii), and can also be deduced from Hedberg's characterization. However, we have not been able to apply Hedberg's condition to the perfect symmetric sets (see below).

COROLLARY 3.4. — *Let $0 < \gamma < 1$ and let $E \subseteq [0,1]$ be a closed set of measure zero. Then $\lambda_\gamma(E)$ is generated by its idempotents.*

The referee has kindly pointed out to us that the corollary can be proved directly quite easily as follows. Let $f \in \lambda_\gamma$ and $\varepsilon > 0$. Choose disjoint, open intervals $V_1, \ldots, V_N$ with $\sum_{n=1}^{N} m(V_n) < \varepsilon$ such that $E \subseteq \bigcup_{n=1}^{N} V_n$. Pick $x_n \in V_n$ and let $g = f(x_n)$ on V_n for $n = 1, \ldots, N$. On each of the contiguous intervals, let $g(x) = f(x) + ax + b$, where a and b are chosen so that g is continuous. Then $g(E)$ is finite and it is easily seen that $\|f - g\|_{\lambda_\gamma} \to 0$ as $\varepsilon \to 0$.

We now wish to prove that the characterization obtained in Theorem 3.3 does not simply give us all closed sets of measure zero or all totally disconnected, closed sets. To this end, we show that, for $0 < \gamma < 1$, there exists a totally disconnected, closed set (necessarily of positive measure) such that $\lambda_\gamma(E)$ is not generated by its idempotents, and that there exists a totally disconnected, closed set of positive measure such that $\lambda_\gamma(E)$ is generated by its idempotents. Examples of both kinds are provided by perfect symmetric sets.

Recall the following definition of these sets from [8], Chapitre I. Let $\underline{\xi} = (\xi_n)$ be a sequence with $0 < \xi_n < \frac{1}{2}$ for $n \in \mathbb{N}$. First, we remove an open interval V_1 of length $(1 - 2\xi_1)$ from the middle of $[0, 1]$. From the middle of each of the two remaining closed intervals $E_{11} = [0, \xi_1]$ and $E_{12} = [(1 - \xi_1), 1]$, we then remove open intervals V_{21} and V_{22} each of length $\xi_1(1 - 2\xi_2)$. In the n'th step, we remove an open interval V_{nk} of length $\xi_1 \cdots \xi_{n-1}(1 - 2\xi_n)$ from the middle of $E_{n-1,k}$ for $k = 1, \ldots, 2^{n-1}$, so that 2^n closed intervals $E_{n1}, \ldots, E_{n2^n}$ each of length $\xi_1 \cdots \xi_n$ remain. For $n \in \mathbb{N}$, let

$$V_n = \bigcup_{k=1}^{2^{n-1}} V_{nk}, \qquad E_n = \bigcup_{k=1}^{2^n} E_{nk},$$

and define

$$E_{\underline{\xi}} = \bigcap_{n=1}^{\infty} E_n = [0, 1] \setminus \bigcup_{n=1}^{\infty} V_n.$$

Then $E_{\underline{\xi}}$ is a perfect, closed set with empty interior and

$$E_{\underline{\xi}} = \left\{ \sum_{n=1}^{\infty} \varepsilon_n \xi_1 \cdots \xi_{n-1}(1 - \xi_n) : \varepsilon_n = 0 \text{ or } 1 \text{ for } n \in \mathbb{N} \right\}.$$

Furthermore,

$$m(E_{\underline{\xi}}) = \lim_{n \to \infty} 2^n \xi_1 \cdots \xi_n.$$

Note that the Cantor set on $[0, 1]$ corresponds to $\xi_n = \frac{1}{3}$ for $n \in \mathbb{N}$. When the k in V_{nk} and E_{nk} is not specified, we often write $V_{n\cdot}$ and $E_{n\cdot}$ instead. Also, for $n \in \mathbb{N}$, let

$$\begin{aligned} l_n &= m(V_{n\cdot}) = \xi_1 \cdots \xi_{n-1}(1 - 2\xi_n), \\ r_n &= m(E_{n\cdot}) = \xi_1 \cdots \xi_n. \end{aligned}$$

We are particularly interested in the case where

$$\xi_n = \frac{1}{2}(1 - c \cdot 2^{-an}) \qquad \text{for } n \in \mathbb{N}$$

for some $a > 0$ and $0 < c < 2^a$. In this case, we write $E(a, c)$ for $E_{\underline{\xi}}$, and we have $m(E(a, c)) > 0$. Also, $l_n \sim 2^{-(a+1)n}$ as $n \to \infty$.

LEMMA 3.5. — *Let $0 < \gamma < 1$, let $E \subseteq [0, 1]$ be a closed set and write $[0, 1] \setminus E = \bigcup_{n=1}^{\infty} V_n$, where (V_n) is a sequence of pairwise disjoint, open intervals. Suppose that there exists a closed interval $F \subseteq [0, 1]$ such that $m(E \cap F) > 0$ and*

$$\sum_{V_n \subseteq F} m(V_n)^{\gamma} < \infty.$$

Then there exists a closed interval $F' \subseteq F$ whose endpoints belong to E such that

$$\sum_{V_n \subseteq F'} m(V_n)^\gamma < m(F')^\gamma.$$

In particular, $\lambda_\gamma(E)$ is not generated by its idempotents.

Proof. — Write $\{V_n : V_n \subseteq F\} = \{V_{n_m} : m \in \mathbb{N}\}$ (with obvious changes in the following if $\{V_n : V_n \subseteq F\}$ is finite). Choose $M \in \mathbb{N}$ such that $\sum_{m=M+1}^{\infty} m(V_{n_m})^\gamma < m(E \cap F)$. Since $[0,1] \setminus \bigcup_{m=1}^{M} V_{n_m}$ consists of $M+1$ closed intervals $F_0, \ldots, F_M$ (two of which are possibly empty) and since

$$\sum_{m=0}^{M} \sum_{V_n \subseteq F_m} m(V_n)^\gamma = \sum_{m=M+1}^{\infty} m(V_{n_m})^\gamma,$$

there exists $m_0 \in \{0, \ldots, M\}$ such that

$$\sum_{V_n \subseteq F_{m_0}} m(V_n)^\gamma < m(E \cap F_{m_0}) \leq m(F_{m_0})^\gamma.$$

Since the endpoints of F_{m_0} belong to E, we have $F_{m_0}^\circ \setminus E = \bigcup_{V_n \subseteq F_{m_0}} V_n$, and the conclusions thus follow from Lemma 3.2 (ii) and Theorem 3.3. □

Example 3.6. — Let $0 < \gamma < 1$ and suppose that a satisfies $\gamma(a+1) > 1$. Then $\lambda_\gamma(E(a,c))$ is not generated by its idempotents.

Proof. — With the above notation, we have

$$\sum_{V_{nk} \subseteq [0,1]} m(V_{nk})^\gamma = \sum_{n=1}^{\infty} 2^{n-1} l_n^\gamma \leq C \sum_{n=1}^{\infty} 2^{(1-\gamma(a+1))n} < \infty$$

(where C is some constant), so it follows from the previous lemma that $\lambda_\gamma(E(a,c))$ is not generated by its idempotents. □

We now wish to prove that $\lambda_\gamma(E(a,c))$ is generated by its idempotents whenever $\gamma(a+1) < 1$. There are a number of preparatory steps.

LEMMA 3.7. — *Let $0 < \gamma < 1$, let $f \in \lambda_\gamma$ be real-valued and suppose that f is linear on $[b,c]$ for some $b, c \in [0,1]$ with $b < c$. Then*

$$p_\gamma(f|_{[a,c]}) \leq \max\left\{p_\gamma(f|_{[a,b]}),\ \sup_{a \leq x \leq b} \frac{|f(c) - f(x)|}{(c-x)^\gamma}\right\} \tag{4}$$

for $0 \leq a \leq b$.

Proof. — Let $0 \le a \le b$ and write C for the right-hand side of (4). We may assume that f is increasing on $[b, c]$ with slope $r \ge 0$. If $b \le x < y \le c$, then

$$\frac{|f(y) - f(x)|}{(y-x)^\gamma} = \frac{r(y-x)}{(y-x)^\gamma} \le r(c-b)^{1-\gamma} = \frac{f(c) - f(b)}{(c-b)^\gamma} \le C.$$

Now let $x \in [a, b)$. If $f(x) \ge f(b)$, then

$$\frac{|f(y) - f(x)|}{(y-x)^\gamma} \le \frac{\max\{f(y) - f(b), f(x) - f(b)\}}{(y-x)^\gamma} \le C \qquad \text{for } y \in [b, c].$$

If $f(x) \le f(b)$, then

$$\frac{|f(y) - f(x)|}{(y-x)^\gamma} = \frac{r(y-b) + f(b) - f(x)}{(y-x)^\gamma} \qquad \text{for } y \in [b, c].$$

Considering this last expression as a function of y, it is easily seen that it does not have a maximum in (b, c), so the result follows. □

The following two results enable us to break up closed intervals into smaller closed intervals, and to reduce the discussion of the set function $\rho_{E,\gamma}$ to these smaller intervals.

LEMMA 3.8. — *Let $0 < \gamma < 1$ and let $E \subseteq [0, 1]$ be a closed set. Let $0 \le a \le b \le c \le 1$ and suppose that the closed intervals $F_1 = [a, b]$ and $F_2 = [b, c]$ satisfy $\rho_{E,\gamma}(F_k) = m(F_k)^\gamma$ for $k = 1, 2$. Then $F = [a, c]$ satisfies $\rho_{E,\gamma}(F) = m(F)^\gamma$.*

Proof. — Let $\varepsilon > 0$ and choose $g_k \in \lambda_\gamma$ real-valued with $g_k(E \cap F_k)$ finite and $p_\gamma(g_k|_{F_k}) \le 1 + \varepsilon$ for $k = 1, 2$, and $g_1(b) - g_1(a) = m(F_1)^\gamma$ and $g_2(c) - g_2(b) = m(F_2)^\gamma$. Let $q_k = (m(F_k)/m(F))^{1-\gamma}$ for $k = 1, 2$ and let $g \in \lambda_\gamma$ be a real-valued function satisfying

$$g(t) = \begin{cases} q_1 g_1(t) & \text{on } F_1 \\ q_2(g_2(t) - g_2(b)) + q_1 g_1(b) & \text{on } F_2. \end{cases}$$

Then $g(E \cap F)$ is finite and $g(c) - g(a) = q_1 m(F_1)^\gamma + q_2 m(F_2)^\gamma = m(F)^\gamma$. If $0 < s \le m(F_1)$ and $0 < t \le m(F_2)$, then

$$\frac{|g(b+t) - g(b-s)|}{(s+t)^\gamma} \le (1+\varepsilon)\frac{q_1 s^\gamma + q_2 t^\gamma}{(s+t)^\gamma} \le 1 + \varepsilon,$$

where the last inequality follows from elementary estimates. Hence we deduce that $p_\gamma(g|_F) \le 1 + \varepsilon$, and the result follows. □

COROLLARY 3.9. — *Let $0 < \gamma < 1$ and let $E \subseteq [0, 1]$ be a closed set. Let $F_n = [x_n, y_n]$ $(n = 1, \ldots, N)$ be closed intervals with $x_1 \ge 0$, $y_n < x_{n+1}$*

for $n = 1, \dots, N-1$ and $y_N \le 1$. Let $V_n = (y_n, x_{n+1})$ and suppose that $V_n \subseteq [0,1] \setminus E$ for $n = 1, \dots, N-1$. If $\rho_{E,\gamma}(F_n) = m(F_n)^\gamma$ for $n = 1, \dots, N$, then $F = [x_1, y_N]$ satisfies $\rho_{E,\gamma}(F) = m(F)^\gamma$.

Proof. — Note that $\rho_{E,\gamma}(\overline{V}) = m(\overline{V})^\gamma$ for every open interval $V \subseteq [0,1] \setminus E$, so the result follows by induction from Lemma 3.8. □

We now return to the perfect symmetric sets. Let $\underline{\xi} = (\xi_n)$ be a sequence with $0 < \xi_n < \frac{1}{2}$ for $n \in \mathbb{N}$ and let $E = E_{\underline{\xi}}$. For $n \in \mathbb{N}$, write $V_{nk} = (a_{nk}, b_{nk})$, with $a_{n1} > 0$, $b_{nk} < a_{n,k+1}$ for $k = 1, \dots, 2^{n-1} - 1$ and $b_{n,2^{n-1}-1} < 1$. For $k = 1, \dots, 2^{n-1} - 1$, let

$$s_E(n,k) = \min\{b_{nk_2} - a_{nk_1} : 1 \le k_1, k_2 \le 2^{n-1} \text{ and } k_2 - k_1 = k\},$$

that is, the minimum distance spanned by k of the intervals $V_{n\cdot}$. (When no misunderstanding is possible, we omit the subscript E.) We aim to prove that, for suitable values of a and c, the set $E = E(a,c)$ satisfies $\rho_{E,\gamma}(F) = m(F)^\gamma$ for every closed interval $F \subseteq [0,1]$ by considering, for $n \in \mathbb{N}$, a function f which is linear on each $V_{n\cdot}$ contained in F and is constant on each of the contiguous intervals. To obtain estimates of $p_\gamma(f|_F)$, we need to establish certain lower bounds for $s(n,k)$. This is done in the following rather technical lemmas.

LEMMA 3.10. — *Let (ξ_n) be a sequence with $0 < \xi_n < \frac{1}{2}$ for $n \in \mathbb{N}$ and consider the perfect symmetric set $E_{\underline{\xi}}$. Let $n \in \mathbb{N}$, $0 \le k \le 2^{n-1} - 1$ and write $k = \sum_{j=0}^{m} \varepsilon_j \cdot 2^j$, where $\varepsilon_j = 0$ or 1 for $j = 0, \dots, m$ and where $\varepsilon_m = 1$ if $k \ne 0$. With $s = s_{E_{\underline{\xi}}}$, we then have*

(5)

$$s(n,k) = \sum_{r=n-m-1}^{n-1} \left(\varepsilon_{n-r-1} + \sum_{j=n-r}^{m} \varepsilon_j \cdot 2^{j-(n-r)} \right) l_r + (k+1) l_n + 2k r_n.$$

In particular,

$$(6) \qquad s(n, 2^m - 1) = \sum_{r=n-m}^{n-1} 2^{r-(n-m)} l_r + 2^m l^n + 2(2^m - 1) r_n$$

for $0 \le m \le n-1$.

Proof. — In general, it is obvious that if $k_2 - k_1 = k$, then (a_{nk_1}, b_{nk_2}) consists of $2k$ of the $E_{n\cdot}$'s, $k+1$ of the $V_{n\cdot}$'s and k of the $V_{m\cdot}$'s with $m < n$ (since there is such an interval between any two of the $V_{n\cdot}$'s). First, suppose that $k = 2^m$ for some m with $0 \le m \le n-2$. It is easily seen that $(a_{n1}, b_{n,2^m+1})$ contains exactly $2^{r-(n-m)}$ of the $V_{r\cdot}$'s for $n-m \le r \le n-1$ and 1 of the $V_{n-m-1\cdot}$'s. Since the sequence (l_n) is decreasing and since there is a $V_{s\cdot}$ between any two of the $V_{r\cdot}$'s with $r > s$, we deduce that $s(n, 2^m) = b_{n,2^m+1} - a_{n1}$ and thus

$$(7) \qquad s(n, 2^m) = l_{n-m-1} + \sum_{r=n-m}^{n-1} 2^{r-(n-m)} l_r + (2^m + 1) l_n + 2 \cdot 2^m r_n.$$

Furthermore, it is not difficult to see that $s(n, 2^m)$ also can be obtained as $b_{n(k_1+2^m)} - a_{nk_1}$, whenever $\varepsilon_m = 0$ in the expansion $k_1 - 1 = \sum_{j=0}^{n-2} \varepsilon_j \cdot 2^j$.

Now let $0 \le k \le 2^{n-1} - 1$ and write $k = \sum_{j=0}^{m} \varepsilon_j \cdot 2^j$, where $\varepsilon_m = 1$ if $k \ne 0$. The k intervals $V_{r\cdot}$ with $r < n$ contained in $(a_{n1}, b_{n,k+1})$, and thus in $(b_{n1}, a_{n,k+1})$, can be divided into those contained in $(b_{n1}, a_{n,1+\varepsilon_m \cdot 2^m})$, those contained in $(b_{n,1+\varepsilon_m \cdot 2^m}, a_{n,1+\varepsilon_m \cdot 2^m + \varepsilon_{m-1} \cdot 2^{m-1}})$, ..., those contained in $(b_{n,k+1-\varepsilon_0}, a_{n,k+1})$, so it follows from (7) and the subsequent remark that

$$(8) \; b_{n,k+1} - a_{n1} = \sum_{j=0}^{m} \varepsilon_j \left(l_{n-j-1} + \sum_{r=n-j}^{n-1} 2^{r-(n-j)} l_r \right) + (k+1) l_n + 2k r_n.$$

Again, it can be seen that $s(n,k)$ equals $b_{n,k+1} - a_{n1}$, so (5) and (6) follow by rewriting (8). □

We are particularly interested in $s(n,k)$, when $k = 2^m - 1$ for $m = 0, \dots, n-1$, because we can express, and later evaluate, these quantities fairly easily.

LEMMA 3.11. — *Let (ξ_n) be a sequence with $0 < \xi_n < \frac{1}{2}$ for $n \in \mathbb{N}$. With $s = s_{E_{\underline{\xi}}}$, we have*

$$s(n, 2^m - 1) = \xi_1 \cdots \xi_{n-m-1} (1 - 2\xi_{n-m} \cdots \xi_n)$$

for $n \in \mathbb{N}$ and $0 \le m \le n-1$.

Proof. — Let $n \in \mathbb{N}$ and $0 \le m \le n-1$. By the previous lemma, we have

$$\begin{aligned} s(n, 2^m - 1) &= \sum_{r=n-m}^{n} 2^{r-(n-m)} l_r + 2(2^m - 1) r_n \\ &= \sum_{r=n-m}^{n} 2^{r-(n-m)} \xi_1 \cdots \xi_{r-1}(1 - 2\xi_r) + 2(2^m - 1)\xi_1 \cdots \xi_n \\ &= \xi_1 \cdots \xi_{n-m-1} - 2^{m+1}\xi_1 \cdots \xi_n + 2(2^m - 1)\xi_1 \cdots \xi_n \\ &= \xi_1 \cdots \xi_{n-m-1}(1 - 2\xi_{n-m} \cdots \xi_n), \end{aligned}$$

as required. □

LEMMA 3.12. — *Let (ξ_n) be an increasing sequence with $0 < \xi_n < \frac{1}{2}$ for $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and let $2^m \leq k \leq 2^{m+1} - 1$ for some m with $0 \leq m \leq n - 2$. With the notation of Lemma 3.10, we then have*

$$\frac{\sum\limits_{r=n-m-1}^{n-1} \left(\varepsilon_{n-r-1} + \sum\limits_{j=n-r}^{m} \varepsilon_j \cdot 2^{j-(n-r)} \right) l_r}{k} \geq \frac{\sum\limits_{r=n-m-1}^{n-1} 2^{r-(n-m-1)} l_r}{2^{m+1} - 1}.$$

Proof. — The result is clearly equivalent to

$$\sum_{r=n-m-1}^{n-1} c_r l_r \geq 0,$$

where

$$\begin{aligned} c_r &= (2^{m+1} - 1) \left(\varepsilon_{n-r-1} + \sum_{j=n-r}^{m} \varepsilon_j \cdot 2^{j-(n-r)} \right) - \sum_{j=0}^{m} \varepsilon_j \cdot 2^j \cdot 2^{r-(n-m-1)} \\ &= \varepsilon_{n-r-1}(2^m - 1) - \sum_{j=0}^{n-r-2} \varepsilon_j \cdot 2^{j+r-(n-m-1)} - \sum_{j=n-r}^{m} \varepsilon_j \cdot 2^{j-(n-r)} \end{aligned}$$

for $n - m - 1 \leq r \leq n - 1$. We have

$$c_{n-m-1} = 2^m - 1 - \sum_{j=0}^{m-1} \varepsilon_j \cdot 2^j = \sum_{\varepsilon_j = 0} 2^j,$$

$c_r \geq 0$ if $\varepsilon_{n-r-1} = 1$ and $c_r \geq -\sum\limits_{j=0}^{m-1} 2^j > -2^m$ for $n - m - 1 \leq r \leq n - 1$, so

$$\sum_{r=n-m-1}^{n-1} c_r l_r \geq \sum_{\varepsilon_j = 0} 2^j \cdot l_{n-m-1} - 2^m \sum_{\varepsilon_j = 0} l_{n-j-1}.$$

Since (ξ_n) is increasing, we have

$$\frac{l_{s+t}}{l_s} = \frac{\xi_1 \cdots \xi_{s+t-1}(1-2\xi_{s+t})}{\xi_1 \cdots \xi_{s-1}(1-2\xi_s)} \leq 2^{-t} \qquad \text{for } s,t \in \mathbb{N},$$

and thus

$$\sum_{r=n-m-1}^{n-1} c_r l_r \geq \sum_{\varepsilon_j=0} 2^j \cdot l_{n-m-1} - 2^m \sum_{\varepsilon_j=0} 2^{-(m-j)} l_{n-m-1} = 0,$$

as required. □

LEMMA 3.13. — *Let $0 < \gamma < 1$ and let $a > 0$ be such that $\gamma(a+1) < 1$. Then there exists c_0 with $0 < c_0 < 2^a$ such that, for $0 < c \leq c_0$, the set $E(a,c)$ satisfies the following condition (with $s = s_{E(a,c)}$): for every $\varepsilon > 0$ and $p \in \mathbb{N}_0$, there exists $N \in \mathbb{N}$ such that*

$$(9) \qquad (k+1)2^{-(n-p-1)} \leq (1+\varepsilon)(\xi_1 \cdots \xi_p)^{-\gamma} s(n,k)^\gamma$$

for $n \geq N$ and $0 \leq k \leq 2^{n-p-1} - 1$.

Proof. — Choose b such that $\gamma < b < 1 - \gamma a$, choose $c_0 \in (0, 2^a)$ such that $\frac{1}{2}(1 - c_0 \cdot 2^{-a}) = 2^{-b/\gamma}$ and choose $M \in \mathbb{N}$ such that $1 - 2^{-M} \geq 2^{-(1-b)/\gamma}$. Let $0 < c \leq c_0$ and let $p \in \mathbb{N}_0$ be given. We have

$$\left(\frac{s(n, 2^m - 1)}{\xi_1 \cdots \xi_p}\right)^\gamma = (\xi_{p+1} \cdots \xi_{n-m-1}(1 - 2\xi_{n-m} \cdots \xi_n))^\gamma$$

for $0 \leq m \leq n - p - 1$. Since $\xi_1 = \frac{1}{2}(1 - c2^{-a}) \geq 2^{-b/\gamma}$, we have

$$\begin{aligned}(\xi_{p+1} \cdots \xi_{n-1}(1 - 2\xi_n))^\gamma \cdot 2^{n-p-1} &\geq \left(\xi_1^{n-p-1} c \cdot 2^{-an}\right)^\gamma \cdot 2^{n-p-1} \\ &\geq c^\gamma \cdot 2^{-(n-p-1)b} \cdot 2^{-\gamma a n} \cdot 2^{n-p-1} \\ &= c^\gamma \cdot 2^{-\gamma a(p+1)} \cdot 2^{(n-p-1)(1-b-\gamma a)},\end{aligned}$$

so it follows that there exists $N_1 \in \mathbb{N}$ such that (9) holds with $\varepsilon = 0$ for $n \geq N_1$ and $k = 0$ (corresponding to $m = 0$). For $1 \leq m \leq M$, we have

$$\begin{aligned}&(\xi_{p+1} \cdots \xi_{n-m-1}(1 - 2\xi_{n-m} \cdots \xi_n))^\gamma \cdot 2^{n-p-m-1} \\ &\qquad \geq \left(2^{-(n-p-m-1)b/\gamma} \cdot \frac{1}{2}\right)^\gamma \cdot 2^{n-p-m-1} = 2^{-\gamma} \cdot 2^{(n-p-m-1)(1-b)},\end{aligned}$$

so we can choose $N_2 \in \mathbb{N}$ such that (9) holds with $\varepsilon = 0$ for $n \geq N_2$ and $k = 2^m - 1$ with $1 \leq m \leq M$. Also, for $M + 1 \leq m \leq n - p - 2$, we have

$$\begin{aligned}&(\xi_{p+1} \cdots \xi_{n-m-1}(1 - 2\xi_{n-m} \cdots \xi_n))^\gamma \cdot 2^{n-p-m-1} \\ &\geq \left(2^{-(n-p-m-1)b/\gamma} \cdot \left(1 - 2^{-(m+1)}\right)\right)^\gamma \cdot 2^{n-p-m-1} \\ &\geq 2^{(n-p-m-2)(1-b)} \\ &\geq 1,\end{aligned}$$

and finally, given $\varepsilon > 0$, there exists $N_3 \in \mathbb{N}$ such that

$$(1 - 2\xi_{p+1}\cdots\xi_n)^\gamma \geq \frac{1}{1+\varepsilon/2} \qquad \text{for } n \geq N_3.$$

We thus deduce that, for $n \geq N_4 = \max\{N_1, N_2, N_3\}$,

$$(10) \qquad 2^{-(n-p-m-1)} \leq (1+\varepsilon/2)(\xi_1\cdots\xi_p)^{-\gamma} s(n, 2^m - 1)^\gamma$$

for $0 \leq m \leq n-p-1$.

Now choose $N_5 \in \mathbb{N}$ such that

$$\frac{l_n}{2r_n} = \frac{1-2\xi_n}{2\xi_n} \leq \left(\frac{\varepsilon}{2}\right)^{1/\gamma} \qquad \text{for } n \geq N_5,$$

and let $N = \max\{N_4, N_5\}$. Let $n \geq N$ and let $2^m \leq k \leq 2^{m+1} - 1$ for some m with $0 \leq m \leq n-p-2$. By the previous lemma and (5), we have

$$\frac{s(n,k)}{k} \geq \frac{s(n, 2^{m+1}-1)}{2^{m+1}-1},$$

so we deduce from (10) that

$$\begin{aligned} k2^{-(n-p-1)} &\leq 2^{-(n-p-1)} s(n,k)^\gamma \frac{2^{m+1}-1}{s(n, 2^{m+1}-1)^\gamma} \\ &\leq (1+\varepsilon/2)(\xi_1\cdots\xi_p)^{-\gamma} s(n,k)^\gamma. \end{aligned}$$

Also, by the choice of N_1, we have

$$\begin{aligned} 2^{-(n-p-1)} &\leq (\xi_1\cdots\xi_p)^{-\gamma} s(n,0)^\gamma = (\xi_1\cdots\xi_p)^{-\gamma} l_n^\gamma \\ &\leq (\varepsilon/2)(\xi_1\cdots\xi_p)^{-\gamma}(2r_n)^\gamma \leq (\varepsilon/2)(\xi_1\cdots\xi_p)^{-\gamma} s(n,k)^\gamma, \end{aligned}$$

and (9) follows. □

LEMMA 3.14. — *Let (ξ_n) be a sequence with $0 < \xi_n < \frac{1}{2}$ for $n \in \mathbb{N}$, let $N \in \mathbb{N}_0$ and define the sequence (η_n) by $\eta_n = \xi_{n+N}$ for $n \in \mathbb{N}$. Then $\lambda_\gamma(E_{\underline{\xi}})$ is generated by its idempotents if and only if $\lambda_\gamma(E_{\underline{\eta}})$ is.*

Proof. — Suppose that $\lambda_\gamma(E_{\underline{\xi}})$ is generated by its idempotents. Let $F = [a,b] \subseteq [0,1]$ be a closed interval and let $\varphi(t) = \xi_1\cdots\xi_N t$ for $t \in [0,1]$. Given $\varepsilon > 0$, there exists $g \in \lambda_\gamma$ real-valued with $g(E_{\underline{\xi}} \cap \varphi(F))$ finite, $p_\gamma(g|_{\varphi(F)}) \leq 1+\varepsilon$ and

$$g(\varphi(b)) - g(\varphi(a)) = (\varphi(b) - \varphi(a))^\gamma = (\xi_1\cdots\xi_N)^\gamma (b-a)^\gamma.$$

Since $\varphi(E_{\underline{\eta}} \cap F) = E_{\underline{\xi}} \cap [0, \xi_1\cdots\xi_N]$, it follows that $g\circ\varphi \in \lambda_\gamma$ is real-valued with $(g\circ\varphi)(E_{\underline{\eta}} \cap F)$ finite, $(g\circ\varphi)(b) - (g\circ\varphi)(a) = (\xi_1\cdots\xi_N)^\gamma (b-a)^\gamma$, and it is easily seen that $p_\gamma(g\circ\varphi|_F) \leq (1+\varepsilon)(\xi_1\cdots\xi_N)^\gamma$. Hence $\rho_{E_{\underline{\eta}},\gamma}(F) = m(F)^\gamma$, so we deduce that $\lambda_\gamma(E_{\underline{\eta}})$ is generated by its idempotents.

Now suppose that $\lambda_\gamma(E_{\underline{\eta}})$ is generated by its idempotents. It follows as in the first part of the proof that $\rho_{E_{\underline{\xi}},\gamma}(F) = m(F)^\gamma$ whenever F is a closed interval contained in $[0, \xi_1 \cdots \xi_N]$. For a closed interval $F \subseteq [0,1]$, it thus follows by similarity that $\rho_{E_{\underline{\xi}},\gamma}(E_{Nk} \cap F) = m(E_{Nk} \cap F)^\gamma$ for $k = 1, \dots, 2^N$ (where E_{Nk} corresponds to the set $E_{\underline{\xi}}$). Hence $\rho_{E_{\underline{\xi}},\gamma}(F) = m(F)^\gamma$ by Corollary 3.9, so $\lambda_\gamma(E_{\underline{\xi}})$ is generated by its idempotents. □

We are now ready to prove the existence of a set E of positive measure for which $\lambda_\gamma(E)$ is generated by its idempotents.

Example 3.15. — Let $0 < \gamma < 1$ and suppose that $a > 0$ satisfies $\gamma(a+1) < 1$. Then $\lambda_\gamma(E(a,c))$ is generated by its idempotents for $0 < c < 2^a$.

Proof. — First, suppose that $c \le c_0$ with c_0 as in Lemma 3.13 and let $E = E(a,c)$. Let $p \in \mathbb{N}$ and let $F = [0, \xi_1 \cdots \xi_p]$. For $n \ge p+1$, the interval F contains V_{nk} for $k = 1, \dots, 2^{n-p-1}$. Let $g_n \in \lambda_\gamma$ be a real-valued function which is linear with increase $2^{-(n-p-1)} m(F)^\gamma$ on each V_{nk} ($k = 1, \dots, 2^{n-p-1}$) and is constant on each of the contiguous intervals. Then $g_n(E \cap F)$ is finite and $g_n(\xi_1 \cdots \xi_p) - g_n(0) = m(F)^\gamma$. Furthermore, it follows from Lemma 3.7 that

$$\begin{aligned} p_\gamma(g_n|_F) &= \sup\left\{ \frac{g_n(b_{nk_2}) - g_n(a_{nk_1})}{(b_{nk_2} - a_{nk_1})^\gamma} : 1 \le k_1, k_2 \le 2^{n-p-1} \right\} \\ &= \sup\left\{ \frac{(k+1)2^{-(n-p-1)} m(F)^\gamma}{s(n,k)^\gamma} : 0 \le k \le 2^{n-p-1} - 1 \right\}. \end{aligned}$$

Given $\varepsilon > 0$, it thus follows from Lemma 3.13 that there exists $N \in \mathbb{N}$ such that

$$p_\gamma(g_n|_F) \le 1 + \varepsilon \qquad \text{for } n \ge N,$$

so we conclude that $\rho_{E,\gamma}(F) = m(F)^\gamma$. Hence, by similarity,

$$\rho_{E,\gamma}(F) = m(F)^\gamma \quad \text{when } F = E_{p\cdot}. \tag{11}$$

Now let $F = [x,y] \subseteq [0,1]$ be a closed interval. If $(x,z) \cap E = \emptyset$ for some $z \in (x,y)$ and $F_1 = [z,y]$, then $\rho_{E,\gamma}(F) = m(F)^\gamma$ follows from $\rho_{E,\gamma}(F_1) = m(F_1)^\gamma$ by Corollary 3.9. Also, since E is perfect, we have $z \in \overline{F_1^\circ \cap E}$. Similarly if $(w,y) \cap E = \emptyset$ for some $w \in (x,y)$. Hence we may assume that $x, y \in \overline{F^\circ \cap E}$. Given $\varepsilon > 0$, we can thus choose $n_1, n_2 \in \mathbb{N}$ such that $V_{n_1\cdot}, V_{n_2\cdot} \subseteq F$ with

$$a_{n_1\cdot} \le x + \varepsilon \quad \text{and} \quad b_{n_2\cdot} \ge y - \varepsilon.$$

With $N = \max\{n_1, n_2\}$ and $U = (a_{n_1\cdot}, b_{n_2\cdot})$, we then have

$$U = \left(\bigcup_{E_{N\cdot} \subseteq U} E_{N\cdot} \right) \cup \left(\bigcup_{n \leq N,\ V_{n\cdot} \subseteq U} V_{n\cdot} \right),$$

so it follows from (11) and Corollary 3.9 that

$$\rho_{E,\gamma}(F) \geq \rho_{E,\gamma}(\overline{U}) = m(\overline{U})^\gamma \geq (m(F) - 2\varepsilon)^\gamma.$$

Hence $\rho_{E,\gamma}(F) = m(F)^\gamma$, so it follows from Theorem 3.3 that $\lambda_\gamma(E)$ is generated by its idempotents.

Finally, let $0 < c < 2^a$ and choose $N \in \mathbb{N}_0$ such that $2^{-aN}c \leq c_0$. Then $\lambda_\gamma(E(a, 2^{-aN}c))$ is generated by its idempotents, so it follows from the previous lemma that the same is true for $\lambda_\gamma(E(a,c))$.

4. INDEMPOTENTS IN RESTRICTIONS OF THE ALGEBRA OF ABSOLUTELY CONTINUOUS FUNCTIONS

We round off the paper by characterizing the closed sets $E \subseteq \mathbb{T}$ for which $\mathcal{AC}(E)$ is generated by its idempotents. The result is not surprising, considering that the norm on $\mathcal{AC}$ can "see" sets of positive measure.

THEOREM 4.1. — *Let $E \subseteq \mathbb{T}$ be a closed set. Then $\mathcal{AC}(E)$ is generated by its idempotents if and only if E is of measure zero.*

Proof. — First, suppose that E is of positive measure. By the Cantor-Bendixson theorem, we can write $E = P \cup C$, where P is a perfect, closed set and C is countable. Let $g \in \mathcal{AC}$ and suppose that $g(E)$ is finite. Then $g' = 0$ on P, so it follows that

$$\mathrm{Var}(e^{it} - g) = \int_{\mathbb{T}} |ie^{it} - g'(t)|\, dt \geq \int_P dt = m(E) > 0.$$

Combined with Lemma 1.2, this shows that $e^{it}|_E$ does not belong to the closed linear span of idempotents in $\mathcal{AC}(E)$.

Conversely, suppose that E is of measure zero and write $\mathbb{T} \setminus E = \bigcup_{n=1}^{\infty} V_n$, where (V_n) is a sequence of pairwise disjoint, open intervals. Let $f \in \mathcal{AC}$ and let $\varepsilon > 0$. Since E is of measure zero, we have

$$\mathrm{Var}(f) = \int_{\mathbb{T}} |f'(t)|\, dt = \sum_{n=1}^{\infty} \mathrm{Var}(f|_{V_n}).$$

Choose $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} \mathrm{Var}(f|_{V_n}) < \varepsilon$, and let $g = f$ on $\bigcup_{n=1}^{N} V_n$. Let $F_1, \ldots, F_N$ be the closed intervals constituting $\mathbb{T} \setminus \bigcup_{n=1}^{N} V_n$ and let $n \in \{1, \ldots, N\}$. If F_n is a singleton, then let $g = f$ on F_n. Otherwise, there exists $m_n \in \mathbb{N}$ such that $F_n \supseteq V_{m_n} = (a_{m_n}, b_{m_n})$. We then let g be the continuous function on $F_n = [x_n, y_n]$ which equals $f(x_n)$ on $[x_n, a_{m_n}]$, equals $f(y_n)$ on $[b_{m_n}, y_n]$ and which is linear on V_{m_n}. Then $\mathrm{Var}(g|_{F_n}) = |f(y_n) - f(x_n)| \leq \mathrm{Var}(f|_{F_n})$. In this way we obtain $g \in \mathcal{AC}$ with $g(E)$ finite. Hence $g|_E$ is a linear combination of idempotents in $\mathcal{AC}(E)$ by Lemma 1.2. Furthermore, since E is of measure zero and since $g = f$ on $\bigcup_{n=1}^{N} V_n$, it follows that

$$\begin{aligned}\mathrm{Var}(f - g) &= \sum_{n=1}^{N} \mathrm{Var}((f-g)|_{F_n}) \leq 2 \sum_{n=1}^{N} \mathrm{Var}(f|_{F_n}) \\ &= 2 \sum_{n=N+1}^{\infty} \mathrm{Var}(f|_{V_n}) < 2\varepsilon.\end{aligned}$$

Also, $\|f - g\|_\infty < 2\varepsilon$, so we deduce that $\mathcal{AC}(E)$ is generated by its idempotents. □

Let $E \subseteq \mathbb{T}$ be a closed set. It follows from Lemma 1.2 that the idempotents in $\mathcal{BVC}(E)$ belong to $\mathcal{AC}(E)$. Since $\mathcal{AC}(E)$ is closed in $\mathcal{BVC}(E)$, we thus deduce from the previous theorem that $\mathcal{BVC}(E)$ is generated by its idempotents if and only if E is of measure zero and $\mathcal{BVC}(E) = \mathcal{AC}(E)$. We shall show that this only holds for closed, countable sets. We need the following result.

LEMMA 4.2. — *Every non-empty, closed, perfect set $P \subseteq \mathbb{T}$ contains a non-empty, closed, perfect set of measure zero.*

Proof. — The result is clear if P contains an interval, so we may assume that P has empty interior. We can then write

$$P = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2^n} P_{nk},$$

where $P_{n1}, \ldots, P_{n2^n}$ are closed, disjoint intervals with $P_{n+1,2k-1} \cup P_{n+1,2k} \subseteq P_{n,k}$ for $k = 1, \ldots, 2^n$ and $n \in \mathbb{N}$, and where $\rho_n = \max\{m(P_{nk}) : k = 1, \ldots, 2^n\} \to 0$ as $n \to \infty$. Choose an increasing sequence (m_n) of natural numbers with $m_1 = 2$ and $m_{n+1} \leq 2m_n$ for $n \in \mathbb{N}$ such that $m_n \to \infty$ and

$m_n\rho_n \to 0$ as $n \to \infty$. For $n \in \mathbb{N}$, choose m_n of the intervals $P_{n1}, \ldots, P_{n2^n}$ (only choosing empty sets if nothing else is left), written $P'_{n1}, \ldots, P'_{nm_n}$, in such a way that each $P'_{n\cdot}$ contains at least one $P'_{n+1,\cdot}$ and such that each $P'_{n+1,\cdot}$ is contained in a $P'_{n\cdot}$. Let

$$P' = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{m_n} P'_{nk}.$$

Then P' is a non-empty, closed, perfect set of measure zero. □

PROPOSITION 4.3. — *Let $E \subseteq \mathbb{T}$ be a closed set. Then $\mathcal{BVC}(E) = \mathcal{AC}(E)$ if and only if E is countable.*

Proof. — First, suppose that E is countable and let $\mathbb{T} \setminus E = \bigcup_{n=1}^{\infty} V_n$, where (V_n) is a sequence of pairwise disjoint, open intervals. Let $f \in \mathcal{BVC}$ and let g be the continuous function on $\mathbb{T}$ that satifies $g = f$ on E and is linear on each V_n. Then $g \in \mathcal{BVC}$ (with $\mathrm{Var}(g) \leq \mathrm{Var}(f)$). Also, if $F \subseteq \mathbb{T}$ is of measure zero, then $g(F \cap E)$ is countable and $g(F \cap V_n)$ is of measure zero (since g is linear on V_n) for $n \in \mathbb{N}$. Hence $g(F)$ is of measure zero, so we conclude that $g \in \mathcal{AC}$. Consequently $f|_E = g|_E \in \mathcal{AC}(E)$. Conversely, suppose that E is uncountable. Then E contains a non-empty, closed, perfect set P which we may assume has measure zero, by the previous lemma. We may also assume that $P \subseteq [0, 2\pi - \varepsilon]$ for some $\varepsilon > 0$. Let f be a Cantor-Lebesgue function for the set P, that is, a real-valued, continuous function on $\mathbb{T}$ which is increasing on $[0, 2\pi - \varepsilon]$, constant on each component of $\mathbb{T} \setminus P$ and satisfies $f(2\pi - \varepsilon) - f(0) > 0$. Then $f \in \mathcal{BVC}$, whereas $f(E) = f(P)$ has positive measure, so that $f|_E \notin \mathcal{AC}(E)$. □

Added in proof: After the submission of this paper it was pointed out to us by J.-P. Kahane and R. Kaufman that Körner ([T.W. Körner, "On the theorem of Ivasev-Musatov.I", Ann. Inst. Fourier 27(3), 1977, pages 97-115], Theorem 1.2) has shown the existence of a perfect, closed set $E \subseteq \mathbb{T}$ of measure zero and a non-zero measure μ with support contained in E such that $\widehat{\mu}(n) = O(|n|^{-1/2})$ and $|n| \to \infty$. Hence Corollary 2.8 remains valid for $\beta = \frac{1}{2}$.

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Manuscrit reçu le 26 juin 1995,
accepté le 12 avril 1996.

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