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DISTRIBUTIVE LAWS AND KOSZULNESS

by Martin MARKL

Introduction.

The basic motivation for our work was the following result of Getzler and Jones [5]. Let $\mathcal{C}_n = \{\mathcal{C}_n(m);\ m \geq 1\}$ be the little n-cubes operad of Boardman and Vogt [8, Definition 4.1] and let $\mathbf{e}_n = \{\mathbf{e}_n(m);\ m \geq 1\}$ be its homology operad, $\mathbf{e}_n(m) := H(\mathcal{C}_n(m))$. Then the operad $\mathbf{e}_n$ is Koszul in the sense of [6, Definition 4.1.3].

Both the little cubes operad and the operad $\mathbf{e}_n$ are intimately related to configuration spaces, namely, $\mathbf{e}_n(m) = H(\mathbf{F}_n(m))$, where $\mathbf{F}_n(m)$ denotes the space of configurations of m distinct points in $\mathbf{R}^n$. In their original proof of the above mentioned statement, Getzler and Jones used the Fulton-MacPherson compactification [3] F_n of $\mathbf{F}_n$. Each $\mathsf{F}_n(m)$ is a real smooth manifold with corners and the basic trick in their proof was to replace, using a spectral sequence associated with the stratification of $\mathsf{F}_n(m)$, the homological definition of the Koszulness by a purely combinatorial property of the structure of the strata of $\mathsf{F}_n(m)$.

The operad $\mathbf{e}_n$ describes so-called n-algebras (in the terminology of [5]) which are, roughly speaking, Poisson algebras where the Lie bracket is of degree $n-1$, in particular, n-algebras are algebras with a *distributive law.* As we already know from our previous work with T. Fox [2], a distributive law induces a spectral sequence for the related cohomology.

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This observation stimulated us to look for an alternative, purely algebraic and more conceptual proof of the above mentioned theorem of Getzler and Jones.

Distributive laws were introduced and studied, in terms of triples, by J. Beck in [1]. They provide a way of composing two algebraic structures into a more complex one. For example, a Poisson algebra structure on a vector space V consists of a Lie algebra bracket $[-,-]$ (denoted sometimes more traditionally as $\{-,-\}$) and of an associative commutative multiplication $\cdot$. These two operations are related by a 'distributive law' $[a \cdot b, c] = a \cdot [b, c] + [a, c] \cdot b$.

Our first aim will be to understand distributive laws in terms of operads. A distributive law for an operad will be given by a certain map which has to satisfy a very explicit and verifiable compatibility condition, see Definition 2.2. We then observe that an algebra over an operad with a distributive law is an algebra with a distributive law in the sense of J. Beck.

Our next aim will be to prove that an operad $\mathcal{C}$ constructed from operads $\mathcal{A}$ and $\mathcal{B}$ via a distributive law is Koszul if $\mathcal{A}$ and $\mathcal{B}$ are (Theorem 4.5). As an immediate corollary we get the above mentioned result of Getzler and Jones (Corollary 4.6).

Statements about operads are usually motivated by the corresponding statements about associative algebras. This phenomenon may work also in the opposite direction: the definition of a distributive law for operads (motivated by the Beck's definition for triples) led us to the definition of a 'distributive law for associative algebras' as of a process which ties together two associative algebras into a third one. This gives a method to construct new examples of Koszul algebras. We do not follow this direction here, a full exposition can be found in the preliminary version of this paper, which is available from `hepth@xxx.lanl.gov` as preprint #9409192.

1. Basic notions.

We will keep the following convention throughout the paper. Capital roman letters (A, B, ...) will denote associative algebras, calligraphic letters ($\mathcal{A}$, $\mathcal{B}$, ...) will denote operads, 'typewriter' capitals ($\mathtt{T}$, $\mathtt{S}$, ...) will denote trees and, finally, 'sans serif' capitals (T, S, ...) will denote triples. All algebraic objects are assumed to be defined over a fixed field $\mathbf{k}$ which is, to make the life easier, supposed to be of characteristic zero.

As the notion of operad and of algebra over an operad have already become common knowledge, we just fix the necessary notation. By a *nonsymmetric operad* we mean a nonsymmetric operad in the monoidal category of graded vector spaces, *i.e.* a sequence $\mathcal{S} = \{\mathcal{S}(n); n \geq 1\}$ of graded vector spaces together with degree zero linear maps

$$\gamma = \gamma_{m_1,\ldots,m_l} : \mathcal{S}(l) \otimes \mathcal{S}(m_1) \otimes \cdots \otimes \mathcal{S}(m_l) \longrightarrow \mathcal{S}(m_1 + \cdots + m_l),$$

given for any $l, m_1, \ldots, m_l \geq 1$, satisfying the usual axioms [8, Definition 3.12]. We also suppose the existence of a unit $1 \in \mathcal{S}(1)$ having the usual property. Similarly, a *symmetric operad* will be an operad in the symmetric monoidal category of graded vector spaces, *i.e.* a structure consisting of the above data plus an action of the symmetric group Σ_n on $\mathcal{S}(n)$ given for any $n \geq 2$, which has again to satisfy the usual axioms [8, Definition 1.1]. We always assume that $\mathcal{S}(1) = \mathbf{k}$ and that algebra structure on $\mathbf{k}$ coincides, under this identification, with the algebra structure on $\mathcal{S}(1)$ induced from the operad structure of $\mathcal{S}$.

We will try to discuss both symmetric and nonsymmetric cases simultaneously whenever possible. We also will not mention explicitly the grading given by the grading of underlying vector spaces if not necessary. For an operad $\mathcal{S}$ we often use also the 'nonunital' notation based on the composition maps $- \circ_i - : \mathcal{S}(m) \otimes \mathcal{S}(n) \to \mathcal{S}(m+n-1)$, given, for any $m, n \geq 1$, $1 \leq i \leq m$, by

$$\mu \circ_i \nu := \gamma(\mu; 1, \ldots, 1, \nu, 1, \ldots, 1) \quad (\nu \text{ at the } i\text{-th place}).$$

These maps again have to satisfy certain axioms which can be found in [7].

By a *collection* we mean a sequence $E = \{E(n);\ n \geq 2\}$ of graded vector spaces; in the symmetric case we suppose moreover that each $E(n)$ is equipped with an action of the symmetric group Σ_n. For any collection E there exists the *free operad* $\mathcal{F}(E)$ on E [6, page 226]; a very explicit description in terms of trees is given in 1.3.

1.1. Suppose the collection E decomposes as $E = \bigoplus_{i=1}^{N} E_i$, meaning, of course, that $E(m) = \bigoplus_{i=1}^{N} E_i(m)$ for each $m \geq 2$, the decomposition being Σ_m-invariant in the symmetric case. Then $\mathcal{F}(E)$ is naturally N-multigraded, $\mathcal{F}(E) = \bigoplus_{i_1,\ldots,i_N} \mathcal{F}_{i_1,\ldots,i_N}(E)$, with the multigrading characterized by the following two properties:

(i) $\mathcal{F}_{0,\ldots,0}(E) = \mathcal{F}(E)(1) = \mathbf{k}$ and $E_i = \mathcal{F}_{0,\ldots,0,1,0,\ldots,0}(E)$ (1 at the i-th place), $1 \leq i \leq N$.

(ii) Let $m, n \geq 1$, $1 \leq l \leq m$ and let $a \in \mathcal{F}_{i_1,\dots,i_N}(E)(m)$, $b \in \mathcal{F}_{j_1,\dots,j_N}(E)(n)$. Then $a \circ_l b \in \mathcal{F}_{k_1,\dots,k_N}(E)(m+n-1)$ with $k_i = i_i + j_i$ for $1 \leq i \leq N$.

Especially, the trivial decomposition of E gives the grading $\mathcal{F}(E) = \bigoplus_{i \geq 0} \mathcal{F}_i(E)$.

1.2. By a *presentation* of an operad $\mathcal{S}$ we mean a collection E and a subcollection $R \subset \mathcal{F}(E)$ such that $\mathcal{S} = \mathcal{F}(E)/(R)$, where (R) is the ideal generated by R in $\mathcal{F}(X)$. We write $\mathcal{S} = \langle E; R \rangle$. An operad $\mathcal{S}$ is *quadratic* if there exists a collection E with $E(n) = 0$ for $n \neq 2$, and a subcollection $R \subset \mathcal{F}(E)(3)$ such that $\mathcal{S} = \langle E; R \rangle$. Quadratic operads are naturally graded, $\mathcal{S} = \bigoplus_{i \geq 0} \mathcal{S}_i$, where the pieces $\mathcal{S}_n$ are described as follows. Let $\tilde{R}$ be an identical copy of the collection R and let $h : \mathcal{F}(E \oplus \tilde{R}) \to \mathcal{F}(E)$ be the map induced by the obvious map $\iota : E \oplus \tilde{R} \to \mathcal{F}(E)$ of collections. Then $\mathcal{S}_n := \mathcal{F}_n(E)/h(\mathcal{F}_{n-2,1}(E \oplus \tilde{R}))$. For a more explicit description, see 1.4.

1.3. There exists a useful way to describe free operads using trees [6], [5]. In the *nonsymmetric case* we shall use the set $\mathcal{T}$ of planar trees. By $\mathcal{T}_n$ we denote the subset of $\mathcal{T}$ consisting of trees having n input edges. Let $\mathbf{v}(\mathrm{T})$ denote the set of vertices of a tree $\mathrm{T} \in \mathcal{T}$ and let, for $v \in \mathbf{v}(\mathrm{T})$, $\mathrm{val}(v)$ denote the number of input edges of v. For a collection $E = \{E(n); n \geq 2\}$ we put

$$E(\mathrm{T}) := \bigotimes_{v \in \mathbf{v}(\mathrm{T})} E(\mathrm{val}(v)).$$

We may interpret the elements of $E(\mathrm{T})$ as 'multilinear' colorings of the vertices of T by the elements of E. The free operad $\mathcal{F}(E)$ on E may be then defined as

$$\mathcal{F}(E)(n) := \bigoplus_{\mathrm{T} \in \mathcal{T}_n} E(\mathrm{T}) \tag{1}$$

with the operad structure on $\mathcal{F}(E)$ given by the operation of 'grafting' trees. In the *symmetric case* we shall work with the set of (abstract) trees with input edges indexed by finite ordered sets. The formulas for $E(\mathrm{T})$ and $\mathcal{F}(E)$ are similar but involve also the symmetric group action, the details may be found in [6], [5]. As mentioned earlier, we try to discuss both the symmetric and nonsymmetric cases simultaneously whenever possible. In the special case when $E(m) = 0$ for $m \neq 2$, the summation in (1) reduces to the summation over the subset $\mathcal{T}_n^2 \subset \mathcal{T}_n$ consisting of *binary* trees, *i.e.* trees T with $\mathrm{val}(v) = 2$ for any vertex $v \in \mathbf{v}(\mathrm{T})$.

1.4. Let $\mathcal{S} = \langle E; R\rangle$ be a quadratic operad as in 1.2 and recall that $\mathcal{S}$ is graded, $\mathcal{S} = \bigotimes_{i\geq 0} \mathcal{S}_i$. Let $\mathcal{T}_n^{2,3_1}$ denote the set of 1-ternary binary n-trees, *i.e.* n-trees whose all vertices have two incoming edges except exactly one which has three incoming edges. Then we have for the collection $\mathcal{F}_{n-2,1}(E \oplus \tilde{R})$ from 1.2

$$\mathcal{F}_{n-2,1}(E \oplus \tilde{R}) = \bigoplus_{\mathrm{S}\in \mathcal{T}_{n+1}^{2,3_1}} \tilde{R}_{\mathrm{S}},$$

where we denoted $\tilde{R}_{\mathrm{S}} := (E \oplus \tilde{R})(\mathrm{S})$. We may interpret the elements of $\tilde{R}_{\mathrm{S}}$ as 'multilinear' colorings of S such that binary vertices are colored by elements of E and the only ternary vertex is colored by an element of R. Denote finally $R_{\mathrm{S}} := h(\tilde{R}_{\mathrm{S}})$. Then $\mathcal{S}_n = \mathcal{F}_n(E)/\operatorname{Span}(R_{\mathrm{S}};\ \mathrm{S} \in \mathcal{T}_{n+1}^{2,3_1})$. This type of description was given, for the symmetric case, in [6, page 262].

1.5. Let U and V be two collections and $E := U \oplus V$. There is an alternative way to describe the free operad $\mathcal{F}(U, V) := \mathcal{F}(U\oplus V)$ resembling the description of the free associative algebra $F(X \oplus Y)$ as the free product of $F(X)$ and $F(Y)$. Let $\mathcal{T}^{wb}$ be the set of 2-colored trees. This means that the elements of $\mathcal{T}^{wb}$ are trees (planar in the nonsymmetric case, abstract in the symmetric case) whose vertices are colored by two colors ('w' from white, 'b' from black). For $\mathrm{T} \in \mathcal{T}^{wb}$ let $\mathbf{v}_w(\mathrm{T})$ (resp. $\mathbf{v}_b(\mathrm{T})$) denote the set of white (resp. black) vertices of T. Let $(U, V)(\mathrm{T})$ be the subset of $E(\mathrm{T})$ defined as

$$(U, V)(\mathrm{T}) := \bigotimes_{v\in \mathbf{v}_w(\mathrm{T})} U(\mathrm{val}(v)) \otimes \bigotimes_{v\in \mathbf{v}_b(\mathrm{T})} V(\mathrm{val}(v)).$$

Then we may define $\mathcal{F}(U, V)(n)$ as $\mathcal{F}(U, V)(n) := \bigoplus_{\mathrm{T}\in\mathcal{T}_n^{wb}} (U, V)(\mathrm{T})$. If the collections U and V are quadratic, the summation reduces to the summation over the subset $\mathcal{T}^{wb,2}$ of 2-colored *binary* trees.

1.6. Recall that a tree is, by definition, an *oriented* graph. Each edge e has an output vertex $\mathrm{out}(e)$ and an input vertex $\mathrm{inp}(e)$. This induces, by $\mathrm{inp}(e) \prec \mathrm{out}(e)$, a partial order $\prec$ on the set $\mathbf{v}(\mathrm{T})$ of vertices of T. For a tree $\mathrm{T} \in \mathcal{T}^{wb}$ define $I(\mathrm{T})$ to be the number of all couples (v_1, v_2), $v_1 \in \mathbf{v}_b(\mathrm{T})$ and $v_2 \in \mathbf{v}_w(\mathrm{T})$, such that $v_2 \prec v_1$.

By a differential graded (dg) collection we mean a collection $E = \{E(n); n \geq 2\}$ such that each $E(n)$ is endowed with a differential $d_E = d_E(n)$ which is, in the symmetric case, supposed to commute with the symmetric group action. For such a dg collection we define its (co)homology collection as $H(E) := \{H(E(n), d_E(n)); n \geq 2\}$. Let $U = \{(U(n), d_U(n)); n \geq 2\}$ and $V = \{(V(n), d_V(n)); n \geq 2\}$ be two dg collections. Let $E = U \oplus V$.

Denote by $U \odot V$ the subcollection of $\mathcal{F}(E)$ generated by (= the smallest subcollection containing) elements of the form $\gamma(u, v_1, \dots, v_m)$, $u \in U(m)$ and $v_i \in V(n_i)$, $1 \leq i \leq m$. Define on $U \odot V$ the differential $d_{U\odot V}$ by

$$d_{U\odot V}(\gamma(u; v_1, \dots, v_k)) := \gamma(d_U(u); v_1, \dots, v_k) + (-1)^{\deg(u)} \sum_{i=1}^{k} (-1)^{i+1} \gamma(u; v_1, \dots, d_V(v_i), \dots, v_k).$$

It can be easily verified that this formula introduces a monoidal structure on the category of dg collections. We formulate the following variant of the Künneth theorem; recall that we assume the ground field $\mathbf{k}$ to be of characteristic zero.

PROPOSITION 1.7. — *There exists a natural isomorphism of collections, $H(U \odot V) \cong H(U) \odot H(V)$.*

Proof. — In the nonsymmetric case we have the decomposition

$$(U \odot V)(m) = \bigoplus (U \odot V)(l; k_1, \dots, k_l), \tag{2}$$

where $(U\odot V)(l; k_1, \dots, k_l) := U(l) \otimes V(k_1) \otimes \cdots \otimes V(k_l)$ and the summation is taken over all $l \geq 2$ and $k_1 + \cdots k_l = m$. The differential $d_{U\odot V}$ obviously preserves the decomposition and agrees on $(U \odot V)(l; k_1, \dots, k_l)$ with the usual tensor product differential on $U(l) \otimes V(k_1) \otimes \cdots \otimes V(k_l)$. The classical Künneth theorem then gives the result.

For the symmetric case we have the same decomposition as in (2), but the summation is now taken over all $l \geq 2$ and $k_1 + \cdots + k_l = m$ *with* $k_1 \leq k_2 \leq \cdots \leq k_l$, and $(U \odot V)(l; k_1, \dots, k_l)$ is defined as $\mathrm{Ind}^{\Sigma_m}_{\Sigma_{k_1} \times \cdots \times \Sigma_{k_l}} (U(l) \otimes V(k_1) \otimes \cdots \otimes V(k_l))$ where $\Sigma_{k_1} \times \cdots \times \Sigma_{k_l}$ acts on Σ_m via the canonical inclusion and $\mathrm{Ind}^{\Sigma_m}_{\Sigma_{k_1} \times \cdots \times \Sigma_{k_l}}(-)$ denotes the induced action. Since $\mathrm{char}(\mathbf{k}) = 0$, 'the (co)homology commutes with finite group actions' and we may use the same arguments as in the nonsymmetric case. $\square$

2. Distributive laws.

2.1. Let $\mathcal{A} = \langle U; S\rangle$ and $\mathcal{B} = \langle V; T\rangle$ be two quadratic operads. Let $V \bullet U$ denote the subcollection of $\mathcal{F}(U, V)$ generated by elements of the form $\gamma(v; u, 1)$ or $\gamma(v; 1, u)$, $u \in U$ and $v \in V$. Clearly $(V \bullet U)(m) = 0$ for $m \neq 3$. The notation $U \bullet V$ has the obvious similar meaning. Suppose we have a map $d : V \bullet U \to U \bullet V$ of collections and let $D := \{z - d(z);\ z \in V \bullet U\} \subset \mathcal{F}(U, V)(3)$ and $\mathcal{C} := \langle U, V; S, D, T\rangle$ (= an abbreviation for $\langle U \oplus V; S \oplus D \oplus T\rangle$). The inclusion $\mathcal{F}(U) \odot \mathcal{F}(V) \subset \mathcal{F}(U, V)$ induces a map $\xi : \mathcal{A} \odot \mathcal{B} \to \mathcal{C}$ of collections. The collection $\mathcal{F}(U, V)$ is bigraded (see 1.1) and the relations S, T and D obviously preserve this bigrading, hence the operad $\mathcal{C}$ is naturally bigraded as well. Also the collection $\mathcal{A}\odot\mathcal{B}$ is bigraded: $(\mathcal{A} \odot \mathcal{B})_{i,j}$ is generated by elements of the form $\gamma(a; b_1, \dots, b_{i+1})$, $a \in \mathcal{A}_i$ $(= \mathcal{A}(i + 1))$ and $b_k \in \mathcal{B}_{j_k}$ $(= \mathcal{B}(j_k + 1))$ for $1 \leq k \leq i + 1$ and $j_1 + \cdots + j_{i+1} = j$. We write more suggestively $\mathcal{A}_i \odot \mathcal{B}_j$ instead of $(\mathcal{A} \odot \mathcal{B})_{i,j}$, abusing the notation a bit.

The map ξ introduced above obviously preserves the bigrading, $\xi(\mathcal{A}_i \odot \mathcal{B}_j) \subset \mathcal{C}_{i,j}$, and we put $\xi_{i,j} := \xi|_{\mathcal{A}_i \odot \mathcal{B}_j}$.

DEFINITION 2.2. — *The map $d : V \bullet U \to U \bullet V$ defines a distributive law if*

(3) $\quad \xi_{i,j} : \mathcal{A}_i \odot \mathcal{B}_j \to \mathcal{C}_{i,j}$ *is an isomorphism for* $(i, j) \in \{(1, 2), (2, 1)\}$.

The main result of this section is the following 'coherence theorem.'

THEOREM 2.3. — *Suppose d is a distributive law as in Definition 2.2. Then the map $\xi_{i,j} : \mathcal{A}_i \odot \mathcal{B}_j \to \mathcal{C}_{i,j}$ is an isomorphism for all (i, j).*

Proof. — It is clear that $\xi_{i,j}$ is an epimorphism, the difficult part is to prove that it is a monomorphism. We must prove that $a \in \mathcal{F}_i(U)\odot\mathcal{F}_j(V) \subset \mathcal{F}_{i,j}(U, V)$ is zero mod (S, D, T) if and only if it is zero mod (S, T).

Let $\mathcal{T}^{wb,2}(i, j)$ denote the subset of $\mathcal{T}^{wb,2}$ consisting of trees having exactly i white and j black vertices; we observe that $\mathcal{F}_{i,j}(U, V) = \bigoplus_{\mathrm{T} \in \mathcal{T}^{wb,2}(i,j)} (U, V)(\mathrm{T})$, see 1.5 for the notation.

Let $\mathrm{T} \in \mathcal{T}^{wb,2}(i, j)$, $v \in \mathbf{v}(\mathrm{T})$ and ϵ be an input edge of v. We say that the couple (v, ϵ) is T-*admissible* if $v \in \mathbf{v}_b(\mathrm{T})$ and $\mathrm{inp}(\epsilon) \in \mathbf{v}_w(\mathrm{T})$.

Sometimes we also say that (v,ϵ) is b-admissible if $b \in (U,V)(\mathrm{T})$ and if (v,ϵ) is T-admissible.

Let us suppose that (v,ϵ) is T-admissible. Let us denote by S the minimal binary subtree of T containing v and $w := \mathrm{inp}(\epsilon)$. Clearly $\mathrm{S} \in \mathcal{T}_3^{wb,2}(1,1)$ and $I(\mathrm{S}) = 1$ (for the definition of $I(\mathrm{S})$ see 1.6). Let $b \in (U,V)(\mathrm{T})$ be of the form $b = \bigotimes_{v\in \mathbf{v}_w(\mathrm{T})} u_v \otimes \bigotimes_{v\in \mathbf{v}_b(\mathrm{T})} v_v$ for some $u_v \in U$ and $v_v \in V$. We call elements of this form monomials and we observe that monomials generate $(U,V)(\mathrm{T})$. For a monomial b as above let $\bar{b}_{\mathrm{S}} \in (U,V)(\mathrm{S}) \subset V \bullet U$ be defined as $\bar{b}_{\mathrm{S}} := \bigotimes_{v\in \mathbf{v}_w(\mathrm{S})} u_v \otimes \bigotimes_{v\in \mathbf{v}_b(\mathrm{S})} v_v$ (observe however that both $\mathbf{v}_w(\mathrm{S})$ and $\mathbf{v}_b(\mathrm{S})$ consist of one element). Let $\Xi := \{\mathrm{R} \in \mathcal{T}_3^{wb,2}(1,1); I(\mathrm{R}) = 0\}$, we note that Ξ consist of exactly two (resp. three) trees in the symmetric (resp. nonsymmetric) case. Then $d(\bar{b}_{\mathrm{S}}) = \sum_{\mathrm{R}\in\Xi} \bar{b}'_{\mathrm{R}}$ for some $\bar{b}'_{\mathrm{R}} \in (U,V)(\mathrm{R}) \subset U \bullet V$. Let T_{R} denote the tree obtained from T by replacing the subtree S by R; observe that $I(\mathrm{T}_{\mathrm{R}}) < I(\mathrm{T})$. Let finally $b'_{\mathrm{R}} \in (U,V)(\mathrm{T}_{\mathrm{R}})$ be, for $\mathrm{R} \in \Xi$, an element obtained by substituting $\bar{b}'_{\mathrm{R}}$ to b at the vertices of R. Let us define $d(v,\epsilon)(b) := \sum_{\mathrm{R}\in\Xi} b'_{\mathrm{R}} \in \mathcal{F}_{i,j}(U,V)$ and let us extend this definition linearly (and equivariantly in the symmetric case) to the whole $(U,V)(\mathrm{T})$. Loosely speaking, $d(v,\epsilon)(b)$ is obtained from b by making the 'surgery' prescribed by the distributive law at the couple (v,ϵ).

Under the above notation the condition $a = 0 \bmod (S,D,T)$ means that there exist a finite set K, trees $\mathrm{T}_\kappa \in \mathcal{T}^{wb,2}(i,j)$, elements $a_\kappa \in (U,V)(\mathrm{T}_\kappa)$, vertices $v_\kappa \in \mathbf{v}(\mathrm{T}_\kappa)$ and edges ϵ_κ such that $(v_\kappa,\epsilon_\kappa)$ is T_k-admissible and

$$a = \sum_{\kappa\in K} (a_\kappa - d(v_\kappa,\epsilon_\kappa)(a_\kappa)) \bmod (S,T). \tag{4}$$

We say that $a = 0 \bmod_N (S,T)$ if $\max\{I(\mathrm{T}_\kappa);\ \kappa \in K\} \le N$. Theorem 2.3 will obviously follow from the following lemma.

LEMMA 2.4. — *If $a = 0 \bmod_N (S,T)$ for some $N \ge 1$, then $a = 0 \bmod_{N-1} (S,T)$.*

Before proving the lemma, we formulate and prove the following statement.

CLAIM 2.5. — *Let $b \in (U,V)(\mathrm{T})$, $I(\mathrm{T}) = N$ and suppose (v_1,ϵ_1) and (v_2,ϵ_2) are two b-admissible couples. Then*

$$b - d(v_1,\epsilon_1)(b) = b - d(v_2,\epsilon_2)(b) \bmod_{N-1} (S,T). \tag{5}$$

Proof of the claim. — The claim is trivial for $(v_1,\epsilon_1)=(v_2,\epsilon_2)$, so suppose $(v_1,\epsilon_1)\neq(v_2,\epsilon_2)$.

Let us discuss the case $v_1\neq v_2$ first. Then (v_2,ϵ_2) is admissible for each monomial in $d(v_1,\epsilon_1)(b)$ because the rearrangements made by $d(v_1,\epsilon_1)$ does not change the vertex v_2 and the edge ϵ_2. Similarly, (v_1,ϵ_1) is admissible for each monomial in $d(v_2,\epsilon_2)(b)$. We also notice that $d(v_1,\epsilon_1)d(v_2,\epsilon_2)(b)=d(v_2,\epsilon_2)d(v_1,\epsilon_1)(b)$. We therefore have

$$\begin{aligned}(\mathbb{1}-d(v_1,\epsilon_1))(b)=(\mathbb{1}-d(v_2,\epsilon_2))(b)+(\mathbb{1}-d(v_1,\epsilon_1))d(v_2,\epsilon_2)(b)\\ -(\mathbb{1}-d(v_2,\epsilon_2))d(v_1,\epsilon_1)(b),\end{aligned}$$

which implies (5).

Suppose $v_1=v_2=:v$, then, of course, $\epsilon_1\neq\epsilon_2$. Let $w_1=\mathrm{inp}(\epsilon_1)$ and $w_2=\mathrm{inp}(\epsilon_2)$. Let T' be the smallest binary subtree of T containing v,w_1 and w_2. Obviously $\mathrm{T}'\in\mathcal{T}_4^{wb,2}(2,1)$ and $I(\mathrm{T}')=2$. Because the surgery made by both $d(v_1,\epsilon_1)$ and $d(v_2,\epsilon_2)$ takes place inside T', we may suppose that in fact $\mathrm{T}'=\mathrm{T}\in\mathcal{T}_4^{wb,2}(2,1)$. Then $b\in\mathcal{F}_1(V)\odot\mathcal{F}_2(U)$ and $d(v_1,\epsilon_1)(b)=\sum_{\mathrm{R}\in\Omega_0}\bar{b}_{\mathrm{R}}+\sum_{\mathrm{R}\in\Omega_1}\bar{b}_{\mathrm{R}}$, where $\Omega_i=\{\mathrm{R}\in\mathcal{T}_4^{wb,2}(2,1);I(\mathrm{R})=i\}$, $\bar{b}_{\mathrm{R}}\in(U,V)(\mathrm{R})$, $i=1,2$. Applying on the summands of the second sum the distributive law once again (which can be done in exactly one way as there is exactly one admissible couple for any $\mathrm{R}\in\Omega_1$) we obtain some $b'\in\mathcal{F}_2(U)\odot\mathcal{F}_1(V)$, $b'=b\bmod(S,D,T)$. By the same process with the rôles of $d(v_1,\epsilon_1)$ and $d(v_2,\epsilon_2)$ interchanged we construct another element $b''\in\mathcal{F}_2(U)\odot\mathcal{F}_1(V)$. But $b'=b''\bmod(S,T)$, by (3) with $(i,j)=(2,1)$. This finishes the proof of the claim. □

Proof of Lemma 2.4. — Relation (5) says that the concrete values of the couples $(v_\kappa,\epsilon_\kappa)$ in (4) are not substantial. Let us denote by $\mathcal{T}^{wb,2,w3_1}$ the set of 2-colored binary 1-ternary trees such that the ternary vertex is white. The notation $\mathcal{T}^{wb,2,b3_1}$ will have the obvious similar meaning.

Let $K_N:=\{\kappa\in K;\ I(\mathrm{T}_\kappa)=N\}$. Then necessarily $a_N:=\sum_{\kappa\in K_N}a_\kappa=0\bmod(S,T)$ which means that $a_N=\sum_{\omega\in\Omega}a^S_\omega+\sum_{\delta\in\Delta}a^T_\delta$, where a^S_ω is an element of S_{S_ω}, $\mathrm{S}_\omega\in\mathcal{T}^{wb,2,w3_1}$ and, similarly, a^T_δ is an element of T_{S_δ}, $\mathrm{S}_\delta\in\mathcal{T}^{wb,2,b3_1}$; see 1.4 for the notation.

Let us discuss the term a^S_ω for a fixed $\omega\in\Omega$ first. Suppose there exists a black vertex $v\in\mathbf{v}_b(\mathrm{S}_\omega)$ and an edge ϵ with $\mathrm{out}(\epsilon)=v$ such that $w:=\mathrm{inp}(\epsilon)$ is white and binary. Then obviously $d(v,\epsilon)(a^S_\omega)$ makes sense and $d(v,\epsilon)(a^S_\omega) = 0 \bmod (S)$ since $d(v,\epsilon)$ does not change the ternary vertex of S_ω. We can delete $a^S_\omega - d(v,\epsilon)(a^S_\omega)$ from the right-hand side of (4).

Suppose that the only edge ϵ of S_ω such that $\mathrm{out}(\epsilon)$ is black is the output edge of the ternary white vertex v_3. Let us pick this edge ϵ and denote $v := \mathrm{out}(\epsilon)$ its black output vertex. Let $\mathsf{R} \in \mathcal{T}_4^{wb,2,w3_1}(1,1)$ be the minimal tree containing v_3 and v. Using the same locality argument as before we may suppose that in fact $\mathsf{S}_\omega = \mathsf{R}$. Then $a^S_\omega \in \mathcal{F}_{2,1}(U,V)$ and we may replace a^S_ω modulo (D) by some ${a'}^S_\omega \in \mathcal{F}_2(U) \odot \mathcal{F}_1(V)$. We infer from (3) with $(i,j) = (2,1)$ that ${a'}^S_\omega = 0 \bmod (S)$, so we may delete a^S_ω from the right-hand side of (4). The discussion of the second type terms is similar. This finishes the proof of Lemma 2.4. □

In the rest of this paragraph we discuss the relation between our definitions and the original triple definition of distributive law. As it can be expected, both definitions coincide; the hard part of this statement is provided by Theorem 2.3.

Recall that each operad $\mathcal{S}$ generates a triple $\mathsf{T} = (\mathsf{T}, \mu^\mathsf{T}, \eta^\mathsf{T})$ on the category of vector spaces having the property that algebras over the operad $\mathcal{S}$ are the same as algebras over the triple T. The details may be found in [5], recall only that the functor T is defined by $\mathsf{T}(V) := \bigoplus_{n\geq 1} (\mathcal{S}(n) \otimes T^n(V))$ in the nonsymmetric case and $\mathsf{T}(V) := \bigoplus_{n\geq 1} (\mathcal{S}(n) \otimes T^n(V))_{\Sigma_n}$, where $(-)_{\Sigma_n}$ indicates the coinvariants of the symmetric group action on the product $\mathcal{S}(n) \otimes T^n(V)$ given by the operad action on the first factor and by permuting the variables of the second factor, in the symmetric case.

Let $\mathsf{S} = (\mathsf{S}, \mu^\mathsf{S}, \eta^\mathsf{S})$ and $\mathsf{T} = (\mathsf{T}, \mu^\mathsf{T}, \eta^\mathsf{T})$ be two triples. Let us recall that a distributive law of S over T is a natural transformation $\ell : \mathsf{TS} \to \mathsf{ST}$ which has to satisfy some functoriality conditions, see [1, page 120] for details. Let $\mathcal{C} = \langle U, V; S, D, T\rangle$ be an operad with a distributive law in the sense of Definition 2.2, $\mathcal{A} := \langle U, S\rangle$ and $\mathcal{B} := \langle V, T\rangle$. Let $\mathsf{S} = (\mathsf{S}, \mu^\mathsf{S}, \eta^\mathsf{S})$ (resp. $\mathsf{T} = (\mathsf{T}, \mu^\mathsf{T}, \eta^\mathsf{T})$) be the triple associated to the operad $\mathcal{A}$ (resp. $\mathcal{B}$). We have, in the nonsymmetric case,

$$\mathsf{ST}(V) := \bigoplus_{n\geq 1}((\mathcal{A}\odot\mathcal{B})(n)\otimes T^n(V)) \text{ and } \mathsf{TS}(V) := \bigoplus_{n\geq 1}((\mathcal{B}\odot\mathcal{C})(n)\otimes T^n(V)),$$

while obvious similar formulas hold, after taking the coinvariants, also in the symmetric case. There is a natural map of collections $\Delta : \mathcal{B}\odot\mathcal{A} \to \mathcal{A}\odot\mathcal{B}$ given as the composition

$$\mathcal{B} \odot \mathcal{A} \xrightarrow{\chi} \mathcal{C} \xrightarrow{\xi^{-1}} \mathcal{A} \odot \mathcal{B}$$

where χ is induced by the inclusion $\mathcal{F}(V) \odot \mathcal{F}(U) \subset \mathcal{F}(U,V)$ and $\xi : \mathcal{A}\odot\mathcal{B} \to \mathcal{C}$ is the map introduced in 2.1. Here we use the nontrivial fact that the map ξ *is an isomorphism* (Theorem 2.3). The map $\Delta : \mathcal{B}\odot\mathcal{A} \to \mathcal{A}\odot\mathcal{B}$ then induces, by $\ell(x\otimes y) := \Delta(x)\otimes y$, $x\otimes y \in (\mathcal{B}\odot\mathcal{A})(n)\otimes T^n(V)$, a natural transformation $\ell : \mathsf{TS} \to \mathsf{ST}$. We leave it to the reader to prove the following proposition.

PROPOSITION 2.6. — *Under the notation above, the transformation* $\ell : \mathsf{TS} \to \mathsf{ST}$ *is a distributive law in the sense of J. Beck.*

3. Examples.

Example 3.1. — Let us consider a vector space V with two associative multiplications, $\cdot$ and $\langle -,-\rangle$, such that

$$(6)\qquad \langle a\cdot b, c\rangle = a\cdot\langle b,c\rangle, \text{ and } \langle a, b\cdot c\rangle = \langle a,b\rangle\cdot c.$$

We call this object a *nonsymmetric Poisson algebra*. Let us describe the corresponding nonsymmetric operad. Let U (resp. V) be the vector space spanned by an element μ (resp. ν). Let $S \subset \mathcal{F}(U)(3)$ be the subspace generated by $\mu\circ_1\mu - \mu\circ_2\mu$ (the associativity) and, similarly, let $T \subset \mathcal{F}(V)(3)$ be generated by $\nu\circ_1\nu - \nu\circ_2\nu$. Define $d : V\bullet U \to U\bullet V$ by $d(\nu\circ_1\mu) := \mu\circ_2\nu$ and $d(\nu\circ_2\mu) := \mu\circ_1\nu$, and let $\mathcal{NP}$ be the nonsymmetric operad $\mathcal{NP} := \langle U,V;S,D,T\rangle$, see 2.1 for the notation. Algebras over this operad are exactly nonsymmetric Poisson algebras, with μ corresponding to $\cdot$ and ν corresponding to $\langle -,-\rangle$. Let us verify the condition of Definition 2.2.

Consider the monomial $\langle a\cdot b, c\cdot d\rangle$, $a,b,c,d \in V$. Relation (6) gives either $\langle a\cdot b, c\cdot d\rangle = a\cdot\langle b, c\cdot d\rangle = a\cdot(\langle b,c\rangle\cdot d)$ or $\langle a\cdot b, c\cdot d\rangle = \langle a\cdot b, c\rangle\cdot d = (a\cdot\langle b,c\rangle)\cdot d$. But $a\cdot(\langle b,c\rangle\cdot d) = (a\cdot\langle b,c\rangle)\cdot d$ by the associativity of $\cdot$, so we did not introduce new relations in this way. Relation (6) gives also $\langle a\cdot(b\cdot c), d\rangle = a\cdot\langle b\cdot c, d\rangle = a\cdot(b\cdot\langle c,d\rangle)$, while $\langle (a\cdot b)\cdot c, d\rangle = (a\cdot b)\cdot\langle c,d\rangle$ (and similarly for $\langle a, b\cdot(c\cdot d)\rangle$ vs. $\langle a, (b\cdot c)\cdot d\rangle$) which means that (6) is compatible with the associativity of $\cdot$. By the same argument, it is compatible with the associativity of $\langle -,-\rangle$. We leave it to the reader to convince him/herself that *we have just proven that the map* $\xi_{i,j}$ *is an isomorphism for* $(i,j) = (1,2),(2,1)$.

Example 3.2. — Let us give another, very strange example of a distributive law. Consider a vector space V and two bilinear operations,

$\cdot$ and $\langle -,- \rangle$, such that, for each $a,b,c \in V$,

$$(a \cdot b) \cdot c = 0,\ \langle a \cdot b, c\rangle = 3a \cdot \langle b,c\rangle,\ \langle a, b \cdot c\rangle = 0 \text{ and } \langle a, \langle b,c\rangle\rangle = 0.$$

We recommend it to the reader to verify, using the same pattern as in Example 3.1, that the condition of Definition 2.2 is satisfied.

Example 3.3. — In this example we give an innocuous generalization of such classical objects as Poisson or Gerstenhaber algebras. Let us fix two natural numbers, m and n. Let U be the graded vector space spanned on an element μ of degree m and let V be the graded vector space spanned on an element ν of degree n. Define Σ_2-actions on U and V by $\sigma\mu := (-1)^m \cdot \mu$ and $\sigma\nu := -(-1)^n \cdot \nu$. Let $S \subset \mathcal{F}(U)(3)$ be the Σ_3-invariant subset generated by $\mu \circ_1 \mu - (-1)^m \cdot \mu \circ_2 \mu$ (the associativity) and let $T \subset \mathcal{F}(V)(3)$ be the Σ_3-invariant subset generated by $\nu \circ_2 \nu + (\nu \circ_1 \nu)(1 \otimes \sigma) + (-1)^n \cdot \nu \circ_1 \nu$ (the Jacobi identity). Finally, let $d : V \bullet U \to U \bullet V$ be given by $d(\nu \circ_2 \mu) := \mu \circ_1 \nu + (-1)^m \cdot (\mu \circ_1 \nu)(1 \otimes \sigma)$. The reader will easily verify that this gives a distributive law.

An algebra over the operad $\mathcal{P}(m,n) := \langle U, V; S, D, T\rangle$ defined above consists of a (graded) vector space P together with two bilinear maps, $- \cup - : P \otimes P \to P$ of degree m, and $[-,-] : P \otimes P \to P$ of degree n such that, for any homogeneous $a,b,c \in P$,

(i) $a \cup b = (-1)^{|a|\cdot|b|+m} \cdot b \cup a$,

(ii) $[a,b] = -(-1)^{|a|\cdot|b|+n} \cdot [b,a]$,

(iii) $- \cup -$ is associative in the sense that

$$a \cup (b \cup c) = (-1)^{m\cdot(|a|+1)} \cdot (a \cup b) \cup c,$$

(iv) $[-,-]$ satisfies the following form of the Jacobi identity:

$$(-1)^{|a|\cdot(|c|+n)} \cdot [a,[b,c]] + (-1)^{|b|\cdot(|a|+n)} \cdot [b,[c,a]] + (-1)^{|c|\cdot(|b|+n)} \cdot [c,[a,b]] = 0,$$

(v) the operations $- \cup -$ and $[-,-]$ are compatible in the sense that

$$(-1)^{m\cdot|a|} \cdot [a, b \cup c] = [a,b] \cup c + (-1)^{(|b|\cdot|c|+m)} \cdot [a,c] \cup b.$$

Following [2] we call these algebras (m,n)-*algebras*. Obviously $(0,0)$-algebras are exactly (graded) Poisson algebras, $(0,-1)$-algebras are Gerstenhaber algebras introduced in [4] while $(0,n-1)$-algebras are the n-algebras of [5]. For a more detailed analysis of this example from an operadic point of view, see [2].

4. Distributive laws and the Koszulness.

For a nonsymmetric collection $\{C(n); n \geq 1\}$ we define the *dual* $\#C$ by $(\#C)(n) := \#(C(n))$. In the symmetric case the definition is the same with the action of Σ_n on $\#C(n)$ being the induced action multiplied by the sign representation. In both cases we have a canonical isomorphism of collections $\#\mathcal{F}(C) = \mathcal{F}(\#C)$. The *Koszul dual* $\mathcal{S}^!$ of a quadratic operad $\mathcal{S} = \langle E; R\rangle$ is then, following [6], defined as $\mathcal{S}^! := \langle \#E; R^\perp\rangle$, where $R^\perp \subset \mathcal{F}(\#E)(3) = \#\mathcal{F}(E)(3)$ is the annihilator of the subspace $R \subset \mathcal{F}(E)(3)$.

LEMMA 4.1. — *Let $R_{\mathrm{S}} \subset \mathcal{F}(E)$ be, for $\mathrm{S} \in \mathcal{T}_n^{2,3_1}$, the same as in 1.4. Then*

$$\#\mathcal{S}^!(n) = \begin{cases} \bigcap\{R_{\mathrm{S}};\ \mathrm{S} \in \mathcal{T}_n^{2,3_1}\}, & \textit{for}\quad n \geq 3, \\ E, & \textit{for}\quad n = 2, \textit{ and} \\ \mathbf{k}, & \textit{for}\quad n = 1. \end{cases}$$

Proof. — An easy linear algebra. The symmetric case of the statement was formulated in [6, page 262]. □

4.2. We are going to define the Koszul complex of an operad, rephrasing, in fact, a definition of [6]. The *Koszul complex* of an operad $\mathcal{S} = \langle E; R\rangle$ is a differential graded collection $K_\bullet(\mathcal{S}) = (K_\bullet(\mathcal{S}), d_{\mathcal{S}})$ with $K_\bullet(\mathcal{S}) := \mathcal{S} \odot \#\mathcal{S}^!$. The component $K_n(\mathcal{S})(m) \subset (\mathcal{S} \odot \#\mathcal{S}^!)(m)$ is generated by elements of the form $\gamma(s; t_1, \dots, t_k)$, $s \in \mathcal{S}(k)$, $t_i \in \#\mathcal{S}^!_{j_i}(m_i)$, $1 \leq i \leq k$, where $m_1 + \cdots + m_k = m$ and $j_1 + \cdots + j_k = n$. As $\#\mathcal{S}^! \subset \mathcal{F}(U)$ by Lemma 4.1, we may in fact suppose that $t_i \in \mathcal{F}(E)$ (or, in a more compact notation, that $K_\bullet(\mathcal{S}) \subset \mathcal{S} \odot \mathcal{F}(E)$). The differential is defined as follows. Let $x = \gamma(s; t_1, \dots, t_k)$, $t_i \in \mathcal{F}(E)(m_i)$ be as above. If $m_i = 1$ put $d_i(x) = 0$. For $m_i > 1$, x can be obviously rewritten as $x = \gamma(s \circ_i r_i; y_1, \dots, y_{k+1})$ with $r_i \in E$ and with some $y_1, \dots, y_{k+1} \in \mathcal{F}(E)$ (in fact, $y_j = t_j$ for $j < i$ and $y_{j+1} = t_j$ for $j > i$). Define then $d_i := \gamma(s \circ_i [r_i]; y_1, \dots, y_{k+1})$, where $[-] : E \to \mathcal{S}$ maps $e \in E$ to its class $[e]$ in $\mathcal{S} = \mathcal{F}(E)/(R)$. Then we put $d(x) := \sum_{1 \leq i \leq k} d_i(x)$. The differential $d_{\mathcal{S}}$ on $K_\bullet(\mathcal{S})$ is defined as the restriction of d to $K_\bullet(\mathcal{S}) \subset \mathcal{S} \circ \mathcal{F}(E)$. We can verify that $d_{\mathcal{S}}^2 = 0$; for the symmetric case it was done in [6], the nonsymmetric case is even easier. As in [6] we say that $\mathcal{S}$ is *Koszul* if the complex $(K_\bullet(\mathcal{S})(m), d_{\mathcal{S}}(m))$ is acyclic for any $m \geq 2$. Observe that, by definition, $K_\bullet(\mathcal{S})(1) = K_0(\mathcal{S})(1) = \mathbf{k}$.

The following lemma was formulated in [2]; the verification is immediate.

LEMMA 4.3. — *Let $\mathcal{C} = \langle U, V; S, D, T\rangle$ be an operad with a distributive law $d : V \bullet U \to U \bullet V$. Let $\#d : \#U \bullet \#V \to \#V \bullet \#U$ be the dual of d and let $D^\perp := \{\alpha \bullet \beta - \#d(\alpha \bullet \beta);\ \alpha \bullet \beta \in \#U \bullet \#V\}$. Then $\mathcal{C}^! = \langle \#V, \#U; T^\perp, D^\perp, S^\perp\rangle$ and $\#d$ is a distributive law.*

Consider the Koszul complex $K_\bullet(\mathcal{C})$ of an operad $\mathcal{C} = \langle U, V; S, D, T\rangle$ with a distributive law. By Lemma 4.3, $\mathcal{C}^! = \langle \#V, \#U; T^\perp, D^\perp, S^\perp\rangle$, and we have the bigrading $\mathcal{C}^!_n = \bigoplus_{i+j=n} \mathcal{C}^!_{i,j}$ which induces the decomposition $\#\mathcal{C}^!_n = \bigoplus_{i+j=n} \#\mathcal{C}^!_{i,j}$ with $\#\mathcal{C}^!_{1,0} = V$ and $\#\mathcal{C}^!_{0,1} = U$ of the dual collection. We may use these data to define the convergent decreasing filtration $F_\bullet K_\bullet(\mathcal{C})$ of $K_\bullet(\mathcal{C})$ as follows. Let $F_pK_n(\mathcal{C}) \subset K_n(\mathcal{C})$ be generated by elements $\gamma(s; t_1, \dots, t_k)$, $s \in \mathcal{C}(k)$, $t_i \in \#\mathcal{C}^!_{a_i,b_i}$, $1 \le i \le k$, $\sum_{i=1}^k b_i \le p$ and $\sum_{i=1}^k (a_i + b_i) = n$. We can easily see that the differential $d_{\mathcal{C}}$ preserves the filtration, $d_{\mathcal{C}}(m)F_pK_n(\mathcal{C})(m) \subset F_pK_{n-1}(\mathcal{C})(m)$, therefore there is a spectral sequence $\mathbf{E}(m) = (E^r_{p,q}(m), d^r(m))$ converging to $H_\bullet(K_\bullet(\mathcal{C}))(m)$, for any $m \ge 1$.

Let us observe that, for any three collections X, Y and Z, the collection $X \odot Y \odot Z$ is naturally bigraded by $(X \odot Y \odot Z)_{p,q} := \mathcal{F}_{1,p,q}(X \oplus Y \oplus Z) \cap (X \oplus Y \oplus Z)$ and we write $X \odot Y_p \odot Z_q$ instead of $(X \odot Y \odot Z)_{p,q}$.

PROPOSITION 4.4. — *For the spectral sequence*

$$\mathbf{E}(m) = (E^r_{p,q}(m), d^r(m))$$

defined above we have, for each $m \ge 1$,

$$(E^0_{p,\bullet}(m), d^0(m)) \cong ((\mathcal{A} \odot K_\bullet(\mathcal{B}) \odot \#\mathcal{A}^!_p)(m), (\mathbb{1} \odot d_{\mathcal{B}} \odot \mathbb{1})(m)).$$

Proof. — By Theorem 2.3 we have the isomorphism of collections $\xi : \mathcal{A} \odot B \to \mathcal{C}$ and, because $\mathcal{C}^!$ is, by Lemma 4.3, also an operad with a distributive law, by the same theorem we have an isomorphism $\xi : \mathcal{B}^! \odot \mathcal{A}^! \to \mathcal{C}^!$ inducing the dual isomorphism $\#\xi_{q,p} : \#\mathcal{C}^!_{q,p} \to \#\mathcal{B}^!_q \odot \#\mathcal{A}^!_p$ of bigraded collections.

We have the identification $E^0_{p,q} = \mathcal{C} \odot \mathcal{C}^!_{q,p}$ (= the space generated by elements $\gamma(s; t_1, \dots, t_k)$ with $s \in \mathcal{C}(k)$, $t_i \in \#\mathcal{C}^!_{a_i,b_i}$, $1 \le i \le k$, $\sum_{i=1}^k a_i = q$ and $\sum_{i=1}^k b_i = p$). We may thus define an isomorphism of collections $\phi_{p,q} : E^0_{p,q} \to \mathcal{A} \odot \mathcal{B} \odot \#\mathcal{B}^!_q \odot \#\mathcal{A}^!_p$ by $\phi_{p,q} := \xi^{-1} \odot \#\xi_{q,p}$. We must show that this map commutes with the differential, *i.e.* that, for $z \in \mathcal{C} \odot \#\mathcal{C}^!_{p,q}$,

$$\phi_{p,q-1}(d^0(z)) = (\mathbb{1} \odot d_{\mathcal{B}} \odot \mathbb{1})(\phi_{p,q}(z)). \tag{7}$$

We have a very explicit description of $\#\xi_{q,p}$. As in 1.4, $\mathcal{F}(V,U) = \bigoplus_{\mathrm{T}\in\mathcal{T}^{wb,2}} (V,U)(\mathrm{T})$ which gives the canonical direct sum decomposition $\mathcal{F}(V,U) = \bigoplus_{l\geq 0} \mathcal{F}(V,U)_{(l)}$ with $\mathcal{F}(V,U)_{(l)} := \bigoplus_{\mathrm{T}\in\mathcal{T}^{wb,2}, I(\mathrm{T})=l} (V,U)(\mathrm{T})$. Observing that $\mathcal{F}(V,U)_{(0)} = \mathcal{F}(V) \odot \mathcal{F}(U)$ we conclude that $\mathcal{F}_q(V) \odot \mathcal{F}_p(U)$ is a canonical direct summand of $\mathcal{F}_{q,p}(U,V)$. Let $\pi : \mathcal{F}_{q,p}(V,U) \to \mathcal{F}_q(V) \odot \mathcal{F}_p(U)$ be the corresponding projection. Using the identification of $\#\mathcal{C}^!_{q,p}$ with a subspace of $\mathcal{F}_{q,p}(V,U)$ provided by Lemma 4.1, the map $\#\xi_{q,p}$ coincides with the restriction of π to $\#\mathcal{C}^!_{q,p}$.

It is obviously enough to prove (7) for elements z of the form $z = \gamma(s; t_1, \dots, t_k)$ with $t_i \in \#\mathcal{C}^!_{a_i,b_i}$ for $1 \leq i \leq k$, $\sum_{i=1}^{k} a_i = q$ and $\sum_{i=1}^{k} b_i = p$. We may also suppose that $s = \gamma(a; b_1, \dots, b_l)$ for some $a \in \mathcal{A}(l)$, $b_i \in \mathcal{B}(m_i)$, for $1 \leq i \leq l$ and $\sum_{i=1}^{l} m_i = k$. We may also take t_i to be of the form $t_i = w_i + r_i$ with $r_i = \sum_{\omega\in\Omega_i} \gamma(v_{i,\omega}; y_{i,\omega,1}, y_{i,\omega,2})$, for $v_{i,\omega} \in V$ and $y_{i,\omega,j} \in \#\mathcal{C}^!_{a_{i,\omega,j},b_{i,\omega,j}}$, $j = 1,2$, with $a_{i,\omega,1} + a_{i,\omega,2} = a_i - 1$ and $b_{i,\omega,1} + b_{i,\omega,2} = b_i$, and $w_i \in U \odot \mathcal{F}_{a_i,b_i-1}(V,U)$. Any other element $z \in \mathcal{C} \odot \#\mathcal{C}^!_{p,q}$ can be expressed as a linear combination (and using the symmetric group action in the symmetric case) of elements of the above form.

We also denote, for each $1 \leq i \leq k$, by $s(i)$ the unique number such that $m_1 + \cdots + m_{s(i)-1} < i \leq m_1 + \cdots + m_{s(i)}$, let us then put $t(i) := i - m_1 + \cdots + m_{s(i)-1}$. For any given i, $1 \leq i \leq k$, we have

$$z = \sum_{\omega\in\Omega_i} \alpha \cdot \gamma(s \circ_i v_{i,\omega}; t_1, \dots, t_{i-1}, y_{i,\omega,1}, y_{i,\omega,2}, t_{i+1}, \dots, t_k) \\ + \gamma(s; t_1, \dots, t_{i-1}, w_i, t_{i+1}, \dots, t_k)$$

therefore

$$\begin{aligned}
&\phi_{p,q-1}(d^0(z)) \\
&= \sum_{\omega\in\Omega_i, 1\leq i\leq k} \alpha\cdot\phi_{p,q-1}(\gamma(s\circ_i[v_{i,\omega}];t_1,\ldots,t_{i-1},y_{i,\omega,1},y_{i,\omega,2},t_{i+1},\ldots,t_k)) \\
&= \sum_{\omega\in\Omega_i, 1\leq i\leq k} \alpha\beta\cdot\gamma(\gamma(a;b_1,\ldots,b_{s(i)-1},b_{s(i)}\circ_{t(i)}[v_{i,\omega}],b_{s(i)+1},\ldots,b_l); \\
&\qquad \#\xi(r_1),\ldots,\#\xi(r_{i-1}),\#\xi(y_{i,\omega,1}),\#\xi(y_{i,\omega,2}),\#\xi(r_{i+1}),\ldots,\#\xi(r_k))
\end{aligned}$$

with $\alpha := (-1)^{|t_1|+\cdots+|t_{i-1}|}$ and $\beta := (-1)^{|b_{s(i)+1}|+\cdots+|b_l|}$. Here we used the clear fact that $\#\xi(t_i) = \#\xi(r_i)$. On the other hand,

$$(\mathbb{1}\odot d_{\mathcal{B}}\odot\mathbb{1})(\phi_{p,q}(z)) = (\mathbb{1}\odot d_{\mathcal{B}}\odot\mathbb{1})\gamma(\gamma(a;b_1,\ldots,b_l);\#\xi(r_1),\ldots,\#\xi(r_k))$$

and this expression coincides, taking into the account the relation

$$\#\xi(r_i) = \sum_{\omega\in\Omega_i}\#\xi(\gamma(v_{i,\omega};y_{i,\omega,1},y_{i,\omega,2})) = \sum_{\omega\in\Omega_i}\gamma(v_{i,\omega};\#\xi(y_{i,\omega,1}),\#\xi(y_{i,\omega,2})),$$

with the right-hand side term of the equation above. □

The following theorem, which is one of the central results of the paper, easily follows from the previous proposition and from the Künneth formula for collections (Proposition 1.7).

THEOREM 4.5. — *Let $\mathcal{C} = \langle U, V; S, D, T\rangle$ be an operad with a distributive law and let $\mathcal{A} := \langle U; S\rangle$ and $\mathcal{B} := \langle V; T\rangle$. If the operads $\mathcal{A}$ and $\mathcal{B}$ are Koszul, then $\mathcal{C}$ is Koszul as well.*

The theorem immediately implies that the operad $\mathcal{NP}$ for non-symmetric Poisson algebras from Example 3.1 is Koszul; in this case $\mathcal{A} = \mathcal{B} = Ass$, the operad for associative algebras, which is known to be Koszul [6, Corollary 4.2.7].

For an operad $\mathcal{S}$, let $\mathrm{s}\,\mathcal{S}$ (the *suspension*) be the operad with $(\mathrm{s}\,\mathcal{S})(n) :=\uparrow^{n-1}\mathcal{S}(n)$, $n\geq 1$, with the composition maps defined in an obvious way; here $\uparrow^{n-1}$ denotes the usual $(n-1)$-fold suspension of a graded vector space. It follows from the computation of [7] that $\mathcal{S}$ is Koszul if and only if its suspension $\mathrm{s}\,\mathcal{S}(n)$ is Koszul.

Let $\mathcal{P}(m,n) = \langle U, V; S, D, T\rangle$ be the operad for (m,n)-algebras as in Example 3.3. It is immediate to see that $\mathcal{A} = \mathrm{s}^n\,\mathit{Comm}$ and that $\mathcal{B} = \mathrm{s}^m\mathit{Lie}$ while both *Comm* (the operad for commutative associative algebras) and *Lie* (the operad for Lie algebras) are well-known to be Koszul, see [6, Corollary 4.2.7]. Theorem 2.3 then implies:

COROLLARY 4.6. — *The operad $\mathcal{P}(m,n)$ for (m,n)-algebras is Koszul for all m and n. In particular, the operads for Poisson, Gerstenhaber and n-algebras are Koszul.*

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