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GALOIS COVERS BETWEEN $K3$ SURFACES

by Gang XIAO

The aim of this note is to classify the actions of finite groups G on Kähler $K3$ surfaces X, i.e., simply connected surfaces over $\mathbb{C}$ with trivial canonical sheaf. Such an action is called *symplectic*, if the (resolution of the) quotient X/G is also a $K3$ surface, or equivalently, if the induced action of G on $H^0(\omega_X)$ is trivial. We will only consider symplectic actions.

This classification is done by Nikulin for the case when G is abelian [N]. Then Mukai [M] gives a complete classification of finite groups G admitting a sympletic action on a $K3$ surface, showing that they are exactly subgroups of 11 maximal groups.

We will give a classification of combinatorial types of the actions, i.e., the numbers of fixed points of each type. Our method is similar to that in [N], that is, we consider the singularities of the quotient X/G, and the sublattice generated by the components of their resolutions. In §1, we generalize Nikulin's argument to non-abelian groups, to formulate a set of criteria on this sublattice. Then an easy computer sieve program using these criteria leads almost directly to the final list of good cases.

Note that we do not use the argument of Mukai (except his examples which are used to prove the existence), therefore providing an independent proof of Mukai's classification.

This combinatorial classification contains much information about the geometry of the symplectic actions. For example, one observes from the list (Table 2) that the moduli space of the action of each of the 11 maximal groups is of dimension 1, due to [N] (see the proof of Lemma 5).

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In particular, there are only a finite number of $K3$ quartics in projective space who admit linearly the symplectic action of a maximal group. It will be interesting to construct the exhaustive list of these quartics.

1. Necessary conditions.

Let X be a smooth minimal Kähler $K3$ surface, with a finite group G acting symplectically on it (i.e., the quotient is also a $K3$ surface). It is well-known that the quotient X/G has at worst rational double points, and the projection $f\colon X \longrightarrow X/G$ is unramified outside these singular points.

Let Y be the minimal resolution of singularities of X/G. For each singular point p_i ($i = 1, \dots, k$) on X/G, its inverse image on Y is a negative definite configuration Σ_i of (-2)-curves, of type A_n, D_n, E_6, E_7, or E_8.

Let Y' be the complement of the points p_i in Y, $X' = f^{-1}(Y')$, $f'\colon X' \longrightarrow Y'$ the induced étale cover. Then $\pi_1(X') = \pi_1(X) = \{1\}$ as X' is the complement of a set of smooth points, hence

$$G = \pi_1(Y').$$

A similar situation holds locally : let G_i be the stabiliser of a point in $f^{-1}(p_i)$, U_i a small topologic neighborhood of p_i. Then

$$G_i \cong \pi_1(U_i \backslash p_i).$$

It is well-known that G_i is the cyclic group C_{n+1} for $\Sigma_i = A_n$; binary dihedral group Q_{4n-8} for $\Sigma_i = D_n$ ($n \geqslant 4$); binary tetrahedral group T_{24} for $\Sigma_i = E_6$; binary octahedral group O_{48} for $\Sigma_i = E_7$; and binary icosahedral group I_{120} for $\Sigma_i = E_8$. By [N], the order of an element of G (hence of G_i) is at most 8, therefore the possible types for Σ_i are:

$$A_1, \dots, A_7,\ D_4,\ D_5,\ D_6,\ E_6,\ E_7\ .$$

Let also c_i be the number of components of Σ_i, $e_i = \chi_{\mathrm{top}}(\Sigma_i) = c_i + 1$, and $N_i = |G_i|$, $N = |G|$.

LEMMA 1. — 1. *$N_i | N$ for every i.*

2. $\displaystyle\sum_{i=1}^{k} (e_i - 1/N_i) = 24(N-1)/N.$

Proof. — 1 is clear as G_i is a subgroup of G, and 2 follows directly from the following computation of topological characters:

$$24 - \sum_{i=1}^{k} N/N_i = \chi_{\mathrm{top}}(X) - \sum_{i=1}^{k} N/N_i = \chi_{\mathrm{top}}(X')$$

$$= N\chi_{\mathrm{top}}(Y') = N\left(24 - \sum_{i=1}^{k} e_i\right). \qquad \square$$

On the other hand, the classes of the (-2)-curves generate a negative definite sublattice L' of rank $c = \sum_{i=1}^{k} c_i$ in $H^2(Y, \mathbb{Z})$ which is an even unimodular lattice of index $(19, 3)$. In particular

$$(1) \qquad c \leqslant 19.$$

Let L be the smallest primitive sublattice of $H^2(Y, \mathbb{Z})$ containing L', L^* (resp. L'^*) the dual lattice of L (resp. of L'). We have natural inclusions

$$L' \subseteq L \subseteq L^* \subseteq L'^*.$$

For a finite abelian group H, let $l(H)$ be the minimal number of generators. Then as L is primitive in $H^2(Y, \mathbb{Z})$, we must have

$$l(L^*/L) \leqslant \mathrm{rank}(H^2(Y, \mathbb{Z}) - \mathrm{rank}(L) = 22 - c$$

(e.g. [D]), in particular for every prime number p,

$$l\left((L^*/L)_p\right) \leqslant 22 - c,$$

where we note by H_p the subgroup of H consisting of elements of order p.

On the other hand, we have $L/L' \cong L'^*/L^*$, hence

$$(2) \qquad 22 - c \geqslant l\left((L^*/L)_p\right) \geqslant l\left((L'^*/L')_p\right) - 2l\left((L/L')_p\right).$$

From a simple computation of the corresponding lattices, it is easy to see that the number $l\left((L'^*/L')_p\right)$, for $p = 2$ or 3, is determined by the configuration of (-2)-curves in the following way:

A configuration of type A_n (n odd), D_5 or E_7 adds 1 to $l\left((L'^*/L')_2\right)$, that of type D_4 or D_6 adds 2 to $l\left((L'^*/L')_2\right)$, and that of type A_2, A_5 or E_6 adds 1 to $l\left((L'^*/L')_3\right)$.

And we can use the following lemma to bound the term $l\left((L/L')_p\right)$.

LEMMA 2. — $L/L' \cong (G/[G,G])^*$.

Proof. — The inclusion of Y' into Y induces a surjective map

$$\alpha\colon H^2(Y,\mathbb{Z}) \cap H^{1,1}(Y) = \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(Y'),$$

with $\mathrm{Ker}(\alpha) = L'$. The image by α of L is the torsion subgroup of Pic (Y'), which is naturally dual to the abelianisation of $\pi_1(Y')$. But $\pi_1(Y') = G$, because $\pi_1(X') = \pi_1(X)$ is trivial. □

COROLLARY. — *If $l\left((L'^*/L')_p\right) + c > 22$, there exists a $K3$ surface W with a generically Galois rational map $\psi\colon W \dashrightarrow Y$ which is étale outside the Σ_i's, with a Galois group C_p^r where*

$$r = \left[\left(l\left((L'^*/L')_p\right) + c - 21\right)/2\right].$$

LEMMA 3. — *Let $Q \cong C_2^r$ be a group acting symplectically on a $K3$ surface W. Then $r \leqslant 4$, and the quotient W/Q has exactly $16 - 2^{4-r}$ singular points of type A_1, forming $2^r - 1$ blocks of 2^{4-r} points each. Two points p_1, p_2 belong to the same block if and only if they are dependent, i.e., the corresponding stabiliser subgroups in Q are the same.*

Similarly, if $Q \cong C_3^r$, then $r \leqslant 2$, and W/Q has $9 - 3^{2-r}$ singular points of type A_2, composed of $3r - 2$ blocks of $2 \cdot 3^{2-r}$ dependent points each.

Proof. — This result can be found in [N], but we give a proof here, because it provides an illustration of our method.

Let $Q \cong C_2^r$. Then the only possible non-trivial stabiliser of a point on W is C_2, in other words W/Q has only singularities of type A_1. In terms of Lemma 1, we have $e_i - 1/N_i = 3/2$, hence $k = 16(N-1)/N$ for $N = |Q|$, so $r \leqslant 4$ and $k = 16 - 2^{4-r}$. Applying the formula to the case $r = 1$, we see that an automorphism of order 2 has 8 fixed points. Because Q is abelian, it permutes these 8 points, giving rise to the number of blocks of dependent points.

The case of C_3^r is similar. □

To apply Lemma 3, consider a normal subgroup H in G such that $Q = G/H \cong C_p^r$, for $p = 2$ or 3 and $r > 0$. Let W be the desingularization of X/H, with the induced rational map ψ: $W \dashrightarrow Y$. Obviously the (-2)-configurations mentioned in Lemma 3 must be subconfigurations of the set $\{\Sigma_i\}$ for the map ψ. Moreover, if a configuration Σ_i contains a non-empty subconfiguration for ψ, then G_i must have a quotient isomorphic to $C_p^{r'}$, for $0 < r' \leqslant r$.

Quotients of this kind for different types of G_i are given in the following table:

type	G_i	quotient	subconfigurations for quotient	kernel
A_{2n-1}	C_{2n}	C_2	n dependent A_1's	A_{n-1}
D_4	Q_8	C_2	2 dependent A_1's	A_3
D_4	Q_8	C_2^2	3 independent A_1's	A_1
D_5	Q_{12}	C_2	2 dependent A_1's	A_5
D_6	Q_{16}	C_2	2 dependent A_1's	A_7
D_6	Q_{16}	C_2	3 dependent A_1's	D_4
D_6	Q_{16}	C_2^2	4 A_1's in 3 blocks $(1,1,2)$	A_3
E_7	O_{48}	C_2	3 dependent A_1's	E_6
A_{3n-1}	C_{3n}	C_3	n dependent A_2's	A_{n-1}
E_6	T_{24}	C_3	2 dependent A_2's	D_4

Table 1

Notation. — We note by $[\Sigma_1, \ldots, \Sigma_m]$ a block of dependent A_1's contained in the configuration $\Sigma_1 + \cdots + \Sigma_m$. The choice of the curves will be clear from the context.

2. The list.

THEOREM 3. — *The action of G falls into one of the following 81 types, where n_i is the number of cyclic subgroups of order i, $Q = G/[G,G]$, and we follow [M] for notations of groups.*

#	N	c	configuration	$n_2, \ldots, n_8$	$[G,G]$	Q	G
1	2	8	$8A_1$	1,,,,,,	0	G	C_2
2	3	12	$6A_2$	,1,,,,,	0	G	C_3
3	4	12	$12A_1$	3,,,,,,	0	G	C_2^2
4	4	14	$4A_3 + 2A_1$	1,,1,,,,	0	G	C_4
5	5	16	$4A_4$	,,,1,,,	0	G	C_5
6	6	14	$3A_2 + 8A_1$	3,1,,,,,	#2	#1	D_6
7	6	16	$2A_5 + 2A_2 + 2A_1$	1,1,,,1,,	0	G	C_6
8	7	18	$3A_6$	,,,,,1,	0	G	C_7
9	8	14	$14A_1$	7,,,,,,	0	G	C_2^3
10	8	15	$2A_3 + 9A_1$	5,,1,,,,	#1	#3	D_8
11	8	16	$4A_3 + 4A_1$	3,,2,,,,	0	G	$C_2 \times C_4$
12	8	17	$2D_4 + 3A_3$	1,,3,,,,	#1	#3	Q_8
13	8	17	$4D_4 + A_1$	1,,3,,,,	#1	#3	Q_8
14	8	18	$2A_7 + A_3 + A_1$	1,,1,,,,1	0	G	C_8
15	9	16	$8A_2$	,4,,,,,	0	G	C_3^2
16	10	16	$2A_4 + 8A_1$	5,,,1,,,	#5	#1	D_{10}
17	12	16	$6A_2 + 4A_1$	3,4,,,,,	#3	#2	$\mathfrak{A}_4$
18	12	16	$A_5 + A_2 + 9A_1$	7,1,,,1,,	#2	#3	D_{12}
19	12	18	$3A_5 + 3A_1$	3,1,,,3,,	0	G	$C_2 \times C_6$
20	12	18	$2D_5 + 2A_3 + A_2$	1,1,3,,1,,	#2	#4	Q_{12}
21	16	15	$15A_1$	15,,,,,,	0	G	C_2^4
22	16	16	$2A_3 + 10A_1$	11,,2,,,,	#1	#9	$C_2 \times D_8$
23	16	17	$4A_3 + 5A_1$	7,,4,,,,	#1	#11	$\Gamma_2 c_1$
24	16	17	$2D_4 + A_3 + 6A_1$	7,,4,,,,	#1	#9	$Q_8 * C_4$
25	16	18	$6A_3$	3,,6,,,,	0	G	C_4^2
26	16	18	$D_4 + A_7 + A_3 + 4A_1$	5,,3,,,,1	#4	#3	SD_{16}
27	16	18	$4D_4 + 2A_1$	3,,6,,,,	#1	#9	$C_2 \times Q_8$

#	N	c	configuration	$n_2, \ldots, n_8$	$[G,G]$	Q	G
28	16	19	$2A_7 + A_3 + 2A_1$	3,,2,,,,2	#1	#11	$\Gamma_2 d$
29	16	19	$2D_6 + D_4 + A_3$	1,,5,,,,1	#4	#3	Q_{16}
30	18	16	$4A_2 + 8A_1$	9,4,,,,,	#15	#1	$\mathfrak{A}_{3,3}$
31	18	18	$2A_5 + 3A_2 + 2A_1$	3,4,,,3,,	#2	#7	$C_3 \times D_6$
32	20	18	$A_4 + 4A_3 + 2A_1$	5,,5,1,,,	#5	#4	$\mathrm{Hol}(C_5)$
33	21	18	$A_6 + 6A_2$	,7,,,,1,	#8	#2	$C_7 \rtimes C_3$
34	24	17	$2A_3 + 3A_2 + 5A_1$	9,4,3,,,,	#17	#1	$\mathfrak{S}_4$
35	24	18	$2A_5 + 2A_2 + 4A_1$	7,4,,,4,,	#3	#7	$C_2 \times \mathfrak{A}_4$
36	24	18	$D_5 + A_5 + A_3 + 5A_1$	9,1,3,,3,,	#7	#3	$C_3 \rtimes D_8$
37	24	19	$E_6 + D_4 + A_5 + 2A_2$	1,4,3,,4,,	#13	#2	T_{24}
38	24	19	$2E_6 + A_3 + 2A_2$	1,4,3,,4,,	#12	#2	T_{24}
39	32	17	$3A_3 + 8A_1$	19,,6,,,,	#3	#9	2^4C_2
40	32	17	$2D_4 + 9A_1$	19,,6,,,,	#1	#21	$Q_8 * Q_8$
41	32	18	$5A_3 + 3A_1$	11,,10,,,,	#3	#11	$\Gamma_7 a_1$
42	32	18	$2D_4 + 2A_3 + 4A_1$	11,,10,,,,	#3	#9	$\Gamma_4 c_2$
43	32	19	$2A_7 + 5A_1$	11,,2,,,,4	#3	#11	$\Gamma_7 a_2$
44	32	19	$D_4 + A_7 + 2A_3 + 2A_1$	7,,8,,,,2	#4	#11	$\Gamma_3 e$
45	32	19	$2D6 + D_4 + 3A_1$	7,,8,,,,2	#4	#9	$\Gamma_6 a_2$
46	36	18	$4A_3 + 2A_2 + 2A_1$	9,4,9,,,,	#15	#4	3^2C_4
47	36	18	$A_5 + 6A_2 + A_1$	3,13,,,3,,	#3	#15	$C_3 \times \mathfrak{A}_4$
48	36	18	$2A_5 + A_2 + 6A_1$	15,4,,,6,,	#15	#3	$\mathfrak{S}_{3,3}$
49	48	17	$6A_2 + 5A_1$	15,16,,,,,	#21	#2	2^4C_3
50	48	18	$2A_3 + 6A_2$	3,16,6,,,,	#25	#2	4^2C_3
51	48	18	$A_5 + 2A_3 + A_2 + 5A_1$	19,4,6,,4,,	#17	#3	$C_2 \times \mathfrak{S}_4$
52	48	19	$3A_5 + 4A_1$	15,4,,,12,,	#3	#19	$2^2(C_2 \times C_6)$
53	48	19	$2D_5 + 2A_3 + A_2 + A_1$	7,4,12,,4,,	#17	#4	2^2Q_{12}
54	48	19	$E_6 + A_7 + A_2 + 4A_1$	13,4,3,,4,,3	#38	#1	T_{48}

#	N	c	configuration	$n_2, \ldots, n_8$	$[G, G]$	Q	G
55	60	18	$2A_4 + 3A_2 + 4A_1$	15,10,,6,,,	G	0	$\mathfrak{A}_5$
56	64	18	$D_4 + 3A_3 + 5A_1$	27,,18,,,,	#9	#9	$\Gamma_{25}a_1$
57	64	18	$3D_4 + 6A_1$	27,,18,,,,	#3	#21	$\Gamma_{13}a_1$
58	64	19	$A_7 + 3A_3 + 3A_1$	19,,14,,,,4	#9	#11	$\Gamma_{22}a_1$
59	64	19	$D_4 + 5A_3$	11,,26,,,,	#11	#11	$\Gamma_{23}a_2$
60	64	19	$2D_6 + A_3 + 4A_1$	19,,14,,,,4	#11	#9	$\Gamma_{26}a_2$
61	72	18	$D_5 + A_3 + 3A_2 + 4A_1$	21,13,9,,3,,	#47	#1	$\mathfrak{A}_{4,3}$
62	72	19	$2A_5 + 2A_3 + 3A_1$	21,4,9,,12,,	#30	#3	N_{72}
63	72	19	$2D_4 + 3A_3 + A_2$	9,4,27,,,,	#30	#3	M_9
64	80	19	$4A_4 + 3A_1$	15,,,16,,,	#21	#5	2^4C_5
65	96	18	$3A_3 + 3A_2 + 3A_1$	27,16,18,,,,	#49	#1	2^4D_6
66	96	19	$2A_5 + A_3 + 2A_2 + 2A_1$	19,16,6,,16,,	#21	#7	2^4C_6
67	96	19	$D_4 + A_7 + 3A_2 + 2A_1$	15,16,12,,,,6	#50	#1	4^2D_6
68	96	19	$D_5 + A_5 + 2A_3 + 3A_1$	27,4,18,,12,,	#35	#3	2^3D_{12}
69	96	19	$2E_6 + 2A_2 + 3A_1$	19,16,6,,16,,	#40	#2	$(Q_8 * Q_8) \rtimes C_3$
70	120	19	$A_5 + A_4 + 2A_3 + A_2 + 2A_1$	25,10,15,6,10,,	#55	#1	$\mathfrak{S}_5$
71	128	19	$D_6 + D_4 + 2A_3 + 3A_1$	35,,38,,,,4	#22	#9	F_{128}
72	144	19	$2A_5 + 4A_2 + A_1$	15,40,,,24,,	#21	#15	$\mathfrak{A}_4^2$
73	160	19	$2A_4 + 3A_3 + 2A_1$	35,,30,16,,,	#64	#1	2^4D_{10}
74	168	19	$A_6 + 2A_3 + 3A_2 + A_1$	21,28,21,,,8,	G	0	$L_2(7)$
75	192	18	$D_4 + 6A_2 + 2A_1$	27,64,18,,,,	#57	#2	$4^2\mathfrak{A}_4$
76	192	19	$D_4 + A_5 + 2A_3 + A_2 + 2A_1$	43,16,42,,16,,	#49	#3	H_{192}
77	192	19	$E_6 + 3A_3 + A_2 + 2A_1$	43,16,42,,16,,	#69	#1	T_{192}
78	288	19	$2D_5 + A_3 + 2A_2 + 2A_1$	51,40,54,,24,,	#72	#1	$\mathfrak{A}_{4,4}$
79	360	19	$2A_4 + 2A_3 + 2A_2 + A_1$	45,40,45,36,,,	G	0	$\mathfrak{A}_6$
80	384	19	$D_6 + 2A_3 + 3A_2 + A_1$	51,64,78,,,,12	#75	#1	F_{384}
81	960	19	$D_4 + 2A_4 + 3A_2 + A_1$	75,160,90,96,,,	G	0	M_{20}

Table 2

Proof. — We implement the following criteria into a mechanical check of all the combinations of (-2)-configurations $\{\Sigma_i\}_{i=1,\ldots,k}$:

1. $c \leqslant 19$.

2. There is no configuration of type A_n for $n > 7$, D_n for $n > 6$, or E_8.

3. There exists an integer N such that Lemma 1 is satisfied.

4. Elements of different orders add up to $N - 1$. The number of elements of order n can be recovered from the number of fixed points on X with a stabiliser containing C_n [N]:

n	2	3	4	5	6	7	8
fixed points of a C_n	8	6	4	4	2	3	2

Table 3

5. By Lemmas 2 and 3, if $r = [(l((L'^*/L')_2) + c - 21)/2] > 0$, $\{\Sigma_i\}$ contains a subconfiguration of $16 - 2^{4-r}$ disjoint A_1's in conformity with Table 1, partitioned into $2^r - 1$ independent blocks of 2^{4-r} curves each.

6. Similarly, if $r = [(l((L'^*/L')_3) + c - 21)/2] > 0$, $\{\Sigma_i\}$ contains a subconfiguration of $9 - 3^{2-r}$ disjoint A_2's in conformity with Table 1, partitioned into $3r - 2$ independent blocks of $2 \cdot 3^{2-r}$ A_2's each.

The result of the check is the cases in Table 2 together with the following list:

#	N	c	configuration	Non-existence
101	8	16	$2D_4 + A_3 + 5A_1$	Too many C_2's (3) for Q_8
102	16	18	$2D_4 + 3A_3 + A_1$	No quotient for Q_8 (#12 or #13)
103	24	18	$4A_3 + 3A_2$	D)
104	24	19	$D_5 + A_5 + 3A_3$	A)
105	24	19	$2D_5 + A_5 + 2A_2$	E)
106	24	19	$3D_5 + A_3 + A_1$	A)
107	32	19	$4D_4 + A_3$	B) $Q = C_2, \Sigma =$#102
108	40	19	$3A_4 + 2A_3 + A_1$	C)
109	48	19	$A_5 + 4A_3 + A_2$	A)
110	64	19	$3D_4 + 2A_3 + A_1$	F)
111	72	18	$4A_3 + A_2 + 4A_1$	B) $Q = C_2$
112	72	18	$A_5 + 5A_2 + 3A_1$	B) $Q = C_3, \Sigma = 3A_2 + 10A_1$
113	120	18	$A_4 + 6A_2 + 2A_1$	B) $Q = C_3, \Sigma = 3A_4 + 6A_1$
114	144	19	$A_5 + 4A_3 + 2A_1$	B) $Q = C_2^2, \Sigma = 2A_2 + 12A_1$
115	144	19	$D_5 + A_5 + A_3 + 2A_2 + 2A_1$	B) $Q = C_2, \Sigma =$#112
116	192	19	$A_7 + A_5 + A_2 + 5A_1$	B) $Q = C_2, \Sigma = 2A_7 + 3A_2$
117	240	19	$3A_4 + 3A_2 + A_1$	C)
118	288	19	$A_5 + 3A_3 + 2A_2 + A_1$	B) $Q = C_2, \Sigma = 2A_3 + 5A_2 + 2A_1$
119	288	19	$D_5 + 4A_3 + 2A_1$	B) $Q = C_2^2, \Sigma = 2A_5 + 8A_1$
120	288	19	$D_5 + A_5 + 4A_2 + A_1$	B) $Q = C_3, \Sigma = 3D_5 + 4A_1$
121	720	19	$A_4 + 4A_3 + A_2 + A_1$	A)
122	720	19	$D_5 + A_4 + A_3 + 3A_2 + A_1$	G)

Table 4

A) For #104,#106,#109 and #121, L'^*/L' contains C_4^l where $l > 22 - c$ (note that a configuration D_5 adds 1 to l), hence by Lemma 2, G has a quotient isomorphic to C_4. From the main list, one sees that a cover of group C_4 must correspond to a configuration of type $4A_3 + 2A_1$. Remarking that only the stabilisers of A_3, A_7 and D_5 allow C_4 quotient (with corresponding subconfiguration A_3, $2A_3$ and $A_3 + A_1$ respectively), it is impossible to get a subconfiguration $4A_3 + 2A_1$ in these cases.

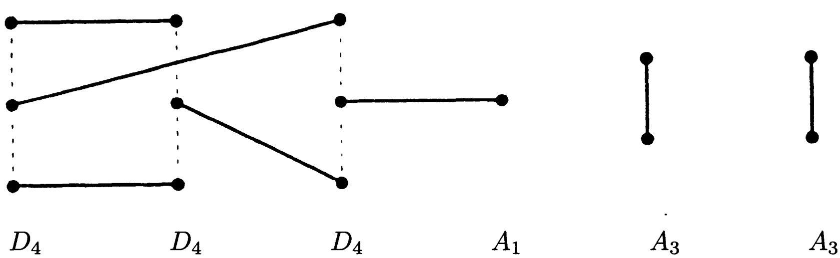
B) For #107,#111-#116 and #118-#120, Lemma 2 imposes a quotient Q of G, for which the kernel corresponds to a configuration Σ which is not in the list. (The 12 A_1's in #111 have two ways to decompose into 3 blocks: $2\,[2A_3]\cup[4A_1]$ or $2\,[A_3, 2A_1]\cup[2A_3]$. The first leads to a $2A_2+12A_1$, the second to $2A_3+2A_2+7A_1$, both not in the list.)

C) #108 and #117 give bad number of 5-Sylow subgroups (6 and 36 respectively).

D) For #103, the action of G on the set of four 3-Sylow subgroups implies a homomorphism $\varphi\colon G \longrightarrow \mathfrak{S}_4$. Its image is clearly either $\mathfrak{A}_4$ or $\mathfrak{S}_4$. But the number of elements of each order in G doesn't make it a $\mathfrak{S}_4$, while the configuration doesn't contain enough A_2 to have a quotient $\mathfrak{A}_4$ which can only be #17.

E) For #105, a group of order 24 always has a cyclic quotient C_2 or C_3. None is allowed by the configuration, due to Lemma 2 and Table 1.

F) #110 : there is only one possible way to decompose the 14 A_1's into 7 blocks, as shown below.



But this decomposition does not verify the following lemma, if we take β_i and β_j to be the 2 blocks linking the first two D_4.

LEMMA 4. — *Let $\psi\colon W \dashrightarrow Y$ be a generically Galois rational map of group $H \cong C_2^3$, with $\{\beta_1, \ldots, \beta_7\}$ the 7 blocks of dependent A_1's on Y. For each pair (i,j), $i \neq j$, there are exactly two double covers $\varphi\colon V \longrightarrow Y$ through which ψ factorises, who are ramified over the 4 (-2)-curves in β_i and β_j. For different pairs (i,j), the sets of 2 double covers are different. There is also exactly one double cover $\varphi'\colon V' \longrightarrow Y$ not ramified over the 4 (-2)-curves.*

Proof. — Each β_i corresponds to an element $\gamma_i \in H$. A double cover φ corresponds to a quotient Q of index 2 of H, and φ is ramified over β_i iff the kernel for Q dos not contain γ_i. The rest is immediate. □

We remark also for the following that two different double covers of the quotient belonging to a group C_2^r have 4 (-2)-curves in common.

G) #122 : Consider a subgroup of order 3, $\langle\gamma_1\rangle$, in the stabiliser of a point in the inverse image of the D_5. Its normaliser N in G has order divisible by 9 (a 3-Sylow subgroup of G must be #15), whose action has a D_5, therefore must be #61 according to the list. As #61 contains only one C_3 in its Q_{12}'s, G has exactly 10 such subgroups of order 3, which are easily seen to be mutually non-normalising.

The action of G on this set S of such subgroups gives $\psi\colon G \longrightarrow \mathfrak{S}_{10}$. Let γ_2 be an element of order 5. γ_2 acts freely on S, decomposing it into two orbits S_1, S_2. Let γ_3 be an element of order 4 in the normaliser of $\langle\gamma_2\rangle$ which must be #32, and consider the actions of γ_3 and γ_3^2 on S. We have $\gamma_3^2\gamma_2\gamma_3^2 = \gamma_2^{-1}$ and S_1, S_2 are stable under the action of γ_3^2, therefore each S_i contains exactly one fixed point of γ_3^2. Now if γ_3 interchanges S_1 and S_2, its action on S is odd, so $\psi^{-1}(\mathfrak{A}_{10})$ is a subgroup of index 2 in G, but we must have $[G,G] = G$ according to Lemma 2; otherwise γ_3 has a fixed point in S_1, so if $\langle\gamma_1\rangle$ is the corresponding subgroup, then γ_3^2 commutes with γ_1. Considering the four orbits of the action of γ_1 on S, one concludes that γ_3^2 should have at least 4 fixed points in S, contradiction.

Now we come to the existence of the cases shown in Table 2.

LEMMA 5. — *Let G be a group acting symplectically on a $K3$ surface, G' a subgroup of G. Let Σ, Σ' be their corresponding configurations, with c, c' the numbers of configurations. Then $c \geqslant c'$.*

Proof. — Let $\mathcal{M}_\Sigma$ be the moduli space of marked $K3$ surfaces having a (-2)-configuration of type Σ. By [N], Proposition 2.9, $\mathcal{M}_\Sigma$ is an analytic space of dimension $20 - c$. As G is the π_1 of the complement of Σ and π_1 is a topological invariant, a union of connected components of $\mathcal{M}_\Sigma$ forms the moduli space of triplets (X, G, Σ), where X is a marked $K3$ surface on which G acts giving rise to a (-2)-configuration Σ on the quotient. Now (X, G', Σ') being a sub-action of (X, G, Σ), we must have $\dim \mathcal{M}_{\Sigma'} \geqslant \dim \mathcal{M}_\Sigma$. □

LEMMA 6. — *For each of the cases in Table 2, the minimal primitive sublattice L is uniquely determined by the configuration and the existence of the group G of order N. In particular, $[G,G]$ and Q are uniquely determined as shown in the table.*

Proof. — For most of the cases, the criterion of §1 gives a unique Q. In case when Q contains a part C_2^2, C_2^3 or C_2^4, Lemma 4 and the requirement that every subgroup of G of index 2 must correspond to a configuration in the table give only one possibility of the decomposition of the A_1's for the quotient, then a unique possibility for L. Also when N is small, the numbers of elements of each order given by the configuration determines G, then Q, and then L. So only the following cases need explanations.

#17 : G must be solvable, and the only cyclic quotient it can have is C_3.

#19 : There is a subgroup of index 2, which must be #7. Then the subgroups of #7 gives quotients of G isomorphic to C_6 and C_2^2. Therefore $[G, G] = \{1\}$. Similar argument works for #31, 32, 35, 43, 46, 52, 53.

#20 : G contains (therefore equals) Q_{12} due to the existence of D_5. This explains the existence of a quotient C_4.

#23 : G has a subgroup H of type #11, hence $H \cong C_2 \times C_4$. The action of $G/H \cong C_2$ on H has a fixed subgroup K, which is either a C_4, or a C_2 not in a C_4. The first case is impossible as there is no configuration for G/K ; in the second case, all the three C_2 of H are normal in G, one of them giving a quotient of type #11.

#24 : $Q \cong C_2^3$. Considering subgroups with quotient C_2^2, one sees that the only possible decomposition of $14A_1$ is $3\,[2D_4] \cup 3\,[2A_1] \cup [A_3]$. Similarly, the decomposition for #27 is $6\,[2D_4] \cup [2A_1]$.

#28 : The C_2 in a C_8 is easily seen to be normal in G, and the quotient is #11. On the other hand, we use [N] to see that G is not abelian, therefore $Q =$#11.

#29 : G equals the stabiliser of a point of type D_6.

#39 : $Q \cong C_2^3$. G has no subgroup of type #25 due to Lemma 5. This gives a unique possibility for L.

#40 : $Q \cong C_2^4$. By Lemma 5, there is no double cover of Y involving $[8A_1]$. Then it is easy to see that there is a unique way to get 15 double covers of Y such that each pair of them has 4 common A_1's: let $\{C_1, C_2, C_3\}$, $\{C_4, C_5, C_6\}$ be the independent A_1's in the two D_4. Each double cover is ramified over either 2 or 4 curves C_i. For each i, there is a subset S_i consisting of 3 of the 9 isolated A_1's, such that:

a) if a double cover is ramified over exactely 2 curves C_i, C_j, then it is ramified over curves in S_i, S_j ;

b) $S_i \cap S_j = \varnothing$ if C_i and C_j belongs to a same D_4, and $|S_i \cap S_j| = 1$ otherwise;

c) if a double cover is ramified over 4 curves C_i, C_j, C_k, C_l, then it is also ramified over the four A_1's which belong to exactly one of the sets S_i, S_j, S_k, S_l.

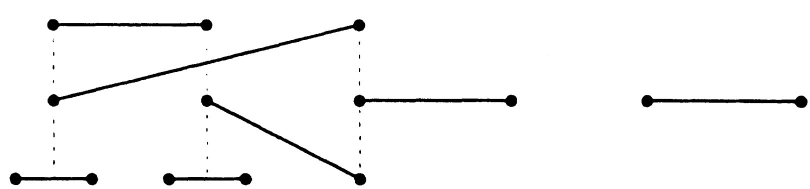
#42 : Lemma 4 gives an exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q' = \langle\gamma\rangle \cong C_2 \longrightarrow 1,$$

where K is of type #25, hence $\cong C_4^2$. From the configurations, it is easy to see that there is a generating set $\{a, b\}$ of K, such that the action of Q' on K is $\gamma(a) = -a$, $\gamma(a+2b) = -a+2b$. This implies that the elements of order 2 in K are fixed under this action, in other words $[G,G] \cong C_2^2$ is in the center of G. In particular there is an element $\alpha \in [G,G]$ whose eight fixed points are above an A_3 of the configuration. Now $G/\langle\alpha\rangle$ is of type #24, whose decomposition fixes the decomposition for #42 to be $3\,[2D_4] \cup 2\,[A_3] \cup 2\,[2A_1]$.

#44 : G has a quotient isomorphic to C_2^2 by Lemma 2, corresponding to an overlattice L'' of L'. But L''^*/L'' has at least 4 generators (3 corresponding to elements of order > 2 in L'^*/L', and one belonging to D_4, which is not touched by L''/L'), therefore L'' is not primitive, and $L \neq L''$. Thus Q can only be #11.

#45 : By Lemma 4, for each D_6 there is a double cover of Y with no ramification over it. The subgroup corresponds to a configuration containing $2D_6$, hence must be #29. This gives a unique possibility of decomposition:



D_6 D_6 D_4 A_1 A_1 A_1

#57 : G contains a central C_2. The quotient H is of type #40, and as the two cases have the same abelianisation, it is easy to see that the set of three A_1 corresponding to the extra D_4 of #57 must intersect each of the set S_j in the proof of #40 by 1. This determines uniquely the overlattice L for #57.

#60 : There are 3 possible decompositions: $2\,[D_6] \cup 2\,[2D_6] \cup [A_3] \cup 2\,[2A_1]$, $2\,[D_6] \cup [2D_6] \cup 2\,[D_6, A_1] \cup [A_3] \cup [2A_1]$, $2\,[D_6] \cup 4\,[D_6, A_1] \cup [A_3]$. The first (only one double cover containing $2\,[2D_6]$) and the last (no double cover not containing the two $[D_6, A_1]$ belonging to the same D_6) are excluded by Lemma 4.

#66 : The subgroup of index 2 contains $6A_2$ hence must be #49. Therefore $[G, G]$ corresponds to $15A_1$.

#71 : In view of Lemma 4, the decomposition is

$$[D_6] \cup [D_6, D_4] \cup [D_6, A_1] \cup 2\,[D_4, A_1] \cup 2\,[A_3] \ . \qquad \square$$

From this lemma, one sees that except for two cases (#12,13 for Q_8, #37,38 for T_{24}), different cases in Table 2 correspond to different groups, due to differences either in number of elements of different order, or in $[G, G]$ and Q. And each such characterization of group corresponds to a subgroup of the 11 maximal groups whose existence is shown by examples in [M] (refer also to [M] for the descriptions of these subgroups). We have therefore only to show the existence for the two duplicate cases.

Among them, #12,13 are commutator subgroups of #38,37 respectively, and #38 is the commutator subgoup of T_{48} (#54).

For #37, consider the stabiliser H of a point in the inverse image of a E_6, in #77 (T_{192}). We show that H cannot be of type #38, hence it is #37.

Assume the contrary. Then H is of index 2 in its normaliser N in $G = T_{192}$, because the configuration of H has two E_6, while that of G has only one. The only possibility for N given by the list is #54, which is impossible because #54 has elements of order 8, but #77 hasn't. $\square$

COROLLARY. — *Let X be a $K3$ surface with a faithful action of a finite group G. Then $|G| \leqslant 5760$.*

Proof. — Let

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be the decomposition of G into a symplectic subgroup K and non-symplectic quotient $Q \cong \mathbb{Z}_n$.

Let Y be minimal resolution of the intemediate quotient X/K, on which we have a purely non-symplectic action of Q. Y is a minimal $K3$

surface. It is well-known that the Euler number $\varphi(n)$ of n must divide the rank of the transcendent lattice of Y, which is $22 - \rho(Y)$. When K is non-trivial, the (-2)-configurations resulting from the resolution of singularities of X/K generates a negative-definite sublattice of $NS(Y)$ of rank c, hence $\rho(Y) \geqslant c + 1$, or $\varphi(n) \leqslant 21 - c$. Now one has only to check the inequality for each case of the list in the theorem. □

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