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CONSTRUCTING GENERIC SMOOTH MAPS OF A MANIFOLD INTO A SURFACE WITH PRESCRIBED SINGULAR LOCI

by
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1. Introduction.

Let $f : M \to N$ be a smooth map of an n-dimensional manifold M ($n \geq 2$) into a surface N. It has been known that if f is generic enough, then f has only folds and cusps as its singularities and that the singular set $S(f)$ is a closed 1-dimensional submanifold of M (see [W], [T], [L2], [L3]). It is a very important problem in the study of the global topology of generic maps to study the position of their singular set (see [T], Chap. IV and V, and [E2], § 1). When M is closed, $S(f)$ represents a 1-dimensional homology class in $\mathbb{Z}_2$-coefficients and this class has been studied by Thom [T], who described its Poincaré dual in terms of the Stiefel-Whitney classes of TM and f^*TN. Furthermore, he showed that the number of cusps has the same parity as the Euler characteristic of the source manifold M when N is orientable. Eliašberg [E1], [E2] studied the problem of realizing a given submanifold as the singular set of a generic map containing only fold singularities and obtained some realization theorems assuming the existence of a certain map between the tangent spaces of M and N.

Our purpose of this paper is to study the singular set $S(f)$ of a generic map $f : M \to N$ as a submanifold of M and to give a complete characterization of such submanifolds. In fact our main theorem is that Thom's homological condition as above suffices to realize a given

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closed 1-dimensional submanifold as the singular set of a generic map (see Theorem 2.2 in § 2).

Given a generic map $f : M \to N$, every singular point has its own absolute index defined via a certain bilinear form associated with the singular point (see [L3]). Consequently, the singular set $S(f)$ is stratified into a finite number of strata on each of which the index is constant, where the 0-dimensional strata correspond to the cusps and the 1-dimensional ones to the folds. These strata naturally satisfy some adjacency conditions with respect to the indices. Our second main theorem is that, given a closed 1-dimensional stratified submanifold satisfying the adjacency conditions together with Thom's homological conditions, we can realize it as the singular set of a generic map consistent with the indexed stratification as long as all the indices appear (see Theorem 2.4 in § 2).

Our idea of the proof is to begin with an arbitrary generic map $g : M \to N$ and to modify it homotopically so that the singular set is isotopic to a given submanifold. For this purpose we study the change of the isotopy class of the singular set in the course of a generic homotopy of smooth maps. We also use the techniques developed in [L3] for eliminating cusps. Note that similar results have been obtained by Eliašberg [E1], [E2] in some cases for smooth maps $f : M \to \mathbb{R}^p$ of n-dimensional manifolds into $\mathbb{R}^p$ $(n \geq p)$ with only fold singular points and that our technique is totally different from his. Furthermore, our result gives a complete characterization of submanifolds arising as the singular set of a generic map, although the target manifolds are restricted to those of dimension 2.

The paper is organized as follows. In § 2, we prepare some known results and notations and state our main theorems precisely. In § 3, we prove some fundamental lemmas for the proof of our main theorems. These results may have already been known for some specialists; however, we included detailed proofs, since there have been no rigorous proofs in the literature. We prove our main theorems in §§ 4 and 5. In § 6, we give some consequences of our results. In particular, we give a complete answer to the problem, posed in [S2], of characterizing 1-dimensional submanifolds of a 3-manifold arising as the singular set of a generic map into the plane. One of the consequences of our results is the remarkable fact that every knot or link in S^3 (or more generally in any homology 3-sphere) is realized as the singular set of a stable map into the plane.

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2. Preliminaries and statement of main theorems.

Let $f : M \to N$ be a smooth map of a connected closed n-dimensional manifold M (where $n \geq 2$) into a connected surface N. We denote by $S(f)$ the singular set of f, *i.e.*, the set of the points in M where the rank of *df* is less than 2. Furthermore, we denote by $J^i(f)$ the induced map of M into $J^i(M, N)$, where $J^i(M, N)$ is the bundle of i-jets of maps of M into N. Then $f : M \to N$ is said to be *generic* if the following conditions are satisfied (see [L3]) :

(1) $J^1(f)$ is transversal to $S_1(M, N)$ and $S_2(M, N)$, and

(2) $J^2(f)$ is transversal to $S_1^2(M, N)$.

(For notations, see [L1], [L2].) This is equivalent to the following normal form interpretation : f is generic if and only if, for each point $p \in S(f)$, one of the following two holds :

(3) We can choose coordinates $(u, z_1, \ldots, z_{n-1})$ centered at p and (U, Y) centered at $f(p)$ so that, in a neighborhood of p, f has one of the forms :

$$U = u, \quad Y = \sum_{i=1}^{n-1} \pm z_i^2.$$

(4) We can choose coordinates $(u, x, z_1, \cdots, z_{n-2})$ centered at p and (U, Y) centered at $f(p)$ so that, in a neighborhood of p, f has one of the forms :

$$U = u, \quad Y = \sum_{i=1}^{n-2} \pm z_i^2 + xu + x^3.$$

If $p \in S(f)$ satisfies the condition (3), we call p a *fold point*, and if it satisfies (4), a *cusp*. It is easy to observe that if f is generic, then the singular set $S(f)$ is a closed 1-dimensional submanifold of M. Note that the set of generic maps is open and dense in the space of all smooth maps $C^\infty(M, N)$ topologized with the C^∞-topology.

The following theorem has been proved by Thom [T].

THEOREM 2.1 (Thom [T]). — *Let $f : M \to N$ be a generic map of a closed n-dimensional manifold M into a surface N. Then the Poincaré dual of the $\mathbb{Z}_2$-homology class represented by $S(f)$ coincides with the degree $(n-1)$-part of $w(M) \cup (f^*w(N))^{-1}$, where w denotes the total Stiefel-Whitney class.*

In particular, if two generic maps $f, g : M \to N$ are homotopic, then their singular sets $S(f)$ and $S(g)$ are $\mathbb{Z}_2$-homologous (see also [T], Thm. 7). Our first main theorem of this paper is, in a sense, a converse of this observation.

THEOREM 2.2. — *Let $f : M \to N$ be a generic map of a closed n-dimensional manifold ($n \geq 3$) into a surface N. Suppose that L is a nonempty closed 1-dimensional submanifold of M which is $\mathbb{Z}_2$-homologous to $S(f)$. Then there exists a generic map $\bar{f} : M \to N$ homotopic to f such that $S(\bar{f}) = L$.*

Recall that every singular point p of a generic map f has its own absolute index $\tau(p)$ (see [L3], p. 273). Note that, if p is a fold point, then

$$n - 1 \geq \tau(p) \geq n - 1 - m,$$

and that if p is a cusp, then

$$n - 2 \geq \tau(p) \geq n - 2 - k,$$

where m is the greatest integer not exceeding $\frac{1}{2}(n-1)$ and k is the greatest integer not exceeding $\frac{1}{2}(n-2)$. Note that $k = m - 1$ when n is odd and that $k = m$ when n is even. Define $F_i(f)$ to be the set of the fold points of absolute index i and $C_j(f)$ to be the set of the cusps of absolute index j. In [L3], it is shown that $F_i(f)$ is a finite disjoint union of open arcs and circles and that $C_j(f)$ consists of a finite number of points. Furthermore, we have the following adjacency conditions:

$$(2.3) \quad \begin{cases} \overline{F}_i(f) \cap \overline{F}_{i+1}(f) = C_i(f) & (i = n-2, \cdots, n-1-m), \\ \operatorname{Int}(\overline{F}_{n-1-m}(f)) - F_{n-1-m}(f) = C_{n-2-k}(f) & (\text{when } n \text{ is even}), \\ \overline{F}_i(f) \cap \overline{F}_j(f) = \emptyset & (\text{when } |i-j| \geq 2), \end{cases}$$

where the overline and Int denote the closure and the interior points in $S(f)$ respectively. Recall that, when n is even and N is orientable, the Euler characteristic of M has the same parity as $\sharp C_{n-2-k}(f)$, where $\sharp$ denotes the cardinality (see [L3], p. 285).

Our second main theorem is stated as follows.

THEOREM 2.4. — *Let $f : M \to N$ be a generic map of a closed orientable n-dimensional manifold ($n \geq 3$) into an orientable surface. Suppose that*

$$L = \Big(\bigcup_{i=n-1-m}^{n-1} F_i \Big) \cup \Big(\bigcup_{j=n-2-k}^{n-2} C_j \Big)$$

is a nonempty closed 1-dimensional stratified submanifold of M satisfying the adjacency conditions similar to (2.3). *When n is even, we also assume that the Euler characteristic of M has the same parity as $\sharp C_{n-2-k}$. If L is $\mathbb{Z}_2$-homologous to $S(f)$ and $F_i \neq \emptyset$ for all i, then there exists a generic map $\bar{f} : M \to N$ homotopic to f such that $S(\bar{f}) = L$, $F_i(\bar{f}) = F_i$ and $C_j(\bar{f}) = C_j$ for all i and j.*

Note that a similar result for $n = 2$ has been obtained by Eliašberg (see [E1], § 4).

Remark 2.5. — In Theorem 2.4, the condition $F_i \neq \emptyset$ for all i is necessary in general. For example, if $F_{n-1} \neq \emptyset$ and $F_{n-2} = \cdots = F_{n-1-m} = \emptyset$, then $\bar{f}$ must be a special generic map. Note that those manifolds which admit special generic maps into orientable surfaces have been characterized in [S1] and that not every manifold admits such a map (see also [BdR], [PF]).

Remark 2.6. — When the target manifold is of dimension 1, say $\mathbb{R}$, a theorem like Theorem 2.4 does not hold, because of the Morse inequality. In other words, Theorem 2.4 shows that we do not have a theorem which corresponds exactly to the classical Morse inequality, for generic maps into surfaces. Compare this with [MPS], § 3. See also [E2], § 1. We note, however, that we have probably a *Morse type inequality* for generic maps into surfaces. For example, when the target manifold is the plane $\mathbb{R}^2$, the composition $\pi \circ f$ of a generic map $f : M \to \mathbb{R}^2$ with a generic projection $\pi : \mathbb{R}^2 \to \mathbb{R}$ is a Morse function and its critical points and their indices are closely related to the singular points of f and their absolute indices respectively.

3. Basic lemmas.

In the following, we assume that $f : M \to N$ is a smooth generic map of a closed n-dimensional manifold M (where $n \geq 2$) into a surface N.

LEMMA 3.1. — *Let $p \in M$ be a regular point of f and U an open neighborhood of p in M with $U \cap S(f) = \emptyset$. Then there exists a smooth homotopy $f_t : M \to N$ (with $t \in [0, 1+\varepsilon]$) for some positive real number ε such that $f_0 = f$, that $f_t = f$ outside of U, that f_t is generic for all $t \neq 1$, and that $U \cap S(f_{1+\varepsilon})$ is an unknotted circle in U.*

Here, an embedded circle in a manifold is *unknotted* if it bounds an embedded 2-disk.

Proof of Lemma 3.1. — Since p is a regular point, there exist local coordinates $(u, x, z_1, \cdots, z_{n-2})$ centered at p and (X, Y) centered at $f(p)$ such that f has the form

$$X = u, \quad Y = x.$$

Since the two maps

$$(u, x, z_1, \cdots, z_{n-2}) \longmapsto (u, x) \quad \text{and}$$
$$(u, x, z_1, \cdots, z_{n-2}) \longmapsto \Big(u, (u^2+1)x + \tfrac{1}{3}x^3 + \sum_{i=1}^{n-2} a_i z_i^2\Big)$$

$(a_i = \pm 1)$ are locally right equivalent, we may assume that f has the form

$$X = u, \quad Y = (u^2+1)x + \frac{1}{3}x^3 + \sum_{i=1}^{n-2} a_i z_i^2,$$

by changing the local coordinates if necessary. There exists a positive real number δ such that

$$B = \big\{|u|, |x|, |z| < \delta\big\} \subset U, \quad \text{where} \quad |z| = \Big(\sum_{i=1}^{n-2} z_i^2\Big)^{1/2}.$$

Take arbitrary real numbers m_1 and m_0 such that $0 < m_1 < m_0 < \delta$. Then there exists a nonnegative C^∞ function $\theta : \mathbb{R} \to \mathbb{R}$ with $\theta'(t) \leq 0$ such that

$$\theta(t) = \begin{cases} 0 & \text{if } t \geq m_0^2, \\ 1 & \text{if } t \leq m_1^2. \end{cases}$$

Set $K = \max_{t \in \mathbb{R}} |\theta'(t)|$ $(K < \infty)$ and take real numbers k_0 and k_1 such that $0 < k_1 < k_0 < \min\{\delta, 1/(2K)\}$. Furthermore, take real numbers r_0 and r_1 such that $0 < r_0 < r_1 < \delta$. There exist nonnegative C^∞ functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ with $\varphi'(t), \psi'(t) \leq 0$ such that

$$\varphi(t) = \begin{cases} 0 & \text{if } t \geq k_0^2, \\ 1 & \text{if } t \leq k_1^2 \end{cases}$$

and

$$\psi(t) = \begin{cases} 0 & \text{if } t \geq r_0^2, \\ 1 & \text{if } t \leq r_1^2. \end{cases}$$

Furthermore, take ε with $0 < \varepsilon < \min\{1, k_1^2, r_1^2\}$. Then for $t \in [0, 1+\varepsilon]$, define $f_t : M \to N$ by

$$X = u, \quad Y = (u^2+1)x + \tfrac{1}{3}x^3 + \sum_{i=1}^{n-2} a_i z_i^2 - tx\,\varphi(x^2)\,\psi(u^2)\,\theta(|z|^2)$$

on B and by $f_t = f$ on $M - B'$, where

$$B' = \{|x| \le k_0, |u| \le r_0, |z| \le m_0\}.$$

It is easy to check that f_t is a well-defined smooth homotopy of maps, $f_0 = f$, and that $f_t = f$ outside of U. In the following, we shall study the behavior of $f_{t|B'}$.

(a) When $|z| \ge m_1$, we see that

$$Y_{z_i} = 2z_i(a_i - tx\,\varphi(x^2)\,\psi(u^2)\,\theta'(|z|^2)) \ne 0,$$

since $|tx\varphi(x^2)\psi(u^2)\theta'(|z|^2)| < 2k_0K < 1$. Hence we have that $Y_{z_i} \ne 0$ for some i and that f_t is nonsingular in this region.

(b) When $|x| \ge k_1$ and $|z| \le m_1$, we see that

$$\begin{aligned} Y_x &= u^2 + 1 + x^2 - t\,\varphi(x^2)\,\psi(u^2) - 2tx^2\varphi'(x^2)\,\psi(u^2) \\ &\ge 1 + x^2 - (1+\varepsilon) \ge k_1^2 - \varepsilon > 0 \end{aligned}$$

and hence that f_t is nonsingular also in this region.

(c) When $|x| \le k_1, |u| \ge r_1$ and $|z| \le m_1$, we see that

$$Y_x = u^2 + 1 + x^2 - t\psi(u^2) \ge u^2 + 1 - (1+\varepsilon) \ge r_1^2 - \varepsilon > 0$$

and hence that f_t is nonsingular also in this region.

(d) When $|x| \le k_1, |u| \le r_1$ and $|z| \le m_1$, we see that f_t is of the form

$$X = u, \quad Y = (u^2+1)x + \tfrac{1}{3}x^3 + \sum_{i=1}^{n-2} a_i z_i^2 - tx$$

and that

$$Y_x = u^2 + 1 + x^2 - t, \quad Y_{z_i} = 2a_i z_i,$$

$$Y_{xx} = 2x, \qquad Y_{xu} = 2u.$$

Thus we see that $J^1(f_t)$ is always transversal to $S_2(M,N)$ and that $J^1(f_t)$ is not transversal to $S_1(M,N)$ at (u,x,z) if and only if $x=u=z_1=\cdots=z_{n-2}=0$ and $t=1$ (see the condition (a″) of [L3], § 3).

Furthermore, we see that, for $0 \le t < 1$, f_t is nonsingular in this region and, for $1 < t \le 1+\varepsilon$, we have

$$S_1(f_t)\cap B' = \{z=0, x^2+u^2=t-1\},$$

which is an unknotted circle in $B' \subset U$, and

$$\begin{aligned} S_1^2(f_t)\cap B' &= \left\{\operatorname{rank}\begin{pmatrix} Y_{xx} & Y_{xz} \\ Y_{zx} & Y_{zz}\end{pmatrix} < n-1\right\} \cap (S_1(f_t)\cap B') \\ &= \{x=z_1=\cdots=z_{n-2}=0,\ u^2=t-1\}, \end{aligned}$$

which consists of two points. (For the image of $S_1(f_t)\cap B'$ by f_t, see Fig. 1.)

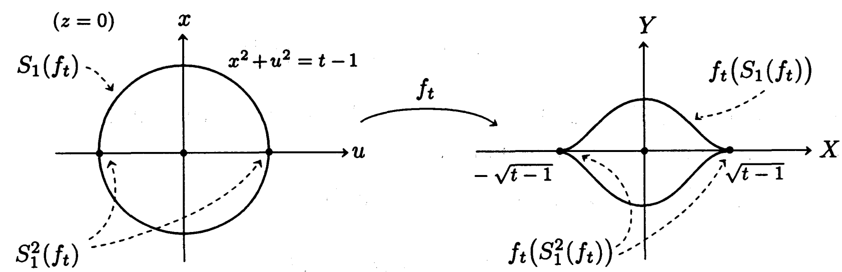


Figure 1.

Furthermore, in the latter case, at the two points, we have $Y_{xxx} \ne 0$. Hence $J^2(f_t)$ is transversal to $S_1^2(M,N)$ (see the condition (b″) of [L3], § 3). Thus f_t is generic for all $t \ne 1$. This completes the proof of Lemma 3.1. □

Remark 3.2. — Note that $(S_1(f_t) - S_1^2(f_t)) \cap B'$ consists of two components for $t > 1$. Choosing $a_i = \pm 1$ appropriately, we can arrange the absolute indices of the two cusps $S_1^2(f_t)$ and the two fold curves as we desire, as long as they satisfy the adjacency conditions as in (2.3). We also note that the homotopy $f_{t|B'}$ constructed above corresponds to the « Lip » of [C] and that $f_{t|(S_1(f_t)-S_1^2(f_t))\cap B'}$ is an embedding for $t > 1$.

We define the vector bundles L and G over $S_1(f)$ by the exactness of

$$0 \to L \longrightarrow TM_{|S_1(f)} \xrightarrow{df} f^*TN_{|S_1(f)} \xrightarrow{\pi} G \to 0.$$

For a point $p \in S_1(f)$, we denote the corresponding fibers over p by L_p and G_p. Note that $\dim L_p = n-1$ and $\dim G_p = 1$.

LEMMA 3.3. — *Let p be a point in $S_1(f) - S_1^2(f)$ and $g(p) \in G_p$ a fixed orientation. We denote by $i(p)$ the index of p with respect to the orientation $g(p)$. We assume that $i(p) \neq n-1$. Let U be a small open neighborhood of p in M such that $(U, U \cap (S_1(f) - S_1^2(f)))$ is diffeomorphic to $(\mathbb{R}^n, x_1\text{-axis})$. Then there exists a smooth homotopy $f_t : M \to N$ $(t \in [0, 1+\varepsilon])$ for some $\varepsilon > 0$ with the following properties:*

(1) $f_0 = f$;

(2) *$f_t = f$ outside of U;*

(3) *f_t is generic for all $t \neq 1$;*

(4) *$(U, U \cap S_1(f_{1+\varepsilon}))$ is diffeomorphic to $(\mathbb{R}^n, x_1\text{-axis})$;*

(5) *$U \cap S_1^2(f_{1+\varepsilon})$ consists of two points p_1 and p_2;*

(6) *let the three components of $U \cap (S_1(f_{1+\varepsilon}) - S_1^2(f_{1+\varepsilon}))$ be denoted by A_1, A_2 and A_3, where $\overline{A}_1 \cap \overline{A}_2 = p_1$ and $\overline{A}_2 \cap \overline{A}_3 = p_2$. Then $G_{|U \cap S_1(f_{1+\varepsilon})}$ has a natural orientation g' induced by $g(p)$ and with respect to this orientation g', the indices of the fold curves A_1, A_2 and A_3 are equal to $i(p), i(p)+1$ and $i(p)$ respectively and the indices of the cusps p_1 and p_2 are both equal to $i(p)$ (see Fig. 2).*

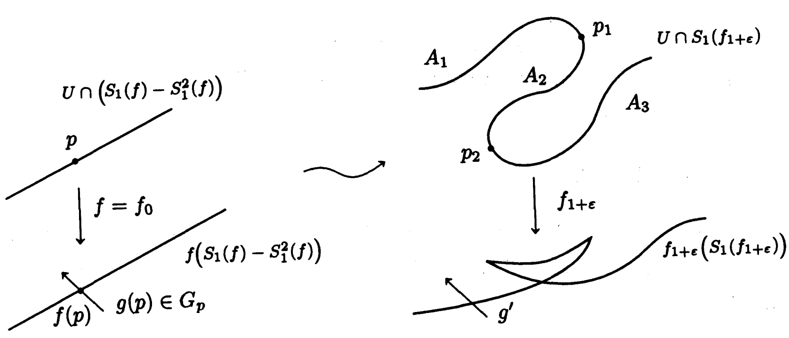


Figure 2.

Proof. — Since $p \in S_1(f) - S_1^2(f)$, there exist local coordinates $(u, x, z_1, \cdots, z_{n-2})$ centered at p and (X, Y) centered at $f(p)$ such that f has the form

$$X = u, \quad Y = Q(z) + x^2$$

and that $\pi(\partial/\partial Y_p) = g(p)$, where $Q(z)$ is a nonsingular quadratic form of index $i(p)$ (*i.e.*, $Q(z) = -z_1 - \cdots - z_{i(p)}^2 + z_{i(p)+1}^2 + \cdots + z_{n-2}^2$). This is true, since $i(p) \neq n-1$. Since this map is locally right-left equivalent to

$$(u, x, z_1, \cdots, z_{n-2}) \longmapsto \big(u, Q(z) + x^4 + ux + x^2\big),$$

we may assume that f is of the form

$$X = u, \quad Y = Q(z) + x^4 + ux + x^2$$

and $\pi(\partial/\partial Y_p) = g(p)$, changing the local coordinates if necessary. There exists a small positive real number $\delta > 0$ such that

$$B = \big\{|x|, |u|, |z| < \delta\big\} \subset U.$$

Take arbitrary real numbers m_1 and m_0 with $0 < m_1 < m_0 < \delta$ and a smooth nonnegative C^∞ function $\theta : \mathbb{R} \to \mathbb{R}$ with $\theta'(t) \leq 0$ as in the proof of Lemma 3.1. Set $K = \max_{t \in \mathbb{R}} |\theta'(t)|$ and take positive real numbers k_0 and k_1 with $0 < k_1 < k_0 < \min\big\{\delta, \sqrt{1/(2K)}\,\big\}$. Furthermore, take positive real numbers r_0, r_1 and ε such that

$$0 < \varepsilon < \min\{2k_1^2, 1\}, \quad 0 < r_1 < r_0 < \delta,$$

$$\tfrac{2}{3}\varepsilon(2k_1^2 + 1)^2 < r_1^2 < r_0^2 < 4k_1^2(2k_1^2 - \varepsilon)^2$$

and $64(\frac{1}{6}\varepsilon)^3 < r_1^2$, and take smooth nonnegative C^∞ functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ with $\varphi'(t), \psi'(t) \leq 0$ as in the proof of Lemma 3.1. Then, for $t \in [0, 1+\varepsilon]$, define $f_t : M \to N$ by

$$X = u, \quad Y = Q(z) + x^4 + ux + x^2 - tx^2\varphi(x^2)\,\psi(u^2)\,\theta\big(|z|^2\big)$$

on B and by $f_t = f$ on $M - B'$, where B' is as in the proof of Lemma 3.1. We see easily that f_t is a well-defined smooth map, $f_0 = f$ and that $f_t = f$ outside of U.

We now study the behavior of $f_{t|B'}$.

(a) When $|z| \geq m_1$, we see that

$$Y_{z_i} = 2z_i\big(\pm 1 - tx^2\varphi(x^2)\,\psi(u^2)\,\theta'(|z|^2)\big) \neq 0,$$

since $|tx^2\varphi(x^2)\psi(u^2)\theta'(|z|^2)| < 2k_0^2K < 1$. Hence f_t is nonsingular in this region.

(b) When $|x| \geq k_1$ and $|z| \leq m_1$, we see that

$$Y_x = u + 2x\big(2x^2 + 1 - t\,\varphi(x^2)\,\psi(u^2) - tx^2\varphi'(x^2)\,\psi(u^2)\big) \neq 0,$$

since

$$\begin{aligned}\big|2x(2x^2+1-t\,\varphi(x^2)\,\psi(u^2)-tx^2\varphi'(x^2)\,\psi(u^2))\big|\\ \geq 2k_1\big(2k_1^2+1-(1+\varepsilon)\big)=2k_1(2k_1^2-\varepsilon)\end{aligned}$$

and $|u|\leq r_0<2k_1(2k_1^2-\varepsilon)$. Hence f_t is nonsingular also in this region.

(c) When $|x|\leq k_1, |u|\geq r_1$ and $|z|\leq m_1$, we see that

$$Y_x=u+2x\big(2x^2+1-t\,\psi(u^2)\big)$$

and hence that

$$\begin{aligned}S_1(f_t)\cap B'&=\big\{Y_x=Y_{z_1}=\cdots=Y_{z_{n-2}}=0\big\}\\&=\big\{u=-2x(2x^2+1-t\psi(u^2)),\ z_1=\cdots=z_{n-2}=0\big\}.\end{aligned}$$

Note that $Y_x<0$ for $x=-k_1$ and $Y_x>0$ for $x=k_1$. Suppose that $Y_x=0$ at a point. Then we have

$$\begin{gathered}u=-2x\big(2x^2+1-t\,\psi(u^2)\big),\\ 0<\big|2x^2+1-t\,\psi(u^2)\big|\leq 2k_1^2+1,\quad |u|\geq r_1\end{gathered}$$

and hence

$$|x|\geq\frac{r_1}{2(2k_1^2+1)}.$$

Then we have

$$Y_{xx}=12x^2+2-2t\,\psi(u^2)\geq\frac{3r_1^2}{(2k_1^2+1)^2}-2\varepsilon>0.$$

Hence, for each hyperplane $\{u=\text{constant}\}$, there exists a unique point in $S_1(f_t)$ and $J^1(f_t)$ is transversal to $S_1(M,N)$ at the point. Furthermore, cusp points of f_t do not exist in this region. Note that the original orientation $g(p)$ induces a natural orientation of G for f_t and that, with respect to this orientation, the index of each point in $S_1(f_t)$ in this region is equal to $i(p)$.

(d) When $|x|\leq k_1, |u|\leq r_1$ and $|z|\leq m_1$; in this region f_t has the form

$$X=u,\quad Y\doteq Q(z)+x^4+ux+x^2-tx^2.$$

Hence we have

$$\begin{aligned}Y_x&=4x^3+u+2(1-t)x, & Y_{z_i}&=\pm 2z_i,\\ Y_{xx}&=12x^2+2(1-t), & Y_{xu}&=1\neq 0,\\ Y_{z_iu}&=Y_{z_ix}=0.\end{aligned}$$

Thus, in this region, $J^1(f_t)$ is transversal to $S_1(M,N)$. Furthermore we have

$$\begin{aligned} S_1^2(f_t)\cap B' &= \left\{\operatorname{rank}\begin{pmatrix} Y_{xx} & Y_{xz} \\ Y_{zx} & Y_{zz}\end{pmatrix} < n-1\right\} \cap \left(S_1(f_t)\cap B'\right) \\ &= \left\{Y_x = Y_{z_i} = Y_{xx} = 0\right\}\cap B' \\ &= \left\{z_i = 0,\ u = 8x^3,\ x^2 = \tfrac{1}{6}(t-1)\right\}\cap B'. \end{aligned}$$

Thus,

- for $0 \le t < 1$, we have $S_1^2(f_t)\cap B' = \emptyset$;
- for $t = 1$, $S_1^2(f_t)\cap B'$ consists of one point $(x,u,z) = (0,0,0)$;
- for $1 < t \le 1+\varepsilon$, we have $|x| = \sqrt{\frac{1}{6}(t-1)} \le \sqrt{\frac{1}{6}\varepsilon} < k_1$, $|u| = |8x^3| < r_1$ and hence

$$S_1^2(f_t)\cap B' = \left\{z_i = 0,\ x = \pm\sqrt{\tfrac{1}{6}(t-1)},\ u = 8x^3\right\},$$

which consists of two points.

At these points, we have $Y_{xxx} = 24x \neq 0$. Hence $J^2(f_t)$ is transversal to $S_1^2(M,N)$ for $t > 1$ and we conclude that f_t is generic for $t \neq 1$. Note that f_1 is not generic.(For the set $S_1(f_t)\cap B'$, see Fig. 3.)

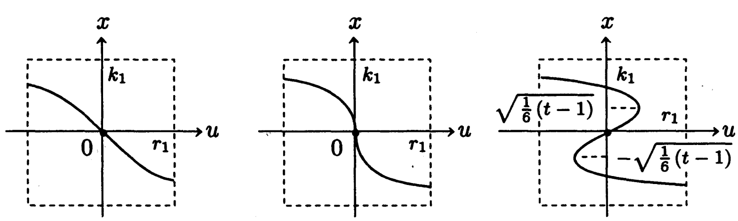


Figure 3.

Furthermore, we have

$$S_1(f_t)\cap B' = \left\{z_i = 0,\ u = -2x(2x^2 + (1-t))\right\}\cap B',$$

which implies the required property (4) of the lemma. Moreover, for $1 < t \le 1+\varepsilon$ and a point in $(S_1(f_t) - S_1^2(f_t))\cap B'$, we have

$$Y_{xx}\begin{cases} > 0 & \text{if } |x| > \sqrt{(t-1)/6}, \\ < 0 & \text{if } |x| < \sqrt{(t-1)/6}. \end{cases}$$

This verifies the required property (6) of the lemma. This completes the proof. $\square$

Remark 3.4. — We note that the homotopy $f_{t|B'}$ constructed above corresponds to the «Swallow-tail» of [C]. Furthermore, we note that $f_{t|S_1(f_t)\cap B'}$ is an embedding for $0 \leq t < 1$, while $f_{t|(S_1(f_t)-S_1^2(f_t))\cap B'}$ is an immersion with normal crossings for $t > 1$.

Let p and p' be distinct cusps of a generic map $f : M \to N$. In [L3], the case where N is orientable has been considered and a definition of a matching pair (p, p') has been given. Here we give a definition of a matching pair which works also in the nonorientable case. Let $\lambda : [0,1] \to M$ be a joining curve in the sense of [L3], (4.4). Then $(f\circ\lambda)^*TN$ is a 2-plane bundle over $[0,1]$ and we fix its orientation. Take $\nu \in T_pM$ which points downward (see [L3], (4.2)), and take $\mu \in T_{f(p)}N$ such that $f_*(\nu)\wedge\mu$ is consistent with the fixed orientation of

$$(f\circ\lambda)^*(T_{f(p)}N) = \big((f\circ\lambda)^*TN\big)_0.$$

Set $\gamma = \pi(\mu)$, where $\pi : f^*TN_{|S_1(f)} \to G$ is the natural projection and we regard $\mu \in (f^*TN)_p$. Note that γ gives an orientation of G_p. We define $I(p)$ to be the index i_γ of p with respect to the orientation γ (see [L3], p. 273). Similarly, we define $I(p')$ using the fixed orientation of $(f\circ\lambda)^*(T_{f(p')}N) = ((f\circ\lambda)^*TN)_1$.

DEFINITION 3.5. — *A pair of cusps (p,p') $(p \neq p')$ is called* a matching pair with respect to a joining curve λ *if* $I(p) + I(p') = n-2$.

Note that the above definition does not depend on the choice of the fixed orientation of $(f\circ\lambda)^*TN$. Note also that, when N is orientable, this definition does not depend on the choice of the joining curve λ and it coincides with the definition given in [L3].

DEFINITION 3.6. — *Let M be an n-dimensional manifold and L a compact 1-dimensional submanifold of M. Furthermore, let $b : J \times J \to M$ $(J = [-1,1])$ be an embedding such that $b(J\times J)\cap L = b(\partial J\times J)$. Such an embedding is called* a band. *Define L' to be the compact 1-dimensional submanifold of M obtained from $(L - b(\partial J\times J))\cup b(J\times\partial J)$ by smoothing the corners (see Fig. 4). Then we say that L' is* obtained from L by a band operation along the arc $b(J\times\{0\})$ (or along the band b). *We also call the arc $b(J\times\{0\})$ the* core of the band b. *Note that this operation is invertible; i.e., if L' is obtained from L by a band operation as above, then L is obtained from L' by a band operation along the band $b\circ\rho$, where $\rho : J\times J \to J\times J$ is defined by $\rho(s,t) = (t,s)$. Note also that the band operation is well-defined up to isotopy.*

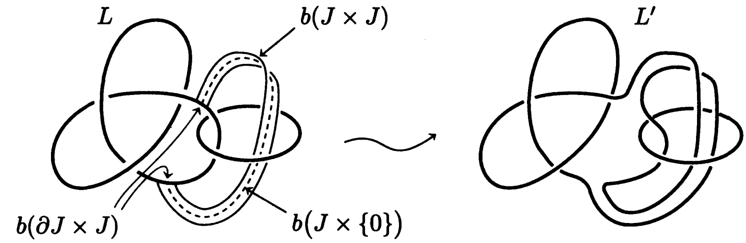


Figure 4.

When L is oriented and the band b is consistent with the orientation (*i.e.*, $b(\partial(J \times J))$ is oriented and the inclusion $b(\partial J \times J) \subset L$ is orientation reversing), we call the above operation an *oriented band operation.*

LEMMA 3.7. — *Let (p, p') be a matching pair of cusps with respect to a joining curve λ and U an open neighborhood of $\lambda([0,1])$ in M. Then there exists a smooth homotopy $f_t : M \to N$ $(t \in [0,1])$ with the following properties:*

(1) $f_0 = f$;

(2) $f_t = f$ *outside of* U;

(3) *there exists a* $t_0 \in [0,1]$ *such that* f_t *is generic for all* $t \neq t_0$;

(4) $\sharp S_1^2(f_t) = \sharp S_1^2(f)$ *for all* t *with* $0 \leq t < t_0$, *where* $\sharp$ *denotes the cardinality*;

(5) $\sharp S_1^2(f_t) = \sharp S_1^2(f) - 2$ *for all* t *with* $t_0 < t \leq 1$;

(6) $S_1(f_1)$ *is obtained from* $S_1(f)$ *by a band operation along* $\lambda([0,1])$.

Proof. — The proof is almost given in [L3], (4.6)–(4.9). The only condition we have to check is our condition (6). Note that, in [L3], Levine constructs the required homotopy in two steps. In his first step (see [L3], Prop. 1, p. 292), the singular set of f_t coincides exactly with that of f (see the equation for $S_1(f_t)$ for this step [L3], p. 291, which does not depend on t). In the following, we will study the change of $S_1(f_t)$ in the course of Levine's second step (see [L3], Lemma, p. 293), using the same notations as Levine's.

First note that, for $|x| \geq k_0$, $|u| \geq r_0$ or $|z| \geq m_0$, we have $f_t = f$ and that for $|x| \geq k_1$, or $|x| < k_1$ and $|z| \geq m_1$, we have $S_1(f_t) = \emptyset$.

(a) When $|x| < k_1$, $r_1 \le |u| < r_0$ and $|z| < m_1$ (*i.e.*, in the region C). In this region, $S_1(f_t) = \{z_i = 0, Y_x = 0\}$. Furthermore, we see that

$$Y_x \begin{cases} > 0 & \text{if } x = 0, \\ < 0 & \text{if } x = \pm k_1 \end{cases}$$

and $Y_{xx} = -2x$. Hence we see that, for each u, there exist exactly two points of $S_1(f_t)$ in this region.

(b) When $|x| < k_1$, $|u| < r_1$ and $|z| < m_1$ (*i.e.*, in the region B_1). First we consider $t > t_0 = \delta^2/(\delta^2 + \varepsilon^2)$. In this case, we have

$$Y_x \begin{cases} > 0 & \text{if } x = 0, \\ < 0 & \text{if } x = \pm k_1 \end{cases}$$

and $Y_{xx} = -2x$. Hence, for each u, there exist exactly two points of $S_1(f_t)$ in this region. When $t < t_0$, we set

$$u_t = \sqrt{\frac{\delta^2 - t(\delta^2 + \varepsilon^2)}{1 - \sigma t}}.$$

For $|u| > u_t$, we have

$$Y_x = \begin{cases} > 0 & \text{if } x = 0, \\ < 0 & \text{if } x = \pm k_1 \end{cases}$$

and $Y_{xx} = -2x$. Hence, for each u with $|u| > u_t$, there exist exactly two points of $S_1(f_t)$ in this region. For $|u| < u_t$, we have $Y_x < 0$ for $x = 0$ and $Y_{xx} = -2x$, and hence there exists no x with $Y_x = 0$ in this region. For $u = \pm u_t$, we have exactly one point of $S_1(f_t)$, which correspond to the cusps in B_1. Hence the set $S_1(f_t) \cap (B_1 \cup C)$ is as illustrated in Fig. 5 and it is now clear that $S_1(f_1)$ is obtained from $S_1(f)$ by a band operation along $\lambda([0,1])$. This completes the proof. □

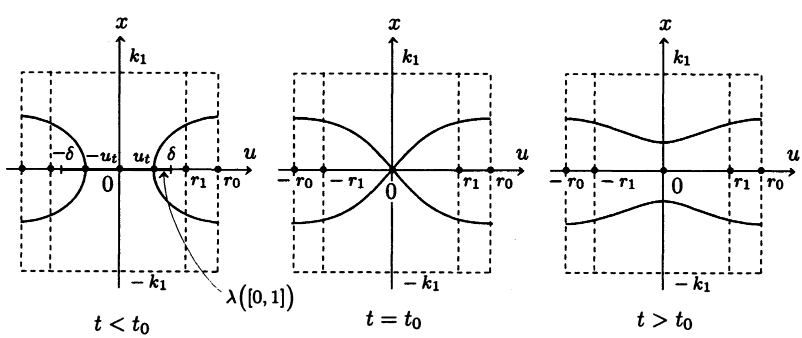


Figure 5.

Remark 3.8. — We note that the homotopy $f_{t|B}$ constructed in [L3], (4.6)–(4.9) corresponds to the «Beak to Beak» of [C].

LEMMA 3.9. — *Let M be a closed n-dimensional manifold ($n \geq 3$) and let L and L' be (nonempty) closed 1-dimensional submanifolds of M. Then L and L' are $\mathbb{Z}_2$-homologous in M if and only if L' is isotopic to a 1-dimensional submanifold obtained from L by a finite iteration of band operations.*

Proof. — The necessity is clear, since a band operation does not change the $\mathbb{Z}_2$-homology class represented by the submanifolds.

Now suppose that L and L' are $\mathbb{Z}_2$-homologous in M. We orient L and L' arbitrarily. Since $H_1(M;\mathbb{Z}_2) \cong H_1(M;\mathbb{Z})/2H_1(M;\mathbb{Z})$, we see that $[L] - [L'] = 2\gamma$ for some homology class $\gamma \in H_1(M;\mathbb{Z})$, where $[L]$ and $[L']$ are the homology classes in $H_1(M;\mathbb{Z})$ represented by the oriented submanifolds L and L' respectively. Then there exists an oriented simple closed curve A_1 in M representing γ. We may assume that $A_1 \cap L = \emptyset$, since $n \geq 3$. Then there exists an oriented band $b_1 : J \times J \to M$ consistent with the orientations of L and A_1 such that $b_1(J\times J)\cap(A_1\cup L) = b_1(\partial J\times J)$, $b_1(\{-1\} \times J) \subset L$, and $b_1(\{1\} \times J) \subset A_1$. Let A_2 be an embedded arc in M which is obtained from $(A_1 - b_1(\{1\} \times J)) \cup b_1(J \times \partial J)$ by smoothing the corners (see Fig. 6). Note that $\partial A_2 \subset L$. Then there exists an oriented band

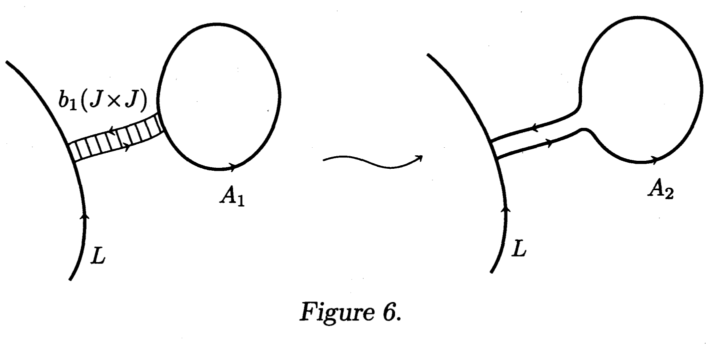


Figure 6.

$b_2 : J \times J \to M$ such that $b_2(J \times J) \cap L = b_2(\partial J \times J)$, $b_2(J \times \{0\}) = A_2$, $b_2(\{-1\} \times J) \subset L$ and $b_2(\{1\} \times J) \subset L$, where the last two inclusions are orientation reversing and preserving respectively (see Fig. 7). Let L_1 be the 1-dimensional submanifold of M obtained from L by the band operation along b_2. For a suitable orientation given to L_1, we have that $[L_1] = [L] + 2\gamma$ in $H_1(M;\mathbb{Z})$. Hence we may assume that $[L] = [L']$ in $H_1(M;\mathbb{Z})$.

By suitable oriented band operations, we may assume that L and

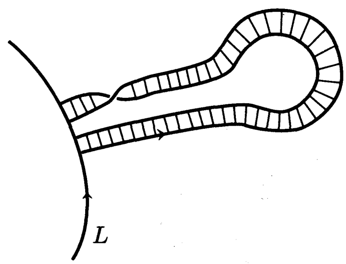


Figure 7.

L' are connected. Take a point $x \in L$ and an arc σ in M connecting x and L' so that we can regard $[L]$ and $[L']$ as elements of $\pi_1(M,x) = G$. Recall that $[L] = [L']$ in $H_1(M;\mathbb{Z}) = G/[G,G]$. Take arbitrary elements $\alpha, \beta \in \pi_1(M,x)$. Then there exists oriented simple closed curves A_3 and A_4 in M such that $A_3 \cap A_4 = A_4 \cap L = L \cap A_3 = \{x\}$, and $[A_3] = \alpha$ and $[A_4] = \beta$ in $\pi_1(M,x)$. Using A_3 and A_4 we can construct two oriented bands $b_3, b_4 : J \times J \to M$ such that $L \cap b_3(J \times J) = b_3(\partial J \times J) \not\ni x$, $L \cap b_4(J \times J) = b_4(\partial J \times J) \not\ni x$, $b_3(J \times J) \cap b_4(J \times J) = \emptyset$, and that the 1-dimensional oriented submanifold L_2 obtained from L by the oriented band operations along b_3 and b_4 satisfies $[L_2] = [L]\alpha\beta\alpha^{-1}\beta^{-1}$ in $\pi_1(M,x)$ (see Fig. 8).

Since the commutator subgroup $[G,G]$ is generated by the elements of the form $\alpha\beta\alpha^{-1}\beta^{-1}$ $(\alpha,\beta \in G)$, we may assume that $[L] = [L']$ in $\pi_1(M,x)$, iterating the above operation finitely many times. In particular, L and L' are freely homotopic. Then by a standard general position argument, we see that L and L' are isotopic for $n \geq 4$. When $n = 3$, taking the free homotopy between L and L' generically, we see that L' is isotopic to a 1-dimensional submanifold of M obtained from L by a finite iteration of crossing changes. Here, a crossing change is a local operation performed on a 1-dimensional submanifold as illustrated in Fig. 9. It is easy to see that a crossing change is realized by two band operations, as is shown in Fig. 10. This completes the proof. □

Remark 3.10. — Alternatively, we could prove Lemma 3.9, using a properly embedded surface F in $M \times [0,1]$ such that $\partial F = L \times \{0\} \cup L' \times \{1\}$ and that $p_{|F} : F \to [0,1]$ is a Morse function, where $p : M \times [0,1] \to [0,1]$ is the projection to the second factor.

Remark 3.11. — Let L be a closed 1-dimensional submanifold of M and $b : J \times J \to M$ a band such that $b(J \times J) \cap L = b(\partial J \times J)$. Then

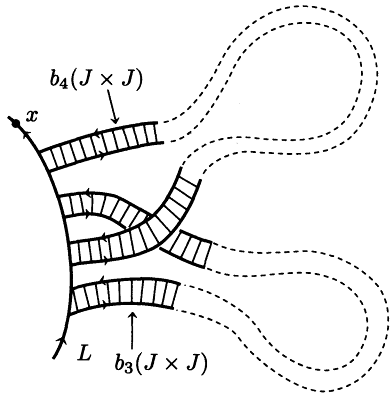


Figure 8.

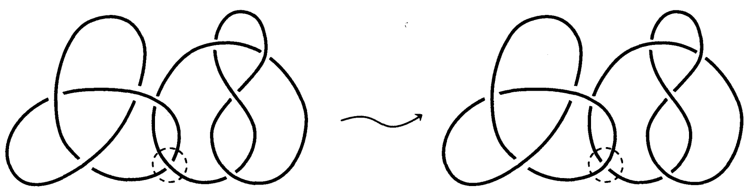


Figure 9.

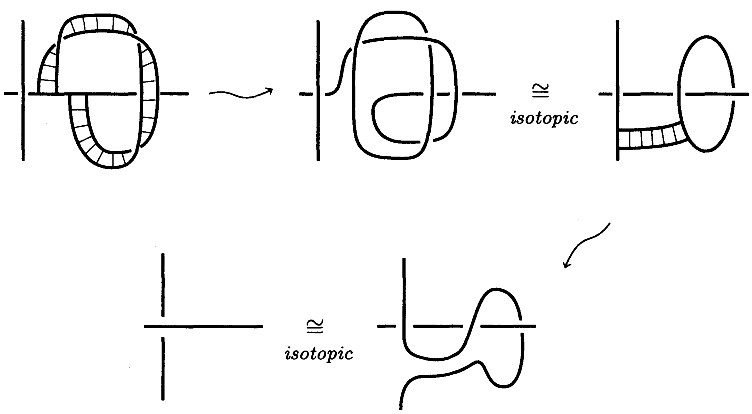


Figure 10.

the isotopy class of the 1-dimensional submanifold obtained by the band operation along b is determined by the following three data.

(1) The isotopy class of the core $b(J \times \{0\})$ in M with the end points contained in L and $b(\text{Int}\, J \times \{0\})$ contained in $M - L$.

(2) With a fixed orientation given to L, whether b is an oriented band or not (there are two possibilities).

(3) The homotopy class of the section of the normal bundle of the core $b(J \times \{0\})$ in M which is canonically determined by the band b, where the section on the boundary $\partial b(J \times \{0\})$ is fixed according to (2). Note that this corresponds exactly to

$$\pi_1\big(\mathrm{SO}(n-1)\big) \cong \begin{cases} \mathbb{Z} & \text{if } n = 3, \\ \mathbb{Z}_2 & \text{if } n \geq 4. \end{cases}$$

4. Proof of Theorem 2.2.

Proof of Theorem 2.2. — Let $f : M \to N$ be a generic map of a closed n-dimensional manifold ($n \geq 3$) into a surface and L a closed nonempty 1-dimensional submanifold of M which is $\mathbb{Z}_2$-homologous to the singular set $S(f)$ of f in M. We will show that there exists a generic map $\bar{f} : M \to N$ homotopic to f whose singular set coincides with L.

By Lemma 3.1, we may assume that $S(f) \neq \emptyset$. Hence by Lemma 3.9, L is isotopic to a 1-dimensional submanifold of M obtained from $S(f)$ by a finite iteration of band operations. By Lemma 3.3, without changing the isotopy type of $S(f)$, we may assume that every component of $S(f)$ contains fold points and cusp points of all absolute indices.

First suppose that n is odd. Let A be a component of $S(f)$ and p a fold point in A of absolute index $\frac{1}{2}(n-1)$. We fix an orientation g of G in a neighborhood of p in A. Furthermore, take a point p' in A sufficiently close to p. Then the index $i(p)$ of p with respect to the orientation $g(p)$ and the absolute index $i'(p')$ with respect to $-g(p')$ are both equal to $\frac{1}{2}(n-1)$. We apply Lemma 3.3 successively to p and p' with respect to the orientations $g(p)$ and $-g(p')$ respectively, and we denote the new cusps corresponding to p by p_1 and p_2 and those corresponding to p' by p'_1 and p'_2. We continue to use the same notation f, A and g for the new generic map, the new component and the new orientation corresponding to the original f, A and g respectively. Note that the indices of p_i and p'_i with respect to g are equal to $\frac{1}{2}(n-1)$ and $\frac{1}{2}(n-3)$ respectively and that those with respect to $-g$ are equal to $\frac{1}{2}(n-3)$ and $\frac{1}{2}(n-1)$ respectively. As to the image $f(A)$, see Fig. 11.

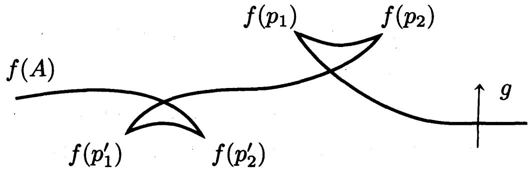


Figure 11.

When n is even, take a point p in A of absolute index $\frac{1}{2}n$. There exists an orientation g for G near p such that the index of p with respect to $g(p)$ is equal to $\frac{1}{2}(n-2)$. We apply Lemma 3.3 to p with the orientation $g(p)$, and we denote the new cusps by p_1 and p_2. We continue to use the same notation f, A and g as in the previous paragraph. Note that the index of p_i with respect to $\pm g$ is equal to $\frac{1}{2}(n-2)$. As to the image $f(A)$, see Fig. 12.

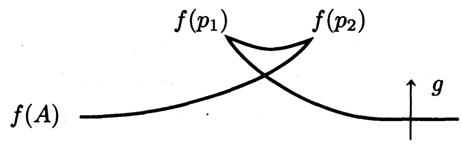


Figure 12.

Now we consider a band operation performed on $S(f)$ along a band $b : J \times J \to M$. We perform the operation described as above near each of the two end points of $\lambda = b(J \times \{0\})$. Changing b by an isotopy, we may assume that each point of $\partial b(J \times \{0\})$ coincides with one of the cusps p_i, p'_i constructed as above. At one of the end points of λ, replacing p_i by p_{3-i} (or p'_i by p'_{3-i}), and changing λ by an isotopy if necessary, we may assume that λ is a joining curve and that the two cusps $\partial\lambda$ are matching with respect to λ. This follows from the fact that the pairs of cusps (p_1, p_2) and (p'_1, p'_2) are matching with respect to local joining curves. Then we apply Lemma 3.9 to f and λ to obtain a generic map $f_1 : M \to N$ homotopic to f such that $S(f_1)$ is obtained from $S(f)$ by a band operation along a band b' whose core coincides with λ. Note that the difference between the band operations along b and b' lies in the data (2) and (3) of Remark 3.11. As to (2), consider the following change of λ. For n odd, at one of the end points of λ, we replace p_i by p'_{3-i} (or p'_i by p_{3-i}). For n even, we replace p_i by p_{3-i}. Then the end points of λ are still a matching pair of cusps with respect to λ and the orientation consistency of the band with respect to a fixed orientation of L changes at the end point. Hence, we may assume that b and b' are the same with respect to the data (2) of Remark 3.11. As to (3), consider the following change of λ. There exists a coordinate neighborhood U of a point in $\operatorname{Int}\lambda$ in M such that $U \cap S(f) = \emptyset, U \cap \partial\lambda = \emptyset$, that there exists a diffeomorphism $\psi : U \to \mathbb{R}^n$ with $\psi(U \cap \lambda([0,1])) = x_1$-axis, that $f(U)$ is an open set of N and that $(f_{|U}) \circ \psi^{-1} : \mathbb{R}^n \to f(U)$ is left equivalent to the natural projection $\mathrm{pr} : \mathbb{R}^n \to \mathbb{R}^2$ defined by $\mathrm{pr}(x_1, \cdots, x_n) = (x_1, x_2)$. We identify U with $\mathbb{R}^n$ by the diffeomorphism ψ. Then we replace $U \cap \lambda([0,1])$ by the curve as illustrated in Fig. 13 in $\mathbb{R}^3 = \{(x_1, x_2, x_3, 0, \cdots, 0)\} \subset \mathbb{R}^n$. Note that the new curve λ' is isotopic to the original λ with the end points fixed, that λ' is still a joining curve and that the end points are a matching pair with respect to λ'. Furthermore, when we apply Lemma 3.9 to λ', the orientation consistency of the band (see (2) of Remark 3.11) involving the band operation of Lemma 3.9 is the same as the original band b'. As to the data (3) of Remark 3.11, it changes by a generator of $\pi_1(\mathrm{SO}(n-1))$. Hence, by performing the change as above finitely many times, we may assume that the band operations along b and b' produce isotopic 1-dimensional submanifolds. Thus, we have shown that every band operation performed on $S(f)$ is realized by a homotopy of f (up to isotopy).

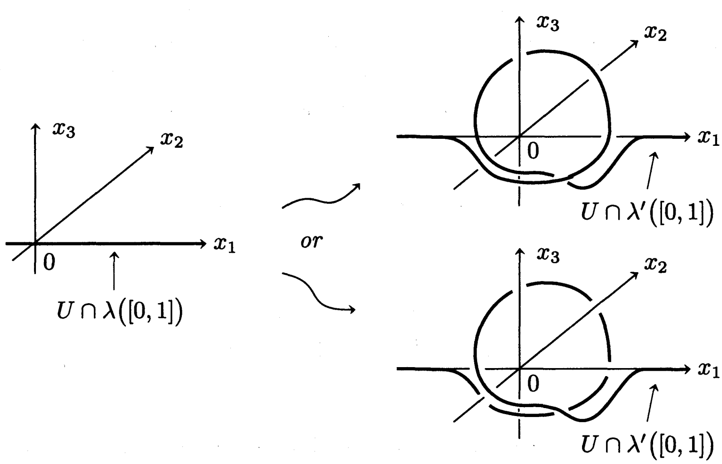


Figure 13.

Since L is obtained from $S(f)$ by a finite number of band operations, we see that there exists a generic map $f_2 : M \to N$ homotopic to f such that $S(f_2)$ is isotopic to L. Let $h_t : M \to M$ $(t \in [0,1])$ be an ambient isotopy such that $h_0 = \mathrm{id}$ and $h_1(L) = S(f_2)$. Then the map $\bar{f} = f_2 \circ h_1 : M \to N$ is the desired generic map. This completes the proof of Theorem 2.2. $\square$

Remark 4.1. — By the above proof, there exists a smooth homotopy $\bar{f}_t : M \to N$ such that $\bar{f}_0 = f$, $\bar{f}_1 = \bar{f}$, and that $\bar{f}_t$ is generic except for a finite number of t. For those t for which $\bar{f}_t$ is not generic, one of «Lip», «Beak to Beak» or «Swallow-tail» occurs (see [C]).

Remark 4.2. — When the original generic map $f : M \to N$ is stable, we can find a stable map as $\bar{f}$. For this, when we apply Lemma 3.1, we choose the point p and the open set U so that $f(p) \notin f(S(f))$ and $U \subset f^{-1}(N - f(S(f)))$. Furthermore, we apply Lemma 3.3 for p with $f(p)$ not being a double point of $f|_{(S_1(f)-S_1^2(f))}$ and we apply Lemma 3.7 for λ with $f \circ \lambda$ and $f|_{(S_1(f)\cup\lambda([0,1])-S_1^2(f))}$ immersions with normal crossings. In this case, as in the previous remark, we have a smooth homotopy $\bar{f}_t : M \to N$ such that $\bar{f}_0 = f$, $\bar{f}_1 = \bar{f}$, and that $\bar{f}_t$ is stable except for a finite number of t. For those t for which $\bar{f}_t$ is not stable, one of «Lip», «Beak to Beak», «Swallow-tail» or the intersection of a cusp and a fold curve occurs (see [C]).

Remark 4.3. — The techniques which appeared in our Lemmas 3.3 and 3.7 are not new ones (see [Bi], [Po]). For example, Porto [Po], Teorema 4.2.8 shows that every closed orientable 3-manifold admits a stable map into $\mathbb{R}^2$ with connected singular set, using a lemma similar to our Lemma 3.3 and the cancellation technique due to Levine [L3].

5. Proof of Theorem 2.4.

Let

$$L = \Big(\bigcup_{i=n-1-m}^{n-1} F_i \Big) \cup \Big(\bigcup_{j=n-2-k}^{n-2} C_j \Big)$$

be a nonempty closed 1-dimensional stratified submanifold of M satisfying the adjacency conditions similar to (2.3). Let $b : J \times J \to M$ be a band such that $b(J \times J) \cap L = b(\partial J \times J)$.

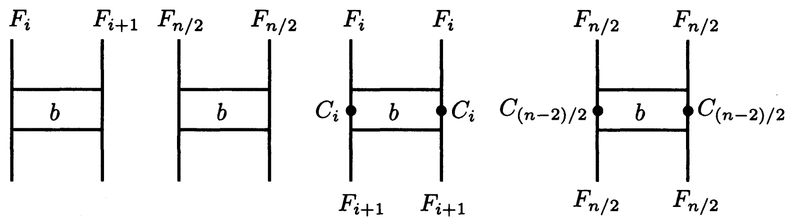
DEFINITION 5.1. — *We say that the band b is* compatible *with the stratified 1-dimensional submanifold L if one of the following four is satisfied (see Fig.* 14):

(1) $b(\{-1\} \times J) \subset F_i$ *and* $b(\{1\} \times J) \subset F_{i+1}$ *for some i, or*

(2) *n is even and* $b(\partial J \times J) \subset F_{n/2}$, *or*

(3) *$b(\partial J \times J) \subset F_{i+1} \cup F_i \cup C_i$ for some i, $b(\{\varepsilon\} \times J) \cap C_i = b(\{\varepsilon\} \times \operatorname{Int} J) \cap C_i$ consists of one point for $\varepsilon = \pm 1$, and $b(-1,-1) \in F_i$ if and only if $b(1,-1) \in F_i$, or*

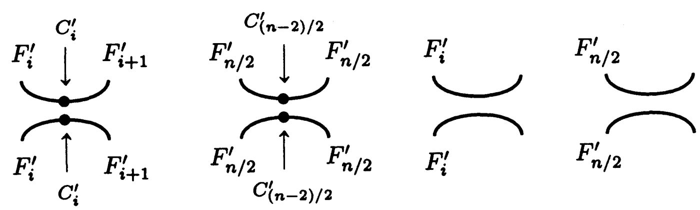
(4) *n is even, $b(\partial J \times J) \subset F_{n/2} \cup C_{(n-2)/2}$, and $b(\{\varepsilon\} \times J) \cap C_{(n-2)/2} = b(\{\varepsilon\} \times \operatorname{Int} J) \cap C_{(n-2)/2}$ consists of one point for $\varepsilon = \pm 1$.*



(1) (2) (3) (4)

Figure 14.

When the band b is compatible with the stratified 1-dimensional submanifold L, the 1-dimensional submanifold L' obtained from L by a band operation along b inherits a stratification which satisfies the adjacency conditions similar to (2.3) as illustrated in Fig. 15. Note that the band operations corresponding to bands as in (1) and (2) are the reverse operations of (3) and (4) respectively. Hence, if L' is obtained from L by band operations along compatible bands, then L is also so obtained from L'.



(1) (2) (3) (4)

Figure 15.

Proof of Theorem 2.4. — Orient L arbitrarily. Then it is easy to see that, performing oriented band operations to L along compatible bands, we obtain a stratified (oriented) 1-dimensional submanifold

$$L' = \Big(\bigcup_{i=n-1-m}^{n-1} F'_i\Big) \cup \Big(\bigcup_{j=n-2-k}^{n-2} C'_j\Big)$$

such that L' is connected. We can construct such L' using only compatible (and oriented) bands of type (1) in Definition 5.1; in other words, we do not need bands of types (2)–(4). Note that L' satisfies the same hypotheses as L stated in Theorem 2.4. Then, by Theorem 2.2, there exists a generic map $f_1 : M \to N$ homotopic to f such that $S(f_1) = L'$. Since N is orientable, the line bundle G over $S(f_1)$ is trivial when n is odd. Hence, when n is odd, applying Lemma 3.3, we may assume that $S(f_1)$ contains fold points of all indices with respect to a fixed orientation g of G. When n is even, we may assume that every absolute index appears.

Now we want to change f_1 homotopically so that the stratification of $S(f_1)$ coincides with that of L'. Take a point $p \in S(f_1)$ which is a fold point of absolute index $\tau \neq n-1$. Starting from p, we go along $S(f_1)$ in a fixed direction and we apply Lemma 3.3 successively to obtain the same stratification as $L' - \{q\}$ for some $q \in F'_\tau$ on an open arc a in $S(f_1)$. Furthermore, apply Lemma 3.3 twice at ∂a and let the new born cusps be denoted by p_i and p'_i $(i = 1, 2)$, where we have p_2, p_1, a, p'_1, p'_2 in this order when we go along $S(f_1)$ in the fixed direction. Here, when n is odd, we apply Lemma 3.3 with respect to the orientation g. Then we see that the pair (p_1, p'_1) is matching by [L3], (4.3) Lemma (b). (Recall that the manifolds M and N are orientable by our hypothesis and that the results of [L3] can be directly applied.) Choosing a joining curve λ for p_1 and p'_1 appropriately which is sufficiently close to $S(f_1) - a$, we apply Lemma 3.7 to obtain a generic map $f_2 : M \to N$ homotopic to f_1 such that $S(f_2)$ consists of two components S_0 and S_1, that S_0 is isotopic to L', that $S_0 (\subset S(f_2))$ has the same stratification as L', and that there exists an embedded 2-disk D in M with $\partial D = S_1$ and $D \cap S_0 = \emptyset$. In other words, S_1 is an unknotted circle and is unlinked with S_0. Note that S_1 contains an even number of cusps of f_2, since the number of cusps of f_2 has the same parity as the Euler characteristic of M, which has the same parity as the number of cusps on S_0 by our assumption. Applying Lemma 3.7 several times to the cusps on S_1 using joining curves close to S_1, we obtain a generic map $f_3 : M \to N$ homotopic to f_2 such that $S(f_3)$ consists of several components $S'_0, S'_1, \dots, S'_r$, that S'_0 coincides with S_0 together with the stratification, that S'_i $(i \geq 1)$ does not contain any cusps, and that $S'_1 \cup \cdots \cup S'_r$ is an unlinked union of unknotted circles unlinked with S'_0. Then we apply Lemma 3.3 so that S'_i $(i \geq 1)$ contains exactly two cusps, which are matching. Here, when n is even, we do this so that a cusp of absolute index $\frac{1}{2}(n-2)$ is not created. Since S'_0 contains fold points of all absolute indices by our hypothesis, one of the cusps on each component S'_i

($i \geq 1$) has a cusp on S'_0 such that they are matching. Then we apply Lemma 3.7 to this matching pair for each S'_i, choosing an appropriate joining curve. Then the resulting generic map $f_4 : M \to N$ is homotopic to f_3 and has connected singular set $S(f_4)$ which coincides with L' together with the stratification.

Finally, we consider the (oriented) band operations along compatible bands which are the reverse operations of those used for converting L to L'; in particular, they are bands of type (3) of Definition 5.1. Note that every pair of cusps of $S(f_4)$ corresponding to the two 0-dimensional strata created when converting L to L' is a matching pair by [L3], (4.3) Lemma (b), and the orientation consistency of the bands. (Note that $S(f_4)$ is connected.) Hence, applying Lemma 3.7 to these matching pairs for appropriate joining curves, we obtain a desired generic map $\bar{f} : M \to N$. This completes the proof of Theorem 2.4. □

We note that the same remarks as in the previous section are valid also for Theorem 2.4.

Remark 5.2. — When the source manifold M is of dimension 2, a result similar to Theorem 2.4 has been obtained by Eliašberg [E1], Thm. 4.8 and 4.9. Note that, in this case, we have a further necessary condition for a stratified 1-dimensional submanifold to be realized as the singular set of a generic map (see [E1], note p. 1131).

Remark 5.3. — In certain special cases, Theorem 2.4 has been obtained by Eliašberg [E2], Cor. 5.7.

6. Consequences.

COROLLARY 6.1. — *Let $f : M \to N$ be a continuous map of a closed n-dimensional manifold ($n \geq 3$) into a surface and let L be a nonempty closed 1-dimensional submanifold of M. Then there exists a generic map $\bar{f} : M \to N$ homotopic to f with $S(\bar{f}) = L$ if and only if the Poincaré dual of the $\mathbb{Z}_2$-homology class $[L]_2$ represented by L is equal to $w_{n-1}(M) + w_{n-2}(M) \cup f^*w_1(N) \in H^{n-1}(M;\mathbb{Z}_2)$.*

Proof. — It is known that there exists a generic map $f_1 : M \to N$ homotopic to f. Then by [T], we see that the Poincaré dual of $[S(f_1)]_2$ is equal to the degree $(n-1)$-part of $w(M) \cup (f^*w(N))^{-1}$. Then the corollary is a consequence of an easy calculation and Theorem 2.2. □

COROLLARY 6.2. — *Let M be a closed n-dimensional manifold ($n \geq 2$) and N an orientable surface. Furthermore, let L be a nonempty closed 1-dimensional submanifold of M. Then there exists a generic map $f : M \to N$ with $S(f) = L$ if and only if the Poincaré dual of the $\mathbb{Z}_2$-homology class $[L]_2$ represented by L is equal to $w_{n-1}(M) \in H^{n-1}(M; \mathbb{Z}_2)$.*

Proof. — For $n \geq 3$, this follows from Corollary 6.1. For $n = 2$, this follows from [E1]. □

Now we consider the 3-dimensional case. The following two corollaries are the complete answers to [S2], Problem 4.12 (1) and (2).

COROLLARY 6.3. — *Let M be a closed orientable 3-dimensional manifold and L a nonempty closed 1-dimensional submanifold of M. Then there exists a generic map $f : M \to \mathbb{R}^2$ with $S(f) = L$ if and only if $[L]_2 = 0$ in $H_1(M; \mathbb{Z}_2)$.*

The above result is a direct corollary of Corollary 6.2.

COROLLARY 6.4. — *Let M be a closed orientable 3-dimensional manifold and L a nonempty closed 1-dimensional submanifold of M. Then there exists a generic map $f : M \to \mathbb{R}^2$ with $S(f) = L$ which contains no cusps if and only if $[L]_2 = 0$ in $H_1(M; \mathbb{Z}_2)$ and one of the following conditions are satisfied:*

(1) *L is disconnected, or*

(2) *L is connected and (M, L) is diffeomorphic to $\partial(F \times D^2, F \times \{0\})$ for some compact orientable surface F with nonempty connected boundary.*

Proof. — First suppose that there exists a generic map $f : M \to \mathbb{R}^2$ as above. When $S(f)$ is connected, $S(f)$ consists of fold points of absolute index 2, since $\mathbb{R}^2$ is an open manifold. In other words, f has only definite fold points and is a special generic map ([BdR], [PF], [S1]). Then by [BdR], [S1], we see that $(M, S(f))$ is diffeomorphic to $\partial(F \times D^2, F \times \{0\})$ for some F. Conversely, suppose that L satisfies the above conditions. If L is disconnected, we can stratify L so that it has no 0-dimensional strata and that it satisfies the conditions in Theorem 2.4. Then the existence of a generic map f follows from Theorem 2.4. When L is connected, we can construct a special generic map $f : M \to \mathbb{R}^2$ with the desired property. This completes the proof. □

Remark 6.5. — Note that a similar result has been obtained by Eliašberg [E2], Cor. 5.7 when $H_1(M;\mathbb{Z}) = 0$.

Corollary 6.3 shows, for example, that every nonempty knot or link in S^3 is realized as the singular set of a generic map of S^3 into the plane. In [Bi], one can find some explicit examples of generic maps of S^3 into the plane with knotted singular set.

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