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EQUIDIMENSIONAL ACTIONS OF ALGEBRAIC TORI

by Haruhisa NAKAJIMA

1. Introduction.

In this paper, we suppose that all algebraic varieties including group varieties are defined over an algebraically closed field K of characteristic zero. Without specifying, varieties are assumed to be irreducible and G (resp. T) will always stand for a connected algebraic group (resp. connected algebraic torus). For an affine variety or scheme X (resp. a closed point $x \in X$), $\mathcal{O}(X)$ (resp. $\mathcal{O}(X)_x$) denotes the coordinate ring of X (resp. the stalk over x) and, for an affine domain R, $\operatorname{Spm} R$ denotes the affine variety defined by R, *i.e.*, the maximal spectrum of R. When a regular action of G on an affine variety X (abbr. (X,G)) (cf. [GM]) is given, we say X is a G-variety and define $\mathcal{O}(X)^G$ to be the K-subalgebra consisting of all invariants of G in $\mathcal{O}(X)$. Recall that X is said to be *conical*, if $\mathcal{O}(X)$ is equipped with a positive graduation $\mathcal{O}(X) = \bigoplus_{i\geq 0} \mathcal{O}(X)_i$ such that $\mathcal{O}(X)_0 = K$. In this case, we say that an action (X,G) is *conical*, if the induced action of G preserves the graduation of $\mathcal{O}(X)$. When $\mathcal{O}(X)^G$ is finitely generated as a K-algebra, we denote by X/G the affine variety associated with $\mathcal{O}(X)^G$, *i.e.*, the algebraic quotient of X under the action of G and by $\pi_{X,G}$ the quotient map $X \to X/G$. In the case where $\mathcal{O}(X)^G$ is affine, the action (X,G) is said to be *cofree* (resp. *equidimensional*), if $\mathcal{O}(X)$ is $\mathcal{O}(X)^G$-free (resp. if $X \to X/G$ is equidimensional). When $\mathcal{O}(X)^G$ is a finite-dimensional

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polynomial ring over K, (X, G) is called *coregular* (*cf.* Remark 5.5). Recall that an affine G-variety X is called a pointed variety with a base point x, if x is G-invariant. A conical affine variety X with a conical action of G is usually regarded as a pointed variety whose base point is associated with the homogeneous maximal ideal $\mathcal{O}(X)_+$. For a pointed affine G-variety with the base point x, we define the nullcone $\mathcal{N}(X, G)$ to be the affine scheme $\operatorname{Spec} \mathcal{O}(X)/\mathcal{O}(X) \cdot \mathfrak{M}_x^G$, where $\mathfrak{M}_x$ denotes the maximal ideal of functions vanishing on x. In the case where V is a representation space of a reductive G, the action (V, G) is cofree if and only if $\mathcal{N}(V, G)$ is a complete intersection. V.L. Popov [P1] proved (by classification) that, for irreducible representations of simple groups, equidimensional representations are cofree, and then he came up with the conjecture that this should be true for all representations of semisimple groups (*cf.* [P2]). Later, V.G. Kac [K] conjectured an argument similar to Popov conjecture for general connected algebraic groups. Several contributions to this conjecture have been done by Popov, G.W. Schwarz, O.M. Adamovich, P. Littelmann and D. Wehlau (see [P3]), however, except [W1], they deal with semisimple groups. For a finite-dimensional representation $G \to GL(V)$ of a non-semisimple reductive G, the quotient $V/[G, G]$ by the commutator subgroup $[G, G]$ is an affine factorial variety with the torus $G/[G, G]$ action. Moreover if the representation is equidimensional, then so is the action $(V/[G, G], G/[G, G])$. Thus, in order to study Popov conjecture for non-semisimple groups, it is natural to ask whether equidimensional conical actions of algebraic tori on conical factorial varieties are cofree. The purpose of this paper is to examine this question. Using theory of associated cones introduced by W. Borho and H. Kraft [BK], [GM], [W2], we give an affirmative answer for stable actions (*cf.* Remarks 5.6 and 5.7). Recall that (X, G) is said to be *stable*, if X contains a non-empty open subset consisting of closed G-orbits. Our main result is

THEOREM 1.1. — *Let X be an affine conical factorial variety with a conical stable action of T and let V be a dual space of a minimal homogeneous T-submodule of $\mathcal{O}(X)$ generating $\mathcal{O}(X)$ as a K-algebra. Then the following conditions are equivalent:*

(1) *(X, T) is equidimensional.*

(2) *(X, T) is cofree.*

(3) *(V, T) is cofree.*

(4) *$\mathcal{N}(X, T)$ is a complete intersection and X is defined by T-invariant*

polynomial functions on V.

The proof of this result will be partially given in Sec. 3 and will be completed in Sec. 4. This seems to be useful in studying on equidimensional representations of non-semisimple reductive algebraic groups (*e.g.*, [N4], [N5]).

Let $\mathfrak{X}(G)$ stand for the rational linear character group of (not necessarily connected) G over K which is regarded as an additive group. For any $\chi \in \mathfrak{X}(G)$, we set

$$\mathcal{O}(X)_\chi := \{x \in \mathcal{O}(X) \mid \sigma(x) = \chi(\sigma) \cdot x \text{ for any } \sigma \in G\},$$

whose elements are called *χ-invariants* or *semi-invariants* of G relative to χ in $\mathcal{O}(X)$. Clearly $\mathcal{O}(X)_\chi$ is an $\mathcal{O}(X)^G$-module.

THEOREM 1.2. — *Let X be an affine conical factorial variety with a conical action of G. Suppose that $\mathcal{O}(X)^{\mathcal{R}_u(G)}$ is noetherian, where $\mathcal{R}_u(G)$ denotes the unipotent radical of G. If the action of G on X is equidimensional, then $\mathcal{O}(X)^G$ is factorial and $\mathcal{O}(X)_\chi$ is $\mathcal{O}(X)^G$-free, for any $\chi \in \mathfrak{X}(G)$ such that $\mathcal{O}(X)_\chi \cdot \mathcal{O}(X)_{-\chi} \neq \{0\}$.*

Wehlau [W1] and, independently, S. Endo (unpublished) show that Popov conjecture for tori is affirmative, and Theorem 1.2 is regarded as a generalization of their result (*cf.* Remark 5.2).

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2. Preliminaries.

Let $\mathbb{Z}_0$ denote the additive monoid of all non-negative integers. For any s and $\mathbf{x} = (x_1, \ldots, x_s) \in \mathbb{Z}_0^s$, put $\operatorname{supp}(\mathbf{x}) = \{i \mid 1 \leq i \leq s, x_i \neq 0\}$. We easily have

LEMMA 2.1. — *Let $\mathbf{a}$, $\mathbf{b}$ be elements in $\mathbb{Z}_0^s$. If $\operatorname{supp}(\mathbf{a}) \supset \operatorname{supp}(\mathbf{b})$, then there are natural numbers m, n such that $m\mathbf{a} - n\mathbf{b} \in \mathbb{Z}_0^s$ and*

$$\operatorname{supp}(\mathbf{a}) \supsetneq \operatorname{supp}(m\mathbf{a} - n\mathbf{b}) \not\supset \operatorname{supp}(\mathbf{b}). \qquad \square$$

Let $\mathrm{Ht}_1(R)$ denote the set consisting of all prime ideals of a ring R of height one. For any subset B of a normal domain R, let $\widetilde{R\cdot B}$ denote the divisorialization of B (or of $R\cdot B$) in R (*i.e.*, $\widetilde{R\cdot B} := \bigcap_{\mathfrak{P}\in\mathrm{Ht}_1(R)} (R\cdot B)_{\mathfrak{P}}$), whose Weil divisor is denoted by $\mathrm{div}_R(B)$, and recall that the ideal $R\cdot B$ is said to be divisorial, if $R\cdot B = \widetilde{R\cdot B}$. As $\mathrm{div}_R(B)$ is regarded as a vector of some $\mathbb{Z}_0^s$, putting

$$\mathrm{supp}_R(B) := \{\mathfrak{p}\in \mathrm{Ht}_1(R) \mid B\subset \mathfrak{p}\},$$

we can identify this with $\mathrm{supp}(\mathrm{div}_R(B))$.

PROPOSITION 2.2. — *Let R be a normal domain and let $\mathfrak{A}$ and $\mathfrak{B}$ be divisorial integral ideals of R. Then the following conditions are equivalent:*

(1) $\sqrt{\mathfrak{A}}\subset\sqrt{\mathfrak{B}}$.

(2) $\mathrm{supp}_R(\mathfrak{A})\supset \mathrm{supp}_R(\mathfrak{B})$.

(3) *There are natural numbers m, n such that $(\widetilde{\mathfrak{A}^m} : \widetilde{\mathfrak{B}^n})\not\subset\sqrt{\mathfrak{B}}$ and $\mathfrak{A}^m\subset\widetilde{\mathfrak{B}^n}$.*

Proof. — The implication (3) $\Longrightarrow$ (1) follows from the assumption that $\mathfrak{B}$ is divisorial. For some $s\in\mathbb{N}$, we can choose $\mathfrak{p}_i\in\mathrm{Ht}_1(R)$ and a_i, $b_i\in\mathbb{Z}_0$ $(1\le i\le s)$ satisfying the following equalities; $\mathrm{div}_R(\mathfrak{A}) = \sum_{i=1}^{s} a_i\cdot \mathrm{div}_R(\mathfrak{p}_i)$ and $\mathrm{div}_R(\mathfrak{B}) = \sum_{i=1}^{s} b_i\cdot\mathrm{div}_R(\mathfrak{p}_i)$. If $\mathfrak{A}^u\subset\mathfrak{B}$ for a $u\in\mathbb{N}$, then $u\cdot a_i\ge b_i$ $(1\le i\le s)$, which shows (1) $\Longrightarrow$ (2). Suppose that $\sqrt{\mathfrak{A}}\subset\sqrt{\mathfrak{B}}$. Putting $\mathbf{a} = (a_1,\dots,a_s)$ and $\mathbf{b} = (b_1,\dots,b_s)$, by (2) we see $\mathrm{supp}(\mathbf{a})\supset\mathrm{supp}(\mathbf{b})$. So we have the natural numbers m, n stated as in Lemma 2.1. Then the assertion (3) follows from the implication (1) $\Longrightarrow$ (2), because $m\cdot\mathrm{div}_R(\mathfrak{A}) - n\cdot\mathrm{div}_R(\mathfrak{B})$ is effective. □

COROLLARY 2.3. — *Let X be a normal affine G-variety and let $\mathfrak{A}$ and $\mathfrak{B}$ be divisorial integral ideals of $\mathcal{O}(X)^G$. Then the following conditions are equivalent:*

(1) $\sqrt{\mathfrak{U}}\subset\sqrt{\mathfrak{B}}$ *in* $\mathcal{O}(X)^G$.

(2) $\mathrm{supp}_{\mathcal{O}(X)}(\mathfrak{A})\supset\mathrm{supp}_{\mathcal{O}(X)}(\mathfrak{B})$.

(3) $\mathrm{supp}_{\mathcal{O}(X)^G}(\mathfrak{A})\supset\mathrm{supp}_{\mathcal{O}(X)^G}(\mathfrak{B})$.

Proof. — The implication (1) $\Longrightarrow$ (2) and the equivalence (1) $\Longleftrightarrow$ (3) are shown in Proposition 2.2 ("divisoriality" is not used in the proof of (1) $\Longrightarrow$ (2)). The implication (2) $\Longrightarrow$ (3) follows from the well known fact (*e.g.* [M]) that, for any $\mathfrak{p} \in \mathrm{Ht}_1(\mathcal{O}(X)^G)$, there exists a prime ideal $\mathfrak{P} \in \mathrm{Ht}_1(\mathcal{O}(X))$ lying over $\mathfrak{p}$. □

LEMMA 2.4. — *Let X be a normal affine G-variety. If x is an element of $\mathcal{O}(X)$ such that $\mathcal{O}(X) \cdot x \cap \mathcal{O}(X)^G \neq \{0\}$, then x is a semi-invariant of G.*

Proof. — Let σ be any element of G and let $\mathfrak{P}_\sigma \in \mathrm{Ht}_1(S)$ to satisfy $\mathfrak{P}_\sigma \ni \sigma(x)$. The ideal $\mathfrak{P}_\sigma$ contains a nonzero element f in $\mathcal{O}(X)^G$. As G is connected, $\mathfrak{P}_\sigma$ is a G-invariant ideal. Hence $\sigma(x) \cdot \mathcal{O}(X)_{\mathfrak{P}_\sigma}$, a power of $\mathfrak{P}_\sigma \mathcal{O}(X)_{\mathfrak{P}_\sigma}$, is G-invariant. Consequently, for any $\mathfrak{P} \in \mathrm{Ht}_1(S)$, $\sigma(x) \in \mathfrak{P}$ if and only if $x \in \mathfrak{P}$, and one sees $\sigma(x) \cdot \mathcal{O}(X)_{\mathfrak{P}} = x \cdot \mathcal{O}(X)_{\mathfrak{P}}$. Since X is normal, $\mathcal{O}(X) \cdot x$ is G-invariant. By [M], Proposition 1, we obtain the assertion of this lemma. □

LEMMA 2.5. — *Let A be a graded affine K-domain $\bigoplus_{i \geq 0} A_i$ defined over $A_0 = K$ and B a graded K-subalgebra of A. Suppose that $B[g_1, \ldots, g_m] = A$ for some homogeneous elements $g_i \in \sqrt{B_+ \cdot A}$ $(1 \leq i \leq m)$, where B_+ denotes the homogeneous maximal ideal of B. Then A is integral over B and B is finitely generated over K.*

Proof. — For any $0 \leq j \leq m$, set $A^{(j)} := B[g_1, \ldots, g_j]$ (especially $A^{(0)} := B$), which is naturally regarded as a graded subalgebra of A. By our assumption, we can fix $0 \leq i < m$ such that $A^{(i+1)}$ is noetherian and $g_j \in \sqrt{B_+ \cdot A^{(i+1)}}$ $(1 \leq j \leq i+1)$. Then

$$g_{i+1}^n = b_1 F_1 + \ldots + b_k F_k$$

for some n, $k \in \mathbb{N}$ and for some homogeneous $b_u \in B_+$ and $F_u \in A^{(i)}[g_{i+1}]$ $(1 \leq u \leq k)$. So $A^{(i+1)}$ is integral over $A^{(i)}$ and, by Eakin-Nagata theorem, $A^{(i)}$ is noetherian. For any maximal ideal $\mathfrak{m}$ of $A^{(i)}$ containing $B_+ \cdot A^{(i)}$, we have a maximal ideal $\mathfrak{M}$ of $A^{(i+1)}$ lying over $\mathfrak{m}$. Then $B_+ \cdot A^{(i+1)} \subseteq \mathfrak{m} \cdot A^{(i+1)} \subseteq \mathfrak{M}$ and, by the choice of i, this implies that $\mathfrak{M}$ is homogeneous. Thus $g_j \in \sqrt{B_+ \cdot A^{(i)}}$ $(1 \leq j \leq i)$, and we inductively get our assertion. □

We require the following elementary fact:

LEMMA 2.6. — *Let χ_i, $1 \leq i \leq n$, (not may be distinct) be linear characters of T. Then $\bigoplus_{1 \leq i \leq n} \chi_i$ is a faithful representation of T if and only* *if* $\sum_{i=1}^{n} \mathbb{Z} \cdot \chi_i = \mathfrak{X}(T)$. □

For any finite generating system Γ of an affine K-domain A, let $\mathcal{F}_\Gamma$ denote the additive free monoid $\bigoplus_{\gamma \in \Gamma} \mathbb{Z}_0 \cdot \gamma$ on Γ. Let $K[\mathcal{F}_\Gamma]$ be the affine semigroup ring over K associated with $\mathcal{F}_\Gamma$ and Φ_Γ the canonical K-algebra map $K[\mathcal{F}_\Gamma] \to A$. Since $K[\mathcal{F}_\Gamma]$ is a K-space with the base $\mathcal{F}_\Gamma$, we naturally regard $\mathcal{F}_\Gamma$ as a (multiplicative) submonoid of $K[\mathcal{F}_\Gamma]$.

LEMMA 2.7. — *Suppose that T is diagonalizable and not necessarily connected. Let X be an affine T-variety and W a finite-dimensional rational T-submodule of $\mathcal{O}(X)$ which generates $\mathcal{O}(X)$ as a K-algebra. Then:*

(1) *If Γ denotes a K-basis of W consisting of semi-invariants of T, then $\mathcal{O}(X)^T$ is generated by some elements in $\Phi_\Gamma(\mathcal{F}_\Gamma) \cap \mathcal{O}(X)^T$ as a K-algebra and, for any $\chi \in \mathfrak{X}(T)$, $\mathcal{O}(X)_\chi$ is generated by some elements in $\Phi_\Gamma(\mathcal{F}_\Gamma) \cap \mathcal{O}(X)_\chi$ as an $\mathcal{O}(X)^T$-module.*

(2) *For a $\chi \in \mathfrak{X}(T)$, if* $\mathrm{Sym}(W)_\chi = \mathcal{O}(W^*)_\chi \neq \{0\}$, *then* $\mathcal{O}(X)_\chi \neq \{0\}$.

Proof. — Let χ be any linear character of T. The K-algebra $\mathrm{Sym}(W)$ can be identified with $K[\mathcal{F}_\Gamma]$ and Φ_Γ induces the surjection $\mathrm{Sym}(W)_\chi \to \mathcal{O}(X)_\chi$. On the other hand, $\mathrm{Sym}(W)_\chi$ is generated by some monomials in $\mathcal{F}_\Gamma \subset K[\mathcal{F}_\Gamma]$ as a $\mathrm{Sym}(W)^T$-module. Thus the assertion in (1) follows from these observations. Suppose $\mathrm{Sym}(W)_\chi \neq \{0\}$. Then this module contains a nonzero monomial M in $\mathcal{F}_\Gamma \subset K[\mathcal{F}_\Gamma]$ and, because X is integral, M does not vanish on X. So we must have the assertion (2). □

The next proposition is a slight modification of a part of [P3], Chap. 4, Theorem 3.3:

PROPOSITION 2.8. — *Suppose that G is reductive and not necessarily connected. Let N be a closed normal subgroup of G and X an affine G-variety. Then:*

(1) *For a point x of X, if $G \cdot x$ is closed in X, the orbit $G/N \cdot \pi_{X,N}(x)$ is closed in X/N.*

(2) *The action (X, G) is stable if and only if both actions (X, N) and $(X/N, G/N)$ are stable.* □

Remark 2.9. — For surjective morphisms $\phi : X \to Y$, $\psi : Y \to Z$

of integral varieties, the equidimensionality of $\psi \circ \phi$ implies one of ψ. As a special case of this, we note the following fact: *Let X be an affine G-variety and N a closed normal subgroup of G. Suppose that both $\mathcal{O}(X)^N$ and $\mathcal{O}(X)^G$ are noetherian. If the action (X,G) is equidimensional, then so is the action $(X/N, G/N)$.*

Almost assertions of the next result can be found in [S] and are known in terms of commutative algebra (*e.g.*, [CM], [LR]):

PROPOSITION 2.10. — *Suppose that G is reductive and let X be an affine conical variety with a conical G-action. Then:*

(1) *If (X,G) is cofree, then it is equidimensional.*

(2) *Suppose that X is Cohen-Macaulay. If (X,G) is coregular and equidimensional, then (X,G) is cofree.*

(3) *Suppose that (X,G) is cofree. Then X is a complete intersection if and only if both X/G and $\mathcal{N}(X,G)$ are complete intersections.*

(4) *Suppose that X is smooth (i.e., an affine space). Then the following three conditions are equivalent:*

(i) *(X,G) is cofree;*

(ii) *(X,G) is coregular and equidimensional;*

(iii) *$\mathcal{N}(X,G)$ is a complete intersection.* □

For any irreducible representation ρ of a reductive G and an affine G-variety X, let $\mu(X,\rho)$ denote the multiplicity of ρ in

$$(\mathcal{O}(X)^G/\mathcal{O}(X)^G_+) \otimes_{\mathcal{O}(X)^G} \mathcal{O}(X) \cong \mathcal{O}(X)/\mathcal{O}(X)^G_+ \cdot \mathcal{O}(X).$$

PROPOSITION 2.11. — *Let $G \to GL(V)$ be a finite-dimensional representation of a reductive G. Suppose that V contains X as a G-stable conical closed subset, where a conical structure of V is given by a weighted polynomial algebra $\mathcal{O}(V)$. Then*

(1) *$\mathcal{N}(V,G)$ is canonically isomorphic to $\mathcal{N}(X,G)$ if and only if $\mu(X,\rho) = \mu(V,\rho)$ for any irreducible representation ρ of G.*

(2) *If X is defined by G-invariant polynomials in V, then the equivalent conditions in (1) are satisfied. Conversely, if (X,G) is cofree and (1) is true, then X is defined by G-invariant polynomials in V.*

Proof. — The canonical epimorphism $\nu : \mathcal{O}(V) \to \mathcal{O}(X)$ induces the K-algebra map

$$\bar{\nu} : \mathcal{O}(V)/\mathcal{O}(V)_+^G \cdot \mathcal{O}(V) \to \mathcal{O}(X)/\mathcal{O}(X)_+^G \cdot \mathcal{O}(X).$$

Because $\bar{\nu}$ is G-equivariant and surjective, the equivalence in (1) follows. Clearly $\operatorname{Ker}\bar{\nu}$ is a quotient of $\operatorname{Ker}\nu/(\operatorname{Ker}\nu)^G \cdot \mathcal{O}(V)$, which implies the first assertion in (2). Suppose that (X, G) is cofree and $\mathcal{N}(V, G)$ is isomorphic to $\mathcal{N}(X, G)$ via $\bar{\nu}$. Then, by a graded version of Nakayama's lemma, we see that any $\mathcal{O}(X)^G$-free basis of $\mathcal{O}(X)$ can be regarded as a system of generators of $\mathcal{O}(V)$ as an $\mathcal{O}(V)^G$-module. From this we derive the last assertion of (2). □

3. Stable actions.

For a (not necessarily connected) G and an affine G-variety or G-scheme X over K, we define the following notations;

$$\mathfrak{X}_X(G) := \{\chi \in \mathfrak{X}(G) \mid \mathcal{O}(X)_\chi \cdot \mathcal{O}(X)_{-\chi} \neq \{0\}\},$$

$$\mathfrak{K}_X(G) := \bigcap_{\chi \in \mathfrak{X}_X(G)} \operatorname{Ker}\chi \subseteq G \text{ and}$$

$$\mathfrak{S}_G(X) := K\Big[\bigcup_{\chi \in \mathfrak{X}_X(G)} \mathcal{O}(X)_\chi\Big] \subseteq \mathcal{O}(X).$$

Recall that, without specifying, G is connected.

LEMMA 3.1. — *Let X be an affine G-variety. Then:*

(1) *$\mathfrak{X}_X(G)$ is a subgroup of $\mathfrak{X}(G)$.*

(2) $\mathfrak{S}_G(X) = \bigoplus_{\chi \in \mathfrak{X}_X(G)} \mathcal{O}(X)_\chi = \bigoplus_{\psi \in \mathfrak{X}_{\operatorname{Spec}\mathfrak{S}_G(X)}(G/\mathfrak{K}_X(G))} \mathfrak{S}_G(X)_\psi.$

(3) *For a G-invariant closed subvariety Y of X, $\mathcal{O}(X)^G \underset{\text{can.}}{\to} \mathcal{O}(Y)^G$ is injective if and only if $\mathfrak{S}_G(X) \underset{\text{can.}}{\to} \mathcal{O}(Y)^G$ induces a monomorphism $\mathfrak{S}_G(X) \to \mathfrak{S}_G(Y)$. In the case where G is reductive, $\mathcal{O}(X)^G \underset{\text{can.}}{\cong} \mathcal{O}(Y)^G$ if and only if $\mathfrak{S}_G(X) \underset{\text{can.}}{\cong} \mathfrak{S}_G(Y)$.*

(4) *If linear characters χ, ψ of G satisfy $\mathcal{O}(X)_\chi \cdot \mathcal{O}(X)_\psi \neq \{0\}$ and $\chi + \psi \in \mathfrak{X}_X(G)$, then both χ and ψ are contained in $\mathfrak{X}_X(G)$.*

(5) *Let L be a normal closed subgroup of G and let Z be an affine G-variety with a G-equivariant dominant morphism $X \to Z$ which satisfies*

$\mathcal{O}(Z) \hookrightarrow \mathcal{O}(X)^L \subset \mathcal{O}(X)$ and $\mathfrak{X}_Z(G/L) = \mathfrak{X}(G/L)$. If linear characters χ, ψ of G satisfy that $\mathcal{O}(X)_\chi \cdot \mathcal{O}(X)_\psi \neq \{0\}$ and $\mathcal{O}(X)_{m(\chi+\psi)} \cap \mathcal{O}(Z) \neq \{0\}$ for some $m \in \mathbb{N}$, then both χ and ψ are contained in $\mathfrak{X}_X(G)$.

Proof. — The proofs of (1) and (4) are easy, and the assertion (3) follows from (2). Since $\bigoplus_{\chi \in \mathfrak{X}_X(G)} \mathcal{O}(X)_\chi$ is a K-subalgebra, the equalities in (2) are clear. The assertion (5) is a consequence of (4) and the fact that $\mathcal{O}(X)$ is a domain, because the inclusion $\mathrm{Ker}(m(\chi+\psi)) \supseteq L$ implies $m(\chi+\psi) \in \mathfrak{X}_X(G)$. □

We denote by $\mathcal{Q}(R)$ the total quotient ring of a ring R. If G acts on R, then it acts naturally on $\mathcal{Q}(R)$.

PROPOSITION 3.2. — *Let X be an affine factorial G-variety. Suppose that $\mathcal{O}(X)^G$ is noetherian. Then the following conditions are equivalent:*

(1) *$\mathcal{O}(X)_\chi$ is $\mathcal{O}(X)^G$-free, for any $\chi \in \mathfrak{X}_X(G)$.*

(2) *$\mathcal{O}(X)^G \to \mathcal{O}(X)$ is no-blowing-up of codimension one (i.e., a restriction of any ideal of $\mathcal{O}(X)$ of height one to $\mathcal{O}(X)^G$ agrees with zero or is of height one) and $\mathcal{O}(X)^G$ is factorial.*

Proof. — First recall that any prime ideal of $\mathcal{O}(X)^G$ of height one is a restriction of a prime ideal of $\mathcal{O}(X)$ of height one (*e.g.* [M]).

(1) $\Longrightarrow$ (2): Let f be a prime element of $\mathcal{O}(X)$ such that $\mathcal{O}(X)\cdot f$ contains a nonzero invariant of G. Then, using Lemma 2.4, we see that $f \in \mathcal{O}(X)_\chi$ for some $\chi \in \mathfrak{X}(G)$. By (1), $\mathcal{O}(X)\cdot f \cap \mathcal{O}(X)^G = f \cdot \mathcal{O}(X)_{-\chi}$ is principal. So $\mathcal{O}(X)^G \to \mathcal{O}(X)$ is no-blowing-up of codimension one. Moreover, since $\mathcal{O}(X)^G$ is noetherian, we also see that it is factorial.

(2) $\Longrightarrow$ (1): Let $\chi \in \mathfrak{X}_X(G)$ and let f be a nonzero element of $\mathcal{O}(X)_{-\chi}$. Clearly $(\mathcal{O}(X)\cdot f)_{\mathfrak{P}} \cap \mathcal{O}(X)^G_{\mathfrak{P}\cap\mathcal{O}(X)^G} = \mathcal{Q}(\mathcal{O}(X)^G)$, for any prime ideal $\mathfrak{P}$ such that $\mathfrak{P} \cap \mathcal{O}(X)^G = \{0\}$. Since $\mathcal{O}(X)^G \to \mathcal{O}(X)$ is no-blowing-up of codimension one,

$$\begin{aligned}\mathcal{O}(X)\cdot f &\supseteq \bigcap_{\mathfrak{P}\in \mathrm{Ht}_1(\mathcal{O}(X))} ((\mathcal{O}(X)\cdot f)_{\mathfrak{P}} \cap \mathcal{O}(X)^G_{\mathfrak{P}\cap\mathcal{O}(X)^G}) \\ &\supseteq \bigcap_{\mathfrak{q}\in \mathrm{Ht}_1(\mathcal{O}(X)^G)} (\mathcal{O}(X)\cdot f \cap \mathcal{O}(X)^G)_{\mathfrak{q}},\end{aligned}$$

which implies that $\mathcal{O}(X)\cdot f \cap \mathcal{O}(X)^G$ is divisorial in $\mathcal{O}(X)^G$. Hence $\mathcal{O}(X)_\chi$ is $\mathcal{O}(X)^G$-free. □

We need a slight modification of A.R. Magid's descent method:

PROPOSITION 3.3 (compare [M], Theorem 6). — *Let X be an affine normal G-variety. Then there is a group E, an epimorphism $E \to \mathrm{Cl}(\mathcal{O}(X)^G)$ and an exact sequence*

$$1 \to F \to E \to \mathrm{Cl}(\mathcal{O}(X)),$$

where F is a subquotient of $\mathfrak{X}_X(G)$.

Proof. — Let χ be a linear character which represents an element of F as in the proof of [M], Theorem 6. We need only to show that F is regarded as a subquotient of $\mathfrak{X}_X(G)$. Let f be a nonzero element of $\mathcal{O}(X)_\chi$ and express

$$\mathcal{O}(X) \cdot f = \mathfrak{P}_1^{(a_1)} \cap \ldots \cap \mathfrak{P}_m^{(a_m)}$$

for some $\mathfrak{P}_i \in \mathrm{Ht}_1(\mathcal{O}(X))$ and $a_i \in \mathbb{N}$ $(1 \leq i \leq m)$, where $\mathfrak{P}_i^{(a_i)}$ denotes the a_i-th symbolic power of $\mathfrak{P}_i$. By the choice of χ, we see that $\mathfrak{P}_i \cap \mathcal{O}(X)^G \neq \{0\}$ $(1 \leq i \leq m)$. So

$$\mathcal{O}(X) \cdot f \cap \mathcal{O}(X)^G \supseteq (\mathfrak{P}_1^{a_1} \cap \mathcal{O}(X)^G) \cap \ldots \cap (\mathfrak{P}_m^{a_m} \cap \mathcal{O}(X)^G) \neq \{0\},$$

which implies that $\chi \in \mathfrak{X}_X(G)$. □

PROPOSITION 3.4. — *Let X be an affine T-variety. Then:*

(1) $\mathfrak{S}_T(X) = \mathcal{O}(X)^{\mathfrak{K}_X(T)^0} = \mathcal{O}(X)^{\mathfrak{K}_X(T)}$.

(2) *If V is a rational T-submodule of $\mathcal{O}(X)$ generating $\mathcal{O}(X)$ as a K-algebra, then*

$$\mathfrak{S}_T(V^*) \underset{\text{can.}}{\cong} \mathrm{Sym}(V \cap \mathfrak{S}_T(X)) = \mathrm{Sym}(V^{\mathfrak{K}_X(T)}),$$

where V^* denotes the dual T-module of V.

(3) $\mathfrak{X}_X(\mathfrak{K}_X(T)) = \mathfrak{X}_X(\mathfrak{K}_X(T)^0) = \{0\}$.

(4) $\mathfrak{X}_X(T) \underset{\text{can.}}{\cong} \mathfrak{X}_{X/\mathfrak{K}_X(T)}(T/\mathfrak{K}_X(T)) = \mathfrak{X}(T/\mathfrak{K}_X(T))$.

(5) *If X is normal, then $\mathrm{Cl}(\mathcal{O}(X)^{\mathfrak{K}_X(T)})$ is isomorphic to a subquotient of $\mathrm{Cl}(\mathcal{O}(X))$.*

Proof. — By the definition, if χ is a linear character in $\mathfrak{X}_X(T)$,

$$\mathcal{O}(X)^{\mathfrak{K}_X(T)} \supseteq \mathfrak{S}_T(X) \supseteq \mathcal{O}(X)_\chi = \mathcal{O}(X/\mathfrak{K}_X(T))_{\chi \bmod \mathfrak{K}_X(T)} \neq \{0\},$$

for any $\chi \in \mathfrak{X}_X(T)$ and we easily see

$$\mathfrak{X}_X(T) \underset{\text{can.}}{\cong} \mathfrak{X}_{X/\mathfrak{K}_X(T)}(T/\mathfrak{K}_X(T)).$$

Since $T/\mathfrak{K}_X(T)$ acts faithfully on $\mathfrak{S}_T(X)$, by Lemma 2.7, any linear character of $T/\mathfrak{K}_X(T)$ is a $\mathbb{Z}$-linear combination of linear characters of $T/\mathfrak{K}_X(T)$ associated with nonzero semi-invariants of $T/\mathfrak{K}_X(T)$ in $\mathfrak{S}_T(X)$. From (1) and (2) of Lemma 3.1, we deduce that $\mathfrak{X}_{\operatorname{Spm}\mathfrak{S}_T(X)}(T/\mathfrak{K}_X(T)) = \mathfrak{X}(T/\mathfrak{K}_X(T))$. Because $\mathfrak{X}_{\operatorname{Spm}\mathfrak{S}_T(X)}(T/\mathfrak{K}_X(T)) \subset \mathfrak{X}_{X/\mathfrak{K}_X(T)}(T/\mathfrak{K}_X(T))$, we must have (4).

Let Γ be a finite generating set of the K-algebra $\mathcal{O}(X)$ consisting of semi-invariants of T and H a closed subgroup of $\mathfrak{K}_X(T)$ of finite index. Let M_i $(i = 1, 2)$ be nonzero elements in $\Phi_\Gamma(\mathcal{F}_\Gamma)$ such that $M_1 \cdot M_2 \in \mathcal{O}(X)^H$ and choose χ_i (i = 1, 2) from $\mathfrak{X}(T)$ in such a way that $M_i \in \mathcal{O}(X)_{\chi_i}$. Because

$$\mathcal{O}(X)^{\mathfrak{K}_X(T)} \cap \mathcal{O}(X)_{[\mathfrak{K}_X(T):H](\chi_1+\chi_2)} \ni (M_1 \cdot M_2)^{[\mathfrak{K}_X(T):H]} \neq \{0\},$$

by (4) of Lemma 3.1 and (4), we see that $\chi_i \in \mathfrak{X}_X(T)$ $(i = 1, 2)$, which implies that $M_i \in \mathcal{O}(X)_{\chi_i} \subseteq \mathfrak{S}_X(T)$. Consequently, by (1) of Lemma 2.7, $\mathcal{O}(X)^H = \mathfrak{S}_X(T)$, which shows (1). Furthermore let ψ be a linear character of H in $\mathfrak{X}_X(H)$, and suppose $N_1 \in \mathcal{O}(X)_\psi$ and $N_2 \in \mathcal{O}(X)_{-\psi}$ belong to $\Phi_\Gamma(\mathcal{F}_\Gamma)$. Then, as in the above discussion on M_i, we similarly infer that $N_i \in \mathfrak{S}_X(T)$ $(i = 1, 2)$. Since $\mathcal{O}(X)_\psi$ is generated by some elements in $\Phi_\Gamma(\mathcal{F}_\Gamma)$ in $\mathcal{O}(X)$ as an $\mathcal{O}(X)^H$-module (*cf.* (2) of Lemma 2.7), we see that $\mathcal{O}(X)_\psi \subseteq \mathfrak{S}_X(T) = \mathcal{O}(X)^H$, which shows $\psi = 0$. Thus (3) has just been proved.

To prove (2), we apply Lemma 2.7, (1), (4) of Lemma 3.1 and (1) to (V, T), and see that

$$\mathfrak{S}_T(V^*) \cong \operatorname{Sym}(V \cap \mathfrak{S}_T(V^*)) = \operatorname{Sym}(V^{\mathfrak{K}_{V^*}(T)}) \cong \operatorname{Sym}(V)^{\mathfrak{K}_{V^*}(T)}.$$

Moreover, using Lemma 2.7 again, we see that $\mathfrak{K}_{V^*}(T) = \mathfrak{K}_X(T)$ and $V \cap \mathfrak{S}_T(X) = V \cap \mathfrak{S}_T(V^*)$.

The assertion (5) follows from Proposition 3.3, (1) and (3). □

Refining [W1], Lemma 2, we obtain

PROPOSITION 3.5. — *Let X be an affine T-variety and W a finite-dimensional rational T-submodule of $\mathcal{O}(X)$ which generates $\mathcal{O}(X)$ as a K-algebra. Then the following three conditions are equivalent:*

(1) *The action of T on X is stable.*

(2) *$\mathcal{O}(X)_{-\chi} \neq \{0\}$ for any $\chi \in \mathfrak{X}(T)$ such that $\mathcal{O}(X)_\chi \neq \{0\}$.*

(3) *The action of T on the affine space W^* dual to W is stable.*

Especially if these conditions are satisfied and (W^*, T) *is cofree, then* (X, T) *is cofree.*

Proof. — Let $\{Y_1, \ldots, Y_n\}$ be a K-basis of W consisting of semi-invariants of T. Putting $V = W^*$ for simplicity, we naturally regard the affine variety X as a T-invariant closed subvariety of an affine space V with the action of T dual to W. We may suppose that $\mathcal{O}(X)^G \neq \{0\}$ and that $\{Y_1, \ldots, Y_m\}$ is a K-basis of $V \cap \mathfrak{S}_T(V)$. Let F denote the linear subspace

$$\{x \in V \mid Y_i(x) = 0 \text{ for } i > m\}$$

of V. By Hilbert-Mumford criterion (*e.g.*, [GM]), we easily see that each T-orbit in $V_{Y_1 \cdot \ldots \cdot Y_m} \cap F$ is closed. Recall that the basic open subset $V_M \cap X$ defined by M is non-empty and T-invariant, for any nontrivial monomial M of $\{Y_1, \ldots, Y_n\}$ in $\mathcal{O}(V)$. So, if $m = n$, both (V, T) and (X, T) are stable. Suppose that $m < n$. Let x be a point of $V_{Y_1 \cdot \ldots \cdot Y_{m+1}}$. Then $x = y + z$ for some $y \in V_{Y_1 \cdot \ldots \cdot Y_m} \cap F$ and $z \in V_{Y_{m+1}}$. Since two distinct T-invariant closed subsets of V can be separated by invariants, from the canonical isomorphism $\mathcal{O}(V)^T \cong \mathcal{O}(F)^T$, we infer that $T \cdot x$ is not closed. Thus $V_{Y_1 \cdot \ldots \cdot Y_{m+1}}$ and $V_{Y_1 \cdot \ldots \cdot Y_{m+1}} \cap X$ are, respectively, non-empty open subsets consisting of non-closed T-orbits. We have just shown the equivalence (1) $\iff$ the condition that $n = m \iff$ (3). The equivalence (2) $\iff$ (3) follows from the observation as above and (1) of Lemma 2.7. Suppose these equivalent conditions are satisfied and that (V, T) is cofree. Then, since (V, T) is stable, $\mu(V, \chi) \leq 1$ for any $\chi \in \mathfrak{X}(T)$, which requires $\mu(V, \chi) = \mu(X, \chi)$. This shows the last assertion. □

COROLLARY 3.6. — *Let X be an affine conical variety with a conical regular stable action of T. Let V be a dual space of a finite-dimensional homogeneous T-submodule of $\mathcal{O}(X)$ which generates $\mathcal{O}(X)$ as a K-algebra. Then (V, T) is cofree if and only if $\mathcal{N}(X, T)$ is a complete intersection and X is defined by T-invariant polynomial functions on V.*

Proof. — Suppose that (V, T) is cofree, *i.e.*, $\mathcal{O}(V)_\chi$ is monogene as an $\mathcal{O}(V)^T$-module for any $\chi \in \mathfrak{X}(T)$. For some χ, assume that $\mathcal{O}(V)_\chi \neq 0$ and $\mathcal{O}(X)_\chi = 0$. Since $\mathcal{O}(V)_\chi$ is generated by a monomial of a basis of V^* in $\mathrm{Sym}(V^*) = \mathcal{O}(V)$ as an $\mathcal{O}(V)^G$-module, this assumption implies that V^* is not canonically embedded in $\mathcal{O}(X)$. Thus we must have $\mu(V, \chi) = \mu(X, \chi)$ for all $\chi \in \mathfrak{X}(T)$. Since (X, T) is cofree, the assertion follows from Proposition 2.10 and Proposition 2.11. Conversely suppose that the latter half of the equivalence holds. Using Proposition 2.11 again, we have an canonical isomorphism $\mathcal{N}(X,T) \cong \mathcal{N}(V,T)$. Since $\mathcal{N}(V,T)$ is a complete intersection, by Proposition 2.10, we see that (V,T) is cofree. □

COROLLARY 3.7. — *Let X be an affine T-variety. Then $\mathfrak{S}_T(X)$ is the maximal T-invariant affine K-subalgebra of $\mathcal{O}(X)$ such that the natural action* $(\operatorname{Spm}\mathfrak{S}_T(X),T)$ *is stable.*

Proof. — By (2) of Lemma 3.1, (1), (4) of Proposition 3.4 and Proposition 3.5, we see that the action $(\operatorname{Spm}\mathfrak{S}_T(X),T)$ is stable. The maximality of $\mathfrak{S}_T(R)$ follows easily from Proposition 3.5. □

Remark 3.8. — Let (X,G) be a stable action of a reductive G on an affine variety. Then, it seems to be well known that $\mathcal{Q}(\mathcal{O}(X))^G = \mathcal{Q}(\mathcal{O}(X)^G)$ and

$$\dim \mathcal{Q}(\mathcal{O}(X))^G = \dim X - \max_{x\in X} \dim G\cdot x.$$

To show this, we may assume that X is factorial and, by Proposition 2.8 and [M], Proposition 1, we can derive the first equality also from Proposition 3.5. The second equality follows from the first one (*e.g.*, [L], [GM]).

For a subgroup H of a group G and a G-module M, we denote by $H|_M$ the image of the natural homomorphism $H \to GL(M)$.

LEMMA 3.9. — *Let X be an affine T-variety and let f_i, $1 \le i \le n$, be semi-invariants of T in $\mathcal{O}(X)$ which generate $\mathcal{O}(X)$ as a K-algebra. Suppose that $\mathcal{O}(X)^T \neq K$ and that $i_j > 0$ $(1 \le j \le n)$, for any $f_1^{i_1}\cdot f_2^{i_2}\cdot\dots\cdot f_n^{i_n} \in \mathcal{O}(X)^T\backslash K$ with $i_j \in \mathbb{Z}_0$. Then:*

(1) *(X,T) is a stable action.*

(2) *$\mathcal{O}(X)$ is a polynomial ring of dimension n.*

(3) $T|_{\sum_{i=1}^n Kf_i} = n-1$.

(4) *For a closed subgroup L of T satisfying $L|_{\mathcal{O}(X)} \neq T|_{\mathcal{O}(X)}$, there is a vector $(k_1,\dots,k_n) \in \mathbb{Z}_0^n$ with $0 <$ card$(\{i \mid k_i > 0\}) < n$ such that $f_1^{k_1}\cdot f_2^{k_2}\cdot\dots\cdot f_n^{k_n} \in \mathcal{O}(X)^L$.*

Proof. — Put $\Gamma = \{f_1,\dots,f_n\}$ and consider the T-equivariant K-epimorphism $\Phi_\Gamma : K[\mathcal{F}_\Gamma] \to \mathcal{O}(X)$. Applyng Lemma 2.7 to our assumption on invariant monomials, we immediately see that $K[\mathcal{F}_\Gamma]^T$ is a one-dimensional polynomial algebra over K isomorphic to $\mathcal{O}(X)^T$. Furthermore, using Lemma 2.7 again and Proposition 3.5, we see that both actions (X,T) and $(\operatorname{Spm} K[\mathcal{F}_\Gamma],\ T)$ are stable. Suppose that Φ_Γ is not an isomorphism. Since $\operatorname{Ker}\Phi_\Gamma$ is T-invariant, there is a nonzero semi-invariant $g \in K[\mathcal{F}_\Gamma]\chi \cap \operatorname{Ker}\Phi_\Gamma$ for a $\chi \in \mathfrak{X}(T)$. Recalling that $(\operatorname{Spm} K[\mathcal{F}_\Gamma],\ T)$ is stable, we have $K[\mathcal{F}_\Gamma]_{-\chi} \neq \{0\}$, which implies $(\operatorname{Ker}\Phi_\Gamma)^T \supset g \cdot K[\mathcal{F}_\Gamma]_{-\chi}$. This contradicts the fact that $(\operatorname{Ker}\Phi_\Gamma)^T = \{0\}$. Hence $X \cong \operatorname{Spm} K[\mathcal{F}_\Gamma]$. Because (X,T) is stable, by Remark 3.8 and the proof of Proposition 3.5,

$$\dim X - \dim X/T = \max_{x \in X} \dim T \cdot x = \dim T|_{\sum_{i=1}^n Kf_i}.$$

Thus the assertions in (1) $\sim$ (3) are shown. The assertion in (4) follows easily from (3). □

4. Equidimensional actions.

Let (Y,T_1) be a conical regular operation of $T_1 \cong K^*$ on a normal affine conical variety Y. Let Ω be a finite gererating system of $\mathcal{O}(Y)$ as a K-algebra consisting of homogeneous semi-invariants of T_1. Fixing an isomorphism $\nu : \mathfrak{X}(T_1) \cong \mathbb{Z}$, we define the following subsets of Ω; $\Omega_+ := \{x \in \Omega \mid x \in \mathcal{O}(Y)_\chi, \nu(\chi) > 0\}$ and $\Omega_- := \{x \in \Omega \mid x \in \mathcal{O}(Y)_\chi, \nu(\chi) < 0\}$. Using Wehlau's result [W2]* on associated cones, we obtain

LEMMA 4.1. — *Let $f \in \Omega_+$ and $g \in \Omega_-$ be elements such that $\sqrt{\mathcal{O}(Y)f}$ (resp. $\sqrt{\mathcal{O}(Y)g}$) is maximal in $\{\sqrt{\mathcal{O}(Y)x} \mid x \in \Omega_+$ (resp. $x \in \Omega_-$)$\}$. If (Y,T_1) is equidimensional and stable, then:*

(1) *$\mathcal{O}(Y)$ is integral over $\mathcal{O}(Y)^{T_1}[f,g]$.*

(2) *$(\operatorname{Spm}\mathcal{O}(Y)^{T_1}[f,g],T_1)$ is cofree.*

(3) *If χ is any non-zero linear character of T_1, then there is a $u \in \mathbb{N}$ depending χ such that $(\mathcal{O}(Y)_\chi)^u \subset \mathcal{O}(Y)\cdot f$ or $(\mathcal{O}(Y)_\chi)^u \subset \mathcal{O}(Y)\cdot g$.*

Proof. — Let h be any element of Ω_- and suppose $h \notin \sqrt{\mathcal{O}(Y)g}$. We choose natural numbers a,b,c and d in such a way that $f^a g^b \in \mathcal{O}(Y)^{T_1}$ and $f^c h^d \in \mathcal{O}(Y)^{T_1}$. Put $x := f^a g^b$ and $y := f^c h^d$.

Suppose that $\sqrt{\mathcal{O}(Y)^{T_1}x} \not\subset \sqrt{\mathcal{O}(Y)^{T_1}y}$. Then, there is a maximal ideal $\mathfrak{M}$ of $\mathcal{O}(Y)^{T_1}$ such that $\mathfrak{M} \not\ni x$ and $\mathfrak{M} \ni y$. Let $\mu : \mathcal{O}(Y)^{T_1} \to \mathcal{O}(Y)^{T_1}/\mathfrak{M} \cong K$ be the canonical K-algebra map. Since

$$\mathfrak{M}\cdot\mathcal{O}(Y) \ni h^{a\cdot d}(x^c - \mu(x^c)) - g^{b\cdot c}y^a,$$

* The author is thankful to Prof. D. Wehlau for sending this preprint to him.

$\sqrt{\mathcal{O}(Y)\cdot\mathfrak{M}} \ni h$. Thus, by [W2], Theorem 2.5, we have

$$h \in \sqrt{\mathcal{O}(Y)\cdot\mathcal{O}(Y)^{T_1}_{+}}.$$

Next suppose $\sqrt{\mathcal{O}(Y)^{T_1}x} \subset \sqrt{\mathcal{O}(Y)^{T_1}y}$. By Proposition 2.2, we can choose n, m from $\mathbb{N}$ and z from $\mathcal{O}(Y)^{T_1}$ in such a way that $x^n = y^m \cdot z$ and $\sqrt{\mathcal{O}(Y)^{T_1}z} \not\subset \sqrt{\mathcal{O}(Y)^{T_1}x}$. Then $f^{a\cdot n}\cdot g^{b\cdot n} = f^{c\cdot m}\cdot h^{d\cdot m}\cdot z$. Assume that $a\cdot n \le c\cdot m$. Since $g^{b\cdot n} = f^{c\cdot m - a\cdot n}\cdot h^{d\cdot m}\cdot z \in h\cdot\mathcal{O}(Y)$, by the maximality of $\sqrt{\mathcal{O}(Y)^{T_1}g}$, we see that $\sqrt{\mathcal{O}(Y)^{T_1}g} = \sqrt{\mathcal{O}(Y)^{T_1}h}$. This contradicts the choice of h (cf. Corollary 2.3). Hence $a\cdot n - c\cdot m - 1$ is non-negative. We express as

$$h^{d\cdot m\cdot a}\cdot z^a = x\cdot f^{a\cdot(a\cdot n - c\cdot m - 1)}\cdot g^{b\cdot(a\cdot n - 1)}.$$

Let $\mathfrak{N}$ be a maximal ideal of $\mathcal{O}(Y)^{T_1}$ satisfying that $\mathfrak{N} \not\ni z$ and $\mathfrak{N} \ni x$ and κ the canonical K-algebra map $\mathcal{O}(Y)^{T_1} \to \mathcal{O}(Y)^{T_1}/\mathfrak{N} \cong K$. Then we see

$$\mathfrak{N}\cdot\mathcal{O}(Y) \ni x\cdot f^{a\cdot(a\cdot n - c\cdot m - 1)}\cdot g^{b\cdot(a\cdot n - 1)} - h^{d\cdot m\cdot a}\cdot(z^a - \kappa(z^a)),$$

which implies $h \in \sqrt{\mathfrak{N}\cdot\mathcal{O}(Y)}$. From [W2], it follows that

$$\sqrt{\mathcal{O}(Y)\cdot\mathcal{O}(Y)^{T_1}_{+}} \ni h.$$

We can continue this procedure, and consequently we conclude that both Ω_- and Ω_+ are contained in $\sqrt{\mathcal{O}(Y)\cdot(\mathcal{O}(Y)^{T_1}[f,g])_+}$. By Lemma 2.5, the first assertion has just been shown.

By Lemma 3.9, the action $(\operatorname{Spm} K[f,g],\ T_1)$ is cofree. Since $(\mathcal{O}(Y)^{T_1}[f,g])_\chi$, $\chi \in \mathfrak{X}(T_1)$, is generated by $K[f,g]_\chi$ as an $\mathcal{O}(Y)^{T_1}$-module, the second assertion follows from this observation.

We now show the last assertion. Let $\chi \in \mathfrak{X}(T)$ satisfying $\mathcal{O}(Y)_\chi \neq \{0\}$ and let v be any nonzero element of $\mathcal{O}(Y)_\chi$. By (1), we can express as

$$v^l + w_1\cdot v^{l-1} + \ldots + w_l = 0$$

for some $w_i \in \mathcal{O}(Y)^{T_1}[f,g]$ and $l \in \mathbb{N}$ and may assume that all w_i are semi-invariants of T_1. Say $\nu(\chi) > 0$. For any $\eta \in \mathfrak{X}(T_1)$ such that $\nu(\eta) > 0$, (2) implies that

$$(\mathcal{O}(Y)^{T_1}[f,g])_\eta = \mathcal{O}(Y)^{T_1}\cdot f^e\cdot g^t$$

for some $e \in \mathbb{N}$ and $t \in \mathbb{Z}_0$, if this module is non-zero. Thus $v^l \in \mathcal{O}(Y)\cdot f$. Because $\mathcal{O}(Y)_\chi$ is finitely generated as an $\mathcal{O}(Y)^{T_1}$-module, for a sufficiently large $u \in \mathbb{N}$, we have $(\mathcal{O}(Y)_\chi)^u \subset \mathcal{O}(Y)\cdot f$. □

From now on, we suppose, in this section, that T is of arbitrary rank and X is an affine *conical normal* T-variety such that the action of T on

X is *conical*. The action (X, T) is said to be *radially-cofree*, if, for any $\chi \in \mathfrak{X}(T)$ with $\mathcal{O}(X)_\chi \neq \{0\}$, there is a natural number m such that $\mathcal{O}(X)_{n \cdot m\chi}$, $n \in \mathbb{N}$, are free as $\mathcal{O}(X)^T$-modules.

THEOREM 4.2. — *Suppose that (X, T) is stable and equidimensional. Then*

(1) *For a linear character χ of T, if $(\mathcal{O}(X)_\chi \cdot \mathcal{O}(X))^\sim = \mathcal{O}(X) \cdot g$ for a nonzero element $g \in \mathcal{O}(X)$, we have $\mathcal{O}(X)_\chi = \mathcal{O}(X)^T \cdot g$. Consequently, under the extra condition that X is factorial, the action (X, T) is cofree.*

(2) *The action (X, T) is radially-cofree.*

Proof. — *(1):* Since $\mathcal{O}(X)_\chi \cdot \mathcal{O}(X)$ is T-invariant and $(\mathcal{O}(X)_\chi \cdot \mathcal{O}(X))^\sim$ is the smallest divisorial ideal of $\mathcal{O}(X)$ containing $\mathcal{O}(X)_\chi \cdot \mathcal{O}(X)$, we can choose ψ from $\mathfrak{X}(T)$ in such a way that $g \in \mathcal{O}(X)_\psi$. Then $\mathcal{O}(X)_\chi = \mathcal{O}(X)_{\chi-\psi} \cdot g$ and $\mathcal{O}(X)_{\chi-\psi} \neq \{0\}$. We assume $\chi - \psi \neq 0$. Clearly $\mathcal{O}(X)^{\mathrm{Ker}(\chi-\psi)}$ is normal and by Proposition 2.8 and Remark 2.9, the natural action of $T/\mathrm{Ker}(\chi - \psi) \cong K^*$ on $\mathcal{O}(X)^{\mathrm{Ker}(\chi-\psi)}$ is stable and equidimensional. Applying the last assertion of Lemma 4.1 to $(X/\mathrm{Ker}(\chi - \psi),\ T/\mathrm{Ker}(\chi - \psi))$, we get an $m \in \mathbb{N}$ and a non-unit element $w \in \mathcal{O}(X)^{\mathrm{Ker}(\chi-\psi)}$ satisfying $(\mathcal{O}(X)_{\chi-\psi})^m \subset w \cdot \mathcal{O}(X)^{\mathrm{Ker}(\chi-\psi)}$. Then

$$\begin{aligned}\mathcal{O}(X) \cdot g^m &= ((\mathcal{O}(X)_\chi \cdot \mathcal{O}(X))^\sim)^m)^\sim = ((\mathcal{O}(X)_\chi \cdot \mathcal{O}(X))^m)^\sim \\ &= ((\mathcal{O}(X)_{\chi-\psi} \cdot g)^m \cdot \mathcal{O}(X))^\sim = ((\mathcal{O}(X)_{\chi-\psi} \cdot \mathcal{O}(X))^m)^\sim \cdot g^m \\ &\subset \mathcal{O}(X) \cdot g^m \cdot w.\end{aligned}$$

This inclusion implies that w is a unit-element of $\mathcal{O}(X)$, *i.e.*, $w \in K^*$, which is a contradiction. So we must have $\chi = \psi$ and the assertion follows from this equality.

(2): Let χ be any non-zero linear character of T such that $\mathcal{O}(X)_\chi \neq \{0\}$. We apply the last assertion of Lemma 4.1 to the induced conical stable and equidimensional action $(X/\mathrm{Ker}\,\chi,\ T/\mathrm{Ker}\,\chi)$, and then, for any $a \in \mathbb{N}$, we can choose a natural number $u(a)$ depending on a and a semi-invariant $f \in \mathcal{O}(X)^{\mathrm{Ker}\,\chi}$ of T not depending on a in such a way that $(\mathcal{O}(X)_{a\chi})^{u(a)} \subset \mathcal{O}(X)^{\mathrm{Ker}\,\chi} \cdot f$. Clearly $T/\mathrm{Ker}\,\chi \cong K^*$ acts faithfully on $\mathcal{O}(X)_\chi$, a submodule of $\mathcal{O}(X)^{\mathrm{Ker}\,\chi}$ and so, by Lemma 2.6,

$$\mathfrak{X}(T/\mathrm{Ker}\,\chi) = \mathbb{Z} \cdot \chi \bmod \mathrm{Ker}\,\chi.$$

Thus f is regarded as an element of $\mathcal{O}(X)_{s\chi}$ for some $s \in \mathbb{N}$. Let b be any natural number. By Lemma 4.1 and Proposition 2.2, we can choose m, $n \in \mathbb{N}$ and a divisorial integral ideal $\mathfrak{J}$ of $\mathcal{O}(X)$ such that

$$(((\mathcal{O}(X)_{b \cdot s\chi} \cdot \mathcal{O}(X))^\sim)^m)^\sim = f^n \cdot \mathfrak{J}$$

and $\mathfrak{J} \not\subset \sqrt{\mathcal{O}(X)f}$. Since

$$(((\mathcal{O}(X)_{b\cdot s\chi} \cdot \mathcal{O}(X))^{\sim})^m)^{\sim} = ((\mathcal{O}(X)_{b\cdot s\chi})^m \cdot \mathcal{O}(X))^{\sim} \subset (\mathcal{O}(X)_{(b\cdot m-n)\cdot s\chi})\mathcal{O}(X))^{\sim} \cdot f^n,$$

we have $\mathfrak{J} \subset (\mathcal{O}(X)_{(b\cdot m-n)\cdot s\chi} \cdot \mathcal{O}(X))^{\sim}$. If $b \cdot m > n$, then

$$(\mathcal{O}(X)_{(b\cdot m-n)\cdot s\chi})^{u((b\cdot m-n)\cdot s)} \subset \mathcal{O}(X) \cdot f,$$

which requires $\mathfrak{J} \subset \sqrt{\mathcal{O}(X) \cdot f}$. Hence $b \cdot m \leq n$ and

$$(\mathcal{O}(X)_{b\cdot s\chi})^m \cdot \mathcal{O}(X) \subset \mathcal{O}(X) \cdot f^{b\cdot m}.$$

Because $(\mathcal{O}(X)_{b\cdot s\chi})^m \ni f^{b\cdot m}$, we see $(\mathcal{O}(X)_{b\cdot s\chi})^m = f^{b\cdot m} \cdot \mathcal{O}(X)^T$ Moreover, since $\mathcal{O}(X)_{b\cdot s\chi} \ni f^b$ and

$$(((\mathcal{O}(X)_{b\cdot s\chi} \cdot \mathcal{O}(X))^{\sim})^m)^{\sim} = ((\mathcal{O}(X)_{b\cdot s\chi})^m \cdot \mathcal{O}(X))^{\sim} = \mathcal{O}(X) \cdot f^{b\cdot m},$$

we must have $(\mathcal{O}(X)_{b\cdot s\chi}\mathcal{O}(X))^{\sim} = \mathcal{O}(X) \cdot f^b$. Thus the assertion follows from (1). □

Let Γ be a minimal homogeneous generating system of $\mathcal{O}(X)$ consisting of semi-invariants of T. The action of T on $\mathcal{O}(X)$ can naturally be lifted to the action on $K[\mathcal{F}_\Gamma]$ and Φ_Γ is T-equivariant. For any subgroup H of T, we denote by $\mathcal{F}_\Gamma^H$ the submonoid $K[\mathcal{F}_\Gamma]^H \cap \mathcal{F}_\Gamma$ of $\mathcal{F}_\Gamma$ and clearly $\mathcal{F}_\Gamma^H$ is a normal affine semigroup (*cf.* [TE]). So its minimal generating system as a semigroup, which is denoted by $\mathrm{FUND}(\mathcal{F}_\Gamma^H)$, is uniquely determined (*cf.* ibid.). Put $\Lambda = \Phi_\Gamma(\mathrm{FUND}(\mathcal{F}_\Gamma^T))$ and let

$$\Lambda_{\min} := \{q \in \Lambda \mid \sqrt{\mathcal{O}(X)q} \text{ is maximal in } \{\sqrt{\mathcal{O}(X)h} \mid h \in \Lambda\}\}.$$

Remark 4.3. — Since any element in $\Phi_\Gamma(\mathcal{F}_\Gamma^T)$ is expressed as a product of elements of Λ, for any $q \in \Lambda$, $q \in \Lambda_{\min}$ if and only if $\sqrt{\mathcal{O}(X)^T q}$ is maximal in $\{\sqrt{\mathcal{O}(X)^T h} \mid h \in \Phi_\Gamma(\mathcal{F}_\Gamma^T)\}$ (*cf.* Corollary 2.3).

PROPOSITION 4.4. — *Let h_i $(i = 1, 2)$ be elements of $\Lambda_{\min}$. Suppose that $h_i \in \sqrt{\mathcal{O}(X)y}$ $(i = 1, 2)$ for a non-unit $y \in \mathcal{O}(X)$ such that $y \in \Phi_\Gamma(\mathcal{F}_\Gamma)$ or $y \notin \mathcal{O}(X)^T$. If (X, T) is stable and equidimensional, then $\sqrt{\mathcal{O}(X)h_1} = \sqrt{\mathcal{O}(X)h_2}$.*

Proof. — By Lemma 2.7 and Theorem 4.2, we note the following fact: For any nonzero $\chi \in \mathfrak{X}(T)$ with $\mathcal{O}(X)_\chi \neq \{0\}$, there are a natural number n and a non-unit $f \in \mathcal{O}(X)_{n\chi} \cap \Phi_\Gamma(\mathcal{F}_\Gamma)$ satisfying $(\mathcal{O}(X)_\chi)^n \subset \mathcal{O}(X)^T \cdot f$.

Suppose that $y \notin \mathcal{O}(X)^T$. Because y is a semi-invariant of T (*cf.* Lemma 2.4), by the fact as above, we see that $y \in \sqrt{\mathcal{O}(X)z}$ for a non-unit $z \in \Phi_\Gamma(\mathcal{F}_\Gamma)\backslash\mathcal{O}(X)^T$. Then there are natural numbers n_i and semi-invariants w_i $(i = 1, 2)$ such that $h_i^{n_i} = z \cdot w_i$. Applying the above fact to the character associated with w_i's again, we can choose a natural number m and a semi-invariant $g \in \Phi_\Gamma(\mathcal{F}_\Gamma)$ in such a way that $w_i^m \in g \cdot \mathcal{O}(X)^T$ $(i = 1, 2)$. Then both h_1 and h_2 belong to $\sqrt{\mathcal{O}(X)z^m \cdot g}$. Since $z^m \cdot g \in \mathcal{O}(X)^T$, exchanging y, we may assume that $y \in \Phi_\Gamma(\mathcal{F}_\Gamma) \cap \mathcal{O}(X)^T$, whether the original y belongs to $\mathcal{O}(X)^T$ or not. The assertion follows from the maximality of $\sqrt{\mathcal{O}(X)h_i}$'s, which is pointed out in Remark 4.3. □

THEOREM 4.5. — *Let V be the dual of a minimal homogeneous T-submodule of $\mathcal{O}(X)$ which generates $\mathcal{O}(X)$ as a K-algebra. Suppose that X is factorial. If the action of T on X is stable and equidimensional, then (V, T) is cofree.*

The rest of this section is devoted to the proof of Theorem 4.5. Hereafter we suppose the following extra conditions are satisfied; X is *factorial* and that the action of T on X is *stable* and *equidimensional*. Clearly $\{\mathrm{div}_{\mathcal{O}(X)}(f) \mid f \in \Gamma\}$ consists of distinct prime divisors. For any $f \in \Phi_\Gamma(\mathcal{F}_\Gamma)$, we put

$$\mathrm{Supp}(f) := \{g \in \Gamma \mid f \in \mathcal{O}(X) \cdot g\},$$

which satisfies $\{\mathrm{div}_{\mathcal{O}(X)}(g) \mid g \in \mathrm{Supp}(f)\} = \mathrm{supp}_{\mathcal{O}(X)}(f)$. By Proposition 4.4, the subsets $\mathrm{Supp}(h)$, $h \in \Lambda_{\min}$, of Γ are pairwisely disjoint. For any subset Δ of Γ, $K[\Delta]$ denotes the K-subalgebra of $\mathcal{O}(X)$ generated by Δ on which T acts naturally. The symbol $\sqcup$ is denoted to one of a disjoint union.

LEMMA 4.6. — *Let h be any element of $\Phi_\Gamma(\mathcal{F}_\Gamma^T)$. Then*

$$\mathrm{Supp}(h) = \bigsqcup_{\substack{g \in \Lambda_{\min} \\ \mathrm{Supp}(g) \subset \mathrm{Supp}(h)}} \mathrm{Supp}(g).$$

Proof. — Suppose that h is a non-unit. Since h is a multiple of elements in Λ,

$$\mathrm{Supp}(h) = \bigcup_{\substack{g \in \Lambda \\ \mathrm{Supp}(g) \subset \mathrm{Supp}(h)}} \mathrm{Supp}(g).$$

So, there is an element v of $\Lambda_{\min}$ satisfying $\mathrm{Supp}(v) \subset \mathrm{Supp}(h)$. Then, because X is factorial, by Proposition 2.2, we can choose natural numbers m, n and an element $h' \in \Phi_\Gamma(\mathcal{F}_\Gamma^T)$ such that $h^m = v^n \cdot h'$ and $\mathrm{Supp}(h') \subsetneq \mathrm{Supp}(h)$. The assertion of this lemma follows inductively from this observation. □

LEMMA 4.7. — $\bigcup_{h\in\Lambda_{\min}} \mathrm{Supp}(h) = \Gamma.$

Proof. — Let f be any element of Γ and χ a linear character of T satisfying $f \in \mathcal{O}(X)_\chi$. Because the stability of (X,T) implies that $\mathcal{O}(X)_{-\chi} \neq \{0\}$ and this $\mathcal{O}(X)^T$-module is generated by $\mathcal{O}(X)_{-\chi} \cap \Phi_\Gamma(\mathcal{F}_\Gamma)$ (*cf.* Lemma 2.7), the element f is a divisor of an element of $\Phi_\Gamma(\mathcal{F}_\Gamma^T)$. Thus the assertion is a consequence of Lemma 4.6. □

Let $M_i \in \mathcal{O}(X)^T$ and $m \in \mathbb{N}$ be defined to satisfy

$$\Lambda_{\min} = \{M_1, \dots, M_m\}.$$

For each M_i, let V_i denote the subspace of $\mathcal{O}(X)$ generated by $\mathrm{Supp}(M_i)$. Clearly V_i is naturally regarded as a T-module of dimension $\mathrm{card}(\mathrm{Supp}(M_i))$. Denote by $\mathcal{K}_i$ the kernel of the representation $T \to GL(V_i)$. The action of $T/\mathcal{K}_i$ on $K[V_i]$, the K-subalgebra of $\mathcal{O}(X)$ gengerated by V_i, is stable (*cf.* Proposition 3.5) and faithful. We easily see that $K[V_i]^T = K[M_i]$ (*cf.* Lemma 2.1 and Lemma 3.9).

LEMMA 4.8. — $T|_{V_i} = \Big(\bigcap_{\substack{j\neq i \\ 1\leq j\leq m}} \mathcal{K}_j \Big)|_{V_i}$ *for all* $1 \leq i \leq m$.

Proof. — Suppose that $T|_{V_1} \neq \mathcal{K}_2|_{V_1}$. By (4) of Lemma 3.9, there is a non-unit $N \in \Phi_\Gamma(\mathcal{F}_\Gamma^{\mathcal{K}_2})$ satisfying $\mathrm{Supp}(N) \subsetneq \mathrm{Supp}(M_1)$, and by the minimality of $\mathrm{Supp}(M_1)$, we see that $N \in \mathcal{O}(X)_\chi$ for some $\chi \in \mathfrak{X}(T)$ vanishing $\mathcal{K}_2$. Since the action of $T/\mathcal{K}_2$ on $\mathrm{Spm}\, K[V_2]$ is faithful and stable, from Lemma 2.6, Proposition 3.5 and Lemma 2,7, we get an element $N' \in K[V_2]_{-\chi} \cap \Phi_\Gamma(\mathcal{F}_\Gamma)$. Then $N \cdot N' \in \Phi_\Gamma(\mathcal{F}_\Gamma^T)$, which contradicts Lemma 4.6. Consequently, for a natural number $2 \leq k \leq m$, we may assume that

$$T|_{V_i} = \Big(\bigcap_{j\in J} \mathcal{K}_j \Big)|_{V_i} \ \ (i \in \{1, \dots, m\} \backslash J),$$

if $J \subset \{1, \dots, m\}$ satisfies $\mathrm{card}(J) < k$. Moreover, for an instance, we suppose that

$$T|V_1 \neq (\mathcal{K}_2 \cap \dots \cap \mathcal{K}_{k+1})|_{V_1}.$$

There is a non-unit

$$N_1 \in \Phi_\Gamma(\mathcal{F}_\Gamma) \cap K[V_1]^{\mathcal{K}_2 \cap \dots \cap \mathcal{K}_{k+1}}$$

satisfying $\mathrm{Supp}(N_1) \subsetneq \mathrm{Supp}(M_1)$ (*cf.* (4) of Lemma 3.9). Clearly $N_1 \in \mathcal{O}(X)_\chi$ for a non-zero $\chi \in \mathfrak{X}(T)$ vanishing on $\mathcal{K}_2 \cap \dots \cap \mathcal{K}_{k+1}$. As

$$(\mathcal{K}_2 \cap \dots \cap \mathcal{K}_k)|_{V_{k+1}} = T|_{V_{k+1}},$$

the closed subgroup

$$(\mathcal{K}_2 \cap \ldots \cap \mathcal{K}_k)/(\mathcal{K}_2 \cap \ldots \cap \mathcal{K}_k \cap \mathcal{K}_{k+1})$$

of $T/(\mathcal{K}_2 \cap \ldots \cap \mathcal{K}_k \cap \mathcal{K}_{k+1})$ is connected and its action on $\operatorname{Spm} K[V_{k+1}]$ is faithful and stable. So, regarding $\chi|_{(\mathcal{K}_2 \cap \ldots \cap \mathcal{K}_k)}$ as a linear character of

$$(\mathcal{K}_2 \cap \ldots \cap \mathcal{K}_k)/(\mathcal{K}_2 \cap \ldots \cap \mathcal{K}_k \cap \mathcal{K}_{k+1}),$$

we choose a nonzero element

$$N_{k+1} \in \Phi_\Gamma(\mathcal{F}_\Gamma) \cap K[V_{k+1}]_{(-\chi)|_{(\mathcal{K}_2 \cap \ldots \cap \mathcal{K}_k)}}$$

(it may be equal to 1). Let ψ be the linear character of T satisfying $N_1 \cdot N_{k+1} \in \mathcal{O}(X)_\psi$. Since

$$(\mathcal{K}_2 \cap \ldots \cap \mathcal{K}_{k-1})|_{V_k} = T|_{V_k},$$

the closed subgroup

$$(\mathcal{K}_2 \cap \ldots \cap \mathcal{K}_{k-1})/(\mathcal{K}_2 \cap \ldots \cap \mathcal{K}_{k-1} \cap \mathcal{K}_k)$$

is connected and its action on $\operatorname{Spm} K[V_k]$ is faithful and stable. Clearly ψ vanishes on the subgroup $\mathcal{K}_2 \cap \ldots \cap \mathcal{K}_k$, and hence we similarly get a nonzero element

$$N_k \in \Phi_\Gamma(\mathcal{F}_\Gamma) \cap K[V_k]_{(-\psi)|_{(\mathcal{K}_2 \cap \ldots \cap \mathcal{K}_{k-1})}}.$$

We continue this procedure and, for any $k > i \geq 2$, consequently choose N_i from $\Phi_\Gamma(\mathcal{F}_\Gamma) \cap K[V_i]$ in such a way that $N_1 \cdot N_2 \cdot \ldots \cdot N_{k+1} \in \mathcal{O}(X)^T$. Then

$$\begin{aligned} \operatorname{Supp}(N_1 \cdot N_2 \cdot \ldots \cdot N_{k+1}) &= \operatorname{Supp}(N_1) \sqcup \operatorname{Supp}(N_2) \sqcup \ldots \sqcup \operatorname{Supp}(N_{k+1}) \\ &\subsetneq \operatorname{Supp}(M_1) \sqcup \operatorname{Supp}(M_2) \sqcup \ldots \sqcup \operatorname{Supp}(M_{k+1}), \end{aligned}$$

which contradicts Lemma 4.6. We inductively get the assertion as desired. □

Proof of Theorem 4.5. — We may regard the dual V^* of V as the subspace of $\mathcal{O}(X)$ generated by Γ. By Lemma 4.7, we have $V^* = \bigoplus_{1 \leq i \leq m} V_i$, and by Lemma 4.8, we see that

$$T|_V = \Big(\bigcap_{\substack{1 \leq j \leq m \\ j \neq 1}} \mathcal{K}_j\Big)|_V \times \Big(\bigcap_{\substack{1 \leq j \leq m \\ j \neq 2}} \mathcal{K}_j\Big)|_V \times \ldots \times \Big(\bigcap_{\substack{1 \leq j \leq m \\ j \neq m}} \mathcal{K}_j\Big)|_V.$$

Since $\operatorname{Sym}(V_i^*)^{\left(\cap_{1 \leq j \leq m, j \neq i} \mathcal{K}_j\right)}$ are of dimension one (*cf.* Lemma 3.9) and $\mathcal{O}(V)^T$ is isomorphic to a tensor product of these algebras, $\Big(V, \bigcap_{1 \leq j \leq m, j \neq i} \mathcal{K}_j\Big)$, $1 \leq i \leq m$, are cofree and hence (V, T) is cofree. □

Consequently, by Corollary 3.6 and the last assertion of Proposition 3.5 , we have just established the equivalence of Theorem 1.1.

5. Additional remarks.

Theorem 1.2 follows from Theorem 1.1, (2) of Lemma 3.1 and the next result:

THEOREM 5.1. — *Under the same circumstances as in Theorem 1.2, if the action (X, G) is equidimensional, then*

$$\operatorname{Spm} \mathfrak{S}_{G/(\mathcal{R}_u(G)\cdot[G,G])}(X/(\mathcal{R}_u(G)\cdot[G,G]))$$

is an affine factorial variety and the induced action of $G/(\mathcal{R}_u(G)\cdot[G,G])$ on this variety is stable and cofree.

Proof. — Putting $\bar{X} := X/\mathcal{R}_u(G)$ and $\bar{G} := G/\mathcal{R}_u(G)$ for simplicity, by [M], $\bar{X}$ is an affine factorial $\bar{G}$-variety. Moreover, using [M] again, $\bar{X}/[\bar{G},\bar{G}]$ is an affine factorial variety with the torus $\bar{G}/[\bar{G},\bar{G}]$-action. Since

$$\operatorname{Spm} \mathfrak{S}_{\bar{G}/[\bar{G},\bar{G}]}(\bar{X}/[\bar{G},\bar{G}]) = (\bar{X}/[\bar{G},\bar{G}])/\mathfrak{K}_{\bar{X}/[\bar{G},\bar{G}]}(\bar{G}/[\bar{G},\bar{G}])$$

(*cf.* (1) of Proposition 3.4), the assertions follow from (5) of Proposition 3.4, Remark 2.9, Corollary 3.7 and Theorem 1.1. □

Remark 5.2. — Let $T \to GL(V)$ be a finite-dimensional rational representation of T. Then, in an appropriate affine space, V/T is defined by polynomial functions in the following form;

$$\text{monomial} - \text{monomial}.$$

Thus we immediately infer that V/T is factorial if and only if it is an affine space over K (*e.g.* [TE]).

Recall that a locality over K (*cf.* [LR]) is said to be a *quotient singularity*, if its completion with respect to its maximal ideal is isomorphic to a ring of invariants of a formal power series ring over K under a linear action of a finite group.

THEOREM 5.3. — *Let G be a (not necessarily connected) reductive algebraic group and (X, x) an affine pointed G-variety with a smooth base point x. Suppose that*

$$\dim \mathcal{N}(X/[G^0, G^0], G) + \dim X/G = \dim X/[G^0, G^0].$$

Then:

(1) $\operatorname{Cl}(\mathcal{O}(X/G)_{\pi_{X,G}(x)}) \cong \mathfrak{X}((G/G^0)/\mathfrak{R}_{X/G^0}(G/G^0))$. *Here $\mathfrak{R}_{X/G^0}(G/G^0)$ denotes the normal subgroup of G/G^0 generated by elements $\sigma \in G/G^0$ such that* $\operatorname{ht}(\mathcal{O}(X/G^0)\cdot(\sigma - 1)(\mathcal{O}(X/G^0))) \leq 1$.

(2) *If* $(X/[G^0, G^0], G)$ *is stable, then* $\mathcal{N}(X/[G^0, G^0], G^0)$ *is a complete intersection and* $\mathcal{O}(X/[G^0, G^0])_{\pi_{X,[G^0,G^0]}(x)}$ *is* $\mathcal{O}(X/G^0)_{\pi_{X,G^0}(x)}$*-free.*

(3) *If* G^0 *is a torus, then* $\mathcal{N}(X, G^0)$ *is a complete intersection and the local ring* $\mathcal{O}(X/G)_{\pi_{X,G}(x)}$ *is a quotient singularity by the action of* G/G^0.

Proof. — As in [L], we similarly have a G-equivariant morphism $\psi : X \to \mathcal{T}_x X$ from X to the Zariski tangent space $\mathcal{T}_x X$ of X at x, étale at x, with $\psi(x) = 0$ such that the tangent map $d\psi_x$ agrees with the natural identification of $\mathcal{T}_x X$ with $\mathcal{T}_0(\mathcal{T}_x X)$. Then ψ induces a G-equivariant isomorphism $\hat{\mathcal{O}}(\mathcal{T}_x X)_0 \cong \hat{\mathcal{O}}(X)_x$ between completions of local rings. For any reductive subgroup H of G, we have the following canonical isomorphisms (*cf.* [N2], Lemma 3.5);

$$\hat{\mathcal{O}}(\mathcal{T}_x X/H)_{\pi_{\mathcal{T}_x X,H}(0)} \cong (\hat{\mathcal{O}}(\mathcal{T}_x X)_0)^H \cong (\hat{\mathcal{O}}(X)_x)^H \cong \hat{\mathcal{O}}(X/H)_{\pi_{X,H}(x)}.$$

Since $\mathcal{N}(X, G^0) \cong \mathcal{N}(\mathcal{T}_x X, G^0)$ (*cf*, [H], Theorem 3.4),

$$\mathcal{N}(X/[G^0, G^0], G^0) \cong \mathcal{N}(\mathcal{T}_x X/[G^0, G^0], G^0)$$

and by our assumption, the action $(\mathcal{T}_x X/[G^0, G^0], G^0)$ is equidimensional. Moreover, applying Theorem 1.2 to this action, we see that $\mathcal{T}_x X/G^0$ is factorial, and so we infer that $\mathcal{T}_x X/G^0$ is an affine space in the case where G^0 is a torus, which shows (3) (*cf.* (4) of Proposition 2.10 and [N2], ibid.). On the other hand, by [N2], we see that

$$\mathrm{Cl}(\mathcal{O}(\mathcal{T}_x X/H)_{\pi_{\mathcal{T}_x X,H}(0)}) \cong \mathrm{Cl}(\hat{\mathcal{O}}(\mathcal{T}_x X/H)_{\pi_{\mathcal{T}_x X,H}(0)}) \cong \mathrm{Cl}(\hat{\mathcal{O}}(X/H)_{\pi_{X,H}(x)}) \cong \mathrm{Cl}(\mathcal{O}(X/H)_{\pi_{X,H}(x)})$$

for the above H. Since $\mathfrak{R}_{X/G^0}(G/G^0) = \mathfrak{R}_{\mathcal{T}_x X/G^0}(G/G^0)$, the assertion (1) follows from this observation, [N1] and [N2], Lemma 3.5.

In order to show (2), we furthermore suppose that $(X/[G^0, G^0], G)$ (and so $(X/[G^0, G^0], G^0)$) is stable. Let $\chi \in \mathfrak{X}(G^0/[G^0, G^0])$ satisfy $\mathcal{O}(\mathcal{T}_x X/[G^0, G^0])_\chi \neq \{0\}$. Then

$$\{0\} \neq (\hat{\mathcal{O}}(\mathcal{T}_x X/[G^0, G^0])_0)_\chi \cong (\hat{\mathcal{O}}(X/[G^0, G^0])_{\pi_{X,[G^0,G^0]}(x)})_\chi,$$

which requires $\mathcal{O}(X/[G^0, G^0])_\chi \neq \{0\}$. By stability, we have

$$\{0\} \neq \mathcal{O}(X/[G^0, G^0])_{-\chi} \hookrightarrow (\hat{\mathcal{O}}(X/[G^0, G^0])_{\pi_{X,[G^0,G^0]}(x)})_{-\chi},$$

which shows that the action $(\mathcal{T}_x X/[G^0, G^0], G^0)$ is stable (*cf.* Proposition 3.5). Applying Theorem 1.1 to this action, we obtain the assertions of (2), since

$$((\mathcal{O}(\mathcal{T}_x X/[G^0, G^0])_0)_\chi)^\wedge \cong ((\mathcal{O}(X/[G^0, G^0])_{\pi_{X,[G^0,G^0]}(x)})_\chi)^\wedge$$

(*cf.* the proof of [N2], Lemma 3.5 and Lemma 3.7). □

By this and the slice étale theorem [L], we have

COROLLARY 5.4. — *Suppose that G is reductive and let X be a smooth affine G-variety. If the action of G on X is equidimensional, then, for any closed point ξ of X/G, $\mathrm{Cl}(\mathcal{O}(X/G)_\xi)$ is isomorphic to a subgroup of $\mathfrak{X}(G_x/G_x^0)$, where x denotes a closed point in the unique closed orbit in X over ξ.* □

Remark 5.5. — In general, *cofree* actions on affine conical varieties do not imply *coregularity.* In fact, we give an example as follows: Let A be an affine graded domain over K and let R be a polynomial ring $A[X, Y]$ with two indeterminates X, Y. Suppose $K^* \ni t$ acts on A-linearly on $A[X,Y]$ in such a way that $t(X) = t^a \cdot X$ and $t(Y) = t^{-b} \cdot Y$ with $a, b \in \mathbb{N}$. Then $(\operatorname{Spm} R, K^*)$ is a cofree action and R^{K^*} is a polynomial ring over A. So $(\operatorname{Spm} R, K^*)$ is coregular if and only if A is regular.

Remark 5.6. — The assumption in Theorem 1.1 that X *is factorial* is needed in view of the following example: Let V be the three-dimensional vector space over K with a basis $\{X, Y, Z\}$. Suppose that $K^* \ni t$ acts on V in such a way that $t(X) = t \cdot X$, $t(Y) = t^{-1} \cdot Y$ and $t(Z) = Z$. Moreover suppose that $\mathbb{Z}/2\mathbb{Z} =< \tau >$ acts on V in such a way that $\tau(X) = -X$, $\tau(Y) = Y$ and $\tau(Z) = -Z$. Put $R := \mathrm{Sym}(V)^{\mathbb{Z}/2\mathbb{Z}}$ and $G := K^* \times \mathbb{Z}/2\mathbb{Z}$. Denote by μ the linear character $K^* \ni t \mapsto t \in K^*$. Clearly $R = K[X^2, Y, Z^2, XZ]$ is a normal graded algebra over K with a grade preserving action of $K^* \cong G/(\mathbb{Z}/2\mathbb{Z})$ and, by Samuel's Galois descent, $\mathrm{Cl}\, R \cong \mathbb{Z}/2\mathbb{Z}$. Since the dual action (V^*, G) is stable and equidimensional (*cf.* Theorem 1.1), so is the action $(\operatorname{Spm} R, K^*)$. However $R_\mu = R^{K^*}(X^2Y, XZ)$, which is not R^{K^*}-free.

Remark 5.7. — Let H be a connected semisimple group and put $G := H \times K^*$. Then there are infinitely many H-isomorphism types of restrictions of representations $G \to GL(V)$ to H such that $V^H = \{0\}$, both (V, G) and $(V/H, G/H)$ are cofree and V/H are not complete intersections. In fact let $\rho_1 : H \to GL(V_1)$ be a cofree representation and $\rho_2 : H \to GL(V_2)$ a representation such that V_2/H is not a complete intersection. Suppose that K^* ($\subset G$) acts trivially on V_1 and it acts on V_2 as homotheties. Then since $(V_1 \oplus V_2)/G \cong V_1/G$, by (2) of Proposition 2.10, Remark 2.9 and [HR], both $(V_1 \oplus V_2, G)$ and $((V_1 \oplus V_2)/[G, G], K^*)$

are cofree. However, since $(V_1 \oplus V_2)/[G,G]$ is not a complete intersection, $\mathcal{N}((V_1 \oplus V_2)/[G,G], K^*)$ is not a complete intersection. Combining [P3], Chap. 4 with [N3], we can show that there are infinitely many isomorphism types of such representations V_2 of H. Thus the assumption in Theorem 1.1 that (X,T) *is stable* is needed in view of these examples.

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