Annales de l'institut Fourier

Nachman Aronszajn
Szeptycki P.

On spaces of potentials connected with L^p classes

Annales de l'institut Fourier, tome 13, nº 2 (1963), p. 211-306
<http://www.numdam.org/item?id=AIF_1963__13_2_211_0>

Ann. Inst. Fourier, Grenoble
13,2 (1963), 211-306.

ON SPACES OF POTENTIALS CONNECTED WITH L^p CLASSES

by N. ARONSZAJN, F. MULLA, P. SZEPTYCKI (1)

(Lawrence, Kansas)

§ 1. Introduction.

There are in existence many classes introduced in view of extending the notion of Bessel potentials of L^2 functions (cf. [2]; classes P^α discussed there were introduced earlier but the theory was not published in extenso).

The most important appear to be the classes often denoted by L^p_α (Calderon [6]), W^α_p (introduced by Gagliardo [11] and Slobodeckii [14] as the extension of classes W^α_p introduced by Sobolev for integral values of α) and $\tilde{\mathfrak{B}}^{\alpha,p}$ (the special case of more general classes introduced by Besov [5]) (2).

These classes are defined essentially as follows (for precise definitions see § 7).

L^p_α is the class of all Bessel potentials of L^p functions, i.e. of all functions u of the form $u = G_\alpha * f$, $f \in L^p$, where G_α is the Bessel kernel of order α (cf. § 2). The norm in L^p_α is defined by $\|u\|_{\alpha,p} = \|f\|_{L^p}$.

W^α_p, for $\alpha > 0$ is defined as the class of all functions which together with all derivatives of order $\leqq \alpha$ (in the sense of the theory of distributions) are in L^p and have finite norm.

$$\left\{ \sum_{l=0}^{[\alpha]} \sum_{|j|=l} \left[\int_{R^n} |D_j u|^p \, dx + \int_{R^n}\int_{R^n} \left| \frac{D_j u(x) - D_j u(y)}{|x-y|^{\alpha-[\alpha]}} \right|^p \frac{dx\, dy}{|x-y|^n} \right] \right\}^{1/p}.$$

(1) Research for this paper was sponsored by the Office of Naval Research, Contract Nonr 583(04), and by the National Science Foundation, Grant G-17057.

(2) We shall not consider here the classes introduced by Nikolskii [12] as they are not so closely related to potentials of L^p functions. The same applies to the general classes of Besov.

In the latter expression the double integrals are to be omitted for α integer. (For a precise definition of the norm in W_p^α see § 3).

$\tilde{\mathfrak{B}}^{\alpha,p}$, $\alpha > 0$, is defined as the class of all functions of L^p with the finite norm

$$|u|_{\alpha,p,k} = \left(\|u\|_{L^p}^p + \int \left\|\frac{\Delta_t^k u}{|t|^\alpha}\right\|_{L_p}^p |t|^{-n} dt\right)^{1/p}, \quad k > \alpha.$$

This expression does not give a standard norm in $\tilde{\mathfrak{B}}^{\alpha,p}$. However, for all integers $k > \alpha$ the corresponding norms $|u|_{\alpha,p,k}$ are equivalent.

In view of different aspects of the theory, each of these classes has its advantages and disadvantages. From the point of view of simplicity of properties the class $\tilde{\mathfrak{B}}^{\alpha,p}$ seems to be the most advantageous; the class L_α^p is the simplest from the point of view of definition and representations of its elements. Class W_p^α is in most cases in a kind of intermediate position between the other two; for α not integer and all p, $1 \leq p \leq \infty$, W_p^α coincides with $\tilde{\mathfrak{B}}^{\alpha,p}$, whereas for α integer $1 < p < \infty$, it coincides with L_α^p. The only cases when W_p^α has a somewhat independent existence are $p = 1$ or $p = \infty$ and α integer. These are actually the cases when the information about W_p^α is the least precise. For this reason, if we were interested in studying these classes in the whole space R^n, there perhaps wouldn't be much point in introducing the classes W_p^α. This study, however, is conceived as an introduction and help to the investigation of the corresponding spaces on domains of the space R^n (as was done in the case of Bessel potentials in [3]). In this connection we immediately come across the question of defining these classes intrinsically for a domain $D \subset R^n$. For W_p^α the answer is immediate. To define $W_p^\alpha(D)$ it suffices to replace R^n by D in all the integrals occurring in the definition of the norm. Such definition is justified by an extension theorem asserting the existence of a simultaneous linear and bounded extension mapping from $W_p^\alpha(D)$ to $W_p^\alpha(R^n)$ for a rather general class of domains (3)

(3) Some details about this question can be found in the revised version of [3] (to appear).

As concerns L^p_α there is no known direct definition of a corresponding class in a domain D.

In the case of $\tilde{\mathfrak{B}}^{\alpha,p}$ there is an intrinsic definition for a domain D proposed by Besov in which the integration of the difference is taken only over the points of D where the difference is defined. However, it is not known, and probably not true that for a general domain D the different norms defining $\tilde{\mathfrak{B}}^{\alpha,p}$ are equivalent. Even if one of them is chosen, the presence of the higher difference occuring in the norm makes it very unwieldy to use it in a domain. In the case of classes W^α_p we know that most of the results of the theory of Bessel potentials of L^2 functions can be extended to $W^\alpha_p(D)$(3). It is not known and seems difficult to extend these results to the proposed classes $\tilde{\mathfrak{B}}^{\alpha,p}$ (D). This is the reason why in the present paper we are stressing the study of the classes W^α_p.

All the classes under consideration can be considered as completions of the class C^∞_0 with corresponding norms (except for $p = \infty$). The classes L^p_α, W^α_p, $\tilde{\mathfrak{B}}^{\alpha,p}$ are such completions relative to the class of sets of Lebesgue measure 0. This approach avoids some essential difficulties, but in some respects it is rather inconvenient, especially if we want to speak about restrictions of these classes to hyperplanes or more general subsets of R^n. Clearly this approach does not allow any insight into pointwise properties of derivatives of functions of the classes under consideration.

Similarly as was done in the case of Bessel potentials of L^2 functions we introduce the perfect functional completions of C^∞_0 with the norms of L^p_α, W^α_p, $\tilde{\mathfrak{B}}^{\alpha,p}$. To distinguish these perfect completions from the imperfect completions we use the symbols $P^{\alpha,p}$ for the perfect completion corresponding to L^p_α (in analogy to the symbol P^α for Bessel potentials of L^2 functions), $\check{P}^{\alpha,p}$ for the perfect completion corresponding to W^α_p (in analogy to $\check{P}^\alpha$ for Bessel potentials intrinsically introduced on domains) and $B^{\alpha,p}$ for the perfect completion corresponding to $\tilde{\mathfrak{B}}^{\alpha,p}$.

It is to be noted that for $p = 2$ all three classes coincide with P^α, and this is the only exponent for which a single class can be defined combining all the advantages of $P^{\alpha,p}$, $\check{P}^{\alpha,p}$ and $B^{\alpha,p}$.

All three families of spaces considered here were extensively investigated by several authors, Besov [5] (see also [12]), Calderon [6], Gagliardo [11], Slobodeckii [14], Stein [7], [8], Taibleson [19] and others, (4) and many of the results presented in this paper were obtained by them. We believe, that in addition to some new results which we obtain here, the most significant contribution made is the introduction of the representation formulas for the study of the spaces under consideration. The method appears to have possible applications in the general study of differential problems.

The basic idea behind the use of representation formulas lies in the fact that they represent a function as an integral transform (or a linear combination of such) applied to expressions whose L^p norms occur in the definitions of the spaces under consideration. For example, the representation formula (c.f. § 5).

$$u(x) = \sum_{l=0}^{m} \sum_{|j|=l} \binom{m}{l} \int_{R^n} D_j^{(y)} G_{2m}(x-y) D_j u(y)\, dy$$

expresses u in terms of all its derivatives of order $\leqq m$; the norm in W_p^m is defined in terms of L^p norms of these derivatives.

We give a general method for obtaining such representation formulas. They are derived from identities written in terms of Fourier transforms, where they appear as quite elementary; the translation of these leads to identities in terms of the original functions, usually in terms of some special integral transformations. This kind of translation has a well determined meaning in terms of tempered distributions, but since we are interested in applying the resulting formulas as bona fide integral transformations, we have to use a relatively simple theorem (§ 5) giving conditions under which the formulas so obtained are valid as integral formulas. These considerations in turn necessitate an analysis of the corresponding integral transformations in order to decide if these transformations are absolutely regular.

In § 6 we give criteria for absolute regularity which were already known for some time to be sufficient (but were not

(4) See [12].

published). Quite recently E. Gagliardo [11 *a*] proved them to be also necessary.

In the introductory chapter we recall the definition of the kernel G_α and some of its properties (§ 2). For functions of C_0^∞ we introduce the standard and approximate norms of W_p^α (§ 3) and the norms $|\ \ |_{\alpha,p,k}$ of $\tilde{\mathfrak{B}}^{\alpha,p}$ (§ 4) and investigate their properties; in particular we prove the equivalence of norms $|\ \ |_{\alpha,p,k}$ with varying k.

The second chapter deals with the imperfect completions. In § 5 we describe the formal way of obtaining all our representation formulas (among these the reproducing formulas and inversion formulas for Bessel potentials). § 6 is to be taken as a brief introduction to the general theory of integral transformations which leads in particular to the notions of semiregular, regular, and absolutely regular transformations and their basic properties. In § 7 we introduce in a precise way the imperfect completions; in § 8 we prove the continuity of the standard norm of W_p^α considered as a function of α. In § 9 we derive various auxiliary inequalities concerning the kernel G_α, its derivatives and differences, which are needed in § 10 where we consider several integral transformations occuring in our representation formulas and analyze them from the point of view of properties described in § 6. Almost all of these transformations turn out to be absolutely regular which allows us to obtain in § 11 all the equalities, isomorphisms and inclusions between the different classes. We show in particular that there is a well-determined space $B^{0,p}$ of tempered distributions such that $\tilde{\mathfrak{B}}^{\alpha,p} = G_\alpha B^{0,p}$ for all $\alpha > 0$. In most cases these results were obtained by other authors by different methods; we were able to make some of them more precise. In § 12 our representation formulas are used to represent the spaces W_p^α, $\tilde{\mathfrak{B}}^{\alpha,p}$ as projections in suitably defined L^p-spaces; this allows us to prove in a simple way that W_p^α, $W_{p'}^\alpha$ and $\mathfrak{B}^{\alpha,p}$ $\mathfrak{B}^{\alpha,p'}$ are conjugate in suitable pairings.

Chapter III deals with the perfect completions $P^{\alpha,p}$, $\check{P}^{\alpha,p}$, and $B^{\alpha,p}$. In § 13 we prove their existence, describe their exceptional classes and show that in almost all cases the representation formulas introduced before give perfect representations of functions in corresponding perfect completions. It is shown further that functions in perfect completions have pointwise defined derivatives (for $p = 1$ the results are somewhat weaker). It is also shown that for every function in any of the imperfect completions we can very easily obtain a corresponding function in the perfect completion by replacing it by the pointwise limit of its regularizations (corrected function) and taking as its exceptional set the set of all points where the limit does not exist or is infinite. (Here again the result is less precise for $\check{P}^{\alpha,1}$, α-integer.)

In the last section we prove theorems about restrictions of functions of our classes to hyperplanes and extensions from hyperplanes to the whole space. We take advantage of the fact that our representation formulas give perfect representations of functions in our classes, and consequently the pointwise restrictions are defined directly by these formulas. The results of § 10 provide an immediate verification that the restrictions so obtained are in suitable classes. The extensions are obtained by again making a suitable use of the representation formulas.

Throughout this paper we shall consistently use the terminology and results of the theory of functional spaces and functional completion; for details we refer the reader to [1].

TABLE OF CONTENTS

CHAPTER I

PRELIMINARIES

§ 2. Notations and Bessel Kernels.

The following notations will be used consistently. x, y, z, ... will denote points of the n-dimensional Euclidean space R^n, $|x-y|$ the Euclidean distance of the points x, y, $|x| = |x-0|$, ξ, η, ... points of the dual space, (ξ, x) the inner product of the vectors ξ and x. The symbol D_i for $i = \{i_1, \ldots, i_l\}$ will denote the operator $\dfrac{\partial^l}{\partial x_{i_1} \ldots \partial x_{i_l}}$, $|i| = l$. $f*g$ will denote the convolution of f and g, $\hat{f}(\xi)$ the Fourier transform of f. We shall denote by $\mathfrak{A}_0$ the class of all sets of Lebesgue measure 0.

In order to avoid any possible misunderstanding, we shall make the following conventions concerning differences. We shall consider only forward differences. The symbol $\Delta^k_{t,a;x}$ will denote the difference of order k with increment t and initial point a, taken with respect to a variable x. In the case when a function preceeded by the symbol $\Delta^k_{t,a;x}$ depends on several variables, then in the operation of taking the difference, all variables other than x are treated as parameters. For example,

$$\Delta_{t,a;x} u(x, x-y, t) = u(a+t, a+t-y, t) - u(a, a-y, t).$$

We will use the following abbreviations systematically. If f is a function of a single variable x (where there is no doubt as to the variable with respect to which the difference is taken) we will write

$$\Delta^k_{t,a;x} f(x) \equiv \Delta^k_{t,a} f(x) \equiv \Delta^k_{t,a} f.$$

We will also write

$$\Delta^k_{t,x;x} \equiv \Delta^k_{t;x}$$

if the difference is applied to a function of several variables, and

$$\Delta^k_{t,x;x} f(x) = \Delta^k_t f.$$

if f is a function of a single variable x.

Concerning mixed differences, we mention only the following evident relations

$$\Delta^k_{t,a;x} \Delta^{k_1}_{t_1,a_1;x_1} = \Delta^{k_1}_{t_1,a_1;x_1} \Delta^k_{t,a;x}$$

if k, t, a, and x are independent of x_1, and k_1, t_1, a_1, and x_1 are independent of x;

$$\Delta^k_{t,x;x} \Delta^{k_1}_{t_1,x;x} = \Delta^k_{t;x} \Delta^{k_1}_{t_1;x} = \Delta^{k_1}_{t_1;x} \Delta^k_{t;x}$$

if k, t, k_1, and t_1 are independent of x.

For $\alpha > 0$ the Bessel kernel of order α,

$$G_\alpha(x - y) = G_\alpha(|x - y|)$$

is defined by the formula (c.f. [2]):

$$(2.1) \qquad G_\alpha(|x|) = \frac{1}{2^{\frac{n+\alpha-2}{2}} \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} K_{\frac{n-\alpha}{2}}(|x|)|x|^{\frac{\alpha-n}{2}},$$

where K_ν denotes the modified Bessel function of the third kind of order ν.

The same formula could be also used for $\alpha < 0$; the resulting function, however, is not locally integrable around the origin and cannot serve to define an integral convolution operator. In some considerations it will be convenient to indicate by $G^{(m)}_\alpha$ the Bessel kernel of order α on the space R^m; thus $G^{(n)}_\alpha = G_\alpha$.

The following properties of the kernels G_α will be needed in the sequel (c.f. [2]).

The Fourier transform of G_α is given by the formula

$$\hat{G}_\alpha(\xi) = \frac{(2\pi)^{-\frac{n}{2}}}{(1 + |\xi|^2)^{\alpha/2}}. \tag{2.2}$$

The kernel G_α is positive and analytic except at $x = 0$; for $x \neq 0$, $G_\alpha(x)$ is an entire function of α. The behavior of G_α is described by the following formulas (all representations being valid uniformly in α for α in any fixed bounded interval).

For $|x| \to 0$:

$$G_\alpha(x) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{n/2} \Gamma\left(\frac{\alpha}{2}\right)} |x|^{\alpha-n} + o\left(|x|^{\alpha-n}\right) \text{ if } \alpha \leqq n-1. \tag{2.3 a}$$

For $n - 1 \leqq \alpha \leqq n$, we have

$$G_\alpha(x) = \frac{1}{2^n \pi^{\frac{n-2}{2}} \Gamma\left(\frac{\alpha}{2}\right) \sin \pi \frac{n-\alpha}{2}} \left[\frac{\left(\frac{1}{2}|x|\right)^{\alpha-n}}{\Gamma\left(\frac{2-n+\alpha}{2}\right)} - \frac{1}{\Gamma\left(\frac{n-\alpha+2}{2}\right)} \right] + 0(1). \tag{2.3 b}$$

The last formula gives, in particular,

$$G_n(x) = \frac{1}{2^{n-1}\pi^{n/2}\Gamma\left(\frac{n}{2}\right)} \left[\log\frac{1}{|x|} + 0(1)\right]. \tag{2.3 c}$$

For $\alpha \geqq n$, we have

$$G_\alpha(x) = \frac{1}{2^n \pi^{\frac{n-2}{2}} \Gamma\left(\frac{\alpha}{2}\right) \sin \pi \frac{\alpha-n}{2}} \left[\frac{1}{\Gamma\left(\frac{2-\alpha+n}{2}\right)} - \frac{\left(\frac{1}{2}|x|\right)^{\alpha-n}}{\Gamma\left(\frac{2+\alpha-n}{2}\right)} \right] + 0\left(|x|^2\left(\log\frac{1}{|x|}\right)\right) \tag{2.3 d}$$

if $0 \leqq \alpha - n \leqq 1$.

(2.3 *e*)

$$G_\alpha(x) = \frac{1}{2^n \pi^{n/2}\Gamma\left(\frac{\alpha}{2}\right)}\left\{\sum_{r=0}^{k-1}\frac{(-1)^r\Gamma\left(\frac{\alpha-n}{2}-r\right)}{r!}\left(\frac{1}{2}|x|\right)^{2r}\right.$$

$$\left.+\frac{1}{\sin\frac{\alpha-n}{2}\pi}\left[\frac{1}{k!\,\Gamma\left(k+1\frac{\alpha-n}{2}\right)}-\frac{\left(\frac{1}{2}|x|\right)^{\alpha-n-2k}}{\Gamma\left(\frac{2+\alpha-n}{2}\right)}\right]\left(\frac{1}{2}|x|\right)^{2k}\right\}$$

$$+\ 0\left(|x|^{2k+2}\log\frac{1}{|x|}\right),$$

for $2k-1 \leqq \alpha - n \leqq 2k+1$, k-integer, $k \geqq 1$.

Hence, for $\alpha - n = 2k$,

(2.3 *f*) $G_{2k+n}(x) =$

$$\frac{1}{2^n\pi^{n/2}\Gamma\left(k+\frac{n}{2}\right)}\left\{\sum_{r=0}^{k-1}\frac{(-1)^r(k-r-1)!}{r!}\left(\frac{1}{2}|x|\right)^{2r}\right.$$

$$\left.+\ 2\frac{\left(\frac{1}{2}|x|\right)^{2k}}{k!}\left[\log\frac{1}{|x|}+o(1)\right]\right\}$$

Formulas (2.3 *a*)-(2.3 *f*) actually give the significant terms of the development of $G_\alpha(x)$ around 0; by differentiation they give the principal part of $D_iG_\alpha(x)$ at 0.

For $|x| \to \infty$,

$$(2.4)\qquad G_\alpha(x) \sim \frac{1}{2^{\frac{n+\alpha-1}{2}}\pi^{\frac{n-1}{2}}\Gamma\left(\frac{\alpha}{2}\right)}|x|^{\frac{\alpha-n-1}{2}}e^{-|x|}.$$

It follows that $G_\alpha \in L^1$ for all $\alpha > 0$; by (2.2) $\int G_\alpha(x)\,dx = 1$. Formula (2.2) also implies the following composition property of the kernel G_α:

$$(2.5)\qquad G_\alpha * G_\beta = G_{\alpha+\beta}.$$

$G_\alpha(x)$ being a function of $|x|$ only, define $G_\alpha(r) = G_\alpha(|x|)$ with $|x| = r$. Then

$$(2.6)\qquad \frac{dG_\alpha(r)}{dr} = \frac{-1}{2^{\frac{n+\alpha-2}{2}}\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}r^{\frac{\alpha-n}{2}}K_{\frac{n-\alpha+2}{2}}(r),$$

and hence $G_\alpha(r)$ is a decreasing function of r.

It will be convenient to introduce on the space $R^n \times R^n$ the measure μ_β, $0 < \beta < 1$ defined by the formula

$$(2.7)\quad d\mu_\beta(x, y) = \frac{1}{C(n, \beta)G_{2n+2\beta}(0)} \frac{G_{2n+2\beta}(x-y)}{|x-y|^n} dx\, dy,$$

where (see (2.3)),

$$(2.8)\qquad G_{2n+2\beta}(0) = \frac{\Gamma\left(\beta + \frac{n}{2}\right)}{2^n \pi^{n/2} \Gamma(n+\beta)},$$

and $C(n, \beta)$ is defined by the formula

$$(2.9)\quad C(n, \beta) = \frac{2^{-2\beta+1}\pi^{\frac{n+2}{2}}}{\Gamma(\beta+1)\Gamma\left(\beta+\frac{n}{2}\right)\sin \pi\beta}$$

$$= \int_{R^n} \frac{|e^{iz_n} - 1|^2}{(z_n^2 + |z'|^2)^{\frac{n+2\beta}{2}}} dz'\, dz_n,$$

where z' denotes the projection of the point $z = (z_1, \ldots, z_n)$ on the hyperplane $z_n = 0$.

It follows from the assymptotic representations of G_α (c.f. formulas (2.3) and (2.4)) that for $\alpha > \frac{n}{p'}$, $G_\alpha(x)$ is an L^p function, $\frac{1}{p} + \frac{1}{p'} = 1$. We will need an estimate for the norm $||G_\alpha||_{L^p}$.

To obtain this estimate we integrate separately over the regions $|x| \geqq 1$ and $|x| \leqq 1$. We use formula (2.4) for $|x| \geqq 1$ and for $|x| \leqq 1$ we estimate (5) : $G_\alpha(x) \leqq \varkappa |x|^{\alpha-n}\left(1 + \log \frac{1}{|x|}\right)$ if

$$\frac{n}{p'} < \alpha \leqq n, \quad G_\alpha(x) \leqq \varkappa\left[1 + \frac{1}{\alpha - n}(1 - |x|^{\alpha-n})\right]$$

for

$$n < \alpha < n+1 \qquad \text{and} \qquad G_\alpha(x) \leqq \varkappa \quad \text{for} \quad \alpha \geqq n+1.$$

(5) $\varkappa$ denotes here a constant (which may differ from one formula to another) depending only on n.

We get

(2.10)

$$\|G_\alpha\|_{L^p} \leq \begin{cases} \dfrac{\varkappa}{\left(\alpha - \dfrac{n}{p'}\right)^{1+1/p}} & \textit{for } \dfrac{n}{p'} < \alpha \leq n \\ \left[(\alpha-n)^{-p-1}B\left(p+1, \dfrac{n}{\alpha-n}\right)\right]^{1/p} & \textit{for } n < \alpha < n+1 \\ \varkappa & \textit{for } \alpha \geq n+1. \end{cases}$$

For $\sigma > 0$, $G_{n+\sigma}(x)$ is a continuous function on R^n. In some instances we shall need an estimate for the difference quotient $\dfrac{\Delta_t^k G_{n+\sigma}(x)}{|t|^\rho}$, $k \geq \rho$, $\rho < \sigma$. From (2.2) we have

$$\Delta_t^k G_{n+\sigma}(x) = (2\pi)^{-n} \int_{R^n} \frac{e^{i(\xi,x)}(e^{i(\xi,t)} - 1)^k}{(1+|\xi|^2)^{\frac{n+\sigma}{2}}} d\xi,$$

and hence

$$(2.11) \quad \frac{1}{|t|^\rho} |\Delta_t^k G_{n+\sigma}(x)| \leq (2\pi)^{-n} \int_{R^n} \frac{2^k \left|\sin \frac{(\xi,t)}{2}\right|^k}{|t|^\rho} \frac{d\xi}{(1+|\xi|^2)^{\frac{n+\sigma}{2}}}$$

$$\leq (2\pi)^{-n} 2^{k-\rho} \int_{R^n} \frac{|\xi|^\rho}{(1+|\xi|^2)^{\frac{n+\sigma}{2}}} d\xi = \pi^{-n} 2^{k-\rho-n-1} \omega_n B\left(\frac{\rho+n}{2}, \frac{\sigma-\rho}{2}\right).$$

§ 3. Standard norm. Approximate norms. Classes $\mathcal{F}^{\alpha,p}$.

In this section we shall define two norms which arise in connection with the generalization of Bessel potentials (c.f. [2]). For this purpose we shall need certain properties of covariant tensors.

Let $V^{(l)}$ denote the linear space of all covariant tensors of order l i.e. of all l-linear complex valued forms $A^{(l)}(\nu_1, \nu_2, \ldots, \nu_l)$ defined on the n-dimensional vector space R^n (of contravariant vectors). In every fixed coordinate system there is a 1-1 correspondence between tensors $A^{(l)}$ and n^l-tuples of their components given by the formula

$$A^{(l)}(\nu_1, \ldots, \nu_l) = A^{(l)}_{i_1,\ldots,i_l} \nu_1^{i_1}, \ldots, \nu_l^{i_l},$$

where $(\varphi_s^1, \ldots, \varphi_s)$ denote the components of the vector φ_s, $s = 1, \ldots, l$ and summation from 1 to n is understood over the repeated indices.

Let Σ denote the surface of the unit sphere in the space R^n, ω_n its area, and Σ^l its l-th cartesian power; let $\theta_{(l)} = (\theta_1, \ldots, \theta_l)$ denote an arbitrary point of Σ^l,

$$|\theta_j| = 1,\ j = 1, \ldots, n \qquad \text{and} \qquad d\theta_{(l)} = d\theta_1 \ldots d\theta_l$$

the element of volume of Σ^l.

Define now for $A^{(l)} \in V^{(l)}$ and $1 \leqq p < \infty$ the standard norm

$$|A^{(l)}|_p^p = \frac{n^l}{\omega_n^l} \int_{\Sigma^l} |A^{(l)}(\theta_{(l)})|^p \, d\theta_{(l)}, \tag{3.1}$$

and the approximate norm (dependent for $p \neq 2$ on the choice of the system of coordinates)

$$|A^{(l)}\urcorner|_p^p = \sum_i |A_i^{(l)}|^p. \tag{3.1'}$$

For $p = \infty$ we put as usual $|A^{(l)}|_\infty = \sup\limits_{\theta_{(l)}} A^{(l)}(|\theta_{(l)})|$ and $|A^{(l)}\urcorner|_\infty = \sup\limits_i |A_i^{(l)}|$.

For any $A^{(l)}$, $B^{(l)} \in V^{(l)}$ we define the corresponding standard and approximate scalar products:

$$(A^{(l)}, B^{(l)}) = \frac{n^l}{\omega_n^l} \int_{\Sigma^l} A^{(l)}(\theta_{(l)})\overline{B^{(l)}(\theta_{(l)})} d\theta_{(l)}, \tag{3.2}$$

$$(A^{(l)}, B^{(l)}\urcorner) = \sum_i A_i^{(l)}\overline{B_i^{(l)}}. \tag{3.2'}$$

Observe that by the orthogonality relation $\int_\Sigma \theta^i\theta^j \, d\theta = \frac{\omega_n}{n}\delta_{ij}$ (where $\theta = (\theta^1, \ldots, \theta^n)$) we get from (3.2)

$$(A^{(l)}, B^{(l)}) = (A^{(l)}, B^{(l)}\urcorner). \tag{3.3}$$

We shall now deduce some inequalities between the norms $|\ \ |_p$ and $|\ \ \urcorner|_p$.

Expanding $A^{(l)}(\theta_{(l)})$ in (3.1) in terms of components, using Hölder inequality and the fact that $\left(\sum\limits_{s=1}^n |\theta^s|^p\right)^{1/p}$ is a decreasing function of p we get

$$\begin{aligned} |A^{(l)}|_p &\leqq n^{l/p}|A^{(l)}\urcorner|_p \qquad \text{if} \quad p \leqq 2 \\ |A^{(l)}|_p &\leqq n^{l/2}|A^{(l)}\urcorner|_p \qquad \text{if} \quad p \geqq 2 \end{aligned} \tag{3.4}$$

(for $p = 2$, $|A^{(l)}|_2 = |A^{(l)}\urcorner|_2$ by (3.3)).

On the other hand, for every $A^{(l)} \in V^{(l)}$ there exists a $B^{(l)} \in V^{(l)}$, $B^{(l)} \neq 0$ and such that $(A^{(l)}, B^{(l)\urcorner}) = |A^{(l)\urcorner}|_p |B^{(l)\urcorner}|_{p'}$. Taking into account (3.3), applying Hölder inequality and using (3.4) we finally obtain

$$\text{(3.5)} \quad \begin{aligned} n^{-l/2}|A^{(l)\urcorner}|_p &\leq |A^{(l)}|_p \leq n^{l/p}|A^{(l)\urcorner}|_p, \qquad p \leq 2, \\ n^{-l/p'}|A^{(l)\urcorner}|_p &\leq |A^{(l)}|_p \leq n^{l/2}|A^{(l)\urcorner}|_p, \qquad p \geq 2. \end{aligned}$$

Denote now for any $u \in C_0^\infty$, by $\nabla^l u(x)$, the (symmetric) tensor of all derivatives of order l of u at the point x, and define for $1 \leq p < \infty$ and $\alpha \geq 0$, $m = [\alpha]$, $\beta = \alpha - m$, $0 \leq \beta < 1$, the *standard norm* of u of order α,

$$\text{(3.6)} \quad |u|_{\alpha,p}^p = \sum_{l=0}^m \binom{m}{l}\left(\frac{2}{p}\right)^l \left[\int_{R^n} |\nabla^l u(x)|_p^p \, dx + \int_{R^n}\int_{R^n} \left|\frac{\nabla^l u(x) - \nabla^l u(y)}{|x-y|^\beta}\right|_p^p d\mu_\beta(x, y)\right].$$

If α is an integer, $\beta = 0$, we omit in (3.6) the double integral (the measure $d\mu_0 = 0$),

$$|u|_{m,p}^p = \sum_{l=0}^m \binom{m}{l}\left(\frac{2}{p}\right)^l \int_{R^n} |\nabla^l u(x)|_p^p \, dx.$$

Similarly, we define for $u \in C_0^\infty$, the *approximate norm* of order α,

$$\text{(3.6')} \quad |u^\urcorner|_{\alpha,p}^p = \sum_{l=0}^m \binom{m}{l}\left(\frac{2}{p}\right)^l \left[\int_{R^n} |\nabla^l u(x)^\urcorner|_p^p \, dx + \int_{R^n}\int_{R^n} \left|\frac{\nabla^l u(x) - \nabla^l u(y)}{|x-y|^\beta}^\urcorner\right|_p^p d\mu_\beta(x, y)\right].$$

If α is an integer, $\beta = 0$, the double integral is to be omitted.

For $p = \infty$ the norms are given by

$$\text{(3.7)} \quad |u|_{\alpha,\infty} = \max_{0 \leqslant l \leqslant m} \left\{ \sup_x |\nabla^l u(x)|_\infty, \sup_{x \neq y} \left|\frac{\nabla^l u(x) - \nabla^l u(y)}{|x-y|^\beta}\right|_\infty \right\},$$

(3.7')

$$|u^\urcorner|_{\alpha,\infty} = \max_{0 \leqslant l \leqslant m} \left\{ \sup_x |\nabla^l u(x)^\urcorner|_\infty, \sup_{x \neq y} \left|\frac{\nabla^l u(x) - \nabla^l u(y)}{|x-y|^\beta}^\urcorner\right|_\infty \right\}.$$

When $\alpha = m$ is an integer, the norms are given by

$$\text{(3.8)} \quad \begin{aligned} |u|_{m,\infty} &= \max_{0 \leqslant l \leqslant m} \{\sup_x |\nabla^l u(x)|_\infty\}, \\ |u^\urcorner|_{m,\infty} &= \max_{0 \leqslant l \leqslant m} \{\sup_x |\nabla^l u(x)^\urcorner|_\infty\}. \end{aligned}$$

Clearly, $|u|_{0,\infty} = |u^\urcorner|_{0,\infty} = \|u\|_{L^\infty}$.

We shall denote by $\mathfrak{F}^{\alpha,p}$, $\alpha \geqq 0$, $1 \leqq p \leqq \infty$ the class of all functions $u \in C_0^\infty$ with the standard norm $|u|_{\alpha,p}$.

For $p = 2$, it is easy to verify (using (3.3)) that both of the norms $|\ \ |_{\alpha,2}$ and $|\ \ \urcorner_{\alpha,2}$ are equal and coincide with the standard norm $|\ \ |_\alpha$ in the space P^α of Bessel potentials. The norm $|\ \ |_\alpha$ is continuous in α; it will be proved in the sequel that so is the standard norm $|\ \ |_{\alpha,p}$. The latter is one of the main reasons for introducing the standard norm (the other being its independence of the choice of a coordinate system). For technical reasons, however, in most of the considerations we shall use the approximate norm $|\ \ \urcorner_{\alpha,p}$, this being justified by the following inequalities which are immediate consequences of (3.5):

$$(3.9)\quad \begin{cases} n^{-m/2}|u\urcorner_{\alpha,p} \leqq |u|_{\alpha,p} \leqq n^{m/p}|u\urcorner_{\alpha,p} & \text{for} \quad p \leqq 2, \\ n^{-m/p'}|u\urcorner_{\alpha,p} \leqq |u|_{\alpha,p} \leqq n^{m/2}|u\urcorner_{\alpha,p} & \text{for} \quad p \geqq 2. \end{cases}$$

We shall now describe some properties of the classes $\mathfrak{F}^{\alpha,p}$ which follow directly from the definition. It is easy to see that $\mathfrak{F}^{\alpha,\infty}$ is a proper functional space whose perfect completion, in the case when α is not an integer, is the proper functional space of all functions of $C^{(m,\beta)}$ which vanish at ∞ with all their derivatives of order $\leqq m$. ($C^{(m,\beta)}$ denotes the class of all functions in C^m satisfying together with all derivatives up to order m uniform Hölder condition with exponent β.) This space will be denoted by $\check{P}^{\alpha,\infty<}$.

For α integer, $\check{P}^{\alpha,\infty<}$ is the space of all the functions u of C^α vanishing at ∞ together with all derivatives of order $\leqq \alpha$.

For $1 \leqq p < \infty$, $\mathfrak{F}^{\alpha,p}$ is a proper normed functional class; it is a proper functional space if $\alpha > n/p$, $p > 1$ and $\alpha \geqq n$, $p = 1$. In all remaining cases it is an (incomplete) functional space rel. $\mathfrak{A}_0$.

§ 4. Classes $\mathfrak{B}^{\alpha,p,k}$.

We shall define in this section the normed functional classes $\mathfrak{B}^{\alpha,p,k}$ which, by completion, will lead to the spaces $B^{\alpha,p}$ mentioned in the Introduction.

Define for $u \in C_0^\infty$, $k > 0$ an integer, $0 \leq \alpha < k$ and $1 \leq p < \infty$,

$$(4.1) \quad \|u\|_{\alpha,p,k}^p = \|u\|_{L^p}^p + \int_{R^n}\int_{R^n} \frac{|\Delta_t^k u(x)|^p}{|t|^{n+p\alpha}}\, dx\, dt = \|u\|_{L^p}^p + \int_{R^n} \frac{\|\Delta_t^k u\|_{L^p}^p}{|t|^{n+p\alpha}}\, dt,$$

and for $p = \infty$,

$$(4.2) \qquad \|u\|_{\alpha,\infty,k} = \max\left\{\sup_x |u(x)|, \sup_{x,t} \frac{|\Delta_t^k u(x)|}{|t|^\alpha}\right\}.$$

Denote by $\mathcal{B}^{\alpha,p,k}$ the class of all functions $u \in C_0^\infty$ with the norm $\|u\|_{\alpha,p,k}$.

We shall first prove that if $k, k_1 > \alpha$, then the norms $\|\ \|_{\alpha,p,k}$ and $\|\ \|_{\alpha,p,k_1}$ are equivalent.

Lemma 4.1. — *Let k, k_1 be two integers, $0 \leq \alpha < k \leq k_1$ and $1 \leq p \leq \infty$. Then for every $u \in C_0^\infty$.*

$$2^{k-k_1}\|u\|_{\alpha,p,k_1} \leq \|u\|_{\alpha,p,k} \leq \frac{7.2^{k-k_1-1}}{(1-2^{\alpha-k})} \frac{\Gamma(k_1)}{\Gamma(k)} \|u\|_{\alpha,p,k_1}.$$

Proof. — The first inequality follows immediately from the remark that

$$|\Delta_t^{k_1} u(x)| = |\Delta_t^{k_1-k}\Delta_t^k u(x)| \leq \sum_{l=0}^{k_1-k} \binom{k_1-k}{l} |\Delta_t^k u(x + lt)|.$$

To prove the second inequality consider first the case when $k_1 = k + 1$. We use the following simple identity,

$$(4.3) \qquad \Delta_t - \frac{1}{2^N}\Delta_{2^N t} = -\frac{1}{2}\sum_{j=0}^{N-1} 2^{-j}\Delta_{2^j t}^2.$$

(4.3) with $N = 1$ applied to the function $\Delta_{2t}^l \Delta_t^{k-l-1} u(x)$, $0 \leq l \leq k-1$, yields

$$\frac{1}{2^l}\Delta_{2t}^l\Delta_t^{k-l}u - \frac{1}{2^{l+1}}\Delta_{2t}^{l+1}\Delta_t^{k-l-1} = -\frac{1}{2^{l+1}}\sum_{s=0}^{l}\binom{l}{s}\Delta_t^{k+1}u(x+st).$$

Adding together the above identities for $l = 0,1, \ldots, k-1$, and dividing both sides of the obtained identity by $|t|^\alpha$ we get

$$\frac{\Delta_t^k u(x)}{|t|^\alpha} - \frac{1}{2^{k-\alpha}}\frac{\Delta_{2t}^k u(x)}{|2t|^\alpha} = -\frac{1}{2}\sum_{l=0}^{k-1}\sum_{s=0}^{l} 2^{-l}\binom{l}{s}\frac{\Delta_t^{k+1}u(x+st)}{|t|^\alpha}.$$

Taking L^p norms of both sides of the last identity, with the measure $\frac{dx\,dt}{|t|^n}$ we get, in view of the invariance of these norms under translations in x and homotetic transformations in t (obvious modification for $p = \infty$):

$$||u||_{\alpha,p,k} \leqq \frac{k}{2(1-2^{\alpha-k})} ||u||_{\alpha,p,k+1}.$$

The result follows now by induction if we observe that

$$\frac{2^{k-k_1} k(k+1)\ldots(k_1-1)}{\prod_{l=k}^{k_1-1}(1-2^{\alpha-l})} < \frac{2^{k-k_1}\Gamma(k_1)}{(1-2^{\alpha-k})\Gamma(k)\prod_{l=1}^{\infty}(1-2^{-l})} < \frac{7.2^{k-k_1-1}\Gamma(k_1)}{(1-2^{\alpha-k})\Gamma(k)}.$$

For $p = \infty$, the class $\mathfrak{B}^{\alpha,\infty,k}$ is a proper functional space. Its (perfect) functional completion will be denoted $B^{\alpha,\infty<}$. (By Lemma, 4.1, $B^{\alpha,\infty<}$ is independent of k.)

Observe that for $1 \leqq p < \infty$, $\mathfrak{B}^{\alpha,p,k}$ is a proper normed functional class and a functional space rel. $\mathfrak{A}_0$.

CHAPTER II

IMPERFECT COMPLETIONS OF $\mathcal{J}^{\alpha,p}$ AND $\mathcal{B}^{\alpha,p,k}$.

§ 5. Some properties of distributions and representation formulas.

We will use the theory of distributions for two purposes: first, to define in the quickest way imperfect completions of the classes $\mathcal{J}^{\alpha,p}$ and $\mathcal{B}^{\alpha,p,k}$ rel. $\mathfrak{A}_0$ (sets of Lebesgue measure 0), and secondly, to establish different representation formulas (such as inversion formulas, reproducing formulas, etc.) which will serve as the main tools in our investigations. The easiest way to obtain these formulas is to write them for tempered distributions [6] in terms of their Fourier transforms; they are obtained then by standard integration techniques. Then, by applying the inverse Fourier transforms we obtain the desired formulas in the form of « integral transforms ». It remains to be shown that when the distribution is a function of some class, its integral transform is also a function of a corresponding class, and that this transform is given by the usual Lebesgue integration, or, in some cases, by singular integrals.

For relevant facts of the theory of distributions we refer to L. Schwartz [13] (we use here the traditional definition of Fourier transform which accounts for some differences in our formulas as compared to [13]). As usual $\mathcal{S}$ denotes the countably normed space of functions of rapid decrease with norms given by

$$(5.1) \qquad ||\varphi||_{(m,k)} = \sup_{\substack{x \\ |i| \leq m}} (1 + |x|^2)^k |D_i \varphi(x)|,$$

[6] Our considerations are still valid for more general classes of distributions, but the greater generality will not be needed in the present paper.

$\mathcal{S}'$ is the space of tempered distributions, $u(\varphi)$ denotes the value of $u \in \mathcal{S}'$ at $\varphi \in \mathcal{S}$.

We use also the derivatives D_i, the differences Δ_t^k and the Fourier transform $\hat{u}$ for $u \in \mathcal{S}'$.

In the following formulas $u \in \mathcal{S}'$ and ν is a distribution of rapid decrease (i.e. Fourier transform of a C^∞ function of slow increase).

$$(5.2) \qquad (u * \nu)^\wedge = (2\pi)^{n/2}\hat{u}\hat{\nu},$$
$$(5.3) \qquad (D_j u)^\wedge = (i\xi)^j\hat{u} \equiv (i\xi_{j_1})(i\xi_{j_2}) \dots \hat{u},$$
$$(5.4) \qquad (\Delta_t^k u(x))^\wedge = (e^{i(\xi,t)} - 1)^k \hat{u}(\xi),$$
$$(5.5) \qquad (G_\alpha(x))^\wedge = (2\pi)^{-n/2}(1 + |\xi|^2)^{-\alpha/2}, \qquad \alpha > 0,$$
$$(5.6) \qquad (D_j G_\alpha(x))^\wedge = (2\pi)^{-n/2}(i\xi)^j (1 + |\xi|^2)^{-\alpha/2}, \qquad \alpha > 0.$$

It should be noted that $D_j G_\alpha(x))$ is a function belonging to L^1 for $|j| < \alpha$. For $|j| \geq \alpha$, it should not be considered as a function but as a distribution-even though for $x \neq 0$, the derivative in the usual sense exists and is an analytic function decreasing exponentially at infinity. We denote this analytic function by $D_j' G_\alpha(x)$. It will be used only for $|j| = \alpha$. In this case the distribution derivative $D_j G_\alpha(x)$ for $\varphi \in \mathcal{S}$ can be written in terms of a singular integral:

$$(5.7) \qquad \int D_j G_{|j|}(x)\, \varphi(x)\, dx = A_j \varphi(0) + \lim_{\varepsilon \searrow 0} \int_{|x| > \varepsilon} D_j' G_{|j|}(x)\, \varphi(x)\, dx,$$

where A_j is a constant determined as follows. Denote by $j_{(k)}$, $k = 1, \dots, n$, the number of differentiations with respect to x_k in D_j; thus $|j| = j_{(1)} + \cdots + j_{(n)}$. Then we have

$$A_j = 0 \quad \textit{if at least one of the } j_{(k)} \textit{ is odd},$$

$$(5.7') \qquad A_j = \frac{2(-1)^{|j|/2}}{\omega_n} \frac{\Gamma\left(\frac{j_{(1)}+1}{2}\right) \dots \Gamma\left(\frac{j_{(n)}+1}{2}\right)}{\Gamma\left(\frac{n+|j|}{2}\right)}$$

if all the $j_{(k)}$ *are even.*

Let T_t be a linear operator $T_t : \mathcal{S} \to \mathcal{S}$ depending on a parameter t varying over some measure space $\mathcal{T}$. We assume that T_t is continuous in $\mathcal{S}$ for almost every $t \in \mathfrak{T}$. Then for almost every $t \in \mathfrak{T}$ the operator

$$(T_t^* u)(\varphi) = u(T_t \varphi) \tag{5.8}$$

is well defined and $T_t^* : \mathcal{S}' \to \mathcal{S}'$. Under the Fourier transform $\mathcal{F}$, T_t and T_t^* give rise to the operators

$$\hat{T}_t = \mathcal{F} T_t \mathcal{F}^{-1}, \qquad \hat{T}_t^* = \mathcal{F} T_t^* \mathcal{F}^{-1}, \tag{5.9}$$

and for every $\varphi \in \mathcal{S}$ and $u \in \mathcal{S}'$,

$$(T_t \varphi)^\wedge = \hat{T}_t \hat{\varphi}, \qquad (T_t^* u)^\wedge = \hat{T}_t^* \hat{u}. \tag{5.10}$$

We will deal with operators of the form

$$T\varphi = \int_{\mathfrak{T}} T_t \varphi \, dt \quad \varphi \in \mathcal{S}, \tag{5.11}$$

and correspondingly we will write

$$T^* u = \int_{\mathfrak{T}} T_t^* u \, dt \quad u \in \mathcal{S}', \tag{5.12}$$

the last integral being defined by

$$(T^* u)(\varphi) = u(T\varphi). \tag{5.13}$$

The following assumptions will be made

A) *For every $\varphi \in \mathcal{S}$ the integral $\int_{\mathfrak{T}} T_t \varphi(x)\, dt$ exists as a Lebesgue integral for every x and represents a function of $\mathcal{S}$. Moreover, the operator* (5.11) *defined by the formula* $(T\varphi)(x) = \int_{\mathfrak{T}} T_t \varphi(x)\, dt$ *is continuous on $\mathcal{S}$.*

B) *For every $\varphi \in \mathcal{S}$ the integral $\int_{\mathfrak{T}} |T_t \varphi(x)|\, dt$ exists for almost all x and as a function of x belongs to* $L^1(R^n) \cap L^\infty(R^n)$.

By virtue of hypotheses (A) and (B) we have the formula

$$\hat{T}\hat{\varphi}(\xi) = \int_{\mathfrak{T}} \hat{T}_t \hat{\varphi}(\xi)\, dt, \qquad \textit{for every } \varphi \in \mathcal{S}. \tag{5.14}$$

The following statement holds.

THEOREM 5.1. — *Let $u \in L^p$ for some $1 \leq p \leq \infty$ and assume that $T_t^* u \in \mathcal{S}'$ satisfies the following conditions:*

$$T_t^* u \textit{ is a function for almost every } t \tag{5.15}$$

(5.16) $\int_{\mathfrak{T}} |T_t^* u(x)|\, dt$ *exists in Lebesgue sense for almost every* x *and as function of* x *is locally integrable.*

Then T^*u *as defined by* (5.13) *is a function and*

$$T^*u(x) = \int T_t^* u(x)\, dt \tag{5.17}$$

almost everywhere.

By our assumption $\int T_t^* u(x)\, dt$ is a function and the only thing to prove is that it is equal to T^*u as defined by (5.13). In fact if $\varphi \in C_0^\infty$, then in view of (B), (5.16) and Fubini's theorem,

$$\int_{R^n} \left(\int_{\mathfrak{T}} T_t^* u(x)\, dt \right) \varphi(x)\, dx = \int_{\mathfrak{T}} \left[\int_{R^n} u(x) T_t \varphi(x)\, dx \right] dt = T^*u(\varphi).$$

Note that the assumption $u \in L^p$ guarantees that

$$(T_t^* u)(\varphi) = u(T_t \varphi) = \int_{R^n} u(x) T_t \varphi(x)\, dx.$$

We shall now proceed according to the following scheme. In terms of Fourier transforms we will write identities which can be proved by standard methods in the form $\hat{T} = \int \hat{T}_t\, dt$. $\hat{T}_t$ will be multiplication operators by functions of C^∞ of slow increase and the same will be true of $\hat{T}$. The same functions will give us the operators $\hat{T}_t^*$ and $\hat{T}^*$ acting on $\mathcal{S}'$. We will then know explicitly the operators T_t^* and T^* as convolution operators; in most cases T_t^* will be a convolution with a function of rapid decrease, at worst it will be a singular integral convolution operator. In every case the verification of conditions (A) and (B) will be immediate. The verification of assumptions (5.15) and (5.16) of our theorem will obviously depend on the function u and we will have to rely on results of forthcoming sections on integral transformations and inequalities to check on the validity of these assumptions for u belonging to different classes of functions in which we are interested.

The formulas we list below are valid under the tacit assumption that (5.15) and (5.16) hold.

The variables t, ξ are n-dimensional vectors, t_0 is real, k is a positive integer, $0 < \beta < k$. Consider the expression

(5.18)

$$I_{k,n,\beta}(\xi) = \int_{-\infty}^{\infty}\int_{R^n} \frac{|e^{it_0+i(t,\xi)} - 1|^{2k}}{(t_0^2 + |t|^2)^{\frac{n+1+2\beta}{2}}} dt\, dt_0$$
$$= \int_{-\infty}^{\infty} \frac{|e^{it_0} - 1|^{2k}}{|t_0|^{1+2\beta}} dt_0 \int_{R^n} \frac{dt}{(1 + |t|^2)^{\frac{n+1+2\beta}{2}}} (1 + |\xi|^2)^{\beta}$$
$$= \frac{(-1)^{k+1}}{2} C(n + 1, \beta)\Delta^{2k}_{1,-k;s}|s|^{2\beta}(1 + |\xi|^2)^{\beta}.$$

On the other hand

$$I_{k,n,\beta}(\xi) = \int_{-\infty}^{\infty}\int_{R^n} (t_0^2 + |t|^2)^{-\frac{n+1+2\beta}{2}} [e^{it_0}(e^{i(t,\xi)} - 1) + (e^{it_0} - 1)]^k$$
$$[e^{-it_0}(e^{-i(t,\xi)} - 1) + (e^{-it_0} - 1)]^k\, dt_0\, dt$$
$$= \int_{-\infty}^{\infty}\int_{R^n} (t_0^2 + |t|^2)^{-\frac{n+1+2\beta}{2}} \sum_{l, l_1=0}^{k} \binom{k}{l}\binom{k}{l_1} e^{i(l-l_1)t_0}$$
$$(e^{it_0} - 1)^{k-l}\,(e^{-it_0} - 1)^{k-l_1}(e^{i(t,\xi)} - 1)^l(e^{-i(t,\xi)} - 1)^{l_1}\, dt_0\, dt$$
$$= \int_{R^n} \sum_{l, l_1=0}^{k} \binom{k}{l}\binom{k}{l_1}(-1)^{k-l_1}\,(\Delta^{2k-l-l_1}_{|t|,(l-k)|t|}2\pi G^{(1)}_{n+1+2\beta})$$
$$|t|^{-n-2\beta}\,(e^{i(t,\xi)} - 1)^l(e^{-i(t,\xi)} - 1)^{l_1}\, dt.$$

The last expression is obtained by integration with respect to t_0. (For a similar reasoning, see[2].)

Changing the kernel $G^{(1)}_{n+1+2\beta}$ to the n-dimensional kernel, we obtain finally

(5.19)
$$1 = \int_{R^n} \frac{1}{C_k(n, \beta)} \sum_{l, l_1=0}^{k} \binom{k}{l}\binom{k}{l_1}(-1)^{k-l_1}$$
$$(\Delta^{2k-l-l_1}_{t,(l-k)t}G_{2n+2\beta}) \frac{(e^{i(t,\xi)} - 1)^l\,(e^{-i(t,\xi)} - 1)^{l_1}}{(1 + |\xi|^2)^{\beta}\,|t|^{n+2\beta}} dt,$$

where

(5.20)
$$C_k(n, \beta) = \frac{(-1)^{k+1}}{2} G_{2n+2\beta}(0)\, C(n, \beta)\, \Delta^{2k}_{1,-k;s}\, |s|^{2\beta}.$$

Of the three factors depending on β in (5.20), the first is a positive decreasing function of β for all $\beta \geq 0$. The second has simple poles for integers $\beta \geq 0$ and no zeros on the positive β-axis. The third is an entire function and has only simple zeros on the interval $0 \leq \beta \leq k$ at integers β, $0 < \beta < k$.

The resulting product $C_k(n, \beta)$ is therefore, for $0 < \beta < k$, a strictly positive analytic function with simple poles at 0 and k.

If we consider the integrand in (5.19) as an operator of multiplication $\hat{T}_t$, thus $\hat{T} = 1$, we obtain by inverse Fourier transform the reproducing formula

$$(5.21) \quad u = \frac{1}{C_k(n, \beta)} \int_{R^n} \sum_{l, l_1 = 0}^{k} \binom{k}{l} \binom{k}{l_1} (-1)^{k-l_1} \frac{\Delta^{2k-l-l_1}_{t, (l-k)t} G_{2n+2\beta}}{|t|^{n+2\beta}} (\Delta^{l_1}_{-t} G_{2\beta}) * (\Delta^l_t u)\, dt.$$

Multiplication of both sides of the identity (5.19) by $(1 + |\xi|^2)^{\alpha/2}$, $0 < \alpha \leqq \beta$, leads to an inversion formula for the operator G_α. We denote the inverse operator of G_α by $G_{-\alpha}$, and we get

$$(5.22) \quad G_{-\alpha} u = \frac{1}{C_k(n, \beta)} \int_{R^n} \sum_{l, l_1 = 0}^{k} \binom{k}{l} \binom{k}{l_1} (-1)^{k-l_1} \frac{\Delta^{2k-l-l_1}_{t, (l-k)t} G_{2n+2\beta}}{|t|^{n+2\beta}} (\Delta^{l_1}_{-t} G_{2\beta-\alpha}) * (\Delta^l_t u)\, dt.$$

Especially simple and interesting is the case when $k = 1$. Then for $0 < \beta < 1$:

$$(5.23) \quad 1 = \frac{1}{(1 + |\xi|^2)^\beta} + \frac{1}{C(n, \beta) G_{2n+2\beta}(0)} \int_{R^n} \frac{G_{2n+2\beta}(t)}{|t|^{n+2\beta}} \frac{(e^{i(t,\xi)} - 1)(e^{-i(t,\xi)} - 1)}{(1 + |\xi|^2)^\beta}\, dt$$

which can be transformed into the reproducing formula

$$(5.24) \quad u = G_{2\beta} * u + \frac{1}{C(n, \beta) G_{2n+2\beta}(0)} \int_{R^n} \frac{G_{2n+2\beta}(t)}{|t|^{n+2\beta}} (\Delta_{-t} G_{2\beta}) * (\Delta_t u)\, dt.$$

Formula (5.24) can also be written in the form

$$(5.25) \quad u(z) = G_{2\beta} * u(z) + \int_{R^n} \int_{R^n} \frac{[G_{2\beta}(z - x) - G_{2\beta}(z - y)]\,[u(x) - u(y)]}{|x - y|^{2\beta}}\, d\mu_\beta(x, y).$$

The corresponding inversion formula for $0 < \alpha \leqq \beta < 1$ is

$$(5.26) \quad G_{-\alpha}u(z) = G_{2\beta-\alpha} * u(z) + \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{[G_{2\beta-\alpha}(z-x) - G_{2\beta-\alpha}(z-y)][u(x)-u(y)]}{|x-y|^{2\beta}} d\mu_\beta(x, y).$$

Multiplying (5.23) by $1 = \frac{1}{(1+|\xi|^2)^m} \sum_{l=0}^{m} \binom{m}{l} \sum_{|j|=l} (-1)^l (i\xi)^j (i\xi)^j$, where m is an integer, $m \geqq 0$ and transforming the result we get, with $\alpha = m + \beta$

$$(5.27) \quad u(z) = \sum_{l=0}^{m} \binom{m}{l} \sum_{|j|=l} \Big\{ \int_{\mathbf{R}^n} D_j^{(x)} G_{2\alpha}(z-x) D_j u(x)\, dx + \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{[D_j^{(x)} G_{2\alpha}(z-x) - D_j^{(y)} G_{2\alpha}(z-y)][D_j u(x) - D_j u(y)]}{|x-y|^{2\beta}} d\mu_\beta(x, y) \Big\}$$

and the corresponding inversion formula for $\gamma \leqq \alpha = m + \beta$,

$$(5.28) \quad G_{-\gamma}u(z) = \sum_{l=0}^{m} \binom{m}{l} \sum_{|j|=l} \Big\{ \int_{\mathbf{R}^n} D_j^{(x)} G_{2\alpha-\gamma}(z-x) D_j u(x)\, dx + \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{[D_j^{(x)} G_{2\alpha-\gamma}(z-x) - D_j^{(y)} G_{2\alpha-\gamma}(z-y)][D_j u(x) - D_j u(y)]}{|x-y|^{2\beta}} d\mu_\beta(x, y) \Big\}.$$

At the end we include the case when $\alpha = m$ is an integer. From the identity $1 = \frac{1}{(1+|\xi|^2)^m} \sum_{l=0} \binom{m}{l} \sum_{|j|=l} (-1)^l (i\xi)^j (i\xi)^j$ mentioned before we then get the following reproducing formula

$$(5.29) \quad u(z) = \sum_{l=0}^{m} \binom{m}{l} \sum_{|j|=l} \int_{\mathbf{R}^n} D_j^{(x)} G_{2m}(z-x) D_j u(x)\, dx,$$

and the corresponding inversion formula

$$(5.30) \quad G_{-m}u(z) = \sum_{l=0}^{m} \binom{m}{l} \sum_{|j|=l} \int D_j^{(x)} G_m(z-x) D_j u(x)\, dx.$$

In the last formula the integrals corresponding to the values $|j| = m$ are understood as singular integrals as explained by formula (5.7).

§ 6. Regular and singular integral transformations.

The purpose of this section is to introduce a terminology concerning integral transformations which will be used throughout this paper.

Let $\{X,\mu\}$, $\{Y,\nu\}$ be measure spaces; denote $L^p(X)=L^p(X,\mu)$, $L^p(Y) = L^p(Y, \nu)$ [7]. u and v will generically denote measurable functions in X and Y respectively. Let $K(x, y)$ be a complex valued function defined on X and Y measurable in $X \times Y$. $K(x, y)$ gives rise to a formal integral transformation defined by the formula

$$(6.1) \qquad v(y) = Ku(y) = \int_X K(x, y)\, u(x)\, d\mu(x).$$

It is defined for all u for which the integral (6.1) exists in Lebesgue sense and is finite for almost all y. Denote by $\mathfrak{D}_K$ the set of all such u. We say that for $u \in \mathfrak{D}_K$ the formal integral transformation K is *properly defined.*

An integral transformation K (or kernel $K(x, y)$) is *p-semiregular* (p-s. r.) if the subspace $\mathfrak{D}_K \cap L^p(X)$ is dense in $L^p(X)$ and is transformed boundedly into $L^p(Y)$, i.e. that there is a constant M_p — the p-bound of K — such that

$$||Ku||_{L^p(Y)} \leqq M_p ||u||_{L^p(X)}.$$

A p-s. r. operator K can be extended by continuity to a unique bounded transformation K_p on the whole of $L^p(X)$, $K_p(L^p(X)) \subset L^p(Y)$. K_p will be called the *p-extension* of K.

The transformation (or kernel) is *p-regular* (p-r.) if $L^p(X) \subset \mathfrak{D}_K$ and $K(L^p(X)) \subset L^p(Y)$. For p-regularity of K it is necessary and sufficient that

$$(6.2) \qquad \int_Y \left(\int_X K(x, y) u(x)\, d\mu(x) \right) v(y)\, d\nu(y) \leqq C ||u||_{L^p} ||v||_{L^{p'}}, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

for any $u \in L^p(X)$, $v \in L^{p'}(Y)$, the integrals being taken in the indicated order, C being a constant independent of u and v.

(7) All measures will be assumed to be σ-finite.

The smallest such constant C is $= M_p$. p-regularity implies p-semi-regularity. K is *p-absolutely-regular* (p.ab.r.) if $|K(x, y)|$ is regular. This is equivalent to the property

$$(6.3) \quad \left| \int_{X\times Y} K(x, y)u(x)v(y)\, d\mu(x)\, dv(y) \right| \leq M_p \|u\|_{L^p} \|v\|_{L^{p'}},$$

for any $u \in L^p(X)$, $v \in L^{p'}(Y)$. Obviously, absolute regularity implies regularity. On the other hand, for non-negative kernels, p-absolute regularity is equivalent to p-semi-regularity.

If a kernel K is p-s.r., p-r. or p-ab.r. for all p, $1 \leq p \leq \infty$, we call it semiregular, regular, or absolutely regular, respectively.

We have the following theorem.

THEOREM 6.1. — *If the transformation* $\int_X K(x, y)u(x)\, d\mu(x)$ *is p-ab. r. then the adjoint transformation* $\int_Y \overline{K(x, y)}v(y)\, dv(y)$ *is* p'-ab.r..

The proof is immediate by (6.3).

THEOREM 6.2. — *Let* K *be a p-ab. r. transformation of* $L^p(X, d\mu)$ *into* $L^p(Y, dv)$ *and* M *be the p-bound of* $|K(x, y)|$. *Consider, moreover, the measures* $d\mu_1(x) = \varphi(x)\, d\mu(x)$ *and* $dv_1(y) = \psi(y)\, dv(y)$ *where* φ *and* ψ *are measurable non-negative functions on* X *and* Y *respectively, satisfying* $\varphi(x) \leq A$ *and* $\psi(y) \leq B$. *Then* K *is p-ab.r. from* $L^p(X, d\mu_1)$ *to* $L^p(Y, dv_1)$ *with bound not exceeding* $MA^{1/p'}B^{1/p}$.

Proof. Observe that for $u_1 \in L^p(X, d\mu_1)$, $v_1 \in L^{p'}(Y, dv_1)$ we have

$$\|\varphi^{1/p}u_1\|_{L^p(X,d\mu)} = \|u_1\|_{L^p(X,d\mu_1)}$$

and

$$\|\psi^{1/p'}v_1\|_{L^{p'}(Y,dv)} = \|v_1\|_{L^{p'}(Y,dv_1)}.$$

Hence for $u_1 \in L^p(X, d\mu_1)$, $v_1 \in L^{p'}(Y, dv_1)$ we have

$$\iint_{X\times Y} |K(x, y)||u_1(x)||v_1(y)|\, d\mu_1(x)\, dv_1(y)$$
$$\leq \iint_{X\times Y} |K(x, y)|\varphi(x)^{1/p'}\psi(y)^{1/p}|u_1(x)|\varphi(x)^{1/p}|v_1(x)|\psi(y)^{1/p'}d\mu(x)\, dv(y)$$
$$\leq MA^{1/p'}B^{1/p}\|u_1\|_{L^p(X,d\mu_1)}\|v_1\|_{L^{p'}(Y,dv_1)}.$$

We are mainly interested in regular integral transforms since we need a pointwise representation of $v(y)$ by the integral

(6.1) for all $u \in L^p(X)$. There are no known direct properties of the kernel $K(x, y)$ characterizing its p-regularity. For p.-ab.r. such properties are well known in the two extreme cases $p = 1$ and $p = \infty$:

(6.4)

$$K \text{ is } 1\text{-ab.r.} \Longleftrightarrow \int_Y |K(x, y)|\, d\nu(y) \leq A = \text{const.} < \infty \text{ a.e. in } x.$$

(6.4')

$$K \text{ is } \infty\text{-ab.r.} \Longleftrightarrow \int_X |K(x, y)|\, d\mu(x) \leq B = \text{const.} < \infty \text{ a.e. in } y.$$

For other values of p the next theorem gives sufficient conditions for p-ab.r.. Quite recently these conditions were proved by E. Gagliardo [11 *a*] to be also necessary.

THEOREM 6.3. — *Let $1 < p < \infty$ and assume that there exist two non-negative measurable kernels K_1 and K_2 such that*

$$|K(x, y)| \leq K_1(x, y)^{1/p} K_2(x, y)^{1/p'} \tag{6.5}$$

and

$$\int_Y K_1(x, y)\, d\nu(y) \leq A \quad \text{a.e. } d\mu \tag{6.6}$$

$$\int_X K_2(x, y)\, d\mu(x) \leq B \quad \text{a.e. } d\nu.$$

Then K is p-ab.r. with bound not exceeding $A^{1/p}B^{1/p'}$.

Proof. — For $u \in L^p(X)$ and $\varphi \in L^{p'}(Y)$, by applying (6.5), Hölder inequality and (6.6), we get

$$\int_X \int_Y |u(x)|\, |K(x, y)|\, |\varphi(y)|\, d\mu(x)\, d\nu(y)$$
$$\leq \left[\int_X \int_Y |u(x)|^p K_1(x, y)\, d\mu(x)\, d\nu(y)\right]^{1/p}$$
$$\left[\int_X \int_Y K_2(x, y)\, |\varphi(y)|^{p'}\, d\mu(x)\, d\nu(y)\right]^{1/p'}$$
$$\leq A^{1/p}B^{1/p'} \|u\|_{L^p} \|\varphi\|_{L^{p'}}.$$

Depending on the nature of the kernel K there are several methods by which we may find kernels K_1 and K_2 that show K to be p-ab.r.. We describe two of these methods which will be used in the sequel.

Method 1. — We find two measurable functions $\varphi(x)$ and $\psi(y)$, positive and finite *a.e.*, and put

$$(6.7)\quad K_1(x, y) = |K(x, y)|\ \psi(y)/\varphi(x)^{p/p'},\qquad K_2(x, y) = |K(x, y)|\ \varphi(x)/\psi(y)^{p'/p}.$$

The functions $\varphi(x)$ and $\psi(y)$ will be called factors. (6.6) now translates into the following conditions for the factors:

$$(6.8)\quad \int_Y |K(x, y)|\psi(y)\, d\nu(y) \leqq A\varphi(x)^{p/p'},\qquad \int_X |K(x, y)|\varphi(x)\, d\mu(x) \leqq B\psi(y)^{p'/p}.$$

Remark 1. — The result of E. Gagliardo mentioned before states that the existence of factors $\varphi(x)$ and $\psi(x)$ satisfying (6.8) is also necessary in order that K be p-absolutely regular. More precisely, it is proved that if K is absolutely regular and M is the p-bound of $|K(x, y)|$ it is possible to find $\varphi \in L^p(X)$ and $\psi \in L^{p'}(Y)$ such that (6.8) is satisfied with $A = B = M + \varepsilon$ for any $\varepsilon > 0$.

Method II. — We find a representation of $K(x, y)$ as a composition of two kernels $\Phi(x, z)$ and $\Psi(z, y)$,

$$(6.9)\qquad K(x, y) = \int_Z \Phi(x, z)\Psi(z, y)\, d\omega(z),$$

where Z is a measure space with measure $d\omega(z)$. We find further an « inner factor » $\lambda(z)$, $0 < \lambda(z) < \infty$ a.e. such that

(6.10)

$$K_1(x, y) = \int_Z |\Phi(x, z)|\lambda(z)^p|\Psi(z, y)|\, d\omega(z) < \infty \quad \textit{a.e. in } x, y,$$

$$K_2(x, y) = \int_Z |\Phi(x, z)|\lambda(z)^{-p'}|\Psi(z, y)|\, d\omega(z) < \infty \quad \textit{a.e. in } x, y.$$

Thus (6.5) is satisfied. The conditions (6.6) now take the form

(6.11)

$$\int_Y\int_Z |\Phi(x, z)|\lambda(z)^p|\Psi(z, y)|\, d\omega(z)\, d\nu(y) \leqq A \quad \textit{a.e. in } x,$$

$$\int_X\int_Z |\Phi(x, z)|\lambda(z)^{-p'}|\Psi(z, y)|\, d\omega(z)\, d\mu(x) \leqq B \quad \textit{a.e. in } y.$$

It is possible to combine the two methods as well as to devise others adapted to special kinds of kernels.

In most cases we will deal with p-absolutely regular kernels. In a few cases, however, we will meet with p-semi-regular kernels; it is therefore of interest to give some information about them. We start with some general remarks.

The subspace $\mathfrak{D}_K$ of measurable functions $u(x)$ for which the integral transform $(Ku)(y)$ is properly defined has the property that with each $u(x)$ it contains all functions $u_1(x)$ majorated by u, i.e. such that $|u_1(x)| \leq |u(x)|$ a.e.

By a simple measure-theoretic argument one proves that there exists a measurable set $A \subset X$, unique up to sets of measure 0, which is the largest among all those sets on which all functions $u \in \mathfrak{D}_K$ vanish a.e.. If $A = X$ we may say that K is singular (such are, for instance, the singular operators of Calderon-Zygmund type); in this case $\mathfrak{D}_K$ reduces to the function 0. If $\mu(X - A) > 0$, but also $\mu(A) > 0$ we may call K partly-singular; in this case, if we replace X by A, the transformation becomes completely singular. Of interest here is the case $\mu(A) = 0$, i.e. essentially $A = 0$; in this case we call K *non-singular* ([8]). A p-semi-regular kernel is certainly non-singular.

The same argument which leads to the existence of the set A shows that for a non-singular K there exists a sequence of measurable sets B_i, $i = 1, 2, \ldots$ such that

$$(6.12) \quad B_i \subset B_{i+1} \subset X, \qquad \mu(B_i) < \infty, \qquad \mu\left(X - \bigcup_1^\infty B_i\right) = 0,$$

the characteristic function of each B_i *belongs to* $\mathfrak{D}_K$.

A simple function is a measurable function taking only a finite number of values and vanishing outside of a set of finite measure. For every function $u(x)$, measurable and finite a.e. a classical standard procedure allows to construct a sequence of simple functions $u_j(x)$ such that $\lim u_j(x) = u(x)$ and $|u_j(x)| \leq |u(x)|$ a.e.. These functions can be chosen so that each $u_j(x)$ vanishes outside some B_i, and hence so that each $u_j \in \mathfrak{D}_K$. In addition, if $u \in L^p(X)$ for some $p < \infty$, then $\lim \|u - u_j\|_{L^p} = 0$.

([8]) The same terminology is used in [21] in a different meaning.

Denote by $\mathfrak{D}'_K$ the class of all simple functions in $\mathfrak{D}_K$. The last remark leads to the following statements.

THEOREM 6.4. — *A non-singular* K *is p-semi-regular for* $p < \infty$ *if and only if* $K(\mathfrak{D}'_K) \subset L^p(Y)$ *and* $\|Ku\|_{L^p(Y)} \geqslant M\|u\|_{L^p(X)}$ *for* $u \in \mathfrak{D}'_K$. K *is p-regular if in addition,* $L^p(X) \subset \mathfrak{D}_K$.

In fact, the above remark shows that $\mathfrak{D}'_K \subset \mathfrak{D}_K \cap L^p(X)$ is dense in $L^p(X)$ and the continuous extension of K from $\mathfrak{D}'_K$ to $L^p(X)$ coincides with K on $\mathfrak{D}_K \cap L^p(X)$ since $(Ku_j)(y)$ converges by dominated convergence to $(Ku)(y)$ for every y where $\int |K(x, y)||u(x)| \, d\mu(x) < \infty$.

THEOREM 6.4′. — *A non-singular* K *is* ∞*-semi-regular if and only if the characteristic function* χ *of* X *belongs to* $\mathfrak{D}_K$, $K(\mathfrak{D}'_K) \subset L^\infty(Y)$ *and* $\|Ku\|_{L^\infty(Y)} \leqq M\|u\|_{L^\infty(X)}$ *for* $u \in \mathfrak{D}'_K$. *The* ∞*-regularity is equivalent to* ∞*-semi-regularity.*

In fact, if $L^\infty(X) \cap \mathfrak{D}_K$ is dense in $L^\infty(X)$, there must be a $u_0 \in \mathfrak{D}_K$ with $\|\chi - u_0\|_{L^\infty(X)} < \frac{1}{2}$, hence $|u_0(x)| > \frac{1}{2}$ a.e. and $\chi \in \mathfrak{D}_K$. On the other hand $\chi \in \mathfrak{D}_K$ implies $L^\infty(X) \subset \mathfrak{D}_K$ (hence the last part of the theorem) and the boundedness of K on $L^\infty(X)$ follows by dominated convergence:

$$(Ku_j)(y) \to (Ku)(y) \text{a.e. in } y,$$
$$|(Ku_j)(y)| \leqq M \sup_x |u_j(x)| \leqq M \sup_x |u(x)|,$$

hence $\sup_y |(Ku)(y)| \leqq M \sup_x |u(x)|$.

Remark 2. — In Theorems 6.4 and 6.4′, the class $\mathfrak{D}'_K$ can be replaced by other subspaces of $\mathfrak{D}_K \cap L^p(X)$ as long as for each $u \in \mathfrak{D}_K \cap L^p(X)$ they contain a sequence u_j converging pointwise a.e. to u, dominated by some $u' \in \mathfrak{D}_K$, and such that $\|u_j\|_{L^p(X)} \leqq c$, c depending on u but not on j. For instance, we may take the class of simple functions vanishing outside of some of the sets B_i (i varying with the function). Another instance of such a change may be of interest if X and Y are euclidean spaces where we would like to replace simple functions by C_0^∞-functions. This is possible if the sets B_i can be chosen to be open.

We turn now to interpolation theorems — the Riesz — Thorin convexity theorem [20].

Let $1 \leq p_0 \leq \infty,\ 1 \leq p_1 \leq \infty,\ 0 \leq \theta \leq 1,$

$$1/p_\theta = (1-\theta)/p_0 + \theta/p_1.$$

THEOREM 6.5. *Let* K *be non-singular. If* K *is p_i-semi-regular* (*or p_i-r., or p_i-*ab.-r.) *for* $i = 0,1$, *then* K *is p_θ-semi-regular* (*or p_θ-r., or p_θ-*ab.-r.) *for* $0 < \theta < 1$. *The p_θ-bound* M_{p_θ} *satisfies* $M_{p_\theta} \leq M_{p_0}^{1-\theta} M_{p_1}^{\theta}$.

Proof. — 1° *Semi-regularity.* By Theorems 6.4 and 6.4′ the question reduces to be boundedness on the subspace of simple functions $\mathfrak{D}'_K$, hence Thorin's proof applies.

2° *Regularity.* Since $L^{p_\theta}(X) \subset L^{p_0}(X) + L^{p_1}(X)$, the result follows from semi-regularity.

3° *Absolute Regularity.* Use 1° for $|K(x, y)|$ and then the fact that for positive kernels ab.-r. is equivalent to s.-r.. If p_i-ab.-r. is established by the kernels K_{1i} and K_{2i} satisfying (6.5) and (6.6) then p_θ-ab.-r. can be established in similar fashion by kernels

$$K_1 = K_{10}^{(1-\theta)p_\theta/p_0} K_{11}^{\theta p_\theta/p_1}, \qquad K_2 = K_{20}^{(1-\theta)p'_\theta/p'_0} K_{21}^{\theta p'_\theta/p'_1}.$$

Remark 3. — The extension of the convexity theorem, due to E. M. Stein (see [15] and [16]), to the case when not only the exponents of the L^p-classes but also the measures μ and ν vary suitably, leads to a similar extension of Theorem 6.5. The proof applies without changes if one notices that if K is non-singular rel. μ and ν then so is the kernel $\varphi(x)K(x, y)\psi(y)$ (φ and ψ finite a.e.) rel. to any two measures μ' and ν' equivalent to μ and ν respectively.

Remark 4. — The notions introduced in this section could easily be extended to integral transforms from $L^p(X)$ to $L^q(Y)$ with $q \neq p$ and even (under suitable restrictions) to transforms between two Banach spaces of measurable functions.

Remark 5. — The terminology we introduced above has not been used before. The notions, however — without being specifically named — were investigated long ago in many special cases. The distinction between semi-regularity and regularity was not so sharply drawn. The p-absolute regularity, especially the first method, was very extensively used as a tool to establish regularity in many special instances (see Hardy, Littlewood, Polya [10], Ch. IX). The criterion of the first method was not put in the general form (6.7), (6.8), but rather in a form adapted to the special cases.

As mentioned before, we deal in this paper with integral transformations which in most cases are p.-ab.r., or at least p-s.r.. In a few cases, however, we meet with a special type of singular integral operator. The pertinent theorems are special instances of theorems of Calderon-Zygmund [7].

We consider kernels of the form $D'_j G_m(x-y)$, $|j| = m$ (see § 5, especially between (5.6) and (5.7)). The following statement holds:

If $u \in L^p(R^n)$, $1 < p < \infty$, *then the limit*

$$\varphi(y) = \lim_{\varepsilon \searrow 0} \int_{|x-y|>\varepsilon} D'_j G_m(x-y) u(x)\, dx \tag{6.13}$$

exist and is finite for almost all y *and* (6.13) *is a bounded transformation of* L^p *into* L^p.

The statement does not hold for $p = 1$ or $p = \infty$. Hence, whenever we have to use singular integrals our results will be restricted to $1 < p < \infty$.

§ 7. The imperfect completions of $\mathcal{F}^{\alpha, p}$, $\mathcal{B}^{\alpha, p, k}$.

As it was remarked in § 3,4 $\mathcal{F}^{\alpha, p}$ and $\mathcal{B}^{\alpha, p, k}$ are functional spaces rel. $\mathfrak{A}_0$. We shall now define their functional completions rel. $\mathfrak{A}_0$ (the imperfect completions).

The norms $|u|_{\beta, p}$, $|u^{-}|_{\beta, p}$, $0 \leqq \beta < 1$ introduced in § 3 have obviously a meaning for any measurable function u (they may be infinite). Let $1 \leqq p < \infty$, $0 \leqq \alpha = m + \beta$, $m = [\alpha]$, $0 \leqq \beta < 1$.

We denote by W^{α}_p the class of all functions $u \in L^p(R^n)$ such that

1. all the distribution derivatives $D_j u$, $|j| \leqq m$ are functions,
2. $|D_j u|_{\beta, p} < \infty$, $0 \leqq |j| \leqq m$.

It is clear that for $u \in W_p^\alpha$ both norms $|u|_{\alpha,p}$ and $|u^-|_{\alpha,p}$ as given by formulas (3.6) and (3.6′) have a meaning and are finite. Also the relations (3.9) hold.

By standard arguments, similar to those in the proof of completeness of L^p spaces, one shows that W_p^α is a complete functional space rel. $\mathfrak{A}_0$ (the class of sets of Lebesgue measure 0). Also a standard argument by regularization (9) shows that $\mathcal{F}^{\alpha,p}$ is dense in W_p^α. Hence we have the following

Theorem 7.1. — *W_p^α is the functional completion of $\mathcal{F}^{\alpha,p}$ rel $\mathfrak{A}_0$.*

For $p = \infty$, we define W_∞^α as the class of all functions u which together with all distribution derivatives of order $\leq \alpha$ belong to L^∞ and, if α is not an integer, satisfy Hölder conditions with exponent β. It is clear that $\mathcal{F}^{\alpha,\infty}$ is contained but not dense in W_∞^α. One shows immediately that each equivalence class of W_∞^α rel. $\mathfrak{A}_0$ contains one and only one function which is continuous and bounded with all its derivatives of orders $< \alpha$ all of these derivatives satisfying a uniform Hölder condition with exponent $\alpha - \alpha^\star$, $\alpha^\star$ being the largest integer $< \alpha$. All such functions form a proper complete functional space $\check{P}^{\alpha,\infty} \subset W_\infty^\alpha$ with the norm of W_∞^α. The space $\check{P}^{\alpha,\infty<}$ (the proper functional completion of $\mathcal{F}^{\alpha,\infty}$ introduced in § 3) is a closed proper subspace of $\check{P}^{\alpha,\infty}$.

We define now $\mathcal{B}^{\alpha,p}$ as the class of all functions $u \in L^p(R^n)$ such that for some integer $k > \alpha$ the norm

$$(7.1) \qquad |u|_{\alpha,p,k}^p = \|u\|_{L^p}^p + \int_{R^n} \frac{\|\Delta_t^k u\|_{L^p}^p}{|t|^{n+p\alpha}} dt$$

is finite.

(9) By regularization we obtain function u_ρ converging to u pointwise almost everywhere and in L^p-norm as $\rho \to 0$. Since $(D_j u)_\rho = D_j u_\rho$ for any regularization, it is sufficient to prove the statement for $0 < \alpha = \beta < 1$. Then

$$|u_\rho - u|_{\beta,p}^p = \|u_\rho - u\|_{L^p}^p + \frac{1}{C(n,\beta)\, G_{2n+2\beta}(0)} \int_{R^n} \frac{G_{2n+2\beta}(t)}{|t|^{n+2\beta}} \|\Delta_t u_\rho - \Delta_t u\|_{L^p}^p \, dt.$$

The integrand in the latter expression is dominated by $\frac{G_{2n+2\beta}(t)}{|t|^{n+2\beta}} 2^p \|\Delta_t u\|_{L^p}^p$ and for fixed t converges to 0 with $\rho \searrow 0$. Taking now a function $\varphi \in C_0^\infty$ which is $= 1$ for $|x| \leq 1$, one proves that for $f \in C^\infty$ with $|f|_{\beta,p} < \infty$, $|f(x) - \varphi(\rho x)\, f(x)|_{\beta,p} \to 0$ as $\rho \searrow 0$. Double integrals in approximate norms are handled in a completely similar way as in the case of Bessel potentials in [3].

The argument used in § 4 to prove that for two integers k, $k_1 > \alpha$, the norms $|\ |_{\alpha,p,k}$, $|\ |_{\alpha,p,k_1}$ are equivalent is still valid in this more general setting, with constants as in Lemma 4.1, which justifies the omission of the index k in the symbol $\tilde{\mathfrak{B}}^{\alpha,p}$.

Using again the standard argument, we have

THEOREM 7.2. — *$\tilde{\mathfrak{B}}^{\alpha,p}$ is the functional completion rel. $\mathfrak{A}_0$ of the class $\mathfrak{B}^{\alpha,p,k}$.*

Similarly as in the case of $\check{P}^{\alpha,\infty}$ we define the proper complete functional space $B^{\alpha,\infty}$ of all continuous function with finite norm $|\ |_{\alpha,p,k}$. Except for vanishing at ∞, the functions of $B^{\alpha,\infty}$ have the same properties as those of $B^{\alpha,\infty<}$.

Let us add the following statement. If $\alpha < \alpha'$ then there is a constant C independent of u such that for every u

$$|u|_{\alpha,p,k} \leq C|u|_{\alpha',p,k'}. \tag{7.2}$$

To prove (7.2) we may restrict ourselves to the case when $k = k'$. Then the integral in the norm (7.1) can be decomposed into two parts: integral over $|t| \leq 1$ and $|t| \geq 1$. The first part is majorated by the corresponding integral in $|u|^p_{\alpha',p,k}$, the second by a constant times $\|u\|^p_{L^p}$.

It follows that

$$\tilde{\mathfrak{B}}^{\alpha,p} \supset \tilde{\mathfrak{B}}^{\alpha',p} \quad \textit{for} \quad \alpha < \alpha'. \tag{7.3}$$

§ 8. Behavior of the standard norm.

The purpose of this section is to describe the behavior of the standard norm $|u|_{\alpha,p}$ for a fixed function u and α varying between two consecutive integers.

Before stating the main theorem of this section we introduce the space $W_1^{m}{}^{>}$, $m > 0$ (m an integer) of all functions of W_1^{m-1}, all of whose derivatives of order m are signed Borel measures of finite absolute mass. In the definition of the norm $|u|_{m,1}$ (see (3.6)) the integral involving the derivative of u of order m is to be replaced by

$$\int_{\Sigma^m}\int_{R^n} |d\mu_{\theta^{(m)}}(x)|\, d\theta^{(m)} \tag{8.1}$$

where $\mu_{\theta^{(m)}} = \dfrac{\partial^m u}{\partial\theta_1 \ldots \partial\theta_m}$.

We shall prove the following theorem:

THEOREM 8.1. — *Let* $1 \leqq p \leqq \infty$ *and* $m \geqq 0$ *be an integer.*

i) *If* $u \in W_p^m$ *then* $\lim_{\alpha \nearrow m+1} |u|_{\alpha,p}$ *exists, possibly* $= +\infty$.

ii) *If* $1 < p < \infty$, *then* $\lim_{\alpha \nearrow m+1} |u|_{\alpha,p} < \infty$ *if and only if* $u \in W_p^{m+1}$; *if* $u \in W_p^{m+1}$ *then* $\lim_{\alpha \nearrow m+1} |u|_{\alpha,p} = |u|_{m+1,p}$.

iii) $\lim_{\alpha \nearrow m+1} |u|_{\alpha,1} < \infty$ *if and only if* $u \in W_{1>}^{m+1}$; if $u \in W_{1>}^{m+1}$ *then* $\lim_{\alpha \nearrow m+1} |u|_{\alpha,1} = |u|_{m+1,1}$.

iv) *If* $1 \leqq p < \infty$, *and* $u \in W_p^{\alpha_0}$, $\alpha_0 > m$, *then* $\lim_{\alpha \searrow m} |u|_{\alpha,p} = |u|_{m,p}$.

v) $\lim_{\alpha \nearrow m+1} |u|_{\alpha,\infty} < \infty$, *if and only if* $u \in \check{P}^{m+1,\infty}$; *if* $u \in \check{P}^{m+1,\infty}$ *then* $\lim_{\alpha \nearrow m+1} |u|_{\alpha,\infty} = |u|_{m+1,\infty}$.

Proof. — It follows from the definition of the standard norm $|\ \ |_{\alpha,p}$ that is is sufficient to consider the case when $m = 0$. Assume first that $1 \leqq p < \infty$. For $0 < \beta < 1$ the standard norm may be written in the form

(8.2)
$$|u|_{\beta,p}^p = ||u||_{L^p}^p + \int_{R^n}\int_{R^n} \left|\frac{u(x) - u(y)}{|x-y|^\beta}\right|^p d\mu_\beta(x, y)$$
$$= ||u||_{L^p}^p + \frac{1}{C(n, \beta) G_{2n+2\beta}(0)} \int_{R^n} \frac{G_{2n+2\beta}(t)}{|t|^{n+p\beta}} ||\Delta_t u||_{L^p}^p \, dt.$$

The expression (8.2) has a meaning (it may be infinite) for every $u \in L^p$.

Observe that for $\beta \nearrow 1$ (see (2.9))

$$\text{(8.3)} \qquad \frac{1}{C(n, \beta)} \to 0; \qquad \frac{1}{(1-\beta)C(n, \beta)} \to \frac{2n}{\omega_n}.$$

Rewrite now the integral in (8.2) for $u \in L^p$ in the form

(8.4)
$$\frac{1}{C(n, \beta) G_{2n+2\beta}(0)} \int_{R^n} \frac{G_{2n+2\beta}(t)}{|t|^{n+p\beta}} ||\Delta_t u||_{L^p}^p \, dt$$
$$= \frac{1}{C(n, \beta) G_{2n+2\beta}(0)} \int_{|t| \leqq 1} \frac{G_{2n+2\beta}(t)}{|t|^{n+p\beta}} ||\Delta_t u||_{L^p}^p \, dt$$
$$+ \frac{1}{C(n, \beta) G_{2n+2\beta}(0)} \int_{|t| \geqq 1} \frac{G_{2n+2\beta}(t)}{|t|^{n+p\beta}} ||\Delta_t u||_{L^p}^p \, dt = I'_\beta(u) + I''_\beta(u).$$

A simple computation yields

$$(8.5)\quad I''_\beta(u) \leq \frac{2^{p-1}}{C(n,\beta)}\|u\|^p_{L^p}\int_{|t|\geq 1}\frac{dt}{|t|^{n+p\beta}} = \frac{2^{p-1}\omega_n}{C(n,\beta)p\beta}\|u\|^p_{L^p},$$

and by (8.3)

$$(8.6)\qquad I''_\beta(u) \to 0 \quad \textit{for} \quad \beta \nearrow 1.$$

According to (8.6), to investigate the behavior of $|u|_{\beta,p}$ as $\beta \nearrow 1$, it is sufficient to determine the behavior of $I'_\beta(u)$ as $\beta \nearrow 1$.

Define now

$$(8.7)\qquad I_\beta(u) = \frac{1}{C(n,\beta)}\int_{|t|\leqslant 1}\frac{\|\Delta_t u\|^p_{L^p}}{|t|^{n+p\beta}}\,dt.$$

Clearly, $I_\beta(u)$ is well defined for all $u \in L^p$; moreover, we have with

$$A_n = \min_{\substack{|t|\leqslant 1\\ 0\leqslant\beta\leqslant 1}}\frac{G_{2n+2\beta}(t)}{G_{2n+2\beta}(0)} = \min_{0\leqslant\beta\leqslant 1}\frac{G^{(1)}_{n+1+2\beta}(1)}{G^{(1)}_{n+1+2\beta}(0)},$$

$$(8.8)\qquad A_n I_\beta(u) \leq I'_\beta(u) \leq I_\beta(u),$$

and therefore $I_\beta(u)$ is finite if and only if $I'_\beta(u)$ is finite.

On the other hand, if $u \in W^{\beta_0}_p$ with $0 < \beta_0 < 1$, $(1-\beta_0)p < 1$ (and consequently $I_{\beta_0}(u) < \infty$) we can write for $\beta \geq \beta_0$,

$$(8.9)\quad |I_\beta(u) - I'_\beta(u)| \leq \frac{C(n,\beta_0)}{C(n,\beta)G_{2n+2\beta}(0)}\max_{|t|\leqslant 1}\frac{G_{2n+2\beta}(0)-G_{2n+2\beta}(t)}{|t|^{p(\beta-\beta_0)}}\,I_{\beta_0}(u),$$

and since (c.f.(2.10)) $\dfrac{G_{2n+2\beta}(0)-G_{2n+2\beta}(t)}{|t|}$ is bounded uniformly with respect to t and β, $\beta_0 \leq \beta \leq 1$, we get by (8,3)

$$(8.10)\qquad |I_\beta(u) - I'_\beta(u)| \to 0 \quad \textit{for} \quad \beta \nearrow 1.$$

$I_\beta(u)$ can now be represented in the form

$$(8.11)\quad I_\beta(u) = \frac{1}{C(n,\beta)}\int_\Sigma\int_0^1 \frac{1}{s^{1+p(\beta-1)}}\varphi(s,\theta)\,ds\,d\theta,$$

where

$$(8.12)\qquad \varphi(s,\theta) = \left\|\frac{\Delta_{s\theta}u}{s}\right\|^p_{L^p}.$$

Since $||\Delta_{s\theta}u||_{L^p} \leq 2||\Delta_{\frac{s}{2}\theta}u||_{L^p}$ we get

$$(8.13) \qquad \varphi(s, \theta) \leq \varphi\left(\frac{s}{2}, \theta\right)^{(10)}$$

for every s.

Rewrite (8.11) in the form

$$I_\beta(u) = \frac{1}{C(n, \beta)} \int_\Sigma \left(\sum_{m=0}^{\infty} \int_{2^{-m-1}}^{2^{-m}} \frac{1}{s^{1+(\beta-1)p}} \varphi(s, \theta)\, ds \right) d\theta$$
$$= \frac{1}{C(n, \beta)} \int_\Sigma \int_{1/2}^{1} \frac{1}{s^{1+(\beta-1)p}} \sum_{m=0}^{\infty} 2^{m(\beta-1)p} \varphi(2^{-m}s, \theta)\, ds.$$

In view of (8.13) the sequence $\{\varphi(2^{-m}s, \theta)\}$ is non-decreasing for every s and θ, therefore applying summation by parts [11] to the series under the sign of the last integral, we get

$$(8.14) \quad I_\beta(u) = \frac{1}{C(n, \beta)\,(1 - 2^{(\beta-1)p})} \int_\Sigma \int_{1/2}^{1} \frac{1}{s^{1+(\beta-1)p}}$$
$$\left\{ \sum_{m=0}^{\infty} 2^{(m+1)(\beta-1)p} [\varphi(2^{-m-1}s, \theta) - \varphi(2^{-m}s, \theta)] + \varphi(s, \theta) \right\} ds\, d\theta.$$

In view of (8.3) we have

$$(8.15) \qquad \lim_{\beta \nearrow 1} \frac{1}{C(n, \beta)\,(1 - 2^{(\beta-1)p})} = \frac{2}{p} \frac{n}{\omega_n} \cdot \frac{1}{\log 2}.$$

On the other hand the integrand in (8.14) is an increasing function of β, $0 \leq \beta \leq 1$ and taking into account (8.10), i) follows.

To prove ii), assume that $1 < p < \infty$ and $\lim\limits_{\beta \nearrow 1} |u|_{\beta,p} < \infty$. Then in view of (8.10) there exists a positive constant M and a set $\Sigma_M \subset \Sigma$ of positive measure such that

$$(8.16) \quad \int_{1/2}^{1} \frac{1}{s^{1+(\beta-1)p}}$$
$$\left\{ \sum_{m=0}^{\infty} 2^{(m+1)(1-\beta)p} [\varphi(2^{-m-1}s, \theta) - \varphi(2^{-m}s, \theta)] + \varphi(s, \theta) \right\} ds \leq M,$$

(10) The idea of introducing the function $\varphi(s, \theta)$ and using the inequality (8.13) is due to E. Gagliardo.

(11) More explicitly we use the following version of the Abel formula: If $a_m \geq 0$, $b_m \geq 0$, $\{a_m\}$-non decreasing, $\sum_{m=0}^{\infty} b_m < \infty$,

then $\sum_{m=0}^{\infty} a_m b_m = a_0 s_0 + \sum_{m=0}^{\infty} (a_{m+1} - a_m) s_{m+1}$ with $s_m = \sum_{l=m}^{\infty} b_l$.

for all $\theta \in \sum_M$ and $\beta < 1$. Invoking now the definition of (8.12) we conclude that for almost every $s \in [1/2, 1]$ and $\theta \in \Sigma_M$ the norms $\left\|\frac{\Delta_{2^{-m}s\theta}u}{2^{-m}s}\right\|_{L^p}$ are uniformly bounded. By reflexivity of the space $L^p(R^n)$ $(1 < p < \infty)$ there exists an increasing sequence of positive integers m_k and a function $u_\theta \in L^p(R^n)$ such that $\frac{\Delta_{2^{-m_k}s\theta}u}{2^{-m_k}s} \to u_\theta$ weakly in L^p. By a standard reasoning in the theory of distributions we conclude that $u_\theta = \frac{\partial u}{\partial \theta}$. Choosing $\theta_1, \ldots, \theta_n \in \Sigma_M$ as any system of linearly independent vectors, we conclude that $\frac{\partial u}{\partial \theta_1}, \ldots, \frac{\partial u}{\partial \theta_n} \in L^p$ and consequently $u \in W_p^1$. Conversely, if $u \in W_p^1$ then applying the Minkowski inequality [12] and Fatou's lemma, we get $\lim_{s \to 0}\left\|\frac{\Delta_{s\theta}u}{s}\right\|_{L^p} = \left\|\frac{\partial u}{\partial \theta}\right\|_{L^p}$ and consequently, taking into account (8.10) and the fact that as $\beta \nearrow 1$ the integral in (8.16) converges increasingly to $\log 2 \lim_{m \to \infty}\left\|\frac{\Delta_{2^{-m}s\theta}u}{2^{-m}s}\right\|_{L^p}$ we get

$$\lim_{\beta \nearrow 1} I_\beta(u) = \frac{2}{p}\,\frac{n}{\omega_n}\int\left\|\frac{\partial u}{\partial \theta}\right\|_{L^p}^p d\theta.$$

This completes the proof of ii).

To prove iii) we use a similar reasoning as in the proof of ii). Assume first that $\lim_{\beta \nearrow 1} |u|_{\beta,1} < \infty$. As in the proof of ii) we conclude that for some sequence $\{s_n\}$, $s_n \searrow 0$ and $\theta \in \Sigma_M$ the norms $\left\|\frac{\Delta_{s_n\theta}u}{s_n}\right\|_{L^1}$, are uniformly bounded. By the theorem about « vague convergence » of Borel signed measures with absolute total mass finite, we can find a subsequence $\{s'_n\} \subset \{s_n\}$ and a measure $d\mu_\theta$ with absolute total mass $|\mu_\theta| \leq \liminf\left\|\frac{\Delta_{s_n\theta}u}{s_n}\right\|_{L}$ such that $\frac{\Delta_{s'_n\theta}u}{s'_n}\,dx$ converges vaguely to $d\mu_\theta$. Using again a standard reasoning from the theory of distributions we conclude that $\mu_\theta = \frac{\partial u}{\partial \theta}$, and consequently for every

(12) See [10], Prop. (203).

$\theta \in \Sigma_{M'}$, $\dfrac{\partial u}{\partial \theta}$ is a signed Borel measure with total absolute mass finite. Therefore $u \in W^1_{1>}$.

Assume now that $u \in W^1_{1>}$. Then, for every, θ, $|\theta| = 1$, $\lim\limits_{s \to 0} \dfrac{u(x + s\theta) - u(x)}{s} = \mu_\theta(x)$, where $\mu_\theta(x)$ is a signed Borel measure with total absolute mass finite, the limit being understood as a vague limit.

Introduce the system of coordinate axes such that the x_n-axis coincides with θ. Then $d\mu_\theta$ is a Borel measure of the form $dx'\, d\nu_{x'}(x_n)$ where the measures $d\nu_{x'}(x_n)$ are of finite total absolute mass on the x_n-axis for almost all x' and such that $|\mu_\theta| = \int_{R^{n-1}} |\nu_{x'}|\, dx'$. $d\nu_{x'}$ is the distribution derivative of the function $u(x', x^n)$ for fixed x'. We can write

(8.17)

$$\left\|\frac{\Delta_{s\theta} u}{s}\right\|_{L^1} = \frac{1}{s}\int_{-\infty}^{\infty}\int_{R^{n-1}} |u(x' + (\tau + s)\theta) - u(x' + \tau\theta)|dx'\, d\tau$$
$$\leq \frac{1}{s}\int_{-\infty}^{\infty}\int_{R^{n-1}}\int_{\tau}^{\tau+s} |d\nu_{x'}(x_n)|dx'\, d\tau = \frac{1}{s}\int_{-\infty}^{\infty} [f(\tau + s) - f(\tau)]d\tau,$$

where $f(\tau) = |\mu_\theta|[-\infty < x_n < \tau] = \int_{R^{n-1}}\int_{-\infty}^{\tau} |d\nu_{x'}(x_n)|dx'$. $f(\tau)$ is an increasing function of τ, such that $f(-\infty) = 0$, $f(\infty) = |\mu_\theta|$, and therefore the last integral in (8.17) yields

$$\lim_{s \to 0}\left\|\frac{\Delta_{s\theta} u}{s}\right\|_{L^1} \leq |\mu_\theta|.$$

The proof of iii) is now completed in exactly the same way as that of ii).

iv) If $u \in W^{\beta_0}_p$ the integral in (8.2) can be estimated for $\beta \leqslant \beta_0$ as follows (c being an absolute constant),

(8.18)
$$\int_{R^n}\int_{R^n}\left|\frac{u(x) - u(y)}{|x - y|^\beta}\right|^p d\mu_\beta(x, y)$$
$$\leq c\,\frac{G_{2n+2\beta_0}(0)C(n, \beta_0)}{G_{2n+2\beta}(0)C(n, \beta)}\iint_{|x-y|\leqslant 1}\left|\frac{u(x) - u(y)}{|x - y|^{\beta_0}}\right|^p d\mu_{\beta_0}(x, y)$$
$$+ \frac{2^{p-1}}{G_{2n+2\beta}(0)C(n, \beta)}\|u\|^p_{L^p}\int_{R^n} G_{2n+2\beta}(t)\, dt.$$

Since for $\beta \searrow 0$, $\dfrac{1}{C(n, \beta)} \to 0$ and all remaining factors are bounded, $\left(\int_{R^n} G_\alpha(t)dt = 1\right)$, iv) follows.

v) follows immediately from the observation that

$$\lim_{\beta \nearrow 1} |u|_{\beta,\infty} = \max\left(\sup_x u(x),\ \sup_{x \neq y}\left|\frac{u(x) - u(y)}{|x - y|}\right|\right).$$

Remark. — If $p = \infty$, iv) is not in general true. We have then

$$\lim_{\beta \searrow 0} |u|_{\beta,\infty} = \max(\sup_x |u(x)|,\ \operatorname{osc}(u))$$

where

$$\operatorname{osc}(u) = \sup_{x,y} |u(x) - u(y)|.$$

COROLLARY. — *If* $0 \leqq \alpha < \alpha'$ *then for every* $u \in W_p^\alpha$, $1 \leqq p \leqq \infty$,

$$|u|_{\alpha,p} \leqq C|u|_{\alpha',p}$$

where $C = \max(1 + 4n,\ 2(0.8)^{-2}A_n^{-1})$, *where* A_n *is the constant of inequality* (8.8). *Consequently,* $W_p^\alpha \supset W_p^{\alpha'}$ for $\alpha' > \alpha$.

Proof. — It is sufficient to consider the case when $0 \leqq \alpha < \alpha' \leqq 1$. Combining (8.4), (8.5), (8.14) and the fact that the integral on the right hand side of (8.14) is an increasing function of β we get for $0 \leqq \beta \leqq \beta' \leqq 1$

$$|u|_{\beta,p} \leqq \left\{\max\left[1 + \frac{2^{p-1}\omega_n}{C(n, \beta)p\beta},\ A_n^{-1}\frac{C(n, \beta')\,(1 - 2^{(\beta'-1)p})}{C(n, \beta)\,(1 - 2^{(\beta-1)p})}\right]\right\}^{1/p} |u|_{\beta',p}$$

and the result follows by an easy estimation of the constant in the latter inequality.

§ 9. Auxiliary inequalities.

In this section we shall establish some inequalities involving kernels G_α which will be needed in the sequel.

We denote by n' a positive integer $n' \leqq n$, $n'' = n - n'$. Unless otherwise indicated $x', y', z', t', \ldots$ will denote projections of points $x, y, z, t, \ldots$ on the hyperplane $R^{n'}$: $x_{n'+1} = \cdots = x_n = 0$, $x'', y'', z'', t'', \ldots$ projections of these points on the hyperplane $R^{n''}$: $x_1 = \cdots = x_{n'} = 0$. Accordingly, dx' and dx'' will denote volume elements of $R^{n'}$ and $R^{n''}$.

The letter c will stand for (in general different) positive constants depending on various parameters. In all considerations we will assume that the orders α of the kernels G_α and orders of occurring differentiations and differences are bounded from above by some fixed but otherwise arbitrary number $M > 0$. The letter $\varkappa$ will be used to denote (in general different) positive constants depending only on n and M. In the cases when behavior of constants is of importance we shall say that c is majorated by $f(\alpha, \beta, \gamma, \ldots)$ if there is a constant $\varkappa$ such that $c \leqq \varkappa f(\alpha, \beta, \gamma, \ldots)$ in the considered region of these parameters.

In several instances we shall use the following

Young's inequality: if $f \in L^p$, $g \in L^q$, $0 \leqq \frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$ then $f * g \in L^r$ and $||f * g||_{L^r} \leqq ||f||_{L^p} ||g||_{L^q}$.

From the differentiation formula (2.6) it can be deduced that for any $\alpha > 0$ and a multi-index j, $|j| < \alpha$,

$$(9.1) \qquad |D_j G_\alpha(x)| \leqq \varkappa \left[G_\alpha(x) + \frac{\alpha}{\alpha - |j|} G_{\alpha-|j|}(x) \right].$$

From series expansions of G_α (see(2.3 a) — (2.3 d)) we also get, with an arbitrary multi-index j,

(9.2)
$$|D_j G_\alpha(x)| \leqq \begin{cases} \varkappa\alpha|x|^{\alpha-n-|j|} \text{ for } \alpha \leqq n + |j| \text{ and } |j| \text{ odd} \\ \dfrac{\varkappa\alpha}{n + |j| - \alpha} |x|^{\alpha-n-|j|} \text{ for } \alpha < n + |j| \text{ and } |j| \text{ even} \\ \varkappa \text{ for } \alpha \geqq n + |j| \text{ and } |j| \text{ odd} \\ \dfrac{\varkappa}{\alpha - n - |j|} \text{ for } \alpha > n + |j| \text{ and } |j| \text{ even.} \end{cases}$$

Also for $|x| \leqq 1$ and even $|j|$,

(9.2′)
$$|D_j G_\alpha(x)| \leqq \begin{cases} \varkappa\alpha|x|^{\alpha-n-|j|}\left(1 + \log\dfrac{1}{|x|}\right) \text{ for } \alpha \leqq n + |j| \\ \varkappa\left(1 + \log\dfrac{1}{|x|}\right) \text{ for } \alpha \geqq n + |j|. \end{cases}$$

For any multi-index j, $|j| < \alpha$, (9.1) implies

$$(9.3) \qquad D_j G_\alpha \in L^1(R^n); \qquad \int_{R^n} |D_j G_\alpha(x)|\, dx \leqq \varkappa\left(1 + \frac{\alpha}{\alpha - |j|}\right).$$

Using (2.2) we easily obtain

$$(9.4)\qquad \int_{R^{n'}} G_\alpha(x)\, dx' = \int_{R^{n'}} G_\alpha(x', x'')\, dx' = (2\pi)^{-n'/2} G_\alpha^{(n'')}(x'').$$

(If $n' = n$, the right-hand side is of course 1.)

Let $\alpha > 0$ and consider the expression $\int_{R^n} |\Delta_t G_\alpha(x)|\, dx$. Choose the coordinates' axes in such a way that the vector t is parallel to the x_n axis. Using the fact that $G_\alpha(x)$ is a decreasing function of $|x|$ we can write ($|t| = t_n > 0$), in view of (9.4),

$$\int_{R^n} |\Delta_t G_\alpha(x)|\, dx = 2\int_{R^{n-1}}\int_{-|t|/2}^{|t|/2} G_\alpha(x)\, dx_n\, dx'$$

$$= 2(2\pi)^{-\frac{n-1}{2}} \int_{-|t|/2}^{|t|/2} G_\alpha^{(1)}(x_n)\, dx_n = 2(2\pi)^{-\frac{n-1}{2}} \int_{-\infty}^{\infty} G_\alpha^{(1)}(x_n)\chi(x_n)\, dx_n,$$

χ being the characteristic function of $\left[-\frac{|t|}{2}, \frac{|t|}{2}\right]$.

By the Parseval equality we get for $\alpha < 1$,

$$\int_{R^n} |\Delta_t G_\alpha(x)|\, dx = 8(2\pi)^{-\frac{n-1}{2}} \int_0^\infty \frac{\sin\frac{\eta|t|}{2}}{\eta(1+\eta^2)^{\alpha/2}}\, d\eta$$

$$= 8(2\pi)^{-\frac{n-1}{2}} |t|^\alpha \int_0^\infty \frac{\sin\eta\, d\eta}{\eta(|t|^2 + 4\eta^2)^{\alpha/2}}$$

$$= 8(2\pi)^{-\frac{n-1}{2}} |t|^\alpha \int_0^\infty 2\sin^2\frac{\eta}{2}\left(\frac{1}{\eta^2(|t|^2+4\eta^2)^{\alpha/2}} + \frac{4\alpha}{(|t|^2+4\eta^2)^{1+\alpha/2}}\right) d\eta$$

$$\leqq 8(2\pi)^{-\frac{n-1}{2}} |t|^\alpha \int_0^\infty \frac{2(1+\alpha)\sin^2\frac{\eta}{2}}{\eta^2(|t|^2+4\eta^2)^{\alpha/2}}\, d\eta,$$

$$(9.5)\qquad \int_{R^n} |\Delta_t G_\alpha(x)|\, dx \leqq \frac{\varkappa}{1-\alpha}|t|^\alpha \qquad \textit{for} \qquad 0 < \alpha < 1.$$

Similarly, one gets

$$(9.6)\qquad \int_{R^n} |\Delta_t G_\alpha(x)|\, dx \leqq \frac{\varkappa}{\alpha-1}|t| \qquad \textit{for} \qquad \alpha > 1.$$

We could also get the inequality

$$\int_{\mathbf{R}^n} |\Delta_t G_1(x)|\, dx \leq \varkappa\left(1 + \log\frac{1}{|t|}\right)|t|, \qquad |t| \leq 1,$$

which, however, will not be used.

Similar inequalities can be obtained for derivatives of the kernel G_α. We have, for $|j| < \alpha$,

$$(9.7) \qquad \int_{\mathbf{R}^n} |\Delta_t D_j G_\alpha(x)|\, dx \leq \frac{\varkappa\alpha}{(\alpha - |j|)(1 - \alpha + |j|)} |t|^{\alpha - |j|}$$

if

$$|j| < \alpha < |j| + 1$$

and

$$(9.7') \quad \int_{\mathbf{R}^n} |\Delta_t D_j G_\alpha(x)|\, dx \leq \frac{\varkappa}{\alpha - |j| - 1} |t| \quad \textit{for} \quad \alpha > |j| + 1.$$

In view of (9.3) it is enough to prove (9.7) and (9.7′) for $|t| < \frac{1}{2}$. For these values of $|t|$, (9.7) and (9.7′) are obtained as follows. The integrals are divided into two parts :

$$\int_{\mathbf{R}^n} = \int_{|x| < 2|t|} + \int_{|x| > 2|t|}.$$

The first integral is evaluated (in (9.7) as well as in (9.7′)) by using (9.2) or (9.2′) and the inequality

$$|\Delta_t D_j G_\alpha(x)| \leq |D_j G_\alpha(x + t)| + |D_j G_\alpha(x)|.$$

To evaluate the second integral we write

$$(*) \qquad |\Delta_t D_j G_\alpha(x)| \leq \int_0^{|t|} \sum_{k=1}^{n} \left| \theta_k \frac{\partial}{\partial x_k} D_j G_\alpha(x + \tau\theta) \right| d\tau,$$

where

$$\theta = \frac{t}{|t|} = (\theta_1, \ldots, \theta_n).$$

To obtain the desired evaluation in (9.7) we use (9.2) for the derivatives of order $|j| + 1$ in $(*)$ and integrate both sides of $(*)$ over $|x| > 2|t|$ $\left(\text{we use here } \frac{1}{2}|x| < |x + \tau\theta| < \frac{3}{2}|x|\right)$. The evaluation in (9.7′) is obtained even more simply by integrating both sides of $(*)$ and using (9.3).

By a similar argument, we get

$$\int_{\mathrm{R}^n} |\Delta_t D_j G_{|j|+1}(x)|\, dx \leqq \varkappa |t| \left(1 + \log \frac{1}{|t|}\right), \quad |t| \leqq 1,$$

but this inequality will not be needed.

We shall now estimate the integral $\int_{\mathrm{R}^{n'}} |\Delta_t G_\alpha(x)|\, dx'$, with $n' < n$, $n'' = n - n' \geqq 1$. We shall restict ourselves to the case when $0 < \alpha < n'' + 1$.

From (9.4) we have (note: $t = t' + t''$, $x = x' + x''$)

$$(9.8) \qquad \int_{\mathrm{R}^{n'}} |\Delta_t G_\alpha(x)| dx' \leqq \int_{\mathrm{R}^{n'}} G_\alpha(x)\, dx' + \int_{\mathrm{R}^{n'}} G_\alpha(x + t)\, dx' = (2\pi)^{-n'/2}[G_\alpha^{(n'')}(x'') + G_\alpha^{(n'')}(x'' + t'')].$$

On the other hand,

$$(9.9) \qquad \int_{\mathrm{R}^{n'}} |\Delta_t G_\alpha(x)|\, dx' \leqq \int_{\mathrm{R}^{n'}} |\Delta_{t'} G_\alpha(x)|\, dx' + \int_{\mathrm{R}^{n'}} |\Delta_{t''} G_\alpha(x)|\, dx'.$$

The first integral on the right hand side of (9.9) can be estimated by an argument similar to that in the derivation of (9.5). Without loss of generality we can assume that t' has the direction of the $x_{n'}$ axis. Integrating separately over the regions where $|x' + t'| \leqq |x'|$ and $|x' + t'| \geqq |x'|$ we get

$$(9.10) \qquad \int_{\mathrm{R}^{n'}} |\Delta_{t'} G_\alpha(x)|\, dx' = 4(2\pi)^{-\frac{{}'-1}{2}} \int_0^{|t'|/2} G_\alpha^{(n''+1)}(x_{n'}, x'')\, dx_{n'}.$$

In view of (9.2) for $|j| = 0$, the latter formula gives

$$(9.11) \qquad \begin{cases} \int_{\mathrm{R}^{n'}} |\Delta_{t'} G_\alpha(x)|\, dx' \leqq [(n'' + 1 - \alpha)(\alpha - n'')]^{-1} |t'|^{\alpha - n''} \\ \quad \textit{if} \quad n'' < \alpha < n'' + 1 \\ \int_{\mathrm{R}^{n'}} |\Delta_{t'} G_\alpha(x)|\, dx' \leqq \varkappa\alpha\, |x''|^{\alpha - n'' - 1}\, |t'| \\ \quad \textit{for} \quad 0 < \alpha \leqq n''. \end{cases}$$

The second integral on the right-hand side of (9.9) can be written in the form

$$(9.12) \qquad \int_{\mathrm{R}^{n'}} |\Delta_{t''} G_\alpha(x)|\, dx' = (2\pi)^{-n'/2}\, |G_\alpha^{(n'')}(x'') - G_\alpha^{(n'')}(x'' + t'')|.$$

Assume that $|x''| \neq 0$ and $|x'' + t''| \neq 0$. Since $G_\alpha^{(n'')}(y'')$ is a function of the radius $r = |y''|$ only, we get from (9.12), using (9.2),

$$\int_{R^{n'}} |\Delta_{t'} G_\alpha(x)|\, dx' \leqq (2\pi)^{-n'/2} \left| \int_{|x''|}^{|x''+t''|} \left| \frac{dG_\alpha^{(n'')}(r)}{dr} \right| dr \right.$$

$$\leqq \varkappa\alpha \left| \int_{|x''|}^{|x''+t''|} \frac{dr}{r^{n''-\alpha+1}} \right|,$$

$$(9.13) \quad \int_{R^{n'}} |\Delta_{t''} G_\alpha(x)|\, dx' \leqq \begin{cases} \varkappa(\alpha - n'')^{-1}\, |t''|^{\alpha-n''} \\ \quad \text{if } \; n'' < \alpha < n'' + 1, \\ \varkappa\alpha[\min(|x''|,\ |x''+t''|)]^{\alpha-n''-1}\, |t''| \\ \quad \text{if } \; 0 < \alpha \leqq n''. \end{cases}$$

The last inequalities combined with the corresponding inequalities (9.11) yield

$$(9.14) \quad \int_{R^{n'}} |\Delta_t G_\alpha(x)|\, dx' \leqq \begin{cases} \varkappa[(n'' + 1 - \alpha)\,(\alpha - n'')]^{-1}\, |t|^{\alpha-n''} \\ \quad \text{if } \; n'' < \alpha < n'' + 1 \\ \varkappa\alpha[\min(|x''|,\ |x''+t''|)]^{\alpha-n''-1}\, |t| \\ \quad \text{if } \; 0 < \alpha \leqq n''. \end{cases}$$

(9.14) is now combined with (9.8) using the following remark. If for positive numbers a, b, c, $a \leqq b$ and $a \leqq c$, then for arbitrary θ, $0 \leqq \theta \leqq 1$, we have also $a \leqq b^\theta c^{1-\theta}$. Applying this remark for $\alpha < n''$ to (9.14) and (9.8), and using the inequality (see (2.3 a) and (2.3 b)),

$$G_\alpha^{(n'')}(x'') \leqq \varkappa\alpha(n'' - \alpha)^{-1}\, |x''|^{\alpha-n''}, \qquad \text{for} \qquad \alpha < n'',$$

we get, with arbitrary θ, $0 \leqq \theta \leqq 1$,

$$(9.15) \quad \int_{R^{n'}} |\Delta_t G_\alpha(x)|\, dx' \leqq \begin{cases} c|t|^{\alpha-n} \quad \text{if } \; n'' < \alpha < n'' + 1; \\ \quad c \leqq \varkappa[(n'' + 1 - \alpha)(\alpha - n'')]^{-1} \\ c|t|^\theta[\min(|x''|,\ |x''+t''|)]^{\alpha-n''-\theta} \\ \quad \text{if } \; 0 < \alpha < n'' \\ \quad c \leqq \varkappa(n'' - \alpha)^{\theta-1}. \end{cases}$$

The following corollary to (9.15) will be needed. If $0 < \alpha < n''$, and $\delta > 0$ is such that $0 < \alpha - \delta < 1$ then

$$(9.16) \qquad \int_{R^n} |\Delta_t G_\alpha(x)|\,|x''|^{-\delta}\, dx \leqq c|t|^{\alpha-\delta};$$
$$c \leqq \varkappa[(n'' - \alpha)(\alpha - \delta)(1 - \alpha + \delta)]^{-1}.$$

We outline briefly the proof. We have

$$\int_{\mathrm{R}^n} |\Delta_t G_\alpha(x)||x''|^{-\delta}\, dx$$
$$= \Big[\int_{|x''|\leq|x''+t''|} + \int_{|x''|\geq|x''+t''|}\Big]\Big[\int_{\mathrm{R}^{n'}} |\Delta_t G_\alpha(x)|\,|x''|^{-\delta}\, dx'\Big]\, dx''.$$

In the first integral on the right-hand side of the formula above we apply the second inequality (9.15) with $\theta = 0$ for $|x''| < |t|$ and $\theta = 1$ for $|x''| \geqq |t|$. We get

$$\int_{|x''|\leq|x''+t''|} \int_{\mathrm{R}^{n'}} |\Delta_t G_\alpha(x)||x''|^{-\delta}\, dx'\, dx''$$
$$\leqq c\Big[\int_{|x''|\leq|t|} |x''|^{\alpha-n''-\delta}\, dx'' + |t| \int_{|x''|\geq|t|} |x''|^{\alpha-n''-\delta-1}\, dx''\Big]$$

and the desired estimate follows. In the integral over $|x''| \geqq |x''+t''|$ we divide the integration over x'' into $|x''+t''| < |t|$ and $|x''+t''| \geqslant |t|$ and proceed similarly.

The previously obtained estimates will now be extended to higher differences. The basic formula will be the following: for $0 \leqq k' \leqq k$, the coordinate axis x_1 being chosen in the direction of the vector $t \neq 0$,

$$(**)\quad \Delta_t^k u(x)$$
$$= |t|^{k'} \Delta_{t;x}^{k-k'} \int_0^1 \cdots \int_0 \frac{\partial^{k'}}{\partial x_1^{k'}} u(x + t(\tau_1 + \cdots + \tau_{k'}))\, d\tau_1 \ldots d\tau_{k'}.$$

Formulas (9.3), (9.5), (9.6), (9,7), and (9.7′) give now for $k \geqq 1$,

(9.17)

$$\int_{\mathrm{R}^n} |\Delta_t^k D_j G_\alpha(x)|dx \leqq \begin{cases} \varkappa\alpha(\alpha-|j|)^{-1}(|j|+k-\alpha)^{-1}|t|^{\alpha-|j|} \\ \quad \text{for } |j| < \alpha < |j|+k \\ \varkappa(\alpha-|j|-k)^{-1}|t|^k \\ \quad \text{for } |j|+k < \alpha. \end{cases}$$

In the first case, if $0 < \alpha - |j| \leqq 1/2$, we write

$$\Delta_t^k D_j G_\alpha = \Delta_t^{k-1}\Delta_t D_j G_\alpha$$

and get by (9.7) the evaluation $\varkappa\alpha(\alpha-|j|)^{-1}|t|^{\alpha-|j|}$. If $1/2 < \alpha - |j| \leqq k - 1/2$, we write

$$\gamma = (\alpha - |j|)/k,\ \Delta_t^k D_j G_\alpha = \Delta_t D_j G_{|j|+\gamma} * \Delta_t G_\gamma * \cdots * \Delta_t G_\gamma$$

and apply (9.5), (9.7) and repeatedly Young's inequality (with $p = q = r = 1$), which leads to the estimate $\varkappa\alpha|t|^{\alpha-|j|}$.

If $k - \frac{1}{2} < \alpha - |j| < k$, we use (**) with $k' = k - 1$ and $u = D_j G_\alpha$ and then apply (9.7) obtaining an evaluation $\varkappa(|j| + k - \alpha)^{-1}|t|^{\alpha-|j|}$.

In the second case, we use (**) with $k' = k$, $u = D_j G_\alpha$ and apply (9.3).

The extensions of formulas (9.15) and (9.16) to higher differences will be needed only for $t = t' \in R^{n'}$. We assume $k \geqq 1$, $n' \geqq 1$, $n'' \geqq 1$, hence $n = n' + n'' \geqq 2$.

$$(9.18) \quad \int_{R^{n'}} |\Delta_{t'}^k G_\alpha(x)|\, dx' \leqq c|t'|^\theta |x''|^{\alpha-n''-\theta}$$
$$\textit{for} \quad \alpha < n'' + k, \ \max\,[(\alpha - n''),\ 0] \leqq \theta \leqq k.$$

The constant c can be expressed in the simplest way by putting $\theta_0 = \max[(\alpha - n''),\ 0]$ and writing $\theta = \theta_0(1 - \tau) + k\tau$, $0 \leqq \tau \leqq 1$. We have then

$$(9.18') \quad \begin{cases} c = \varkappa\alpha|n'' - \alpha|^{\tau-1}(n'' + k - \alpha)^{-\tau} \\ \quad \textit{for} \quad \alpha \neq n'', \quad k > 1 \quad \textit{and} \quad \alpha - n'' \leqq k - 1 \\ c = \varkappa(n'' + k - \alpha)^{-2\tau} \\ \quad \textit{for} \quad k > 1 \ \textit{and} \ k - 1 < \alpha - n'' < k \\ c = \varkappa\alpha|n'' - \alpha|^{\tau-1}(n'' + 1 - \alpha) \\ \quad \textit{for} \quad \alpha \neq n'' \ \textit{and} \ k = 1 \\ c = \varkappa\theta^{-1} \\ \quad \textit{for} \quad \alpha = n'' \ \textit{and} \text{ any } k \geqq 1. \end{cases}$$

One should notice that for $\alpha = n''$, θ has to be strictly positive.

The inequality (9.18) for $\theta = k$ is obtained by using (**) with $k' = k$ and $u = G_\alpha(x)$, then applying (9.2) and integrating over $R^{n'}$. The resulting constant c is $\varkappa\alpha(n'' + k - \alpha)^{-1}$.

When $\alpha \neq n''$, we can take the other extreme value of θ, $\theta_0 = \max[(\alpha - n''),\ 0]$. For $\alpha < n''$, this means $\theta_0 = 0$. We write then $\Delta_{t'}^k G_\alpha = \Delta_{t'}^{k-1}(\Delta_{t'} G_\alpha)$ and the inequality is given by (9.8) with $t'' = 0$ and with constant $\varkappa\alpha(n'' - \alpha)$. For $\alpha > n''$, $\theta_0 = \alpha - n''$. If $n'' < \alpha < n'' + 1$ and $k = 1$, the inequality is given by (9.11). If $n'' < \alpha < n'' + 1$ and $k \geqq 2$, we write $\Delta_{t'}^k G_\alpha = \left(\Delta_{t'} G_{\frac{\alpha+n''}{2}}\right) * \left(\Delta_{t'}^{k-1} G_{\frac{\alpha-n''}{2}}\right)$, integrate with respect to x' and apply (9.11) and for the second integration (over R^n) use (9.17) with $j = 0$. Finally, for $\alpha \geqq n'' + 1$, which implies $k \geqq 2$, we write $\Delta_{t'}^k G_\alpha = (\Delta_{t'} G_{n''+\gamma}) * (\Delta_{t'}^{k-1} G_{\alpha-n''-\gamma})$

with $\gamma = \frac{1}{2}$ if $\alpha - n'' \leqq k - 1$ and $\gamma = \frac{\alpha - n''}{k}$ if $\alpha - n'' > k - 1$ and argue as in the preceding case.

In all previous cases we obtain (9.18) by combining the evaluations A and B corresponding to $\theta = \theta_0$ and $\theta = k$ into $A^{1-\tau}B^{\tau}$. The remaining case $\alpha = n''$ is dealt with presently.

We write $\int |\Delta_{t'}^k G_{n''}(x)| \, dx' \leqq \varkappa \int |\Delta_{t'} G_{n''}(x)| \, dx'$. By (9.10) this is majorated for $0 < \theta < 1$ by

$$\varkappa \int_0^{|t'|/2} G_{n''}^{(n''+1)}(x_{n'}, x'') \, dx_{n'} \leqq \varkappa \int_0^{|t'|/2} (s^2 + |x''|^2)^{-1/2} \, ds < \varkappa |x''|^{-\theta} \int_0^{|t'|/2} s^{-1+\theta} \, ds = \varkappa \theta^{-1} |t'|^{\theta} |x''|^{-\theta}.$$

and the result is obtained by combining the latter inequality with that for $\theta = k$.

We next extend formula (9.16)

(9.19)

$$\int_{R^n} |\Delta_{t'}^k G_\alpha(x)| \, |x''|^{-\delta} \, dx \leqq c |t'|^{\alpha - \delta} \quad \text{for} \quad \delta < n'', \; 0 < \alpha - \delta < k,$$

$$c = \varkappa \max[(k + \delta - \alpha)^{-1}(k + n'' - \alpha)^{-2}, \; \alpha |n'' - \alpha|^{-1}(\alpha - \delta)^{-1}, \; |n'' - \alpha|^{-1}(n'' - \delta)^{-1}] \quad \textit{for} \quad \alpha \neq n''$$

$$c = \varkappa (n'' - \delta)^{-2} \quad \textit{for} \quad \alpha = n''.$$

The proof is completely similar to the one of (9.16) using (9.18) instead of (9.15).

Remark. — The constants in (9.18) and (9.19) are not the best possible; they become infinite when $\alpha \to n''$ for fixed $\theta > 0$ in (9.18) or fixed $\delta < n''$ in (9.19) which should not happen in view of the evaluation for $\alpha = n''$. In the present work we shall not need better evaluations. It would not be difficult, however, to improve them by making more thorough use of the exact formula (9.10).

Our next two formulas concern differences with respect to two different increments t and t_1.

(9.20)

$$\begin{cases} \text{For} \quad 0 < \beta < k, \quad 0 < \beta_1 < k_1, \quad \beta + \beta_1 \leqq \alpha - |j|, \\ \int_{R^n} |\Delta_t^k \Delta_{t_1}^{k_1} D_j G_\alpha(x)| \, dx \leqq \varkappa \left(1 + \frac{2|j|}{\beta + \beta_1}\right) (k - \beta)^{-1}(k_1 - \beta_1)^{-1} |t|^{\beta} |t_1|^{\beta_1}. \end{cases}$$

Decompose $j = i \cup i'$, hence $|j| = |i| + |i'|$. Write then

$$\int_{R^n} |\Delta_t^k \Delta_{t_1}^{k_1} D_j G_\alpha(x)|\, dx \leq \int_{R^n}\int_{R^n}\int_{R^n} |\Delta_t^k D_i G_{|i|+\beta}(x-y)| \, |\Delta_{t_1}^{k_1} D_{i'} G_{|i'|+\beta_1}(y-z)| G_{\alpha-|j|-\beta-\beta_1}(z)\, dx\, dy\, dz.$$

If $\alpha = |j| + \beta + \beta_1$ we have only a double integral. Apply then Young's inequality and (9.17) to obtain (9.20), at first with a constant depending on $|i|$ and $|i'|$. Making the two extremal choices $|i| = 0$ and $|i'| = 0$ and combining the resulting evaluations, one obtains the desired constant.

For $n' \leq n$, $0 < \beta < k$, $0 < \beta_1 < k_1$, $\beta + \beta_1 \leq \alpha - |j|$,

$$(9.21) \qquad \int_{R^{n'}}\int_{R^n} |t_1'|^{-n'-\beta_1} |\Delta_t^k \Delta_{t_1}^k D_j G_\alpha(x)|\, dx\, dt_1' \leq c|t|^\beta,$$

with $c = \varkappa[\min(\beta, \beta_1, k-\beta, k_1-\beta_1)]^{-1} (k-\beta)^{-1}(k_1-\beta_1)^{-1}\left(1 + \frac{2|j|}{\beta+\beta_1}\right)$.

In the proof we divide the integration relative to t_1' into $|t_1'| < |t|$ and $|t_1'| > |t|$. For $|t_1'| < |t|$ we apply (9.20) with β and β_1 replaced by $\beta - \varepsilon$ and $\beta_1 + \varepsilon$ respectively, where $\varepsilon = 1/2 \min\ (\beta,\ \beta_1,\ (k-\beta),\ (k_1-\beta_1))$. For $|t_1'| > |t|$ we apply again (9.20) but with β and β_1 replaced $\beta + \varepsilon$ and $\beta_1 - \varepsilon$ respectively.

We finish this section with the following inequality

$$(9.22) \qquad \int_{R^{n'}}\int_{R^n} \frac{|\Delta_{t'}^k D_j G_\alpha(x)|}{|t'|^{\gamma+n''}}\, dx\, dt' \leq c$$

for $n' \leq n$, $\gamma > 0$, *and* $\min[\alpha - |j| - \gamma,\ k - \gamma] \equiv \tau > 0$, $c = \varkappa[\tau\gamma(k-\gamma)]^{-1}$.

Integration over t' is divided into $|t'| < 1$ and $|t'| > 1$. In the first part we write

$$|\Delta_{t'}^k D_j G_\alpha(x)| \leq \int |\Delta_{t'}^k D_j G_{\alpha'}(x-z)|\ G_{\alpha-\alpha'}(z)\, dz$$

with $\alpha' = |j| + \gamma + \tau/2$. Integrating over x (where we apply (9.17)), then over z and finally over $|t'| < 1$ we obtain an evaluation $\leq \varkappa(\gamma + \tau/2)^{-1}(k-\gamma)^{-1}\tau^{-1}$. In the second part we write $\int |\Delta_{t'}^k D_j G_\alpha(x)|\, dx \leq \varkappa \int |D_j G_\alpha(x)|\, dx$ which by (9.3) gives, after integration over $|t'| > 1$,

$$\varkappa\left(1 + \frac{\alpha}{\alpha - |j|}\right)\gamma^{-1} < \varkappa(\gamma + \tau/2)^{-1}\,\gamma^{-1}.$$

§ 10. Special integral transformations.

In this section we will describe certain regularity properties of integral transformations occuring in connection with the representation formulas of § 5.

The properties established here (in particular in propositions 1 and 2) imply that for $u \in W_p^\alpha$ and $u \in \tilde{\mathfrak{B}}^{\alpha, p}$ with suitable α, the integrals occuring in the representations formulas of section 5 considered as integral transformations applied to u, its derivatives, difference quotients of u and its derivatives are p-absolutely regular (in some exceptional cases p-semi-regular). Consequently, for u in a suitable class W_p^α or $\tilde{\mathfrak{B}}^{\alpha,p}$ the corresponding identities are valid pointwise almost everywhere. Further consequences of this fact will be presented in sections 11 and 12.

We use the same notations as in § 9: n' is an integer, $0 < n' \leqq n$, $n'' = n - n'$, $d\mu(x, t) = \dfrac{dx\, dt}{|t|^n}$, $d\mu'(x', t') = \dfrac{dx'\, dt'}{|t'|^{n'}}$.

We recall (c.f. § 6) that the statement $K(x, y)$ is p-s.r., p-r or p-ab.r. with measure spaces $\{X, d\mu\}$, $\{Y, d\nu\}$ means that the transformation $\int_X K(x, y)u(x)d\mu$ is p-s.r., p-r., or p-ab.r., respectively.

Proposition 10.1. — *If* $\alpha - |j| - \dfrac{n''}{p} > 0$, *then the kernel* $K(y, x') = D_j^{(y)} G_\alpha(x' - y)$ *with measure spaces* $\{R^n, dy\}$, $\{R^{n'}, dx'\}$ *is p-ab.r.. For* $\alpha - |j| - n'' > 0$ *it is ab.-r..*

Proof. — For $n = n'$ the proposition follows directly from (9.3) and Young's inequality; the bound for the transformation K is in this case majorated by $\dfrac{\varkappa}{\alpha - |j|}$.

For $n' < n$ we consider first the cases when $p = 1$ and $p = \infty$. For $p = 1$, $\alpha - |j| > n''$ and condition (6.4) must be verified. By (9.1), (9.4) and (2.3 d),

$$\int_{R^{n'}} |D_j G_\alpha(x' - y)|\, dx' \leqq \varkappa[G_\alpha^{(n'')}(y'') + \frac{\alpha}{\alpha - |j|} G_{\alpha - |j|}^{(n'')}(y'')] \leqq \frac{\varkappa}{\alpha - |j| - n''}.$$

If $p = \infty$, $\alpha - |j| > 0$, (6.4′) has to be checked. By (9.3)

$$\int |D_j G_\alpha(x' - y)|\, dy \leqq \frac{\varkappa}{\alpha - |j|}.$$

Let now $1 < p < \infty$, $\dfrac{n''}{p} < \alpha - |j|$. In this case we apply Method I of § 6 with

$$\varphi(y) = [\alpha G^{n''}_{\alpha - |j|}(y'') + (\alpha - |j|) G^{(n'')}_\alpha(y'')]^{p'/p} \quad \text{and} \quad \psi(x') = 1.$$

By (9.1) and (9.4) we get

$$\int_{R^n} |D_j G_\alpha(x' - y)|\, dx' \leqq \frac{\varkappa}{\alpha - |j|} [\alpha G^{(n'')}_{\alpha - |j|}(y'') + (\alpha - |j|) G^{(n'')}_\alpha(y'')]$$

$$= \frac{\varkappa}{\alpha - |j|} \varphi(y)^{p/p'}.$$

On the other hand, using again (9.1) and (9.4) we get

$$\int_{R^n} |D_j G_\alpha(x' - y)| [\alpha G^{(n'')}_{\alpha - |j|}(y'') + (\alpha - |j|) G^{(n'')}_\alpha(y'')]^{p'/p}\, dy'\, dy''$$

$$\leqq \frac{\varkappa}{\alpha - |j|} \int_{R^{n''}} [\alpha G^{(n'')}_{\alpha - |j|}(y'') + (\alpha - |j|) G\alpha^{(n'')}_\alpha(y'')]^{p'}\, dy''.$$

In view of (2.10) this is

$$\leqq \frac{\varkappa}{(\alpha - |j|)(\alpha - |j| - n''/p)^{p'+1}} \psi(x')^{p'/p},$$

and the proposition follows from Theorem 6.3 with the p-bound of the transformation majorated by

(10.1) $$M_p \leqq \varkappa(\alpha - |j|)^{-1} \left(\alpha - |j| - \frac{n''}{p}\right)^{-1}.$$

Proposition 10.2. — *Let k be an integer, $k > \gamma > 0$ and let $\alpha - |j| - \dfrac{n''}{p} \geqq \gamma$ then the kernel*

$$K(y, x', t') = \frac{\Delta^k_{t' ; x'} D^{(y)}_j G_\alpha(x' - y)}{|t'|^\gamma}$$

with measure spaces $\{R^n;\, dy\}$, $\{R^{n'} \times R^{n'},\, d\mu'(x', t')\}$ has the following properties

i) *If $\alpha - |j| - \dfrac{n''}{p} > \gamma$ then K is p-ab.r. for $1 \leqq p \leqq \infty$.*

ii) *If* $n'' > 0$, $|j| = 0$, $\alpha - \frac{n''}{p} = \gamma$ *then* K *is p.-ab.r. for* $1 < p \leqq \infty$.

iii) *If* $n'' = 0$, *and* $\alpha - |j| = \gamma$ *then* K *is p-s.r. for* $2 \leqq p \leqq \infty$ *and its adjoint kernel is p-s.r., for* $1 \leqq p \leqq 2$.

Proof. — i) We write, using the composition property of G_α,

$$|\Delta^k_{t';x'} D^{(y)}_j G_\alpha(x'-y)| = \left|\int_{R^n} \Delta^k_{t';z} D^{(z)}_j G_{|j|+\gamma+\varepsilon}(z-y) G_{\alpha-|j|-\gamma-\varepsilon}(x'-z)\, dz\right| = \left|\int_{R^n} \Delta^k_{t';y} D^{(y)}_j G_{|j|+\gamma+\varepsilon}(z-y) G_{\alpha-|j|-\gamma-\varepsilon}(x'-z)\, dz\right|,$$

with $\varepsilon = \frac{1}{2}\min(k-\gamma,\ \alpha - |j| - \gamma - n''/p) > 0$. We apply now Method II of § 6 with inner factor

$$\lambda(z) = [G^{(n'')}_{\alpha-|j|-\gamma-\varepsilon}(z'')]^{-1/p}.$$

By (9.4) and (9.22) we have

$$A = \int_{R^{n'}}\int_{R^n}\int_{R^{n'}} \frac{|\Delta^k_{t';z} D^{(z)}_j G_{|j|+\gamma+\varepsilon}(z-y)|}{|t'|^{\gamma+n'}} \lambda(z)^p G_{\alpha-|j|-\gamma-\varepsilon}(x'-z)\, dx'\, dz\, dt' = \int_{R^{n'}}\int_{R^n} \frac{|\Delta^k_{t';z} D^{(z)}_j G_{|j|+\gamma+\varepsilon}(z)|}{|t'|^{\gamma+n'}}\, dz\, dt' \leqq \varkappa(\varepsilon\gamma(k-\gamma))^{-1}.$$

By (9.17), (9.4), and (2.10) we get

$$\begin{aligned} B &= \int_{R^n}\int_{R^n} \frac{|\Delta^k_{t';y} D^{(y)}_j G_{|j|+\gamma+\varepsilon}(z-y)|}{|t'|^{\gamma}} \lambda(z)^{-p'} G_{\alpha-|j|-\gamma-\varepsilon}(x'-z)\, dy\, dz \\ &\leqq \int_{R^n}\int_{R^n} \frac{|\Delta^k_{t';y} D^{(y)}_j G_{|j|+\gamma}(z-y)|}{|t'|^{\gamma}} \lambda(z)^{-p'} G_{\alpha-|j|-\gamma-\varepsilon}(x'-z)\, dy\, dz \\ &< \varkappa(|j|+\gamma)\gamma^{-1}(k-\gamma)^{-1} \int_{R^{n''}}\int_{R^{n'}} [G^{(n'')}_{\alpha-|j|-\gamma-\varepsilon}(z'')^{p'/p} G_{\alpha-|j|-\gamma-\varepsilon}(z)\, dz'\, dz'' \\ &< \varkappa(|j|+\gamma)\gamma^{-1}(k-\gamma)^{-1} \int_{R^{n''}} [G^{(n'')}_{\alpha-|j|-\gamma-\varepsilon}(z'')]^{p'}\, dz'' \\ &\leqq \varkappa(|j|+\gamma)[\gamma(k-\gamma)(\alpha-|j|-\gamma-n''/p)^{p'+1}]^{-1} \end{aligned}$$

It follows by Theorem 6.3 that the bound M_p of our transformation can be evaluated in the present case by

(10.2 i)
$$M_p \leqq \varkappa(|j|+\gamma)^{1/p'}[\gamma\varepsilon^{1/p}(k-\gamma)(\alpha-|j|-\gamma-n''/p)^{1+1/p'}]^{-1},$$

where

$$\varepsilon = \frac{1}{2}\min[(k-\gamma), (\alpha-|j|-\gamma-n''/p)].$$

ii) In this case we shall apply Method I of § 6 with the factors $\varphi(y) = |y''|^{-n''/p}$ and $\psi(x', t') = 1$.

We have, by (9.19), with $\alpha = \gamma + n''/p$, $\delta = n''/p$,

$$\int_{R^n} |K(y, x', t')| \varphi(y)\, dy = \int_{R^n} \frac{|\Delta^k_{t';x'} G_\alpha(x' - y)|\, |y''|^{-n''/p}}{|t'|^{\alpha - n''/p}}\, dy \leq c.$$

On the other hand, by (9.18) we have, for $|t'| \leq |y''|$

$$\int_{R^{n'}} |\Delta^k_{t'} G_\alpha(x' - y)|\, dx' \leq c|y''|^{\alpha - n'' - k} |t'|^k$$

and for $|t'| \geq |y''|$,

$$\int_{R^{n'}} |\Delta^k_{t'} G_\alpha(x' - y)|\, dx' \leq \begin{cases} c|y''|^{\alpha - n''} & \text{if } \alpha < n'' \\ c|y''|^{\frac{-n''}{2p}} |t'|^{\frac{r''}{2p'}} & \text{if } \alpha = n'' \\ c|t'|^{\alpha - n''} & \text{if } \alpha > n''. \end{cases}$$

Therefore

$$\int_{R^{n'}} \int_{R^{n'}} \frac{|\Delta^k_{t';x'} G_\alpha(x' - y)|}{|t'|^{\alpha - \frac{n''}{p} + n'}}\, dx'\, dt'$$
$$= \left(\int_{|t'| \leq |y''|} + \int_{|t'| \geq |y''|} \right) \int_{R^{n'}} \frac{|\Delta^k_{t';x'} G_\alpha(x' - y)|}{|t'|^{\alpha - \frac{n''}{p} + n'}}\, dx'\, dt'$$
$$\leq c|y''|^{-n''/p'} = c\varphi(y)^{p/p'},$$

which completes the proof of ii). An evaluation of the bound M_p can be obtained from the constants in (9.18) and (9.19).

iii) With $\alpha - |j| = \gamma$, $n'' = 0$, $x' = x$, $t' = t$, we get, using (9.17)

$$\int_{R^n} \frac{|\Delta^k_{t;x} D^{(\gamma)}_j G_\alpha(x - y)|}{|t|^{\alpha - |j|}}\, dy \leq \varkappa\alpha(\alpha - |j|)^{-1}(k + |j| - \alpha)^{-1},$$

and hence K is ∞-ab.r. and the adjoint of K is 1-ab.r.. We shall prove now the 2-semi-regularity of K and its adjoint. By Theorem 6.4 it is sufficient to verify that

$$(*) \qquad \left\| \int_{R^n} K(y, x, t) u(y)\, dy \right\|_{L^2(R^n \times R^n, d\mu)} \leq c\|u\|_{L^2(R^n)},$$

for all simple functions u on R^n with some constant c independent of u and that

$$(**) \quad \left\| \int_{R^n} \int_{R^n} K(y, x, t) w(x, t) \, d\mu(x, t) \right\|_{L^2(R^n)} \leqq c \|w\|_{L^2(R^n \times R^n, d\mu)}$$

for all simple functions on $\{R^n \times R^n, d\mu\}$ with a constant c independent of u. To prove $(*)$ observe that for any simple function u,

$$w(x, t) = \int_{R^n} \frac{\Delta_{t,x}^k D_j^\gamma G_\alpha(x-y)}{|t|^{\alpha-|j|}} u(y) \, dy = \mathfrak{F}^{-1}(v_t(\xi)),$$

where

$$v_t(\xi) = \frac{(e^{-i(\xi,t)} - 1)^k}{(1+|\xi|^2)^{\alpha/2}|t|^{\alpha-|j|}} (-i\xi)^j \hat{u}(\xi).$$

Hence, using Parseval equality and (5.18) we get

$$\|w\|^2_{L^2(R^n \times R^n, d\mu)} = \int_{R^n}\int_{R^n} |v_t(\xi)|^2 d\mu(\xi, t)$$
$$\leqq \frac{(-1)^{k-1}}{2} C(n, \alpha - |j|) \Delta^{2k}_{1, -k;s} |s|^{2(\alpha-|j|)} \|u\|^2_{L^2(R^n)}.$$

Similarly, if w is a simple function on $\{R^n \times R^n; d\mu\}$ and

$$u(y) = \iint \frac{\Delta_{t;x}^k D_j^{(\gamma)} G_\alpha(x-y)}{|t|^{\alpha-|j|}} w(x, t) \, d\mu(x, t)$$

then

$$\hat{u}(\xi) = \frac{1}{(1+|\xi|^2)^{\alpha/2}} (-i\xi)^j \int_{R^n} \frac{(e^{-i(t,\xi)} - 1)^k}{|t|^{n+\alpha-|j|}} \hat{w}(\xi, t) \, dt$$

where $\hat{w}(\xi, t)$ is the Fourier transform of $w(x, t)$ with respect to x. Using Schwartz' inequality, Parseval equality and (5.18) we get $(**)$ with the same constant as in $(*)$.

iii) follows now by interpolation (see Theorem 6.5). For $2 \leqq p \leqq \infty$, the p-bound M_p of the transformation is equal to the p'-bound $M'_{p'}$ of the adjoint transformation and they are both evaluated by

$$(10.2 \text{ iii}) \quad M_p = M'_{p'} \leqq \varkappa \alpha^{1-2/p} (\alpha - |j|)^{-1} (k + |j| - \alpha)^{-1}.$$

PROPOSITION 10.3. — *Let* $k > \gamma > 0$, *and* $\alpha - |j| - \frac{n''}{p'} > \gamma$. *Then the kernel* $K(x', y, t) = \frac{\Delta_t^k D_j^{(\gamma)} G_\alpha(x'-y)}{|t|^\gamma}$, *with measure spaces* $\{R^{n'}, dx'\}$ *and* $\{R^n \times R^n, d\mu(y, t)\}$ *is p-ab.r.*, $1 \leqq p \leqq \infty$.

The proof is completely similar to the one in Proposition 10.2 i). We choose $\varepsilon = \frac{1}{2}\min[(k-\gamma), (\alpha - |j| - \gamma - n''/p')]$ and apply the second method with the inner factor

$$\lambda(z) = [G^{(n'')}_{\alpha-|j|-\gamma-\varepsilon}(z'')]^{1/p'}.$$

The p-bound of the present transformation is equal to the p'-bound in (10.2 i).

Proposition 10.4. — Let

$$k > \gamma > 0,\ k' > \gamma' > 0,\ \alpha - |j| - \frac{n''}{p} \geqq \gamma + \gamma',$$

$1 \leqq p \leqq \infty$. *The kernel*

$$K(y, t, x', t'_1) = \frac{\Delta^{k'}_{t'_1;x'}\Delta^{k}_{t;y}D^{(y)}_j G_\alpha(x'-y)}{|t'_1|^{\gamma'}|t|^{\gamma}}$$

with measure spaces

$$\{R^n \times R^n, d\mu(y, t)\}, \qquad \{R^{n'} \times R^{n'}, d\mu'(x', t'_1)\}$$

is p-ab.r.

Proof. — Consider first the case when $n'' = 0$. Then by (9.21) K satisfies conditions (6.4) and (6.4') with constants $A = B$. Hence K is 1-ab. r. and ∞-ab.r. and by interpolation, (Theorem 6.5), it is ab.r. with p-bound $= A = B$ given by

(10.4 *a*)

$$M_p = \varkappa[(k-\gamma)(k'-\gamma')\min(\gamma, \gamma', k-\gamma, k'-\gamma')]^{-1}\left(1 + \frac{|j|}{\gamma+\gamma'}\right).$$

Consider next $n' < n$ and $1 < p < \infty$. We use now the general criterion of Theorem 6.3 with kernels

$$K_1(y, t, x', t'_1) = |t|^{-\gamma}|t'_1|^{-\gamma'}\int |\Delta^{k'}_{t'_1;x'}G_{n''/p+\gamma'\pm\varepsilon}(x'-z)|\ |z''|^{n''/p'}|\Delta^{k}_{t;y}D^{(y)}_j G_{\beta\mp\varepsilon}(z-y)|\,dz,$$

$$K_2(y, t, x', t'_1) = |t|^{-\gamma}|t'_1|^{-\gamma'}\int |\Delta^{k'}_{t'_1;x'}G_{n''/p+\gamma'\pm\varepsilon}(x'-z)|\ |z''|^{-n''/p}|\Delta^{k}_{t;y}D^{(y)}_j G_{\beta\mp\varepsilon}(z-y)|\,dz.$$

We have put here $\beta = \alpha - n''/p - \gamma'$; $\varepsilon = \frac{1}{4}\varepsilon_0$ or $= \frac{3}{4}\varepsilon_0$ depending on whether

$$|\gamma' - n''/p'| \geqq \varepsilon_0/2 \qquad \text{or} \qquad |\gamma' - n''/p'| < \varepsilon_0/2$$

$\left(\text{so that } |\gamma' - n''/p' \pm \varepsilon| \geqq \frac{1}{4}\varepsilon_0\right)$ with

$$\varepsilon_0 = \min(\gamma, \gamma', k - \gamma, k' - \gamma');$$

the upper or lower sign accompanying ε is chosen depending on whether $|t'_1| \leqq |t|$ or $|t'_1| > |t|$ ([13]),

Condition (6.5) is checked immediately. The first inequality in (6.6) is obtained as follows:

$$\int_{\mathrm{R}^{n'}} \int_{\mathrm{R}^{n'}} \mathrm{K}_1(y, t, x', t'_1)\, |t'_1|^{-n'}\, dx'\, dt'.$$
$$= \int_{\mathrm{R}^{n'}} \int_{\mathrm{R}^{n}} \int_{\mathrm{R}^{n'}} |t|^{-\gamma} |t'_1|^{-\gamma' - n'} |\Delta^{k'}_{t'_1;\, x'} \mathrm{G}_{n''/p + \gamma' \pm \varepsilon}(x' - z)|\, |z''|^{n''/p'}$$
$$\left| \int_{\mathrm{R}^n} \Delta^k_{t;\, z_1} \mathrm{D}^{(z_1)}_j \mathrm{G}_{|j| + \gamma \mp \varepsilon}(z - z_1) \mathrm{G}_{\beta - |j| - \gamma}(z_1 - y)\, dz_1 \right| dx'\, dz\, dt'_1 \text{ ([14])}.$$

We integrate first with respect to x' applying (9.18) with $\theta = \gamma' \pm \varepsilon$, and then integrate with respect to z, applying (9.17), and then with respect to z_1. We end with integrals with respect to t'_1 of the form

$$c \int_{|t'_1| \leqslant |t|} |t|^{-\varepsilon} |t'_1|^{\varepsilon - n'}\, dt'_1 + c \int_{|t'_1| > |t|} |t|^{\varepsilon} |t'_1|^{-\varepsilon - n'}\, dt'_1 \leqq \varkappa \varepsilon_0^{-4} = \mathrm{A}.$$

We treat similarly the second inequality in (6.6) where in the integral

$$\int_{\mathrm{R}^n} \int_{\mathrm{R}^n} \mathrm{K}_2(y, t, x', t'_1)\, |t|^{-n}\, dy\, dt = \int_{\mathrm{R}^n} \int_{\mathrm{R}^n} \int_{\mathrm{R}^n} \int_{\mathrm{R}^n} \dots dy\, dz_1\, dz\, dt$$

we apply (9.17) for integration with respect to z_1 and (9.19) when integrating over z, and end again with integrals over $|t| \geqq |t'_1|$ and $|t| < |t'_1|$ similar to those above. For the constant B we get the evaluation $\varkappa \varepsilon_0^{-4} (n''/p')^{-1}$. For the bound M_p we obtain thus

(10.4 *b*)
For $n' < n$, $\mathrm{M}_p \leqq \mathrm{A}^{1/p} \mathrm{B}^{1/p'} \leqq \varkappa[\min(\gamma, \gamma', k - \gamma, k' - \gamma')]^{-4}$.

(13) The proof could also be obtained by applying the second method of § 6 separately to the two components K′ and K″ of our kernel $\mathrm{K} = \mathrm{K}' + \mathrm{K}''$ where $\mathrm{K}' = \mathrm{K}$ for $|t'_1| \leqslant |t|$ and $\mathrm{K}' = 0$ for $|t'_1| > |t|$.

(14) If $\beta - |j| - \gamma = 0$ the last integral $\int \dots dz_1$ is replaced by $\Delta^k_{t;\, y} \mathrm{D}^{(y)}_j \mathrm{G}_{|j| + \gamma \mp \varepsilon}(z - y)$.

This evaluation is at first obtained for $1 < p < \infty$. However, since it is independent of p it is also valid for $p = 1$ or $p = \infty$ (one could obtain similar evaluations more directly by using (6.4) or (6.4')).

Proposition 10.5. — *Let* $t \in R^n$ *be fixed,* $0 < \beta < k$, $0 < \gamma < k'$, $\alpha \geq \beta + \gamma$.

i) *The kernel* $|t|^{-\beta}\Delta^k_{t;z}G_\alpha(z - x)$ *is ab.-r. for measure spaces* $\{R^n; dx\}$ *and* $\{R^n; dz\}$ *with bounds independent of* t.

ii) *The kernel* $|t|^{-\beta}|t_1|^{-\gamma}\Delta^{k'}_{t_1;z}\Delta^k_{t;z}G_\alpha(z - x)$ *is ab.-r. for measure-spaces* $\{R^n; dx\}$ *and* $\{R^n \times R^n; d\mu(z, t_1)\}$ *with bounds independent of* t.

Proof. — We show that the kernels are 1-ab.-r. and ∞-ab.-r. by finding evaluations A and B for the corresponding integrals (6.4) and (6.4'). In case i) we apply (9.17) with $|j| = 0$ by writing $K(z, x) = \int |t|^{-\beta}\Delta^k_{t;z}G_\beta(z - y)\, G_{\alpha-\beta}(y - x)\,dy$ to obtain A and $K(z, x) = \int G_{\alpha-\beta}(z - y)\, |t|^{-\beta}\Delta^k_{-t;x}G_\beta(y - x)\,dy$ to obtain B. The p-bound so obtained is

(10.5 i)
$$M_p \leq \varkappa(k - \beta)^{-1} \quad \textit{for} \quad 1 \leq p \leq \infty.$$

In case ii) we apply (9.21) to obtain A and (9.20) to obtain B. The p-bound so obtained is

(10.5 ii)
$$M_p \leq \varkappa[\min(\beta, \gamma, k - \beta, k' - \gamma)]^{-1/p}(k - \beta)^{-1}(k' - \gamma)^{-1}$$
for $1 \leq p \leq \infty$.

Remark 1. — Statements in Propositions 10.1. — 10.4 pertaining to p-ab. regularity of an integral transformation are equivalent to p'-ab. regularity of the corresponding adjoint transformation. When we refer to such a statement about the adjoint transformation we will write « *adjoint proposition* » (e.g., adjoint Prop. 10.2).

Remark 2. — In the preceding propositions we considered only the measure $d\mu(x, t)$ or $d\mu'(x', t')$. In the following sections we will often need these propositions with the measure $d\mu_\beta(x, t) = \dfrac{1}{C(n, \beta)} \dfrac{G_{2n+2\beta}(t)}{G_{2n+2\beta}(0)}\, d\mu(x, t)$ (or $d\mu'_\beta(x', t')$) replacing $d\mu(x, t)$. Whenever the statements pertain to p-ab. regularity, by virtue of Theorem 6.1, we still have p-ab. regularity with the new measure, with bound $M_p^{(\beta)} \leqq (C(n, \beta))^{-1/p'} M_p$ or $\leqq (C(n, \beta))^{-1/p} M_p$ depending on whether the measure is changed in the domain-space or the range-space. The only case when we deal with p-s.-regularity is in Prop. 10.2 iii). By checking directly the proof in this case (especially for the 2-s.-regularity) one verifies immediately that p-s.-regularity is maintained with $d\mu_\beta$ replacing $d\mu$, the evaluation of the bound being changed as above.

§ 11. Inclusions. W_p^α and $\tilde{\mathfrak{B}}^{\alpha,p}$ as spaces of potentials.

In this section we give a description of inclusions between spaces W_p^α, L_α^p, and $\tilde{\mathfrak{B}}^{\alpha,p}$. We also derive some representation formulas for functions of W_p^α and $\tilde{\mathfrak{B}}^{\alpha,p}$ which allow us to characterize those spaces as spaces of Bessel potentials of certain classes of distributions.

It will be convenient to introduce the space

$$(11.1)\quad \Lambda_\alpha^p = \underbrace{[L^p(R^n) \times L^p(R^n \times R^n, d\mu_\beta)] \times \cdots \times [L^p(R^n) \times L^p(R^n \times R^n, d\mu_\beta)]}_{\frac{n^{m+1}-1}{n-1}\ times}$$

if α is not an integer, $\alpha = m + \beta$, $m = [\alpha]$, $0 < \beta < 1$, and

$$(11.1')\quad \Lambda_m^p = \underbrace{L^p(R^n) \times \cdots \times L^p(R^n)}_{\frac{n^{m+1}-1}{n-1}\ times}$$

if $\alpha = m$ is an integer.

Elements of Λ_α^p will be denoted by $\{v_j, w_j\}$ or by $\{v_j\}$ if α is an integer, j being a multiindex, $0 \leqq |j| \leqq m$. The norm in Λ_α^p is defined by the formula

$$(11.2)\quad \|\{v_j, w_j\}\|_p^p = \sum_{l=0}^{m} \binom{m}{l} \left(\frac{2}{p}\right)^l \sum_{|j|=l} [\|v_j\|_{L^p(R^n)}^p + \|w_j\|_{L^p_{\mu_\beta}}^p].$$

Clearly, W_p^α is boundedly imbedded in Λ_α^p (with approximate norm $|\quad\bar{}|_{\alpha,p}$ isometrically imbedded), the imbedding being defined by

$$(11.3)\qquad v_j = D_j u;\ w_j(x,t) = \frac{\Delta_t D_j u(x)}{|t|^\beta}\ (u \in W_p^\alpha).$$

W_p^α can be therefore considered as a (closed) subspace of Λ_α^p.

L_α^p will denote the space of Bessel potentials of order α of functions in L^p, saturated rel. $\mathfrak{A}_0$, i.e. the space of all functions u for which there exists a function $f \in L^p(R^n)$ such that $u(x) = G_\alpha * f(x)$ almost everywhere. The standard norm of u is defined by

$$(11.4)\qquad ||u||_{\alpha,p} = ||f||_{L^p(R^n)}.$$

The space L_α^p was investigated by Calderon [6]. An equivalent definition of L_α^p as a space of distributions is that L_α^p is the space of tempered distributions u whose inverse potential of order α, $G_{-\alpha}u$, is in L^p (15).

The space L_α^p, for $p < \infty$, will be considered as an *imperfect completion* of the space C_0^∞ with norm given by

$$||u||_{\alpha,p} = ||G_{2m-\alpha} * (1 - \Delta)^m u||_{L^p},$$

where m is an integer $\geq \alpha/2$. For $p = \infty$, the imperfect completion leads to the space $L_\alpha^{\infty <}$; this is the space of all bounded functions u such that $G_{-\alpha}u$ is continuous in $R^n \cup (\infty)$ and vanishes at ∞. Obviously $L_\alpha^{\infty <} \subset L_\alpha^\infty$. For $p = 1$ we introduce also $L_\alpha^{1>}$ as the space of tempered distributions u such that $G_{-\alpha}u$ is a Borel measure of finite absolute mass; we put $||u||_{\alpha,1} = |G_{-\alpha}u|(R^n)$. Obviously again $L_\alpha^1 \subset L_\alpha^{1>} \subset L_\beta^1$ for $0 \leq \beta < \alpha$.

The perfect completions corresponding to spaces L_α^p will be introduced in § 13 and denoted by $P^{\alpha,p}$.

As concerns inclusions between spaces W_p^α and L_p^α we have the following theorem (16).

THEOREM 11.1 — i) *If α is an integer then $L_\alpha^p = W_p^\alpha$ for $1 < p < \infty$.* ii) *If α is not an integer then $L_\alpha^p \supset W_p^\alpha$ for $1 \leq p \leq 2$ and $L_\alpha^p \subset W_p^\alpha$ for $2 \leq p \leq \infty$.* iii) *If $\alpha' > \alpha > 0$, then $W_p^{\alpha'} \subset L_\alpha^p$ and $L_{\alpha'}^p \subset W_p^\alpha$.*

Proof. — i) Let $\alpha = m$ be an integer. If $u \in L_m^p$, $1 < p < \infty$, then $u = G_m * f$, $f \in L^p(R^n)$ and therefore by (5.7) and (6.13)

(15) $G_{-\alpha}u$ is given in terms of Fourier transforms by $(G_{-\alpha}u)^\wedge = (1 + |\xi|^2)^{\alpha/2}\hat{u}$.

(16) This theorem is contained in the results of Taibleson [19].

the distribution derivatives $D_j u$, $|j| \leq m$, are in $L^p(R^n)$ and there is a constant C independent of u such that

$$|u^-|_{m,p} \leq C\|f\|_{L^p} = C\|u\|_{m,p}.$$

Conversely if $u \in W_p^m$ then (5.30) gives for $f = G_{-m}u$ the expression $f = \sum_{l=0}^{m} \binom{m}{l} (-1)^l \sum_{|j|=l} D_j[G_m * D_j u]$ in sense of distributions and therefore by (5.7) and (6.13), $f \in L^p(R^n)$ and there is a constant C independent of u such that $\|f\|_{L^p} \leq C|u|_{m,p}$.

ii) Let $1 \leq p \leq 2$, $\alpha = m + \beta$, $m = [\alpha]$, $0 < \beta < 1$, and $u \in C_0^\infty$. Then $G_{-\alpha}u$ is clearly defined pointwise by formula (5.28). We write this formula in the form

$$(11.5) \quad G_{-\alpha}u(z) = \sum_{l=0}^{m} \binom{m}{l} \sum_{|j|=l} [(-1)^l D_j G_\alpha * \varphi_j(z) + \int_{R^n} \int_{R^n} \frac{\Delta_{t;\,x} D_j^{(x)} G_\alpha(z-x)}{|t|^\beta} \psi_j(x,t)\, d\mu_\beta(x,t)$$

with φ_j, ψ_j as in (11.3). $G_{-\alpha}$ can then be interpreted as the result of a transformation of an element of Λ_α^p. In view of the propositions 10.1 (for $n' = n$), the adjoint Prop. 10.2 iii) and Remark 2, § 10, there is a constant C independent of u such that $\|G_{-\alpha}u\|_{L^p} \leq C|u|_{\alpha,p}$ for $1 \leq p \leq 2$.

Let $2 \leq p \leq \infty$ and $u = G_\alpha * f$, $f \in L^p$. Then by Prop. 10.1, $D_j u \in L^p$, $|j| \leq m$, and there is a constant C independent of u such that $\|D_j u\|_{L^p} \leq C\|f\|_{L^p}$. On the other hand the expression $\psi_j = \dfrac{\Delta_t D_j u(x)}{|t|^\beta}$ is the result of the integral transformation of Prop. 10.2 ($n = n'$) applied to f (with measure $d\mu$ replaced by $d\mu_\beta$) and by Prop. 10.2 and Remark 2, § 10, there is a constant C independent of u such that $\|\psi_j\|_{L^p(R^n \times R^n,\, d\mu_\beta)} \leq C\|f\|_{L^p}$. This completes the proof of ii).

iii) Let $u \in W_p^{\alpha'}$. Since W_p^α with increasing α form a decreasing sequence of spaces we may assume without loss of generality that α' is not an integer, $\alpha' = m' + \beta'$, $m' = [\alpha']$, $0 < \beta' < 1$. Then by (5.28), $u = G_\alpha * f$ where

$$(11.6) \quad f(z) = \sum_{l=0}^{m} \binom{m}{l} \sum_{|j|=l} [(-1)^l D_j G_{2\alpha'-\alpha} * \varphi_j(z) + \int_{R^n} \int_{R^n} \frac{\Delta_{t;\,x} D_j^{(x)} G_{2\alpha'-\alpha}(z-x)}{|t|^{\beta'}} \psi_j(x,t)\, d\mu_{\beta'}(x,t)],$$

φ_j and ψ_j as in (11.3) (with β' instead of β). By virtue of Propositions 10.1 (with $n' = n$), 10.2 i) (adjoint, with $n' = n$) and Remark 2, § 10, formula (11.6) is valid pointwise almost everywhere and $f \in L^p$.

On the other hand, if $u \in L^p_{\alpha'}$, $u = G_{\alpha'} * f$, $f \in L^p$, then by Proposition 10.1, $D_j u \in L^p$, $|j| < \alpha'$, and iii) is proved for α integer If α is not an integer, $\alpha = m + \beta$, then the expression $\dfrac{\Delta_t D_j u(x)}{|t|^\beta}$ for $|j| \leqq m$ belongs to $L^p(R^n \times R^n, d\mu_\beta)$ by Proposition 10.2 *i*) (with $n' = n$) and Remark 2, § 10, with norm bounded by $C\|f\|_{L^p}$ with C independent of f.

Remark. — It can be proved by examples that the inclusions in ii) are proper for $p \neq 2$. It is well known that W^α_2 and L^2_α coincide for every $\alpha > 0$ (c.f. [2]).

We now proceed to prove the following theorem.

THEOREM 11.2. — *If $\alpha > \gamma$ and both α and $\alpha - \gamma$ are not integers, then $W^\alpha_p = G_\gamma W^{\alpha-\gamma}_p$, $1 \leqq p \leqq \infty$. More explictly, the space, W^α_p consists of all functions u of the form $u = G_\gamma * \varphi$, $\varphi \in W^{\alpha-\gamma}_p$, and there are constants C_1, $C_2 > 0$, independent of u such that*

$$C_1|\varphi|_{\alpha-\gamma,p} \leqq |u|_{\alpha,p} \leqslant C_2|\varphi|_{\alpha-\gamma,p}. \tag{11.7}$$

Proof. — Let $u \in W^\alpha_p$. By propositions 10.2) and the last remark of § 10, the inversion formula (5.28) is valid pointwise almost everywhere if $\gamma < \alpha$ and α is not an integer. Let $\alpha = m + \beta$, $m = [\alpha]$, $0 < \beta < 1$, $\alpha - \gamma = m' + \beta'$, $m' = [\alpha - \gamma]$, $0 < \beta' < 1$. Then for $|j'| \leqq m'$,

$$\begin{aligned} &D_{j'} G_{-\gamma} u(z) \\ &\quad = (-1)^{|j'|} \sum_{l=0}^{m} \binom{m}{l} \sum_{|j|=l} \Big[\int_{R^n} D^{(x)}_{j \cup j'} G_{2\alpha-\gamma}(z - x) \varphi_j(x) dx \\ &\quad + \int_{R^n}\int_{R^n} \frac{\Delta_{t;x} D^{(x)}_{j\cup j'} G_{2\alpha-\gamma}(z-x)}{|t|^\beta} \psi_j(x, t)\, d\mu_\beta(x, t) \Big], \end{aligned} \tag{11.8}$$

$$\begin{aligned} &\frac{\Delta_{t_1} D_{j'} G_{-\gamma} u(z)}{|t_1|^{\beta'}} \\ &= (-1)^{|j'|} \sum_{l=0}^{m} \binom{m}{l} \sum_{|j|=l} \Big[\int_{R^n} \frac{\Delta_{t_1;z} D^{(x)}_{j\cup j'} G_{2\alpha-\gamma}(z-x)}{|t_1|^{\beta'}} \varphi_j(x)\, dx \\ &+ \int_{R^n}\int_{R^n} \frac{\Delta_{t_1;z}\Delta_{t;x} D^{(x)}_{j\cup j'} G_{2\alpha-\gamma}(z-x)}{|t|^\beta |t_1|^{\beta'}} \psi_j(x, t)\, d\mu_\beta(x, t) \Big], \end{aligned} \tag{11.9}$$

where v_j, w_j have a meaning as in formula (11.3)

Noticing that $d\mu_\beta(x, t) \leqq \frac{1}{C(n, \beta)} d\mu(x, t)$ and recalling that $\frac{1}{C(n, \beta)} \sim \beta(1-\beta)$ for $0 < \beta < 1$, using Propositions 10.1, 10.2 i) adjoint 10.2 ii), 10.4 and Remark 2, § 10, we get $G_{-\gamma}u \in W_p^{\alpha-\gamma}$ and $|G_{-\gamma}u|_{\alpha-\gamma,p} \leqq C|u|_{\alpha,p}$ with

$$C \leqq \varkappa\{\min(\beta, 1-\beta, \beta', 1-\beta')[\beta(1-\beta)]^{1/p}[\beta'(1-\beta')]^{1/p'}\}^{-1}.$$

Conversely, if $v \in W_p^{\alpha-\gamma}$ then $G_\gamma v$ is given pointwise almost everywhere by the formula

$$G_\gamma v(z) = \sum_{l=0}^{m'} \binom{m}{l} \sum_{|j|=l} \Big[\int_{R^n} D_j^{(x)} G_{2\alpha-\gamma}(z-x) D_j v(x)\, dx + \int_{R^n}\int_{R^n} \frac{\Delta_{t:x} D_j^{(x)} G_{2\alpha-\gamma}(z-x)}{|t|^{\beta'}} \frac{\Delta_t D_j v(x)}{|t|^{\beta'}} d\mu_{\beta'}(x, t)\Big].$$

Using the same reasoning as above we conclude that $G_\gamma v \in W_p^\alpha$ and $|G_\gamma v|_{\alpha,p} \leqq C|v|_{\alpha-\gamma,p}$ with

$$C \leqq \varkappa\{\min(\beta, 1-\beta, \beta', 1-\beta')[\beta(1-\beta)]^{1/p'}[\beta'(1-\beta')]^{1/p}\}^{-1}.$$

This completes the proof.

In particular it follows from Theorem 11.2 that

$$W_p^{m+\beta} = G_m W_p^\beta \quad \text{for} \quad 0 < \beta < 1$$

and m integer, and there is a constant $C > 0$ such that

$$C^{-1}|v|_{\beta,p} \leqq |G_m v|_{m+\beta,p} \leqq C|v|_{\beta,p}.$$

It follows from the estimates indicated in the proof that the constant C increases unboundedly as $\beta \to 0$ or $\beta \to 1$. For $1 < p < \infty$, this result can be improved by using singular integrals. This is done by means of the following proposition.

Proposition A. — *If* $K(x-y)$ *is a kernel such that for* $f \in L^p$ *the integral* $Kf(x) = \int_{R^n} K(x-y)f(y)\, dy$ *(possibly understood as singular integral) exists pointwise almost everywhere and there is a constant* C *independent of* f *such that*

$$\|Kf\|_{L^p} \leqq C\|f\|_{L^p}, \tag{11.10}$$

then for every $v \in W_p^\beta$, $0 < \beta < 1$, $Kv \in W_p^\beta$ *and* $|Kv|_{\beta,p} \leqq C|v|_{\beta,p}$ *with the same constant* C *as in* (11.10).

The proof follows immediately if we notice that $\Delta_t K f = K \Delta_t f$ and that

$$|u|^p_{\beta,p} = ||u||^p_{L^p} + \int_{R^n} \frac{G_{2n+2\beta}(t)}{C(n,\beta)\, G_{2n+\beta}(0)\, |t|^{n+\beta p}} ||\Delta_t u||^p_{L^p}\, dt.$$

We can now state the partial improvement of Theorem 11.2 :

THEOREM 11.2'. — *There exists a constant* C *depending only on p, n, the positive integer m and an upper bound of* α *such that for* $1 < p < \infty$

$$C^{-1}|\varphi|_{\alpha,p} \leqq |G_m \varphi|_{\alpha+m,p} \leqq C|\varphi|_{\alpha,p}.$$

Proof. — Obviously it is enough to consider the case $0 \leqq \alpha \leqq 1$, $m = 1$. Put $u = G_1\varphi$, $\frac{\partial}{\partial x_k} u = \frac{\partial}{\partial x_k} G_1 * \varphi$. By (5.30) with $m = 1$, we have

$$\varphi(z) = (G_1 * u)(z) - \sum_{k=1}^{n} \left(\frac{\partial}{\partial z_k} G_1 * \frac{\partial u}{\partial x_k}\right)(z).$$

As in Theorem 11.1 i), this gives our present theorem for $\alpha = 0$ and, by Prop. A, also for $0 \leqq \alpha < 1$ with the same constant C. We use then Theorem 8.1 ii) to extend it to $\alpha = 1$.

The next theorem ([17]) is a counterpart of Theorem 11.2 for spaces $\tilde{\mathfrak{B}}^{\alpha,p}$. In its proof we will use the following obvious propositions

PROPOSITION B. — *Consider two measure-spaces* $\{X, d\mu\}$, $\{Y, d\nu\}$ *and a kernel* $K(x, y)$ *p-ab.-r. with p-bound* M_p *for* $|K(x, y)|$. *Let*

$$K'(x, y) = A(x, y)K(x, y) \quad \textit{with} \quad |A(x, y)| \leqq C = \textit{const.}$$

for all x, y. Then K' *is p-ab.-r. with p-bound* $\leqq CM_p$.

PROPOSITION C. — *Consider three measure-spaces* $\{X, d\mu\}$, $\{Y, d\nu\}$, $\{T, d\omega\}$, *and a kernel* $K(x, y, t)$ $x \in X, y \in Y, t \in T$ *measurable in the product space* $X \times Y \times T$. *Suppose that for each fixed t,* $K(x, y, t)$ *is p-ab. -r. with p-bound for* $|K(x, y, t)|$ *uniformly bounded by* M. *Then, if the total mass* $\omega(T)$ *is finite, the kernel* $\int K(x, y, t)\, d\omega(t)$ *is p-ab. -r. with p-bound* $\leqq M\omega(T)$.

([17]) This theorem is a particular case of a result of Taibleson [19].

THEOREM 11.3. — *If* $\alpha > \gamma > 0$, $1 \leqq p \leqq \infty$, *then*

$$G_{\alpha-\gamma}\tilde{\mathfrak{B}}^{\gamma,p} = \tilde{\mathfrak{B}}^{\alpha,p}.$$

More explicitly, $\tilde{\mathfrak{B}}^{\alpha,p}$ *is the space of all functions* u *of the form* $u = G_{\alpha-\gamma}\varphi$ *with* $\varphi \in \tilde{\mathfrak{B}}^{\gamma,p}$ *and there exist constants* C, $C' > 0$ *depending on* α, γ, k, k' (k, k' *are integers,* $k' > \gamma$, $k > \alpha$) *such that*

$$(11.11) \qquad C|\varphi|_{\gamma,p,k'} \leqq |G_{\alpha-\gamma}\varphi|_{\alpha,p,k} \leqq C'|\varphi|_{\gamma,p,k'}.$$

Proof. — By Lemma 4.1 we may assume without loss of generality that $k = [\alpha] + 4$ and we may choose then k' so that $k - k' \geqq \alpha - \gamma + 1$ and $k' \geqq \gamma + 1$.

If $\varphi \in \tilde{\mathfrak{B}}^{\gamma,\beta}$ then by Young's inequality we get $G_{\alpha-\gamma}\varphi \in L^p$ and $||G_{\alpha-\gamma}\varphi||_{L^p} \leqq ||\varphi||_{L^p}$. Furthermore, for every t,

$$\Delta_t^k G_{\alpha-\gamma}\varphi = \Delta_t^{k-k'} G_{\alpha-\gamma} * \Delta_t^{k'}\varphi.$$

Applying (9.17) (with $|j| = 0$) we get

$$\int_{R^n} |t|^{-\alpha+\gamma}|\Delta_t^{k-k'}G_{\alpha-\gamma}(x)|\, dx < \varkappa$$

and hence, by Young's inequality

$$\int_{R^n} |t|^{-n}|| |t|^{-\alpha}\Delta_t^k G_{\alpha-\gamma}\varphi||_{L^p}^p\, dt \leqq \int_{R^n} |t|^{-n}\varkappa^p|| |t|^{-\gamma}\Delta_t^{k'}\varphi||_{L^p}^p\, dt,$$

which completes the proof of he second inequality in (11.11) with $C' \leqq \varkappa$.

Put now $u = G_{\alpha-\gamma}\varphi$. Hence $\varphi = G_{\gamma-\alpha}u$. We use the formula (5.22) which at first we know only to be valid in sense of distributions (we replace β by α and α by $\alpha - \gamma$). By shifting a suitable number of differences from $G_{\alpha+\gamma}$ to u (or vice-versa) in the convolutions we can rewrite the formula (still in sense of distributions) as follows

$$(11.12) \quad G_{\gamma-\alpha}u(z) = \frac{1}{C_k(n, \alpha)}\Big\{\sum_{\substack{l,\, l'=0 \\ l+l' \leqslant k}}^{k} \binom{k}{l}\binom{k}{l'}(-1)^k \int_{R^n}\int_{R^n} \frac{\Delta_{t,\,(l-k)t;\,t}^{2k-l-l'} G_{2n+2\alpha}(t)}{|t|^{n+2\alpha}} \Delta_{t,z-l't;\,z}^{l+l'} G_{\alpha+\gamma}(z-x)u(x)\, dx\, dt$$
$$+ \sum_{\substack{l,\, l'=0 \\ l+l' > k}}^{k} \binom{k}{l}\binom{k}{l'} \int_{R^n}\int_{R^n} \frac{\Delta_{t,\,(l-k)t:\,t}^{2k-l-l'} G_{2n+2\alpha}(t)}{|t|^{n+2\alpha}} \Delta_{t,\,z-l't;\,z}^{l+l'-k} G_{\alpha+\gamma}(z-x)\Delta_{l;\,x}^{k} u(x)dxdt\Big\}.$$

We have here a linear combination with constant coefficients of formal integral transformations. Our aim is to show that when $|u|_{\alpha,p,k} < \infty$ each of these transforms is in $L^p(R^n; dz)$ and when we apply $|t_1|^{-\gamma}\Delta^{k'}_{t_1;z}$ to them we obtain functions in $L^p(R^n \times R^n; d\mu(z, t_1))$.

Consider first the transforms in (11.12) in the first sum when $l + l' \leqq k$. Their kernels can be written in the form

$$(11.13) \qquad \int_{R^n} K(x, z, t)\, d\omega(t)$$

with

$$\left.\begin{aligned} K(x, z, t) &= \Delta^{l+l'}_{t,z-l't;z} G_{\alpha+\gamma}(z - x) \\ d\omega(t) &= |t|^{-n-2\alpha}\Delta^{2k-l-l'}_{t,(l-k)t;t} G_{2n+2\alpha}(t)\, dt \end{aligned}\right\} \quad \text{for} \quad l + l' \leqq 1$$

$$\left.\begin{aligned} K(x, z, t) &= |t|^{-\beta}\Delta^{l+l'}_{t,z-l't;z} G_{\alpha+\gamma}(z - x) \\ d\omega(t) &= |t|^{-n-2\alpha+\beta}\Delta^{2k-l-l'}_{t,(l-k)t;t} G_{2n+2\alpha}(t)\, dt \\ \beta &= \min(l + l' - 1, \alpha) \end{aligned}\right\} \quad \text{for} \quad 2 \leqq l + l' \leqq k$$

By (2.11) and in view of the exponential decrease at ∞ of $G_{2n+2\alpha}$, $d\omega(t)$ has a finite total mass $\leq \varkappa$. The kernels $|K(x, z, t)|$ are *p-ab.- r.* for $(R^n; dx)$ and $(R^n; dz)$ by virtue of Prop. 10.1 and 10.5 i) with bounds $\leq \varkappa$ independent of t. Furthermore, the kernels $|t_1|^{-\gamma}|\Delta^{k'}_{t;z}K(x, z, t)|$ are p-ab. -r. by Prop. 10.2 i) and 10.5 ii) for $(R^n; dx)$ and $(R^n \times R^n; d\mu(z, t_1))$ with bounds independent of t. Hence, by Proposition C above, the transforms in the first sum in (11.12) have norms $|\ \ |_{\gamma,p,k'}$ bounded by $c\|u\|_{L^p}$.

Consider now the second sum in (11.12) where $l + l' \geqq k + 1$. The corresponding transforms can be written

$$(11.14) \qquad \int_{R^n}\int_{R^n} A(t)K(x, t, z)w(x, t)\, d\mu(x, t)$$

where we put

$$w(x, t) = |t|^{-\alpha}\Delta^k_{t;x}u(x - l't),$$

$$K(x, t, z) = |t|^{-\frac{\alpha(l+l'-k)}{k}}\Delta^{l+l'-k}_{t;z}G_{\alpha+\gamma}(z - x)$$

$$A(t) = |t|^{-\frac{\alpha(2k-l-l')}{k}}\Delta^{2k-l-l'}_{t,(l-k)t;t}G_{2n+2\alpha}(t)$$

We have here $|A(t)| \leq \varkappa$ (by (2.11)), $K(x, t, z)$ is p-ab. -r. for $(R^n \times R^n; d\mu(x, t)$ and (R^n, dz) (by adjoint Prop. 10.2 i) for $n' = n$ and $|j| = 0$) and $|t_1|^{-\gamma}\Delta^{k'}_{t_1;z}K(x, z, t)$ is p-ab. -r. for $(R^n \times R^n; d\mu(x, t))$ and $(R^n \times R^n; d\mu(z, t_1))$ (by Prop. 10.4

with $n' = n$ and $|j| = 0$). By Proposition B, this finishes the proof of the first inequality in (11.11). By checking on the bounds in all the propositions used in our proof we find the following evaluations for the constants C and C′ in (11.11) [18]:

$$(11.15)\quad 1/C \leqq \varkappa\gamma^{-2}, \qquad C' \leqq \varkappa \qquad \text{for} \qquad 1 \leqq p \leqq \infty.$$

THEOREM 11.4 [19]. — *If α is not an integer then $\tilde{\mathfrak{B}}^{\alpha,p} = W^{\alpha}_{p}$, $1 \leqq p \leqq \infty$. If α is an integer then $\tilde{\mathfrak{B}}^{\alpha,p} \subset W^{\alpha}_{p}$ for $1 \leqq p \leqq 2$ and $W^{\alpha}_{p} \subset \tilde{\mathfrak{B}}^{\alpha,p}$ for $2 \leqq p \leqq \infty$.*

Proof. — The first part follows directly from Theorems 11.2 and 11.3 and the remark that for $0 < \beta < 1$, $\tilde{\mathfrak{B}}^{\beta,p} = W^{\beta}_{p}$, $1 \leqq p \leqq \infty$. To prove the second part, observe that if $u \in \tilde{\mathfrak{B}}^{\alpha,p}$, α-integer, then $u = G_{\alpha-\varepsilon} f_\varepsilon$, $f_\varepsilon \in \tilde{\mathfrak{B}}^{\varepsilon,p}$, $0 < \varepsilon < 1$, and the norms $|u|_{\alpha,p,k}$ and $|f_\varepsilon|_{\varepsilon,p}$ are equivalent. By the reproducing formula (5.24) (with $\beta = \varepsilon$) and Propositions 10.1, 10.2 i) adjoint, we also have pointwise a.e..,

$$u(x) = \int_{\mathbf{R}^n} G_{\alpha+\varepsilon}(x-y) f_\varepsilon(y)\, dy + \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{\Delta_{t;\,y} G_{\alpha+\varepsilon}(x-y)}{|t|^\varepsilon} \frac{\Delta_t f_\varepsilon(y)}{|t|^\varepsilon}\, d\mu_\varepsilon(y,t),$$

Therefore derivatives $D_j u$, $|j| \leqq \alpha$ are given by the formula

$$D_j u(x) = \int_{\mathbf{R}^n} D^{(x)}_j G_{\alpha+\varepsilon}(x-y) f_\varepsilon(y) + \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{\Delta_t D^{(x)}_j G_{\alpha+\varepsilon}(x-y)}{|t|^\varepsilon} \frac{\Delta_t f_\varepsilon(y)}{|t|^\varepsilon}\, d\mu_\varepsilon(y,t).$$

The right-hand side of the last expression can be interpreted as the sum of results of two integral transformations applied to f_ε and $w_\varepsilon = \dfrac{\Delta_t f_\varepsilon(y)}{|t|^\varepsilon}$ respectively. By Propositions 10.1, 10.2 i) adjoint, and 10.2 iii), the first transformation is absolutely regular for $|j| \leqq \alpha$, the second is absolutely regular for $|j| < \alpha$ and p-s.r., $1 \leqq p \leqq 2$ if $|j| = \alpha$. Thus $\tilde{\mathfrak{B}}^{\alpha,p} \subset W^{\alpha}_{p}$, if α is an integer and $1 \leqq p \leqq 2$.

[18] On the assumption that k and k' are chosen as they were at the beginning of the proof. For other choices of k and k' the evaluations should be changed by using Lemma 4.1.

[19] Besov obtained this theorem for $1 < p < \infty$. The first part was obtained by Taibleson without restrictions.

To prove the opposite inclusion for $2 \leqq p \leqq \infty$, we remark that if $u \in W_p^\alpha$ then by (5.29) (with $m = \alpha$) we have, at first in sense of distributions $u = G_{\alpha-\varepsilon} * f_\varepsilon$ where

$$(11.16) \quad f_\varepsilon(y) = \sum_{l=0}^{\alpha} \binom{\alpha}{l} \sum_{|j|=l} \int_{R^n} D_j^{(z)} G_{\alpha+\varepsilon}(y-z) D_j u(z)\, dz.$$

Applying Prop. 10.1 we prove that this is a bona fide integral representation, that $f_\varepsilon \in L^p$ and is given by (11.16) a.e.. By Theorem 11.3 it is sufficient to prove that $f_\varepsilon \in \tilde{\mathfrak{B}}^{\varepsilon,p}$, $2 \leqq p \leqq \infty$. We know already that $f_\varepsilon \in L^p$; on the other hand $\frac{\Delta_t f_\varepsilon}{|t|^\varepsilon}$ can be written as a linear combination of terms $w_j(y, t)$ given by the formula

$$w_j(y, t) = \int_{R^n} \frac{\Delta_{t;y} D_j^{(z)} G_{\alpha+\varepsilon}(y-z)}{|t|^\varepsilon} D_j u(z)\, dz.$$

By Proposition 10.2, for $|j| < \alpha$, $w_j(y, t)$ is the result of an absolutely regular integral transformation applied to $D_j u$; for $|j| = \alpha$ and $2 \leqq p \leqq \infty$ it is the result of a p-s.r. transformation. Hence $w_j \in L^p(R^n \times R^n, d\mu)$ which completes the proof.

If for fixed $k_0 > \alpha_0 > 0$ we choose a norm $||u||_{B^{\alpha_0,p}}$ on $\tilde{\mathfrak{B}}^{\alpha_0,p}$ equivalent to $|u|_{\alpha_0,p,k_0}$ and then define

$$||u||_{B^{\alpha,p}} = ||G_{\alpha_0-\alpha} u||_{B^{\alpha_0,p}}$$

for $u \in \tilde{\mathfrak{B}}^{\alpha,p}$, $\alpha \geqq 0$, this norm, by Theorem 11.3 will be equivalent to $|u|_{\alpha,p,k}$ for $\alpha > 0$. If we restrict the choice of $||u||_{B^{\alpha_0,p}}$ by the additional requirement that for $p = 2$ it coincides with $|u|_{\alpha_0} \equiv |u|_{\alpha_0,2}$ we shall call the resulting norm, $||u||_{B^{\alpha,p}}$ a *standard norm* on $\tilde{\mathfrak{B}}^{\alpha,p}$. The simplest such choices of $||u||_{B^{\alpha_0,p}}$ seem to be the two following norms: the first, for $\alpha_0 = 1$, leads to the standard norm:

$$(11.17) \quad ||u||_{B^{\alpha,p}}^p = ||G_{1-\alpha} u||_{L^p}^p + \frac{\Gamma(1+n/2)}{4\pi^{n/2} \log 2} \int_{R^n} |t|^{-n-p} ||\Delta_t^2 G_{1-\alpha} u(x)||_{L^p}^p\, dt,$$

the second, for $\alpha_0 = 1/2$, defined by $||u||_{B^{1/2,p}} = |u|_{1/2,p}$ leads to

$$(11.17') \quad ||u||_{B^{\alpha,p}}^p = ||G_{1/2-\alpha} u||_{L^p}^p + \frac{2^{n-1}\Gamma(n+1/2)}{\pi^{1/2}} \int_{R^n} |t|^{-n-p/2} G_{2n+1}(t) ||\Delta_t G_{1/2-\alpha} u||_{L^p}^p\, dt.$$

Recapitulating, we can state

THEOREM 11.5. — *Consider* $\tilde{\mathfrak{B}}^{\alpha,p}$ *with a standard norm for* $\alpha \geqq 0$. *The potential operator* G_γ *is then an isometric isomorphism of* $\tilde{\mathfrak{B}}^{\alpha,p}$ *onto* $\tilde{\mathfrak{B}}^{\alpha+\gamma,p}$. *For* $p = 2$, $\tilde{\mathfrak{B}}^{\alpha,2} = W_2^\alpha = L_\alpha^2$, *with equality of standard norms in all these spaces.*

Remark. — For any norm $||u||_{B^{\alpha,p}}$ as defined above, and function $u(y)$ we can consider the function $\Phi(\alpha) = ||u||_{B^{\alpha,p}}$ ($= \infty$ if $u \notin \tilde{\mathfrak{B}}^{\alpha,p}$) for $\alpha \geqq 0$. Obviously $\Phi(\alpha) < \infty$ implies $\Phi(\alpha') < \infty$ for $\alpha' < \alpha$. It can be proved without much difficulty that 1° for all α, $\Phi(\alpha)$ is continuous to the left; 2° if $\Phi(\alpha) < \infty$, for $0 \leqq \alpha \leqq \alpha'$ then Φ is continuous on this interval. If we take for $||u||_{B^{\alpha,p}}$ the norm (11.17) or (11.17′), then $\Phi(\alpha)$ is non-decreasing.

Consider the inverse potential operator $G_{-\alpha}$ applied to $\tilde{\mathfrak{B}}^{\alpha,p}$. This gives a space of distributions $G_{-\alpha}(\tilde{\mathfrak{B}}^{\alpha,p})$ which, by Theorem 11.3, is independent of α. We will denote this space by $B^{0,p}$. Hence

$$(11.18) \qquad \tilde{\mathfrak{B}}^{\alpha,p} = G_\alpha(B^{0,p}) \quad \textit{for} \quad \alpha \geqq 0.$$

Since for $0 < \beta < 1$, $\tilde{\mathfrak{B}}^{\beta,p} = W_p^\beta$, we obtain by Theorem 11.1 ii) in view of the fact that $G_{-\alpha}(L_\alpha^p) = L^p$,

$$(11.19) \quad B^{0,p} \subset L^p \;\; \textit{for} \;\; 1 \leqq p \leqq 2, \qquad B^{0,p} \supset L^p \;\; \textit{for} \;\; 2 \leqq p \leqq \infty.$$

As a consequence, we have also

$$(11.20) \quad \tilde{\mathfrak{B}}^{\alpha,p} \subset L_\alpha^p \;\; \textit{for} \;\; 1 \leqq p \leqq 2, \qquad \tilde{\mathfrak{B}}^{\alpha,p} \supset L_\alpha^p \;\; \textit{for} \;\; 2 \leqq p \leqq \infty.$$

§ 12. A projection formula and conjugate spaces.

In this section we shall need some results of the theory of pairings and associated norms (c.f. [4]). Let A and B be complex Banach spaces and $\langle v, w \rangle$ be a bilinear hermitian complex valued form on $A \times B$ (i.e. linear in v, antilinear in w). The system $[A, B, \langle\ ,\ \rangle]$ is called a *pairing*. A pairing is *proper* if $\langle v_0, w \rangle = 0$ for all $w \in B$ implies $v_0 = 0$ and $\langle v, w_0 \rangle = 0$

for all $v \in A$ implies $w_0 = 0$. The norms in A and B are *admissible* with respect to the pairing $[A, B, \langle\ ,\ \rangle]$ if $\langle v, w\rangle$ is a bounded functional on A for every fixed $w \in B$ and a bounded functional on B for every fixed $v \in A$. Let $[A, B, \langle\ ,\ \rangle]$ be a proper pairing and norms in A and B be admissible.

The correspondence $v \to f(v) = \langle v, w\rangle$ is a canonical linear continuous mapping $A \to B^*$ where B^* is the anticonjugate of B, i.e. the space of antilinear continuous functionals on B. Similarly, $w \to \langle v, w\rangle$ is the canonical mapping of B into A^*. We say that in this pairing B is *canonically isomorphic* with A^* if every linear functional $\varphi \in A^*$ can be represented in the form $\varphi(v) = \langle v, w^\varphi\rangle$ with some fixed $w^\varphi \in B$ (since the pairing is proper this w^φ is clearly unique). A bounded operator $P^* : B \to B$ is called *adjoint* of a bounded operator $P : A \to A$ with respect to the pairing $[A, B, \langle\ ,\ \rangle]$ if $\langle Pv, w\rangle = \langle v, P^*w\rangle$ for all $v \in A$ and $w \in B$.

The adjoint may not exist for some operators in some pairings. In the pairing $[A, B, \langle\ ,\ \rangle]$ every bounded operator on A will possess an adjoint if and only if B is canonically isomorphic to A^*.

If A_0 is a closed subspace of a Banach space A then we say that an operator $P : A \to A_0$ is a *projection* of A onto A_0 if P is bounded, $P(A) = A_0$ and $P^2 = P$.

If a projection P of A onto A_0 has an adjoint P^* then P^* is also a projection.

THEOREM 12.1. — *Let* $[A, B, \langle\ ,\ \rangle]$ *be a proper pairing of Banach spaces,* A_0, B_0 *be closed subspaces of* A *and* B, *and* P, P^* *be adjoint projections of* A *onto* A_0 *and* B *onto* B_0 *respectively. Then*

i) *The pairing* $[A_0, B_0, \langle\ ,\ \rangle]$ *is proper.*

ii) *If* B *is canonically isomorphic with the conjugate space of* A (*in the pairing* $[A, B, \langle\ ,\ \rangle]$) *then* B_0 *is canonically isomorphic with the conjugate space of* A_0 (*in the pairing* $[A_0, B_0, \langle\ ,\ \rangle]$.

Proof. — i) Let $v_0 \in A_0$ and $\langle v_0, P^*w\rangle = 0$ for all $w \in B$. Then by definition $\langle v_0, P^*w\rangle = \langle Pv_0, w\rangle = \langle v_0, w\rangle = 0$ for all $w \in B$ and since the pairing is proper, $v_0 = 0$. The proof is similar for $w_0 \in B_0$.

ii) Let φ be any bounded linear functional on A_0. By the

Hahn-Banach theorem φ can be extended to some bounded linear functional $\tilde{\varphi}$ on A. By assumption there is an element $w^{\varphi} \in B$ such that $\tilde{\varphi}(v) = \langle v, w^{\varphi}\rangle$ for all $v \in A$. Hence for $v \in A_0$, $\varphi(v) = \langle v, w^{\varphi}\rangle = \langle Pv, w^{\varphi}\rangle = \langle v, P^* w^{\varphi}\rangle = \langle v, w_0^{\varphi}\rangle$,

$$w_0^{\varphi} = Pw^{\varphi} \in B_0.$$

By i) w_0^{φ} is unique.

We proceed now to apply Theorem 12.1 to the case when $A = \Lambda_\alpha^p, B = \Lambda_\alpha^{p'}$ (c.f. § 11). For $\{v_j, w_j\} \in \Lambda_\alpha^p$ and $\{v_j', w_j'\} \in \Lambda_\alpha^{p'}$ and for $\{v_j\} \in \Lambda_\alpha^p$ and $\{v_j'\} \in \Lambda_\alpha^{p'}$, if α is an integer, the bilinear form $\langle\ ,\ \rangle_\alpha$ is defined by the formulas

(12.1)

$$\langle\{v_j, w_j\}, \{v_j', w_j'\}\rangle_\alpha = \sum_{l=0}^{m}\binom{m}{l}\sum_{|j|=l}\Big[\int_{R^n} v_j(x)\overline{v_j'(x)}\,dx + \int_{R^n}\int_{R^n} w_j(x, t)\overline{w_j'(x, t)}d\mu_\beta(x, t)\Big]$$

for α not integer, $m = [\alpha]$, $\beta = \alpha - [\alpha]$, and

$$(12.1')\quad \langle\{v_j\}, \{v_j'\}\rangle_m = \sum_{l=0}^{m}\binom{m}{l}\sum_{|j|=l}\int_{R^n} v_j(x)\overline{v_j'(x)}\,dx$$

if α is an integer $\alpha = m$.

The pairing

$$(12.2)\qquad [\Lambda_\alpha^p, \Lambda_\alpha^{p'}, \langle\ ,\ \rangle_\alpha]$$

is clearly proper, the norms in Λ_α^p and $\Lambda_\alpha^{p'}$ are admissible and for $1 \leqq p < \infty$ $\Lambda_\alpha^{p'}$ is in this pairing canonically isomorphic to the conjugate space of Λ_α^p.

As indicated in § 11, for every p, $1 \leqq p \leqq \infty$, the space W_p^α with norm $|\ \ \overline{\ }|_{\alpha,p}$ can be isometrically imbedded in the space Λ_α^p, the imbedding $E_{\alpha,p}: W_p^\alpha \to \Lambda_\alpha^p$ being given by the formulas

$$(12.3)\quad v_j(x) = D_j u(x), \qquad w_j(x, t) = \frac{\Delta_t D_j u(x)}{|t|^\beta},$$
$$u \in W_p^\alpha, \qquad |j| \leqq m = [\alpha], \qquad \beta = \alpha - [\alpha],$$

if α is not an integer, and

$$(12.3')\quad v_j(x) = D_j u(x), \qquad u \in W_p^m, \qquad |j| \leqq m,$$

if α is an integer, $\alpha = m$.

Consider now, for $\{\varphi_j, \psi_j\} \in \Lambda^p_\alpha$ or for $\{\varphi_j\} \in \Lambda^p_m$ if $\alpha = m$ is an integer, the transformation $T_{\alpha,p}$ defined by the formula

$$(12.4) \quad T_{\alpha,p}\{\varphi_j, \psi_j\}(z) = \sum_{l=0}^{m} \binom{m}{l} \sum_{|j|=l} \left[\int_{R^n} D_j^{(x)} G_{2\alpha}(z-x)\varphi_j(x) + \int_{R^n}\int_{R^n} \frac{\Delta_{t,x:x} D_j^{(x)} G_{2\alpha}(z-x)}{|t|^\beta} \psi_j(x,t)\, d\mu_\beta(x,t) \right]$$

for α not an integer, and

$$(12.4') \quad T_{m,p}\{\varphi_j\}(z) = \sum_{l=0}^{m} \binom{m}{l} \sum_{|j|=l} \int_{R^n} D_j^{(x)} G_{2m}(z-x)\varphi_j(x)\, dx$$

for α integer, $\alpha = m$.

If $u \in W^\alpha_p$ then the reproducing formulas (5.27), (5.29) and Propositions 10.1 and 10.2 give

$$(12.5) \qquad T_{\alpha,p}E_{\alpha,p}u(x) = u(x) \textit{ almost everywhere.}$$

Using propositions 10.1, 10.2, and 10.4 we conclude that for α not an integer and $1 \leq p \leq \infty$, $T_{\alpha,p}\{\varphi_j, \psi_j\} \in W^\alpha_p$ if $\{\varphi_j, \psi_j\} \in \Lambda^p_\alpha$ and there is a constant C independent of $\{\varphi_j, \psi_j\}$ such that

$$(12.6) \qquad |T_{\alpha,p}\{\varphi_j, \psi_j\}^\neg|_{\alpha,p} \leq C\|\{\varphi_j, \psi_j\}\|_{\Lambda^p_\alpha} \qquad \textit{for} \qquad 1 \leq p \leq \infty.$$

On the other hand if α is an integer, $\alpha = m$, then from (5.7) and (6.13) it follows that $\{\varphi_j\} \in \Lambda^p_m$ implies $T_{m,p}\{\varphi_j\} \in W^m_p$ and there is a constant C independent of $\{\varphi_j\}$ such that

$$(12.6') \qquad |T_{m,p}\{\varphi_j\}^\neg|_{m,p} \leq C\|\{\varphi_j\}\|_{\Lambda^p_m} \qquad \textit{for} \qquad 1 < p < \infty.$$

We easily verify that

$$(12.7) \qquad (E_{\alpha,p}T_{\alpha,p})^* = E_{\alpha,p'}T_{\alpha,p'}$$

in the pairing $[\Lambda^p_\alpha, \Lambda^{p'}_\alpha, \langle\ ,\ \rangle_\alpha]$.

Taking into account (12.5), (12.6), (12.6') and (12.7) we get

THEOREM 12.2. — *If either α is not an integer and $1 \leq p \leq \infty$, or α is an integer and $1 < p < \infty$, then the operator $P_{\alpha,p} = E_{\alpha,p}T_{\alpha,p}$ is a projection of Λ^p_α onto the subspace $E_{\alpha,p}(W^\alpha_p)$. In the pairing* (12.2), *$P_{\alpha,p'}$ is the adjoint operator* of $P_{\alpha,p}$.

Pairing (12.2) induces a corresponding pairing of the spaces W_p^α and $W_{p'}^\alpha$,

$$(12.8)\qquad [W_p^\alpha,\ W_{p'}^\alpha,\ (\ ,\ )_\alpha]$$

with

$$(12.9)\qquad (u,\ v)_\alpha = \langle E_{\alpha,p}u,\ E_{\alpha,p'}v\rangle_\alpha$$

for $u \in W_p^\alpha$, $v \in W_{p'}^\alpha$.

Hence, using Theorems 12.1 and 12.2, we get

THEOREM 12.3. — *If either α is not an integer and $1 \leq p < \infty$ or α is an integer and $1 < p < \infty$, then in the pairing* (12.8) *the space $W_{p'}^\alpha$ is canonically isomorphic to the conjugate space of W_p^α.*

Similar results can be obtained for spaces $\tilde{\mathfrak{B}}^{\alpha,p}$. To obtain an isomorphism of $\tilde{\mathfrak{B}}^{\alpha,p'}$ with $(\tilde{\mathfrak{B}}^{\alpha,p})^*$ we have to choose a suitable pairing (the isomorphism obviously depends on the pairing). The quickest way is to use the isomorphism $G_{-\alpha+1/2}$ between $\tilde{\mathfrak{B}}^{\alpha,p}$ and $W_p^{1/2}$ (see theorems 11.4 and 11.5) and take advantage of the pairing $[W_p^{1/2},\ W_{p'}^{1/2},\ (\ ,\ )_{1/2}]$ (see (12.8) and (12.9)). We obtain thus the pairing

$$(12.10)\qquad [\tilde{\mathfrak{B}}^{\alpha,p},\ \tilde{\mathfrak{B}}^{\alpha,p'},\ (G_{-\alpha+1/2}v,\ G_{-\alpha+1/2}w)_{1/2}]$$

and the theorem

THEOREM 12.4. — *For $1 \leq p < \infty$, $\tilde{\mathfrak{B}}^{\alpha,p'}$ is canonically isomorphic to $(\tilde{\mathfrak{B}}^{\alpha,p})^*$ in pairing* (12.10).

Remark. — In analogy with our procedure in the case of spaces W_p^α it would seem more natural to use the following construction for spaces $\tilde{\mathfrak{B}}^{\alpha,p}$. Put $\mathcal{L}^p = L^p(R^n) \times L^p(R^n \times R^n, d\mu)$. For $\{v, w\} \in \mathcal{L}^p$ define $\|\{v,\ w\}\|_{\mathcal{L}^p}^p = \|v\|_{L^p}^p + \|w\|_{L^p(R^n\times R^n,\, d\mu)}^p$. For $\alpha > 0$, the space $\tilde{\mathfrak{B}}^{\alpha,p}$ with norm $|\ \ |_{\alpha,p,k}$, $k > \alpha$, is then isometrically imbedded in $\mathcal{L}^p$ by the mapping $E_{\alpha,p}^{(1)}$:

$$u \to \{u,\ |t|^{-\alpha}\Delta_t^k u\}.$$

The spaces $\mathcal{L}^p$ and $\mathcal{L}^{p'}$ are in natural pairing with scalar product $\langle\{v,\ w\},\ \{v',\ w'\}\rangle = \int v\bar{v}'\,dx + \iint w\bar{w}'\,d\mu(x,t)$. We would expect now to find suitable adjoint projections of

$\mathscr{L}^p$ onto $E^{(1)}_{\alpha,p}(\tilde{\mathscr{B}}^{\alpha,p})$ and of $\mathscr{L}^{p'}$ onto $E^{(1)}_{\alpha,p'}(\tilde{\mathscr{B}}^{\alpha,p'})$. These will be obtained if we get suitable reproducing formulas for the $\tilde{\mathscr{B}}^{\alpha,p}$ which would play in the present case the same role as the formulas (5.27) and (5.29) played in the case of spaces W^α_p when we constructed the transformations $T_{\alpha,p}$ and the projections $E_{\alpha,p}T_{\alpha,p}$. Such reproducing formulas exist; they require the use of the reproducing (or pseudo-reproducing) kernel for the space $\tilde{\mathscr{B}}^{\alpha,2}$ with the norm $| \quad |_{\alpha,2,k}$ (for W^α_p we used the reproducing kernel $G_{2\alpha}(x-y)$ of the space W^α_2 with norm $| \quad |_{\alpha,2}$, this space being essentially the space P^α). The required reproducing kernel is the inverse Fourier transform of $(2\pi)^{-n/2}(1+C|\xi|^{2\alpha})^{-1}$ with $C=\dfrac{(-1)^{k+1}}{2}C(n,\alpha)\Delta^{2k}_{1,-k;s}|s|^{2\alpha}$. The reason why we did not use this approach is that we would need many properties of this kernel which are not readily available.

CHAPTER III

PERFECT COMPLETION OF $\mathfrak{F}^{\alpha,p}$ AND $\mathfrak{B}^{\alpha,p,k}$.

§ 13. The spaces $\check{\mathrm{P}}^{\alpha,p}$ and $\mathrm{B}^{\alpha,p}$.

In this section we prove the existence of perfect functional completions of $\mathfrak{F}^{\alpha,p}$ and $\mathfrak{B}^{\alpha,p,k}$ which will be denoted $\check{\mathrm{P}}^{\alpha,p}$ and $\mathrm{B}^{\alpha,p}$ respectively. We give also a description of the exceptional sets of these classes and differentiability properties (in the ordinary sense) of functions in these classes.

We recall that a functional space $\overline{\mathfrak{F}}$ rel. $\overline{\mathfrak{A}}$ is the perfect completion of a normed functional class $\mathfrak{F}$ rel. $\mathfrak{A}$, $\overline{\mathfrak{A}} \subset \mathfrak{A}$, if $\overline{\mathfrak{F}}$ is a functional completion of $\mathfrak{F}$ rel. $\overline{\mathfrak{A}}$ and $\overline{\mathfrak{A}}$ is contained in the exceptional class of any functional completion of $\mathfrak{F}$. A perfect functional completion, if such exists, is always unique.

We remind the reader that $\mathfrak{F}^{\alpha,p}$ and $\mathfrak{B}^{\alpha,p,k}$ are formed by functions in C_0^∞ with norms $|u|_{\alpha,p}$ or $|u|_{\alpha,p,k}$ and their imperfect completions (rel. $\mathfrak{A}_0$) are W_p^α and $\tilde{\mathfrak{B}}^{\alpha,p}$ respectively. We also consider the class C_0^∞ with the norm $||u||_{\alpha,p}$ as defined in L_α^p. We define its perfect completion, which will be denoted $\mathrm{P}^{\alpha,p}$ (L_α^p is its imperfect completion rel $\mathfrak{A}_0$).

Since for α non-integer the norm in W_p^α is equivalent to the one in $\tilde{\mathfrak{B}}^{\alpha,p}$ (see Theorem 11.4) we will have

$$(13.1) \qquad \mathrm{B}^{\alpha,p} = \check{\mathrm{P}}^{\alpha,p} \quad \textit{for } \alpha \textit{ non-integer.}$$

Since for integer m and $1 < p < \infty$ the norm in W_p^m is equivalent to the one in L_m^p (see Theorem 11.1) we will have

$$(13.2) \quad \mathrm{P}^{m,p} = \check{\mathrm{P}}^{m,p} \quad \textit{for integer } m \textit{ and } 1 < p < \infty .$$

It is therefore enough to prove the existence of $\mathrm{B}^{\alpha,p}$ and $\mathrm{P}^{\alpha,p}$ in order to have $\check{\mathrm{P}}^{\alpha,p}$ except when α is an integer and

$p = 1$. We will show that $\check{P}^{1,1}$ exists, but the problem, of existence of $\check{P}^{m,1}$ for m integer > 1 remains open.

For $p = \infty$ all our incomplete spaces are proper functional spaces and, as mentioned before, have proper functional completions denoted by $P^{\alpha,\infty<}$, $\check{P}^{\alpha,\infty<}$ and $B^{\alpha,\infty<}$ contained in $P^{\alpha,\infty}$, $\check{P}^{\alpha,\infty}$ and $B^{\alpha,\infty}$ respectively.

The exceptional classes for $P^{\alpha,p}$ and $B^{\alpha,p}$ will be denoted $\mathfrak{A}^{\alpha,p}$ and $\mathfrak{B}^{\alpha,p}$ respectively. Since for $0 \leqq \alpha_1 < \alpha_2 < \alpha_3$ we have $L^p_{\alpha_1} \supset \tilde{\mathfrak{B}}^{\alpha_2,p} \supset L^p_{\alpha_3}$ (see Theorem 11.1 iii)), the corresponding norms on C_0^∞ satisfy $||u||_{\alpha_1,p} \leqq c|u|_{\alpha_2,p,k} \leqq c'||u||_{\alpha_3,p}$ with positive constants c, c'. Hence

$$(13.3) \qquad P^{\alpha_1,p} \supset B^{\alpha_2,p} \supset P^{\alpha_3,p} \qquad \textit{and} \qquad \mathfrak{A}^{\alpha_1,p} \supset \mathfrak{B}^{\alpha_2,p} \supset \mathfrak{A}^{\alpha_3,p}$$

for $0 \leqq \alpha_1 < \alpha_2 < \alpha_3$.

Since we will prove the existence of $\check{P}^{1,1}$, the exceptional class of which will be denoted $\check{\mathfrak{A}}^{1,1}$ we have also

$$(13.3') \qquad P^{\alpha_1,1} \supset \check{P}^{1,1} \supset P^{\alpha_2,1}, \qquad \mathfrak{A}^{\alpha_1,1} \supset \check{\mathfrak{A}}^{1,1} \supset \mathfrak{A}^{\alpha_2,1}$$

for $0 \leqq \alpha_1 < 1 < \alpha_2$.

The existence of $\check{P}^{m,1}$ for m an integer > 1 not being proved as yet, we will use an « almost perfect » completion of $\mathcal{F}^{m,1}$ which we will denote here (improperly !)$\check{P}^{m,1}$ and which will have an exceptional class given by

$$(13.4) \qquad \check{\mathfrak{A}}^{m,1} = \bigcap_{\alpha < m} \mathfrak{A}^{\alpha,1}.$$

This class is much smaller than $\mathfrak{A}_0$. The existence of a completion of $\mathcal{F}^{m,1}$ rel. $\check{\mathfrak{A}}^{m,1}$ is assured by the fact that there exists a completion of $\mathcal{F}^{m,1}$ rel. $\mathfrak{A}^{\alpha,1}$ for every $\alpha < m$ ([20]), hence also rel. $\check{\mathfrak{A}}^{m,1}$ (see Prop. 6, § 4 of [1]).

We can therefore write, extending (13.3'),

$$(13.5) \qquad P^{\alpha_1,1} \supset \check{P}^{m,1} \supset P^{\alpha_2,1}, \qquad \mathfrak{A}^{\alpha_1,1} \supset \check{\mathfrak{A}}^{m,1} \supset \mathfrak{A}^{\alpha_2,1}$$

for m an integer and $0 \leqq \alpha_1 < m < \alpha_2$.

([20]) This follows from the fact that there exists a completion rel. $\mathfrak{A}_0$, namely W^m_1, and that there exists a completion of C_0^∞ with the weaker norm $||u||_{\alpha,1}$ rel. $\mathfrak{A}^{\alpha,1} \subset \mathfrak{A}_0$.

To simplify some statements we will use the notation $\check{\mathfrak{A}}^{\alpha,p}$ for the exceptional class of $\check{\mathrm{P}}^{\alpha,p}$ even in cases when $\check{\mathrm{P}}^{\alpha,p}$ coincides with $\mathrm{P}^{\alpha,p}$ or $\mathrm{B}^{\alpha,p}$ respectively. (However, $\check{\mathrm{P}}^{\alpha,p}$ will be considered with its own standard norm $|\ \ |_{\alpha,p}$.)

We shall need the following facts from the theory of functional spaces and functional completion.

A normed functional class $\mathcal{F}$ rel. $\mathfrak{A}$ with the norm $\|\ \ \|$ is said to have the *global majoration property* if there is a constant $\mathrm{M} \geq 1$ such that for every $u \in \mathcal{F}$ there exists a $u' \in \mathcal{F}$ such that $\mathrm{Re}\,u'(x) \geq |u(x)|$ exc. $\mathfrak{A}$ and $\|u'\| \leq \mathrm{M}\|u\|$. If $\mathrm{M} = 1$ this property is referred to as the *strong majoration property.*

Denote by $\mathfrak{B}$ the class of all sets $\mathrm{B} \subset \mathrm{E}$($\mathrm{E}$ — set of definition of $\mathcal{F}$) for which there exists a $u \in \mathcal{F}$ such that $|u(x)| \geq 1$ on B exc. $\mathfrak{A}$; let $\mathfrak{B}_\sigma$ be the class of all countable unions of sets of $\mathfrak{B}$. For $\mathrm{B} \in \mathfrak{B}$ we define $\delta(\mathrm{B}) = \inf\|u\|$, with inf extended over all $u \in \mathcal{F}$, $|u(x)| \geq 1$ on B exc. $\mathfrak{A}$. For $\mathrm{B} \in \mathfrak{B}_\sigma$ the capacity $c_1(\mathrm{B})$ is defined by $c_1(\mathrm{B}) = \inf \Sigma\delta(\mathrm{B}_k)$, the inf being extended over all $\{\mathrm{B}_k\} \subset \mathfrak{B}$ such that $\bigcup \mathrm{B}_k \supset \mathrm{B}$.

We have the following propositions:

Proposition A. — *If the normed functional class $\mathcal{F}$ satisfies the global majoration property and has some functional completion, then it has a perfect functional completion relative to the exceptional class of all sets* B *with* $c_1(\mathrm{B}) = 0$. (c.f. [1], Th. 6.3.).

Proposition B. — *Let $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_0 \subset \mathcal{F}_1$, be two normed functional classes rel. $\mathfrak{A}$ such that:*

1° *For every $f \in \mathcal{F}_0$, the norms of f in $\mathcal{F}_0$ and $\mathcal{F}_1$ coincide.*

2° *For every $f \in \mathcal{F}_1$, there exists a sequence $\{f_n\} \subset \mathcal{F}_0$ such that* $\lim\limits_{n\to\infty}\|f_n - f\| = 0$ *and* $\lim\limits_{n\to\infty} f_n(x) = f(x)$ exc. $\mathfrak{A}$.

Then $\mathcal{F}_0$ and $\mathcal{F}_1$ have the same functional completions.

The proof of Prop. B is simple and we omit it.

We turn now to the proof of existence of $\mathrm{P}^{\alpha,p}$, $\mathrm{B}^{\alpha,p}$, $1 \leq p < \infty$, and $\check{\mathrm{P}}^{1,1}$. We will notice first that in all our imperfect completions L^p_α, $\tilde{\mathcal{B}}^{\alpha,p}$, and W^α_p, if a function $u(x)$ belongs to one of them, then so do all regularizations $u_\rho = u * e_\rho$ with some fixed regularizing function e and u_ρ converges strongly to u in the corresponding norm. Furthermore for a function $\varphi \in \mathrm{C}_0^\infty$ such that $\varphi(x) = 1$ when $|x| \leq 1$, $\varphi(\sigma x)u_\rho(x)$

belongs to the same space and converges in norm to $u_\rho(x)$ when $\sigma \searrow 0$. It follows that we can choose $\rho_k \searrow 0$ and $\sigma_k \searrow 0$ such that $\varphi(\sigma_k x) u_{\rho_k}(x)$ converge in norm to $u(x)$. Moreover, if $u(x)$ is continuous $\varphi(\sigma_k x) u_{\rho_k}(x)$ will converge pointwise *everywhere* to $u(x)$.

To abbreviate, we will denote by $\tilde{\mathfrak{F}}$ any of the imperfect completions L^p_α, $\tilde{\mathfrak{B}}^{\alpha,p}$ and W^α_p and by $\|\ \ \|$ the corresponding norm. What has been said above implies

1) *A continuous function belonging to* $\tilde{\mathfrak{F}}$ *must belong to any functional completion of* C_0^∞ *with norm* $\|\ \ \|$.

We have furthermore

2) *If for each* $u(x) \in \tilde{\mathfrak{F}}$ *the function* $u'(x) = |u(x)|$ *also belongs to* $\tilde{\mathfrak{F}}$ *and* $\|u'\| \leq \|u\|$ (21) *then* C_0^∞ *with norm* $\|\ \ \|$ *has a perfect functional completion rel. to an exceptional class* $\mathfrak{A}$ *formed by sets* A *for which there exists an increasing Cauchy sequence of positive continuous functions* $f_n \in \tilde{\mathfrak{F}}$ *such that* $f_n(x) \nearrow \infty$ *for* $x \in A$.

Proof. — By Prop. B the class $\mathfrak{F}$ of continuous functions belonging to $\tilde{\mathfrak{F}}$ has the same functional completions as C_0^∞. Since $\mathfrak{F}$ has the strong majoration property there exists by Prop. A a common perfect completion of C_0^∞ and $\mathfrak{F}$. Also the exceptional sets A for this completion are those of capacity $c_1(A) = 0$. Since the sets of the class $\mathfrak{A}$ are obviously exceptional for any functional completion it remains to show that if $c_1(A) = 0$ then $A \in \mathfrak{A}$. In fact, $c_1(A) = 0$ means that for every k there exist sets $A_n^{(k)}$ and functions $f_n^{(k)} \in \mathfrak{F}$ such that

$$A \subset \bigcup_{n=1}^{\infty} A_n^{(k)}, \qquad \sum_{n=1}^{\infty} \|f_n^{(k)}\| < 2^{-k} \quad \text{and} \quad |f_n^{(k)}(x)| \geq 1 \quad \text{for} \quad x \in A_n^{(k)}.$$

The sequence of functions $f_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{n} |f_i^{(k)}(x)|$ shows that $A \in \mathfrak{A}$.

THEOREM 13.1 — *The perfect completions* $B^{\alpha,p}$ *for* $0 < \alpha < 1$ *and* $\check{P}^{\alpha,p}$ *for* $0 < \alpha \leq 1$ *exist and their exceptional classes* $\mathfrak{B}^{\alpha,p}$ *and* $\check{\mathfrak{A}}^{\alpha,p}$ *are determined as in Prop.* 2).

(21) This is a special form of strong majoration property.

For $\tilde{\mathfrak{B}}^{\alpha,p}$, $\alpha < 1$, we can take the norm $|u|_{\alpha,p,1}$ and the condition in Prop. 2) is obviously satisfied since for

$$u'(x) = |u(x)|, \qquad |\Delta_t u'(x)| \leqq |\Delta_t u(x)|.$$

The only remaining case of W_p^1 is settled by noticing that if $u(x)$ is absolutely continuous in any variable x_k on an interval, so is $|u(x)|$ and $\left|\frac{\partial}{\partial x_k}|u(x)|\right| = \left|\frac{\partial u(x)}{\partial x_k}\right|$ almost everywhere on the interval.

Remark 1. — The exceptional class $\check{\mathfrak{A}}^{1,1}$ was investigated by W. H. Fleming [8] who proved that it is the class of sets of $(n-1)$-dimensional Hausdorff measure 0.

We will need the following mean-value theorems for Bessel potentials, similar to Frostman's theorems for Riesz potentials; the theorems were proved in [2].

For any $g(x) \geqq 0$, $g \in L^1_{\text{loc}}$ we will consider the function

$$u(x) = G_\alpha g(x) = \int G_\alpha(x-y) g(y)\, dy$$

as defined everywhere by the integral — infinite when the integral is infinite.

Mean Value Theorems. — *There exists a constant* C *depending only on* α *and* n *such that for each sphere* $S(x, r)$, $r \leqq 1$,

i) $\dfrac{1}{|S(x, r)|}\displaystyle\int_{S(x,r)} G_\alpha(z-y)\, dy \leqq CG_\alpha(z-x)$ *for every* z.

ii) $\dfrac{1}{|S(x, r)|}\displaystyle\int_{S(x,r)} G_\alpha g(y)\, dy \leqq CG_\alpha g(x)$ *for every* x *when* $g \in L^1_{\text{loc}}$ *and* $g \geqq 0$.

iii) $\displaystyle\lim_{r \searrow 0} \frac{1}{|S(x, r)|}\int_{S(x,r)} G_\alpha g(y)\, dy = G_\alpha g(x)$ *for every* x *when* $g \in L^1_{\text{loc}}$ *and* $g \geqq 0$.

iv) $\displaystyle\lim_{\rho \searrow 0} (e_\rho * G_\alpha g)(x) = \lim_{\rho \searrow 0} G_\alpha g_\rho(x) = G_\alpha g(x)$ *for every* x *when* $g \in L^1_{\text{loc}}$ *and* $g \geqq 0$ *where* e *is any regularizing function.*

Our next proposition will settle the question of existence of $P^{\alpha,p}$ and $B^{\alpha,p}$ in all the remaining cases.

3) *Consider two of our imperfect completions* $\tilde{\mathfrak{F}}$ *and* $\tilde{\mathfrak{F}}_1$ *such that for some* $\alpha > 0$, $G_\alpha(\tilde{\mathfrak{F}}_1) = \tilde{\mathfrak{F}}$ *and*

$$C^{-1}||f||_1 \leqq ||G_\alpha f|| \leqq C||f||_1$$

for every $f \in \tilde{\mathfrak{F}}_1$ *with a constant* $C > 0$. *Suppose further that* $\tilde{\mathfrak{F}}_1$ *satisfies the global majoration property in the form*

(*) *For every* $f \in \tilde{\mathfrak{F}}_1$ *there exists* $f' \in \tilde{\mathfrak{F}}_1$ *such that* $f'(x) \geqq |f(x)|$ *a.e. and* $\|f'\|_1 \leqq M\|f\|_1$ *with* M *independent of* f.

Then: 1° $\tilde{\mathfrak{F}}$ *has property* (*); 2° C_0^∞ *in the norm* $\|\ \ \|$ of $\tilde{\mathfrak{F}}$ *has a perfect functional completion* $\overline{\mathfrak{F}}$ rel. $\mathfrak{A}$ *where* $\mathfrak{A}$ *is the class of sets* A *for which there exists a function* $g \in \tilde{\mathfrak{F}}_1$, $g \geqq 0$ *with* $G_\alpha g(x) = \infty$ *for* $x \in A$; 3° $\overline{\mathfrak{F}}$ *is formed by all functions defined exc.* $\mathfrak{A}$ *by the integrals* $\int G_\alpha(x-y)f(y)\,dy$ *with* $f \in \tilde{\mathfrak{F}}_1$.

Proof. — 1° For $u \in \tilde{\mathfrak{F}}$ take $f \in \tilde{\mathfrak{F}}_1$ with $u = G_\alpha f$, then f' by (*) and put $u' = G_\alpha f'$. Obviously $u' \geqq |u|$ and $\|u'\| \leqq MC^2\|u\|$.

2° We show first that $\mathfrak{A}$ is σ-additive. If $A = \bigcup A_k$, $A_k \in \mathfrak{A}$ and g_k is the corresponding function, then $g = \Sigma 2^{-k}\|g_k\|_1^{-1} g_k$ corresponds to A. Next we show that every $A \in \mathfrak{A}$ must be an exceptional set for any completion of C_0^∞ in the norm of $\tilde{\mathfrak{F}}$. To this effect consider the function $g \in \tilde{\mathfrak{F}}_1$, $g \geqq 0$, $G_\alpha g(x) = \infty$ for $x \in A$. As before, we can find a sequence of functions $\varphi(\sigma_k x)(e_{\rho_k} * G_\alpha g) \in C_0^\infty$ which converge in norm of $\tilde{\mathfrak{F}}$ to $G_\alpha g$. By Mean-Value Theorem iv) these functions converge pointwise to $G_\alpha g(x) = \infty$ for $x \in A$.

To finish the proof of 2° and 3° we remark that each

$$u(x) = \int G_\alpha(x-y)f(y)\,dy$$

in $\overline{\mathfrak{F}}$ is finite exc. $\mathfrak{A}$, namely outside of the set A where

$$\int G_\alpha(x-y)f'(y)\,dy = \infty$$

(f' corresponds to f by (*)). It follows that in each equivalence class rel. $\mathfrak{A}_0$ of $\tilde{\mathfrak{F}}$ there exists one and only one equivalence class of $\overline{\mathfrak{F}}$ rel. $\mathfrak{A}$. Taking $\overline{\mathfrak{F}}$ with the norm of $\tilde{\mathfrak{F}}$ we see that $\overline{\mathfrak{F}}$ is a functional class $\subset \tilde{\mathfrak{F}}$ forming a Banach space isometrically isomorphic to the one formed by $\tilde{\mathfrak{F}}$; hence $\overline{\mathfrak{F}}$ is complete. Since $C_0^\infty \subset \overline{\mathfrak{F}}$ [22] it remains only to show that $\overline{\mathfrak{F}}$ is a functional space rel. $\mathfrak{A}$. In fact, if $\{u_n\} \subset \overline{\mathfrak{F}}$ and $\|u_n\| \to 0$ we choose

[22] The simplest way to see this is to write for $u \in C_0^\infty$, $f = G_{-\alpha}u = G_{2l-\alpha}(1-\Delta)^l u$ where Δ is the Laplacian, l an integer $> \alpha/2$.

u_{n_k} so that $\Sigma\|u_{n_k}\| < \infty$. If $f_{n_k} \in \tilde{\mathfrak{F}}_1$ with $u_{n_k} = G_\alpha f_{n_k}$, f'_{n_k} corresponds to f_{n_k} by $(*)$ and $g = \Sigma f'_{n_k}$ then $u_{n_k}(x) \to 0$ outside of the set A where $G_\alpha g(x) = \infty$.

THEOREM 13.2. — *The perfect completions* $P^{\alpha,p}$ and $B^{\alpha,p}$ *exist for all* $\alpha > 0$ *and* $p \geq 1$. *The exceptional classes* $\mathfrak{A}^{\alpha,p}$ *and* $\mathfrak{B}^{\alpha,p}$ *are determined as in Prop.* 3, 2° *by taking in case of* $P^{\alpha,p}$ *the isomorphism* $G_\alpha : L^p \to L^p_\alpha$ *and in case of* $B^{\alpha,p}$ *the isomorphism* $G_{\alpha-\gamma} : \tilde{\mathfrak{B}}^{\gamma,p} \to \tilde{\mathfrak{B}}^{\alpha,p}$ *with any* γ, $0 < \gamma < \alpha$.

A comment should be made in case of $B^{\alpha,p}$. We first use $\gamma < 1$ to be assured of the strong majoration property in $\tilde{\mathfrak{B}}^{\gamma,p}$ as in Prop. 2). Then by Prop. 3) 1° we obtain the global majoration property for all $\tilde{\mathfrak{B}}^{\gamma,p}$. Obviously, the perfect completion and its exceptional class are independent of the choice of γ.

Remark 2. — The classes $\mathfrak{A}^{\alpha,2} = \check{\mathfrak{A}}^{\alpha,2} = \mathfrak{B}^{\alpha,2}$ were studied extensively in [2]. Classes $\mathfrak{A}^{\alpha,p}$ for $p \neq 2$ were investigated by B. Fuglede [9].

For a function $u \in L^1_{loc}$ *the Lebesgue set* is the set of points x such that there exits a number $u^L(x)$ with

$$\lim_{r \searrow 0} \frac{1}{|S(x, r)|} \int_{S(x,r)} |u(y) - u^L(x)|\, dy = 0.$$

The complement A_u of the Lebesgue set is the *Lebesgue exceptional set* (L.-exc. set) of u on which the function $u^L(x)$ is not defined (see the corresponding developments in [3]).

With an arbitrary bounded function g vanishing outside of a compact and satisfying $\int g\, dx = 1$ define

$$u^g(x) = \lim_{\rho \searrow 0} \int \rho^{-n} g\left(\frac{x-y}{\rho}\right) u(y)\, dy,$$

wherever the limit exists. The points x where the limit does not exist form the exceptional set of u^g- *the corrected function* of u by g. The Lebesgue function u^L serves as a « minimal » corrected function since every u^g is an extension of u^L.

$$u^L(x) = u(x) \text{a.e.}$$

and the L.exc.set A_u has measure 0.

The following remark concerning the function u^L is of importance to us; it is an immediate consequence of the mean value theorem i) (c.f. [3]).

Remark 3. — *a) If $u(x)$ is represented a.e. by the integral $\int G_\alpha(x-y)f(y)\,dy$ then the integral represents $u^L(x)$ at every point x where the integral exists and is finite.*

b) More generally if u is represented a.e. by the integral $\int D_j G_\alpha(x-y)f(y)\,dy$, $|j| < \alpha$, then the integral represents $u^L(x)$ wherever $\int [G_{\alpha-|j|}(x-y) + G_\alpha(x-y)]f(y)\,dy$ exists.

THEOREM 13.3. — *i) If u belongs to L^p_α or $\tilde{\mathfrak{B}}^{\alpha,p}$ then u^L and every correction u^g belong to $P^{\alpha,p}$ or $B^{\alpha,p}$ respectively.* ii) *If $u \in W^m_1$, m an integer, u^L and every correction u^g belong to the almost perfect completion $\check{P}^{m,1}$ rel. $\bigcap_{\alpha<m} \mathfrak{A}^{\alpha,1}$.*

Proof. — Part i) follows immediately from the Remark 3 and the representation of the functions in perfect completion. given in Prop. 3) 3°. Part ii) follows from i) since $\check{P}^{m,1} \subset \bigcap_{\alpha<m} P^{\alpha,1}$. For $m = 1$ it is an open problem if actually u^L is in the perfect completion $\check{P}^{1,1}$ and if the L. exc. set is in $\check{\mathfrak{A}}^{1,1}$.

Remark 4. — The corrected functions and the minimal corrected function were introduced with the idea of recapturing the « true » value of a function which might be « incorrectly » defined on a set of measure 0. The above theorem shows that there is some factual background in this heuristic idea. The corrections most often used are by spherical means ($g = \omega_n/n$ for $|x| < 1$, $= 0$ for $|x| > 1$) or by regularizations ($g = e$).

From now on we consider a (non-singular) integral transformation as defining a function wherever the integrals occurring exist and are finite. An integral representation of functions in an imperfect completion $\tilde{\mathfrak{F}}$ will be called *perfect* if it actually defines functions in the perfect completion $\bar{\mathfrak{F}}$.

In the preceding section we considered several representation formulas which represented almost everywhere, by integrals, functions in different imperfect completions $\tilde{\mathcal{F}}$. It is important to know if these integrals give actually a perfect representation of the corresponding functions in the perfect completion $\bar{\mathcal{F}}$. This is true in most cases and the key to this result lies in the following theorem.

THEOREM 13.4 — *As in Prop.* 3) *consider two spaces* $\tilde{\mathcal{F}} = G_\alpha(\tilde{\mathcal{F}}_1)$ *where* $\tilde{\mathcal{F}}$ *is* L^p_α *or* $\tilde{\mathfrak{B}}^{\alpha+\varepsilon,p}$ *and* $\tilde{\mathcal{F}}_1$ *is* L^p *or* $\tilde{\mathfrak{B}}^{\varepsilon,p}$ *with* $0 < \varepsilon < 1$. *Suppose further that an integral transform* K *from some measure space* $\{Z, d\omega(z)\}$ ([23]) *to* $\{R^n, dy\}$ *transforms p-ab. regularly* $L^p(Z, d\omega(z))$ *into* $\tilde{\mathcal{F}}_1$ ([24]). *Then for any function* $w(z) \in L^p(Z, d\omega(z))$ *the integral*

$$(**) \qquad \iint G_\alpha(x-y)K(z, y)w(z)\, d\omega(z)\, dy$$

represents perfectly a function $u(x) \in \bar{\mathcal{F}}$ *outside of a set of the corresponding class* $\mathfrak{A}$.

Proof. — By Prop. 3) 3° it is enough to show that $f(y) = \int |K(z, y)||w(z)|d\omega(z)$ is in $\tilde{\mathcal{F}}_1$. When $\tilde{\mathcal{F}}_1 = L^p$ this follows from p-ab. regularity of K. When $\tilde{\mathcal{F}}_1 = \tilde{\mathfrak{B}}^{\varepsilon,p}$ one has also that $|t|^{-\varepsilon}\Delta_{t,y;y}K(z, y)$ is p-ab. -r. and since

$$|\Delta_{t,y;y}|K(z, y)|| \leq |\Delta_{t,y;y}K(z, y)|,$$

the kernel $|t|^{-\varepsilon}\Delta_{t,y;y}|K(z, y)|$ is p-ab. -*r*. too.

Remark 5. — As examples of formulas to which our theorem applies we note the reproducing formulas (5.21) (especially as rearranged in (11.12)) (5.25), (5.27), (5.29), inversion formulas (5.22) (rearranged as in (11.12)), (5.26), (5.28), the operator (12.4) in the projection $E_{\alpha,p}T_{\alpha,p}$ and many others. However it does not apply to (5.30) or (12.4′) since these contain some singular integral operators.

We pass now to differentiability of functions in our classes. There are three basic questions in this connection.

I) *Existence of distribution-derivatives as functions in the right classes.*

([23]) $\{Z, d\omega(z)\}$ may be $\{R^m, dz\}$ or $\{R^m \times R^m, d\mu(x, t)\}$ and so on with dimension m possibly different from n.

([24]) This means when $\tilde{\mathcal{F}}_1 = \tilde{\mathfrak{B}}^{\varepsilon,p}$ not only that K is *p.-ab.-r.* but also that the kernel $|t|^{-\varepsilon}\Delta_{t,y;y}K(z, y)$ is also *p.-ab.-r.* from $\{Z, d\omega(z)\}$ to $\{R^n \times R^n, d\mu(y, t)\}$.

We may consider the imperfect completions $\tilde{\mathfrak{F}}$. The right class for derivatives D_j of functions in $\tilde{\mathfrak{F}}$ is the class of the same type (L, W or $\tilde{\mathfrak{B}}$) with the same exponent p and with order α diminished by $|j|$ units.

a) *Classes* W_p^α. — These are the best from the present point of view. Their definition implies that $D_j(W_p^\alpha) \subset W_p^{\alpha-|j|}$ for all p, $1 \leqq p \leqq \infty$ and all j with $|j| \leqq \alpha$.

b) *Classes* $\tilde{\mathfrak{B}}^{\alpha,p}$. — Practically as good as the preceding. By theorems 11.3 and 11.4 we have with

$$\varepsilon = \frac{1}{2}\min\left(\frac{1}{2}, \alpha - |j|\right), \qquad \tilde{\mathfrak{B}}^{\alpha,p} = G_{\alpha-\varepsilon}\tilde{\mathfrak{B}}^{\varepsilon,p} = G_{\alpha-2\varepsilon-|j|}G_{|j|+\varepsilon}\tilde{\mathfrak{B}}^{\varepsilon,p}$$

and

$$D_j\tilde{\mathfrak{B}}^{\alpha,p} = G_{\alpha-2\varepsilon-|j|}D_jW_p^{|j|+2\varepsilon} \subset G_{\alpha-2\varepsilon-|j|}W_p^{2\varepsilon} = \tilde{\mathfrak{B}}^{\alpha-|j|,p}.$$

With our definition of $B^{0,p}$ (see § 11) the inclusion is true even for $|j| = \alpha$ but $B^{0,p}$ is a functional space only for $p \leqq 2$ and for $p > 2$ it contains distributions that are not functions.

c) *Classes* L_α^p. — Everything is right for $1 < p < \infty$. For $u \in L_\alpha^p$ we use the representation

$$D_ju(x) = \int G_{\alpha-|j|}(x-y)\int D_jG_{|j|}(y-z)f(z)\,dz\,dy$$

for $f \in L^p$. The inner integral is a singular integral (see (5.7) and (6.13)). Hence $D_j(L_\alpha^p) \subset L_{\alpha-|j|}^p$ for $1<p<\infty$, $|j| \leqq \alpha$. But when $p = 1$, or $p = \infty$, the inclusion is never valid. We have still obviously $D_j(L_\alpha^p) \subset \bigcap_{\beta<\alpha-|j|} L_\beta^p = \bigcap_{\beta<\alpha-|j|} W_p^\beta$ for $|j| < \alpha$; also $D_jL_\alpha^\infty \subset \tilde{\mathfrak{B}}^{\alpha-|j|,\infty}$. For $|j| = \alpha$, $D_j(L_{|j|}^1)$ contains distributions which are not functions, whereas

$$D_j(L_{|j|}^\infty) \subset \bigcap_{1\leqslant q<\infty} L_{\text{loc}}^q.$$

II) *Representation of derivatives by differentiation under integral sign. Perfect representation.*

If the function u is represented by one of our integral transforms, which, by our theorems, puts it in one of the classes L, W, $\tilde{\mathfrak{B}}$, of order α at most, then we cannot apply D_j to the kernel for $|j| \geqq \alpha$ and obtain still a non-singular integral transform. (Sometimes, when $|j| = \alpha$ we get a singular integral transform of the type (5.7)). Therefore we will assume $|j| < \alpha$. Our considerations are valid also for $|j| = 0$.

The case $1 < p < \infty$. — The only relevant classes are L^p_α and $\tilde{\mathfrak{B}}^{\alpha,p}$. If $u \in \tilde{\mathfrak{B}}^{\alpha,p}$ (or L^p_α) then a.e. $u = G_{\alpha-\varepsilon}*f$ and

$$D_j u = D_j G_{\alpha-\varepsilon}*f$$

with $f \in \tilde{\mathfrak{B}}^{\varepsilon,p}$, $0 < \varepsilon < \min(1, \alpha - |j|)$, (or $u = G_\alpha * f$ and $D_j u = D_j G_\alpha * f$ with $f \in L^p$); in both cases the representation of $D_j u$ is perfect in view of Remark 3 *b* and Theorem 13.3 i).

The case $p = 1$. — If $u \in \tilde{\mathfrak{B}}^{\alpha,1}$ the results are exactly the same as in the preceding case.

If $u \in W^\alpha_1$, α an integer, we do not know if the representation is of the kind treated in Theorem 13.4. However, we know that $D_j u \in W_1^{\alpha-|j|}$ and the representation is almost perfect, i.e. valid outside of a set in $\bigcap\limits_{\beta < \alpha - |j|} \mathfrak{A}^{\beta,1}$.

If $u \in L^1_\alpha$ we know that in general $D_j u \notin L^1_{\alpha-|j|}$. However, if the representation is $u = G_\alpha f$, $f \in L^1$, we get, in view of inequality (9.1) that $D_j u$ is defined by the integral outside of a set $\in \mathfrak{A}^{\alpha-|j|,1}$.

The case $p = \infty$. — In this case all functions in our classes and all their derivatives of order $< \alpha$ are continuous and bounded. The derivatives are represented by the corresponding integrals everywhere.

III) *Pointwise differentiation.*

We will introduce a notion of pointwise derivative, somewhat more restrictive than usual. We will say that u defined outside of some exceptional set A has a pointwise derivative in some direction, say the direction of x_n — axis, at the point y if in some interval $y_n - a < x_n < y_n + a$, $a > 0$, $u(y', x_n)$ *is defined and absolutely continuous* and

$$D_{x_n} u(y) = \lim_{h \to 0} \frac{1}{h} \Delta_{h;y_n} u(y', y_n)$$

exists and is finite. If $u \in L^1_{\text{loc}}$ and the so defined $D_{x_n} u$ exists a.e. and $D_{x_n} u \in L^1_{\text{loc}}$ then $D_{x_n} u$ is the distribution derivative of u.

By repeating the operation we obtain any higher order pointwise derivative $D_j u$. It is clear that it is necessary to define u much more precisely than exc. $\mathfrak{A}_0$ in order that the derivatives $D_j u$ exist in pointwise sense.

We will consider the perfect completions $P^{\alpha,p}$, $\check{P}^{\alpha,p}$, and $B^{\alpha,p}$ and prove that for u in any one of them the pointwise derivatives $D_j u$ exist for $|j| \leq \alpha$ outside of a set of the corresponding class $\mathfrak{A}^{\alpha-|j|,p}$, $\check{\mathfrak{A}}^{\alpha-|j|,p}$ or $\mathfrak{B}^{\alpha-|j|,p}$ and belong to $P^{\alpha-|j|,p}$, $\check{P}^{\alpha-|j|,p}$ and $B^{\alpha-|j|,p}$ respectively. The only exceptions will be $p = 1$ for all classes and $p = \infty$ for $P^{\alpha,p}$.

We prove first a few inclusions

(13.6) *For* $0 < \alpha' < \alpha$ *and* $\dfrac{1}{p} > \dfrac{1}{q} > \dfrac{1}{p} - \dfrac{\alpha - \alpha'}{n}$,

$$P^{\alpha,p} \subset P^{\alpha',q}, \quad \mathfrak{A}^{\alpha,p} \subset \mathfrak{A}^{\alpha',q}.$$

In fact, by Young's inequality (see [2], § 10, Prop. 1)) we have $G_{\alpha-\alpha'} f \in L^q$ if $f \in L^p$, hence $G_\alpha f = G_{\alpha'} * (G_{\alpha-\alpha'} f) \in P^{\alpha',q}$. The inclusion between exceptional classes follows from the one between the spaces.

(13.7) *For* $p < q$, $\mathfrak{A}^{\alpha,p} \supset \mathfrak{A}^{\alpha,q}$.

It is enough to prove this for bounded sets. Suppose $A \subset S(0, R)$ and $A \in \mathfrak{A}^{\alpha,q}$. It follows from Prop. 3) 2° for the isomorphism $G_\alpha: L^q \to L^q_\alpha$, that $A \subset [x: G_\alpha f(x) = \infty]$ for some $f \in L^q$, $f \geq 0$. Let $\chi(x)$ be the characteristic function of $S(0, R)$. Put $f_1 = \chi f$, $f_2 = (1 - \chi) f$. Then $G_\alpha f_2$ is a regular analytic function in $S(0, R)$, and hence $A \subset [x: G_\alpha f_1(x) = \infty]$. Since $f_1 \in L^p$, (13.7) follows.

LEMMA. — 1° *Let* $A \in \mathfrak{A}^{\alpha,p}$ *(or* $A \in \mathfrak{B}^{\alpha,p}$*),* $\alpha > 1$. *Then all straight lines parallel to the* x_n*-axis and meeting* A *form a set* $\in \mathfrak{A}^{\alpha-1,p}$ (or $\in \mathfrak{B}^{\alpha-1,p}$).

2° *Let* $A \in \check{\mathfrak{A}}^{1,p}$. *Then all straight lines parallel to the* x_n*-axis and meeting* A *form a set of Lebesgue measure* 0.

Proof. — 1° By proposition 3) 2° there exists a function $\varphi \geq 0$ such that $A = [x: G_\alpha \varphi(x) = \infty]$ with $\varphi \in L^p$ or $A = [x: G_{\alpha-\varepsilon} \varphi(x) = \infty]$ with $\varepsilon = \min\left[\dfrac{\alpha - 1}{2}, \dfrac{1}{2}\right]$ and $\varphi \in \tilde{\mathfrak{B}}^{\varepsilon,p}$.

Put $\varphi_1(x', x_n) = \int_{-N}^{N} \varphi(x', x_n + \tau)\, d\tau$ for a positive integer N. We have

$$\|\varphi_1\|_{L^p} \leqq 2N\|\varphi\|_{L^p}, \qquad \|\Delta_t\varphi_1\|_{L^p} \leqq 2N\|\Delta_t\varphi\|_{L^p}.$$

Therefore $\varphi_1 \in L^p$ or $\varphi_1 \in \tilde{\mathfrak{B}}^{\varepsilon,p}$ respectively. Put

$$A_1^{(N)} = [x : G_\alpha\varphi_1(x) = \infty]$$

and $A_2^{(N)} = [x : G_{\alpha-1}\varphi_1(x) = \infty]$ (or $A_1^{(N)} = [x : G_{\alpha-\varepsilon}\varphi_1(x) = \infty]$ and $A_2^{(N)} = [x : G_{\alpha-\varepsilon-1}\varphi_1(x) = \infty]$). Then $A_1^{(N)} \in \mathfrak{A}^{\alpha,p}$ and $A_2^{(N)} \in \mathfrak{A}^{\alpha-1,p}$ (or $\mathfrak{B}^{\alpha,p}$ and $\mathfrak{B}^{\alpha-1,p}$ respectively). Consider a point $y \notin A \cup A_1^{(N)} \cup A_2^{(N)}$. By (9.1) we have

$$\left|\frac{\partial}{\partial x_n} G_\alpha(x-y)\right| \leqq c[G_\alpha(x-y) + G_{\alpha-1}(x-y)]$$

hence for any h, $|h| < N$,

$$\begin{aligned}
|G_\alpha\varphi(y', y_n+h) - G_\alpha\varphi(y', y_n)| &\leqq \int_{R^n}\int_0^h \left|\frac{\partial}{\partial x_n} G_\alpha(y'-x', y_n - x_n + \tau)\right| \varphi(x', x_n)\, d\tau\, dx \\
\leqq c\Bigg[\int_{R^n} G_\alpha(y'-x', y_n - x_n) &\int_0^h \varphi(x', x_n+\tau)\, d\tau\, dx \\
&+ \int_{R^n} G_{\alpha-1}(y-x)\int_0^h \varphi(x', x_n+\tau)\, d\tau\, dx\Bigg] \\
&\leqq c[G_\alpha\varphi_1(y) + G_{\alpha-1}\varphi_1(y)] < \infty
\end{aligned}$$

(or similarly

$$|G_{\alpha-\varepsilon}\varphi(y, y_n+h) - G_{\alpha-\varepsilon}\varphi(y)| \leqq c[G_{\alpha-\varepsilon}\varphi_1(y) + G_{\alpha-\varepsilon-1}\varphi_1(y)] < \infty).$$

It follows that for y outside of the set

$$A \cup \bigcup_{N=1}^{\infty} (A_1^{(N)} \cup A_2^{(N)}) \in \mathfrak{A}^{\alpha-1,p}$$

(or $\mathfrak{B}^{\alpha-1,p}$) the whole straight line parallel to x_n-axis and passing through y lies outside of A.

2° By Prop. 2) there exists an increasing sequence of continuous positive functions u_k forming a Cauchy sequence in W_p^1 such that $A \subset [x : u_k(x) \nearrow \infty]$. Since the u_k are continuous we can find a set A_1 of measure 0 formed by straight lines parallel to x_n-axis such that

$$u_k(x', x_n+h) - u_k(x) = \int_0^h \frac{\partial}{\partial x_n} u_k(x', x_n+\tau)\, d\tau$$

for all k, h and x outside of A_1.

If there was a set of positive measure of straight lines parallel to x_n-axis and meeting A there would be also a set of positive measure of such lines on which $\int_{-\infty}^{\infty}\left|\frac{\partial}{\partial x_n}u_k(x', x_n+\tau)\right|^p d\tau < M^p$ for some constant M and all k $\left(\text{since } \frac{\partial}{\partial x_n}u_k \text{ is a}\right.$ Cauchy sequence in L^p). Also in this last set there would have to be a point y where $|u_k(y)| < N$ for all k. On the corresponding line we would have

$$|u_k(y', y_n + h)| < N + M|h|^{1/p}$$

and the line would not meet A.

THEOREM 13.5. — 1° *The case* $1 < p < \infty$. *If* $u \in P^{\alpha,p}$ (*or* $B^{\alpha,p}$) *and* $|j| < \alpha$ *the pointwise derivative* $D_j u$ *exists exc.* $\mathfrak{A}^{\alpha-|j|,p}$ (*or* $\mathfrak{B}^{\alpha-|j|,p}$) *and belongs to* $P^{\alpha-|j|,p}$ (*or* $B^{\alpha-|j|,p}$); *if* $|j| = \alpha$, $D_j u$ *exists exc.* $\mathfrak{A}_0$ *and* $\in L^p$ *for* $u \in P^{\alpha,p} = \check{P}^{\alpha,p}$.
2° *The case* $p = 1$. *If* $u \in P^{\alpha,1}$, $\check{P}^{\alpha,1}$, *or* $B^{\alpha,1}$, *and* $|j| < \alpha$, $D_j u$ *exists exc.* $\bigcap_{\beta<\alpha-|j|} \mathfrak{A}^{\beta,1}$ *and belongs to* $\bigcap_{\beta<\alpha-|j|} P^{\beta,1}$; *if* $|j| = \alpha = 1$ *and* $u \in \check{P}^{1,1}$, $D_j u$ *exists exc.* $\mathfrak{A}_0$ *and belongs to* L^1. 3° *The case* $p = \infty$. *If* u *belongs to* $P^{\alpha,\infty}$, $\check{P}^{\alpha,\infty}$, *or* $B^{\alpha,\infty}$ *and* $|j| < \alpha$, $D_j u$ *exists everywhere and belongs to* $B^{\alpha-|j|,\infty}$, $P^{\alpha-|j|,\infty}$, *or* $B^{\alpha-|j|,\infty}$ *respectively*; *if* $|j| = \alpha$, *and* $u \in \check{P}^{\alpha,\infty}$, *then* $D_j u$ *exists exc.* $\mathfrak{A}_0$ *and belongs to* L^∞.

Proof. — 1° Clearly it is enough to consider the case $|j| = 1$. Suppose first $1 < \alpha$. We confine ourselves to the case $u \in B^{\alpha,p}$ (the case $u \in P^{\alpha,p}$ is slightly simpler, both are similar to the case $p = 2$ treated in [2]). Since $u(x) = G_{\alpha-\varepsilon}f(x)$ exc. $\mathfrak{B}^{\alpha,p}$ with $2\varepsilon = \min(\alpha - 1,1)$ and $f \in \tilde{\mathfrak{B}}^{\varepsilon,p}$ we can take the set $A \in \mathfrak{B}^{\alpha-1,p}$ of straight lines parallel to x_n-axis such that

$$u(x) = G_{\alpha-\varepsilon}f(x)$$

outside of A as in the above Lemma; then we write

$$\frac{1}{h}(u(x', x_n + h) - u(x', x_n))$$
$$= \int_{R^n}\int_0^h \frac{1}{h}\frac{\partial}{\partial x_n}G_{\alpha-\varepsilon}(x' - y', x_n - y_n)f(y', y_n + \tau)\, d\tau\, dy.$$

The integrand is majorated by

$$c\frac{1}{h}\left[G_{\alpha-\varepsilon}(x'-y', x_n-y_n)+G_{\alpha-1-\varepsilon}(x'-y', x_n-y_n)\right]f(y', y_n+\tau).$$

Introducing

$$\bar{f}(y', y_n) = \sup \frac{1}{h}\int_0^h |f(y', y_n+\tau)|\, d\tau$$

we check immediately that

$$|\Delta_t\bar{f}(y, y_n)| \leqq \sup \frac{1}{h}\int_0^h |\Delta_t f(y', y_n+\tau)|\, d\tau.$$

Applying Hardy-Littlewood inequality we get $\bar{f} \in \tilde{\mathfrak{B}}^{\varepsilon,p}$ hence outside the set where $G_{\alpha-\varepsilon}\bar{f}(x) + G_{\alpha-\varepsilon-1}\bar{f}(x) = \infty$ and set A - which form a set in $\mathfrak{B}^{\alpha-1,p}$ - $\dfrac{\partial}{\partial x_n}u(x)$ exists and is given by $\left(\dfrac{\partial}{\partial x_n}G_\alpha\right)*f$ which is a perfect representation of a function in $B^{\alpha-1,p}$.

If $\alpha = 1$, we use a sequence $\{\varphi_k\} \subset C_0^\infty$ converging in $\check{P}^{1,p}$ to u exc. $\check{\mathfrak{A}}^{1,p}$. For almost all lines $\dfrac{\partial}{\partial x_n}\varphi_k$ converges in L^p-norm. If we assume that $\Sigma|\varphi_k-\varphi_{k+1}|_{1,p} < \infty$ the convergence is dominated by

$$\Sigma\left|\frac{\partial}{\partial x_n}\varphi_k(x)-\frac{\partial}{\partial x_n}\varphi_{k-1}(x)\right|+\left|\frac{\partial}{\partial x_n}\varphi_1(x)\right| \in L^p,$$

hence almost everywhere

$$\lim_{h=0}\frac{1}{h}\left(u(x', x_n+h)-u(x', x_n)\right) = \lim_{k=\infty}\lim_{h=0}\frac{1}{h}\left(\varphi_k(x', x_n+h)-\varphi_k(x', x_n)\right)$$

which finishes this part of the proof.

2° We use the preceding part and the inclusions (13.6) and (13.7) to show that $D_j u$ for $|j| < \alpha$ exists exc. $\bigcap_{\beta<\alpha-|j|}\mathfrak{A}^{\beta,1}$ and is represented by any of the relevant representation formulas differentiated under the sign of integral; but such a differentiated formula in all cases represents a function in $\bigcap_{\beta<\alpha-|j|} P^{\beta,1}$. For $|j| = \alpha = 1$ and $u \in \check{P}^{1,1}$ the proof is as in case 1°.

3° This is obvious except when $|j| = \alpha$ and $u \in \check{P}^{\alpha,\infty}$ when we proceed as in 1°.

§ 14. Restrictions and extensions of functions of $P^{\alpha,p}$, $B^{\alpha,p}$.

We shall apply here the results of § 10 and § 13 to characterize the restrictions of functions of $B^{\alpha,p}$ and $P^{\alpha,p}$ to hyperplanes and extensions of functions of $B^{\alpha,p}$ from hyperplanes to the whole space. Results presented here were obtained in a somehow less precise form by Besov [5] (for $B^{\alpha,p}$) and Stein [18] (for $P^{\alpha,p}$). The corresponding results for $\check{P}^{\alpha,p}$ can be obtained from the ones described here, in view of its inclusion relations with $B^{\alpha,p}$ and $P^{\alpha,p}$ (§ 13).

We begin with the characterization of restrictions of functions of $B^{\alpha,p}$.

By Theorem 13.2, if $u \in B^{\alpha,p}$ and γ is a fixed number, $0 < \gamma < \min\,(1, \alpha)$, (25) then $u = \int G_{\alpha-\gamma}(x-y)f(y)\,dy$ exc. $\mathfrak{B}^{\alpha,p}$ with $f \in \tilde{\mathfrak{B}}^{\gamma,p}$ $(= W_p^\gamma)$ and the norms $|f|_{\gamma,p}$ and $|u|_{\alpha,p,k}$ $(k > \alpha)$ are equivalent. For almost all z we have

$$f(z) = G_{2\gamma} * f(z) + \int_{R^n}\int_{R^n} \frac{\Delta_{t;y} G_{2\gamma}(z-y)}{|t|^\gamma}\, w(y,t)\, d\mu_\gamma(y,t)$$

where $w(y,t) = |t|^{-\gamma}\Delta_t f(y)$, and consequently,

$$(14.1)\quad u(x) = \int_{R^n} G_{\alpha+\gamma}(x-y)f(y)\,dy + \int_{R^n}\int_{R^n} \frac{\Delta_{t;y} G_{\alpha+\gamma}(x-y)}{|t|^\gamma}\, w(y,t)\, d\mu_\gamma(y,t),$$

the latter formula being valid in view of Theorem 13.4 exc. $\mathfrak{B}^{\alpha,p}$. Formula 14.1 is suitable for defining restrictions of u to hyperplanes. As before, for n'-integer, $0 < n' < n$, x' will denote the projection of the point x onto the hyperplane

(25) We could put $\gamma = \frac{1}{2}\min\,(1, \alpha)$.

$x_{n'+1} = \cdots = x_n = 0$, $n'' = n - n'$. Assume that $\alpha > \frac{n''}{p}$, $1 \leq p \leq \infty$ and define the restriction of u to $R^{n'}$,

$$(14.2) \quad u'(x') = \int_{R^n} G_{\alpha+\gamma}(x'-y) f(y)\, dy + \int_{R^n}\int_{R^n} \frac{\Delta_{t;y} G_{\alpha+\gamma}(x'-y)}{|t|^\gamma} w(y, t)\, d\mu_\gamma(y, t),$$

with f and w as in formula (14.1)

Hence u' is the sum of results of integral transformations of Props. 10.1 and 10.3 adjoint applied to $f \in L^p(R^n)$ and $w \in L^p[R^n \times R^n, d\mu_\gamma(y, t)]$ respectively. By Props. 10.1, 10.3 adjoint, and Remark 2 of § 10, we conclude that u' is defined a.e. on $R^{n'}$, belongs to $L^p(R^{n'})$ and $|u'|_{L^p(R^{n'})} \leq c|f|_{\gamma,p}$ with a constant c independent of f. Similarly, the difference quotient $w'(t_1', x') = |t_1'|^{\frac{n''}{p}-\alpha} \Delta_{t_1'}^{k'} u'(x')$, $k' > \alpha - \frac{n''}{p}$, is the sum of results of the transformations of Props. 10.2 and 10.4 applied to f and w respectively, and by Props. 10.2 i), 10.4, and Remark 2 of § 10, it belongs to $L^p(R^{n'} \times R^{n'}, d\mu'(x', t_1')]$ and

$$|w'|_{L^p(\mu')} \leq c|f|_{\gamma,p}$$

with some constant independent of f. We conclude that $u' \in \tilde{\mathfrak{B}}^{\alpha-n''/p,p}(R^{n'})$ and

$$(14.3) \qquad |u'|_{\alpha-n''/p,p,k'} \leq c|u|_{\alpha,p,k}$$

with $k > \alpha$ and some constant c independent of u.

It remains to prove that $u' \in B^{\alpha-n''/p,p}(R^{n'})$. In fact, $u(x)$ is a pointwise limit outside of $A \in \mathfrak{B}^{\alpha,p}$ of a Cauchy sequence of continuous functions $u_k \in \tilde{\mathfrak{B}}^{\alpha,p}(R^n)$. Hence their restrictions u_k' form by (14.3) a Cauchy sequence of continuous functions in $\tilde{\mathfrak{B}}^{\alpha-n''/p,p}(R^{n'})$ converging pointwise to u' outside of $A \cap R^{n'}$. We must now prove that $A \cap R^{n'} \in \mathfrak{B}^{\alpha-n''/p,p}(R^{n'})$. In the proof of Prop. 3), 2°, § 13, it was shown that there exists a sequence $\{v_k\} \subset C_0^\infty$ Cauchy in $\tilde{\mathfrak{B}}^{\alpha,p}(R^n)$ such that $A \subset [x : \lim v_k(x) = \infty]$. Their restrictions form a Cauchy sequence of continuous functions $v_k' \in \tilde{\mathfrak{B}}^{\alpha-n''/p,p}(R^{n'})$ and on $A \cap R^{n'}$, $v_k'(x') \to \infty$, hence $A \cap R^{n'} \in \mathfrak{B}^{\alpha-n''/p,p}(R^{n'})$. We have proved thus

THEOREM 14.1. — *If* $u \in B^{\alpha,p}(R^n)$, $\alpha > \frac{n''}{p}$, $1 \leq p \leq \infty$, *then the pointwise restriction* u' *of* u *to* $R^{n'}$ *belongs to* $B^{\alpha - n''/p, p}(R^{n'})$ *and the restriction mapping is linear and bounded.*

We shall prove now that this restriction mapping is a mapping onto. Let $u'(x') \in B^{\beta,p}(R^{n'})$. Similarly as in (14.1) we can write with some γ, $0 < \gamma < \min(1, \beta)$ [26] and an $f' \in W_p^{\gamma}(R^{n'})$ (with the norms $|u'|_{\beta,p,k'}$ and $|f'|_{\gamma,p}$ equivalent),

$$(14.4)\quad u'(x') = G^{(n')}_{\beta+\gamma} * f(x') + \int_{R^{n'}}\int_{R^{n'}} \frac{\Delta_{t';y'} G^{(n')}_{\beta+\gamma}(x'-y')}{|t|^{\gamma}} w'(y', t')\, d\mu_\gamma(y', t')$$

exc. $\mathfrak{B}^{\beta,p}$ (in $R^{n'}$) where $w'(y', t') = |t'|^{-\gamma}\Delta_{t'}f'(y')$. Observe, that by the definition of the kernel $G^{(n')}_\alpha$ we have

$$(14.5)\quad G^{(n')}_\alpha(x') = G^{(n')}_\alpha(|x'|) = (4\pi)^{n''/2} \frac{\Gamma\left(\frac{\alpha+n''}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} G_{\alpha+n''}(|x'|) = c_{n'',\alpha} G_{\alpha+n''}(|x'|).$$

where $G_{\alpha+n''}$ denotes the usual n-dimensional kernel.

Define now the extention u of the function u' by the formula

$$(14.6)\quad u(x) = c_{n'',\beta+\gamma}\Big[\int_{R^{n'}} G_{n''+\beta+\gamma}(x-y')f'(y')\,dy' + \int_{R^{n'}}\int_{R^{n'}} \frac{\Delta_{t';y'} G_{n''+\beta+\gamma}(x-y')}{|t'|^{\gamma}} w'(y', t')\, d\mu'_\gamma(y', t')\Big].$$

Clearly u is analytic outside the hyperplane $R^{n'}$ and $u(x') = u'(x')$ exc. $\mathfrak{B}^{\beta,p}$ (in $R^{n'}$).

Let $\alpha = \beta + \frac{n''}{p}$ and k be an integer, $k > \alpha$. Applying Props. 10.1 adjoint, 10.2 i) adjoint, and Remark 2 of § 10, we verify that $u \in L^p(R^n)$ and $\|u\|_{L^p} \leq c|u'|_{\beta,p,k'}$ $(k' > \beta)$, with some constant c independent of u'. Similarly, by Prop. 10.3 and 10.4 adjoint, the difference quotient $|t|^{-\alpha}\Delta_t^k u(x) = w(x, t)$ is in $L^p[R^n \times R^n, d\mu(x, t)]$ and

[26] We could put $\gamma = \frac{1}{3}\min(1, \beta)$.

$\|w\|_{L^p(d\mu)} \leq c|u'|_{\beta,p,k'}$ with c independent of u'. Since (14.6) is of type (**) of Theorem 13.4, this proves

THEOREM 14.2. — *If $u' \in B^{\beta,p}(R^{n'})$, $\beta > 0$, $1 \leq p \leq \infty$, then u' can be canonically extended by* (14.6) *to a function $u \in B^{\beta+n''/p,p}(R^n)$, the extension mapping being linear and bounded.*

We state now the following theorem concerning spaces $P^{\alpha,p}$:

THEOREM 14.3. — i) *If $u \in P^{\alpha,p}$, $\alpha > \frac{n''}{p}$, $n'' > 0$, $1 < p \leq \infty$, then the restriction u' of u to $R^{n'}$ belongs to $B^{\alpha-n''/p,p}(R^{n'})$ the, restriction mapping being linear and bounded.*

ii) *If $u' \in B^{\beta,p}(R^{n'})$, $\beta > 0$, $n'' > 0$, $1 \leq p < \infty$, then u' can be extended to a function $u \in P^{\beta+n''/p,p}$ the extension mapping being linear and bounded.*

Proof. — Let $u \in P^{\alpha,p}(R^n)$, then by Theorem 13.2,

$$u(x) = \int_{R^n} G_\alpha(x-y)f(y)\,dy$$

exc. $\mathfrak{A}^{\alpha,p}$, $f \in L^p$.

Define

$$u'(x') = \int_{R^n} G_\alpha(x'-y)f(y)\,dy.$$

By Prop. 10.1, u' is defined a.e. on $R^{n'}$, belongs to $L^p(R^{n'})$ and $\|u'\|_{L^p} \leq c\|f\|_{L^p}$ with a constant c independent of f. On the other hand, by Prop. 10.2 ii) for $k' > \alpha - \frac{n''}{p}$ the difference quotient

$$w'(x', t') = |t'|^{\frac{n''}{p}-\alpha} \Delta_{t'}^{k'} u'(x') = \int_{R^n} \frac{\Delta_{t'}^{k'} G_\alpha(x'-y)}{|t'|^{\alpha-n''/p}} f(y)\,dy$$

belongs to $L^p[R^{n'} \times R^{n'}, d\mu'(x', t')]$ and $\|w'\|_{L^p(d\mu')} \leq c\|f\|_{L^p}$ with some constant c independent of f. This proves that $u' \in \tilde{\mathfrak{B}}^{\alpha-n''/p,p}(R^n)$. To show that u' is actually in $B^{\alpha-n''/p,p}(R^{n'})$ we proceed as in the last part of Theorem 14.1

ii) Let $u' \in B^{\beta,p}(R^{n'})$ and let u be given by (14.6). Then $u = G_{\beta+n''/p}f$ with

$$f(x) = c_{n'',\beta+\gamma}\Big[\int_{R^{n'}} G_{\gamma+n''/p'}(x-y')f'(y')\,dy' + \int_{R^{n'}}\int_{R^{n'}} \frac{\Delta_{t';y'} G_{\gamma+n''/p'}(x-y')}{|t'|^\gamma} w'(y, t')\,d\mu'_\gamma(y', t')\Big],$$

and by Prop. 10.1 adjoint, 10.2 ii) adjoint, and Remark 2 of § 10, $f \in L^p$ and $\|f\|_{L^p} \leqq c|f'|_{\gamma,p}$. In view of the definition of f' (as in (14.6)) this completes the proof.

We mention finally the case of the spaces $\check{P}^{m,1}$ m-integer, about which no information can be obtained from the theorems proved above. E. Gagliardo proved (c.f. [11]) that restrictions of functions of $\check{P}^{1,1}(R^n)$ to R^{n-1} are in $L^1(R^{n-1})$. His reasoning can be extended (by completion of C_0^∞) to prove that restrictions of functions of $\check{P}^{m,1}(R^n)$ to $R^{n'}$ are in $\check{P}^{m-n'',1}(R^{n'})$, $m - n'' \geqq 0$, $\check{P}^{0,1} = L^1$.

BIBLIOGRAPHY

[1] N. Aronszajn and K. T. Smith, Functional spaces and functional completion, *Ann. de l'Inst. Fourier*, Vol. VI, 1956, pp. 125-185.

[2] N. Aronszajn and K. T. Smith, *Theory of Bessel Potentials*, Part I, *Ann. de l'Inst. Fourier*, Vol. XI, 1961, pp. 385-475.

[3] N. Aronszajn and K.T. Smith, *Theory of Bessel Potentials*, Part II, Technical Report 26, University of Kansas 1961.

[4] N. Aronszajn. Associated spaces, interpolation theorems and the regularity of solutions of differential problems, *Proc. of Symp. in Pure Math.*, Vol. IV, Partial Differential Equations 1961, pp. 23-32.

[5] O. V. Besov, On a family of functional spaces. Theorems about restrictions and extensions. *Dokl. Ak. Nauk SSSR*, 126, 6, 1959, pp. 1163-1165.

[6] A. P. Calderon, Lebesgue spaces of differentiable functions and distributions, *Proc. of Symp. in Pure Mathematics*, Vol IV, Partial Differential Equations 1961, pp. 33-49.

[7] A. P. Calderon and A. Zygmund, On singular integrals, *Am. Jour. Math.*, Vol. 28, 2, 1956, pp. 289-309.

[8] W. Fleming, Functions whose partial derivatives are measures, *Bull. Am. Math. Soc.*, 64, 1958, pp. 364-366.

[9] B. Fuglede, Extremal length and functional completion, *Acta Math.* Vol. 98, 1957, pp. 171-219.

[10] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge, 1959.

[11] E. Gagliardo, Caratterizzazioni della tracce sulla frontiera relative ad alcune classi di funzioni in n variabili, *Rendiconti Sem. Mat. Padova*, Vol. 27, 1957, pp. 284-305.

[11*a*] E.Gagliardo, On integral transformations with positive kernel, to appear in *Proc. A. M. S.*

[12] S. M. Nikolskii, Theorems about restrictions, extensions, and approximations of differentiable functions of several variables (survey article), *Usp. Mat. Nauk.*, Vol. 16, 5, 1961, pp. 63-114.

[13] L. Schwartz, *Théorie des distributions*, Vol I, II, Paris, 1950-1951.
[14] L. I. I. Slobodeckii, Spaces of S. L. Sobolev of fractional order, *Dokl. Akad. Nauk. SSSR.*, Vol. 118, 1958, pp. 243-246.
[15] E. M. Stein, Interpolation of linear operators, *Trans. Amer. Math. Soc.*, Vol. 83 1956,.
[16] E. M. Stein and G. Weiss, Interpolation of operators with change of measures, *Trans. Am. Math. Soc.* Vol. 87, 1958.
[17] E. M. Stein, The characterization of functions arising as potentials I, *Bull. Am. Math. Soc.*, Vol. 67, 1961, pp. 102-104.
[18] E. M. Stein, The characterization of functions arising as potentials II, *Bull. Am. Math. Soc.*, Vol. 68, 1962, pp. 577-584.
[19] M. H. Taibleson, Lipschitz classes of functions and distributions in E_n, *Bull. A.M.S.*, 69, 4, 1963.
[20] O. Thorin, *Convexity theorems*, Upsala, 1948.
[21] A. C. Zaanen, *Linear Analysis*, Amsterdam Groningen, 1953.