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THE MODIFIED DIAGONAL CYCLE ON THE TRIPLE PRODUCT OF A POINTED CURVE

by B.H. GROSS and C. SCHOEN

0.

Let k be a field, and let X be a smooth, projective, geometrically connected curve over k. Let $Y = X^3$ be the triple product of X over k, and let

$$\Delta_{123} = \{(x, x, x) : x \in X\}$$

denote the image of X in Y under the diagonal embedding.

More generally, if e is a k-rational point of X, define the following subvarieties of codimension 2 in Y :

$$\begin{aligned} \Delta_{12} &= \{(x, x, e) : x \in X\} \\ \Delta_{13} &= \{(x, e, x) : x \in X\} \\ \Delta_{23} &= \{(e, x, x) : x \in X\} \\ \Delta_{1} &= \{(x, e, e) : x \in X\} \\ \Delta_{2} &= \{(e, x, e) : x \in X\} \\ \Delta_{3} &= \{(e, e, x) : x \in X\}. \end{aligned}$$

In this paper, we will study the codimension 2 cycle Δ_e on Y, which is defined by

$$\Delta_e = \Delta_{123} - \Delta_{12} - \Delta_{13} - \Delta_{23} + \Delta_1 + \Delta_2 + \Delta_3. \tag{0.1}$$

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We call Δ_e the modified diagonal cycle, and show it is homologous to zero on Y. We also show that the class of Δ_e in the Chow group $CH^2(Y)$ depends only on the class of e in $CH^1(X) = \mathrm{Pic}(X)$, and the class of Δ_e in the Griffiths group $\mathrm{Gr}^2(Y)$ is independent of the choice of e.

The plan of this paper is as follows. In §1 we review the equivalence relations on cycles and operations on Chow groups. In §2 and §3 we study a natural projector P_e on the Chow group of the product variety $Y = X^n$ over k; this depends on the choice of a k-rational point e on X, and we define Δ_e as the image under P_e of the diagonal cycle. In §4 we study Δ_e on $Y = X^n$ when X is a rational, elliptic, or hyperelliptic curve, and in §5 consider the associated 1-cycle in the Jacobian J of X. For several reasons, the case $n = 3$ is the most interesting one.

The remainder of this paper is concerned with the case where k is either a number field, or the function field of an algebraic curve S. If k is a number field, we let $S = \mathrm{Spec} A$ where A is the ring of integers of k. We assume that the curve X over k has a regular, semi-stable model $\mathfrak{X}$ over S in which each fibral component is non-singular. In §6 we construct a regular model $\mathcal{Y}$ over S with general fibre $Y = X^3$, via an explicit desingularization of the triple product $\mathfrak{X}^3$ over S. In §7 we show that the modified diagonal cycle Δ_e on Y can be extended to a codimension 2 cycle on $\mathcal{Y}$ which is numerically equivalent to zero in the normalization of each fibre $\mathcal{Y}_s$.

In §8 we use these results to show that the Beilinson-Bloch height pairing $\langle \tau_* \Delta_e, \tau'_*(\Delta_e)\rangle$ is well defined, where τ and τ' are self-correspondences of Y over k, and depends only on the actions of τ_* and τ'_* on the regular 3-forms of Y. Some conjectures on this pairing, when X is a Shimura curve, were presented in [GK], §13.

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Notational convention : Given a field k, we denote by $\bar{k}$ a separable closure of k.

1. Chow groups.

Let Y be a smooth, projective, geometrically connected variety over k. We review the various equivalence relations of cycles on Y [Fu] 1.3, 10.3, 19.3.

For $r \geq 0$, let $Z^r(Y)$ be the free abelian group generated by the irreducible subvarieties of codimension r on Y over k. Let

$$(1.1) \qquad Z^r(Y)_{\mathrm{rat}} \subset Z^r(Y)_{\mathrm{alg}} \subset Z^r(Y)_{\mathrm{hom}} \subset Z^r(Y)$$

be the subgroups of cycles which are rationally equivalent to zero, algebraically equivalent to zero, and homologically equivalent to zero respectively.

We recall that, informally speaking, a cycle is rationally equivalent to zero if it is of the form $\Gamma_0 - \Gamma_\infty$, where Γ_t is a family of effective codimension r cycles on Y parametrized by $t \in \mathbb{P}^1$. Similarly, a cycle is algebraically equivalent to zero if it is of the form $\Gamma_a - \Gamma_b$, where Γ_t is a family of effective codimension r cycles on Y parametrized by points on an irreducible curve over k. If $k = \mathbb{C}$, a cycle is homologous to zero if it is in the kernel of the cycle class map to integral cohomology

$$c\ell_{\mathbb{C}} : Z^r(Y) \longrightarrow H^{2r}(Y, \mathbb{Z}(r)).$$

In general, the homologically trivial cycles are those in the kernel of the ℓ-adic cycle class mappings [Mi], VI.9,

$$Z^r(Y) \longrightarrow \prod_{\ell \neq \mathrm{char}(k)} H^{2r}(Y \otimes \bar{k}, \mathbb{Z}_\ell(r)).$$

If we take the quotients of the groups in (1.1) by the subgroup $Z^r(Y)_{\mathrm{rat}}$, we get the associated filtration of the Chow group :

$$(1.2) \qquad 0 \subset CH^r(Y)_{\mathrm{alg}} \subset CH^r(Y)_{\mathrm{hom}} \subset CH^r(Y).$$

The quotient

$$(1.3) \qquad \mathrm{Gr}^r(Y) = CH^r(Y)_{\mathrm{hom}}/CH^r(Y)_{\mathrm{alg}}$$

is called the Griffiths group of codimension r cycles. We also define $CH_s(Y) = CH^r(Y)$ and $\mathrm{Gr}_s(Y) = \mathrm{Gr}^r(Y)$, where $s + r = \dim(Y)$.

If $f : Y \to Y'$ is a proper morphism, it induces a push-forward map $f_* : CH_s(Y) \to CH_s(Y')$ [Fu], 1.4. If $g : Y' \to Y$ is a flat morphism, it induces a pull-back map $g^* : CH^r(Y) \to CH^r(Y')$ [Fu] 1.7. Both f_* and g^* preserve the filtrations (1.2) of the respective Chow groups [Fu] 10.3.

The intersection product

$$\begin{aligned} CH^r(Y) \otimes CH^s(Y) &\longrightarrow CH^{r+s}(Y) \\ \alpha \otimes \beta &\longmapsto (\alpha \cdot \beta) \end{aligned} \tag{1.4}$$

defined in [Fu] 6.1, gives $CH^\bullet(Y) = \bigoplus_{r \geq 0} CH^r(Y)$ the structure of a commutative, associative ring, with unit the class of Y in $CH^0(Y) = \mathbb{Z}$. The pull-back $g^* : CH^\bullet(Y) \longrightarrow CH^\bullet(Y')$ is a ring homomorphism.

2. A projector on product varieties.

Let X be a smooth, projective, geometrically connected curve over k. Let $n \geq 0$ be an integer, and let $Y = X^n$ be the n-fold product of X over k. Then Y is smooth and projective of dimension n; by convention X^0 is the point Spec k.

Let e be a k-rational point of X. Our aim is to define a cycle P_e in $Z^n(Y \times Y)$, which acts as a projector on the Chow group of Y. For any subset T of $\{1, 2, \cdots, n\}$, let T' be the complementary set $\{1, 2, \cdots, n\} - T$. Write $p_T : X^n \to X^{\mathrm{Card}(T)}$ for the usual projection, and let $q_T : X^{\mathrm{Card}(T)} \to X^n$ be the inclusion which uses the point e to fill in the missing coordinates. For example, if $T = \{1, 2\}$ we have

$$\begin{cases} p_T(x_1, x_2, \cdots, x_n) &= (x_1, x_2) \\ q_T(x_1, x_2) &= (x_1, x_2, e, e, \cdots, e). \end{cases}$$

Let P_T be the graph of the morphism $q_T \circ p_T : Y \to Y$, viewed as a cycle of codimension n on $Y \times Y$, and define

$$P_e = \sum_T (-1)^{\mathrm{Card}\ T'} P_T \tag{2.1}$$

in $Z^n(Y \times Y)$. The sum is taken over the 2^n distinct subsets T of $\{1, 2, \cdots, n\}$. For example, when $n = 3$, we find

$$P_e = P_{123} - P_{12} - P_{13} - P_{23} + P_1 + P_2 + P_3 - P_\phi. \tag{2.2}$$

The Chow group $CH^n(Y \times Y)$ is an associative ring under the operation of composition $\alpha \circ \beta$ of correspondences. Any class $\alpha \in CH^n(Y \times$

Y) gives an endomorphism of the graded abelian group $CH^\bullet(Y)$, by the formula $\alpha_* = (\mathrm{pr}_2)_*(\alpha \cdot \mathrm{pr}_1^*)$, and the map $CH^n(Y \times Y) \to \mathrm{End}(CH^\bullet(Y))$ is a ring homomorphism [Fu] 16.1.2.

PROPOSITION 2.3. — *We have $P_e \circ P_e = P_e$ in $CH^n(Y \times Y)$, hence $(P_e)_*$ is a projector on the graded Chow group $CH^\bullet(Y) = CH^\bullet(X^n)$.*

Proof. — If S and T are subsets of $\{1, \cdots, n\}$, we have $P_S \circ P_T = q_S \circ p_S \circ q_T \circ p_T = q_{S\cap T} \circ p_{S\cap T} = P_{S\cap T}$. Hence $P_S \circ P_e = \sum_T (-1)^{\mathrm{Card}\ T'} \cdot P_{S\cap T}$. This sum is zero, except in the case when $S = \{1, 2, \cdots, n\}$. Hence $P_e \circ P_e = P_e$.

We now compute the action of $(P_e)_*$ on the homology of $Y = X^n$, using the Künneth decomposition $H_\bullet(Y) = H_\bullet(X)^{\otimes n}$.

PROPOSITION 2.4. — *Let $h = h_1 \otimes h_2 \otimes \cdots \otimes h_n$ be a homology class on Y, where $h_i \in H_{d_i}(X)$. Then*

$$(P_e)_* h = \begin{cases} h & \text{if } d_1 d_2 \cdots d_n \neq 0 \\ 0 & \text{if } d_1 d_2 \cdots d_n = 0. \end{cases}$$

Proof. — We sketch the argument when $k = \mathbb{C}$ and $H_\bullet$ is singular homology. In this case, we may represent the classes h_i by oriented, compact manifolds M_i in $X(\mathbb{C})$ with dim $(M_i) = d_i$. If p_T maps the product $M_1 \times \cdots \times M_n$ to a manifold of smaller dimension, then $(P_T)_* h = 0$. Otherwise, $(P_T)_* h = h$.

Let S be the subset of $\{1, \cdots, n\}$ consisting of those indices where $d_i = \dim M_i = 0$. The above argument gives

$$(P_T)_* h = \begin{cases} 0 & \text{if } T' \cap S' \text{ is non-empty} \\ h & \text{if } T' \subset S. \end{cases}$$

Thus $(P_e)_* h = \sum_{T : T' \subset S} (-1)^{\mathrm{Card}\ T'} \cdot h$. This sum is zero unless S is the empty set, i.e., $d_1 d_2 \cdots d_n \neq 0$.

COROLLARY 2.5. — *$(P_e)_*$ annihilates the homology groups $H_0(Y)$, $H_1(Y), H_2(Y), \ldots, H_{n-1}(Y)$ and maps $H_n(Y)$ onto the Künneth summand $H_1(X)^{\otimes n}$.*

By Poincaré duality, we may identify $H_i(Y)$ with $H^{2n-i}(Y)$.

COROLLARY 2.6. — *$(P_e)_*$ annihilates the cohomology groups $H^{2n}(Y)$, $H^{2n-1}(Y), \cdots, H^{n+1}(Y)$, and maps $H^n(Y)$ onto the Künneth summand $H^1(X)^{\otimes n}$.*

3. The modified diagonal cycle.

Let $\Delta(X)$ denote the image of X in $Y = X^n$ under the diagonal embedding. If T is a non-empty subset of $\{1, 2 \cdots, n\}$, $\Delta_T = (P_T)_*(\Delta(X))$ is a 1-cycle on Y. The modified diagonal cycle is defined by

$$\Delta_e = \sum_{\substack{T \\ \text{non-empty}}} (-1)^{\text{Card } T'} \cdot \Delta_T.$$

When $n = 3$, this agrees with the definition (0.1), using (2.2).

PROPOSITION 3.1. — *For all $n \geq 3$, the cycle Δ_e is homologous to zero on $Y = X^n$.*

Proof. — The class of Δ_e in $CH_1(Y) = CH^{n-1}(Y)$ is equal to $(P_e)_*(\Delta(X))$, as $(P_\phi)_*$ annihilates 1-cycles. But the cycle class mapping

$$c\ell : CH^{n-1}(Y) \longrightarrow H^{2n-2}(Y, \mathbb{Z}_l(n-1))$$

commutes with the action of $(P_e)_*$. Thus

$$\begin{aligned} c\ell(\Delta_e) &= c\ell((P_e)_*(\Delta(X))) \\ &= (P_e)_* c\ell(\Delta(X)). \end{aligned}$$

Since $(P_e)_*$ anihilates H^{2n-2} once $n \geq 3$ by (3.1), this class is zero.

Note 3.2. — When $n = 2$ the cycle $\Delta_e = \Delta_{12} - \Delta_1 - \Delta_2$ has cohomology class in $H^1(X)^{\otimes 2}$, by (2.6). When $n \geq 4$, an argument similar to the proof of Proposition 3.1 shows that, when $k = \mathbb{C}$, Δ_e has trivial Abel-Jacobi class in the intermediate Jacobian associated to the cohomology group $H^{2n-3}(Y)$.

Note 3.3. — If $y = (x_1, \ldots, x_n)$ is a k-rational point of Y, the 0-cycle $(P_e)_*(y)$ is homologically equivalent to zero on Y once $n \geq 1$, and has trivial class in the Albanese variety of Y once $n \geq 2$.

Finally, we investigate the dependence of the class Δ_e in $CH_1(Y) = CH^{n-1}(Y)$ on the choice of a point e of X. First, note that there is a canonical cycle P of codimension n on $X \times Y \times Y = X^{2n+1}$ such that

$$P\Big|_{e\times Y\times Y} = P_e \qquad \text{on} \quad Y \times Y. \tag{3.4}$$

Indeed, let $P_T(x)$ be the map $q_T \circ p_T : Y \to Y$, defined using a variable point x for e. Then the T^{th}-component of P is the subvariety $x \times (\text{graph of } P_T(x))$ of $X \times Y^2$.

Write $\mathrm{pr}_2 : X \times Y^2 \to Y$ (respectively $\mathrm{pr}_{13} : X \times Y^2 \to X \times Y$) for projection on the second (respectively first and third) factors. From P and the diagonal $\Delta(X)$ in Y, we construct a class

$$\Delta = (\mathrm{pr}_{13})_*(P \cdot \mathrm{pr}_2^*(\Delta(X)))$$

of dimension 2 on $X \times Y$, with the property

$$\Delta\Big|_{e\times Y} = \Delta_e \quad \text{on } Y.$$

The class Δ in turn determines a homomorphism

$$\Delta_* : CH^1(X) = \mathrm{Pic}(X) \longrightarrow CH^{n-1}(Y) \tag{3.5}$$

defined by $\Delta_* = (\mathrm{pr}_Y)_*(\Delta \cdot \mathrm{pr}_X^*)$. We have $\Delta_*(e) = \Delta_e$. Since Δ_* is compatible with algebraic equivalence, it maps the subgroup $CH^1(X)_{\mathrm{alg}} = \mathrm{Pic}^0(X)$ to $CH^{n-1}(Y)_{\mathrm{alg}}$. We have therefore shown.

Proposition 3.6. — *Let $d = \sum m(e)e$ be a divisor on X. The class of the cycle $\Delta_d = \sum m(e)\Delta_e$ in $CH_1(Y)$ depends only on the class of d in* Pic(X). *The class of Δ_d in the Griffiths group* $\mathrm{Gr}_1(Y) = CH_1(Y)_{\mathrm{hom}}/CH_1(Y)_{\mathrm{alg}}$ *depends only on the integer* $\deg d = \sum m(e)$.

By the Proposition, once $n \geq 3$ there is always a canonical cyclic subgroup in the Griffiths group $\mathrm{Gr}_1(Y) = \mathrm{Gr}^{n-1}(X^n)$ generated by the class of Δ_d for d a k-rational divisor class of minimal degree. An interesting geometric question is to determine the locus of curves X where this cyclic group has finite order.

4. Rational, elliptic and hyperelliptic curves.

We retain our previous notation : X is a curve with a k-rational point e, $Y = X^n$, and Δ_e is the modified diagonal 1-cycle on Y.

PROPOSITION 4.1. — *Assume that $n \geq 2$. If X has genus zero, the cycle Δ_e is rationally equivalent to zero on Y.*

Proof. — For $n \geq 3$ we have seen that Δ_e is homologous to zero. When $H^1(X) = 0$, this is also true for $n = 2$ by (3.2). Since X has a k-rational point, $X = \mathbb{P}^1$ and $Y = (\mathbb{P}^1)^n$. Writing $X = e \cup \mathbb{A}^1$ and taking products gives rise to a "cellular decomposition" of Y in the sense of [Fu] 1.9.1. It follows that $CH^\bullet(Y)$ is generated by the classes of the product subvarieties $V_1 \times ... \times V_n$, where each V_i is either e or $\mathbb{P}^1$. There are 2^n such subvarieties. By the Künneth formula, their cohomology classes form a basis for $H^{2\bullet}(Y \otimes \bar{k}, \mathbb{Z}_\ell(\bullet))$. (Here ℓ is a prime distinct from the characteristic of k.) Thus the cycle class map

$$CH^\bullet(Y) \longrightarrow H^{2\bullet}(Y \otimes \bar{k}, \mathbb{Z}_\ell(\bullet))$$

is injective. The proposition follows.

Before considering the case when X is elliptic or hyperelliptic, we remark that Δ_e is closely related to a cycle Γ_e on the n^{th} symmetric product $S^n X$. Let

$$f : X^n \longrightarrow S^n X \tag{4.2}$$

be the covering map, which is Galois with group the symmetric group Σ_n on n letters, and define a 1-cycle on $S^n X$ by

$$\Gamma_e = f_*(\Delta_e). \tag{4.3}$$

We then have the formula

$$f^*(\Gamma_e) = f^*(f_*\Delta_e) = n! \cdot \Delta_e \tag{4.4}$$

in $Z^{n-1}(Y)$.

PROPOSITION 4.5. — *Assume that $n \geq 3$. If X has genus one, then Γ_e is rationally equivalent to zero on $S^n X$. If X has genus two, then Γ_e is algebraically equivalent to zero on $S^n X$.*

Proof. — Write $J = \mathrm{Pic}^0(X)$ and view S^nX as the space of effective degree n divisors on X. When $n \geq 2g-1$ the morphism

$$\pi_n : S^nX \longrightarrow J, \qquad \pi_n(D) = \mathcal{O}_X(D - ne)$$

is a $\mathbb{P}^{n-g}$-bundle [Fu] 4.3.3. We have $CH^\bullet(S^nX) = CH^\bullet(J)[\xi]/P(\xi)$, where $\xi \in CH^1(S^nX)$ is represented by the image of

$$h : S^{n-1}X \longrightarrow S^nX, \qquad h(D) = D + e$$

and P is a monic polynomial of degree $n - g + 1$ [Fu] 3.3(b). Since $CH^0(J) = \mathbb{Z}$ and $CH^1(J) = \mathrm{Pic}(J)(k)$, we find :

$$\begin{aligned} & CH^{n-1}(S^{n-1}X) = \xi^{n-2}\mathrm{Pic}(J)(k) && g=1 \\ (4.6) \qquad & CH^{n-1}(S^nX) = \xi^{n-1}\cdot\mathbb{Z} \oplus \xi^{n-2}\mathrm{Pic}(J)(k) && g=1 \\ & CH^{n-1}(S^nX) = \xi^{n-2}\mathrm{Pic}(J)(k) \oplus \xi^{n-3}CH^2(J) && g=2. \end{aligned}$$

Since dim $J \leq 2$, homological and algebraic equivalence coincide in $CH^\bullet(J)$. Hence they coincide in $CH^\bullet(S^nX)$. Since Γ_e is homologically trivial by Proposition 3.1, it is algebraically equivalent to zero on S^nX.

To treat the case $g = 1$ define

$$W = \{(x_1, ..., x_n) \in X^n : x_i = e \quad \text{for some } i\}$$

and consider the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{i} & X^n \\ \downarrow f|_W & & \downarrow f \\ S^{n-1}X & \xrightarrow{h} & S^nX \\ \downarrow \pi_{n-1} & & \downarrow \pi_n \\ X & = & X \ . \end{array}$$

The class $\xi^{n-2} \in CH_1(S^{n-1}X)$ is represented by the section of π_{n-1} which maps a point x to the divisor $x + (n-2)e$. Thus (4.6) implies that

$$\begin{aligned} (\pi_{n-1})_* &: CH^{n-1}(S^{n-1}X)_{\mathrm{hom}} \to CH^1(X)_{\mathrm{hom}} \\ \text{and} \qquad h^* &: CH^{n-1}(S^nX)_{\mathrm{hom}} \to CH^{n-1}(S^{n-1}X)_{\mathrm{hom}} \end{aligned}$$

are isomorphisms. Intersection theory [Fu], 6.2(a) gives

$$(\pi_{n-1})_* \circ h^*(\Gamma_e) = (\pi_{n-1})_* \circ h^* \circ f_*(\Delta_e) = (\pi_{n-1})_* \circ (f|_W)_* \circ i^*(\Delta_e).$$

But $i^*(\Delta_e) = 0$ since the restriction of Δ_e to each irreducible component of W is zero.

COROLLARY 4.7. — *If $n \geq 3$ and $g = 1$, then $n!\Delta_e \equiv 0$ in $CH_1(Y)$. If $n \geq 3$ and $g = 2$, then $n!\Delta_e \equiv 0$ in $\mathrm{Gr}_1(Y)$.*

We can make these results slightly more precise when $n = 3$, using an explicit construction, which also applies to hyperelliptic curves.

PROPOSITION 4.8. —*Assume that the curve X has an involution σ over k which fixes the point e, and that the quotient curve has genus zero. Then $\Gamma_e \equiv 0$ in $CH^2(S^3X)$.*

Proof. — Let $u : (X, e) \longrightarrow (\mathbb{P}^1, \infty)$ be the associated covering of degree 2, with Galois group $\langle 1, \sigma \rangle$. We define three functions on three surfaces in S^3X, such that the sum of their divisors is equal to the cycle Γ_e.

If we view points of S^kX as effective divisors of degree k on X, the three surfaces are the images of the maps :

$$\begin{array}{rccc} r_1 : & S^2X & \longrightarrow & S^3X \\ & x_1 + x_2 & \longmapsto & x_1 + x_2 + e \end{array}$$

$$\begin{array}{rccc} r_2 : & X^2 & \longrightarrow & S^3X \\ & (x_1, x_2) & \longmapsto & x_1 + 2x_2 \end{array}$$

$$\begin{array}{rccc} r_3 : & X \times \mathbb{P}^1 & \longrightarrow & S^3X \\ & (x, t) & \longmapsto & x + u^*(t). \end{array}$$

The respective functions are given by

$$\begin{array}{lcll} f_1(x_1 + x_2) & = & (u(x_1) - u(x_2))^{-2} & \text{on } S^2X \\ f_2(x_1, x_2) & = & u(x_1) - u(x_2) & \text{on } X^2 \\ f_3(x, t) & = & (u(x) - t)^{-1} & \text{on } X \times \mathbb{P}^1. \end{array}$$

Let $D_i = (r_i)_*(\mathrm{div}(f_i))$ be the associated 1-cycles on S^3X.

We find that

$$D_1 = 4\{x + 2e\} - 2\{x + x^\sigma + e\} - \{2x + e\}$$

$$D_2 = \{3x\} + \{2x + x^\sigma\} - 2\{2x + e\} - 2\{x + 2e\}$$

$$D_3 = \{x + 2e\} + 2\{x + x^\sigma + e\} - \{2x + x^\sigma\}.$$

Hence $D_1 + D_2 + D_3 = \{3x\} - 3\{2x + e\} + 3\{x + 2e\} = \Gamma_e$. Since the D_i are rationally equivalent to zero in $CH^2(S^3X)$, so is Γ_e.

COROLLARY 4.9. — *If the curve X is hyperelliptic, then $6\Delta_e \equiv 0$ in $\mathrm{Gr}^2(X^3)$, for all points e on X.*

Following [Ce] and [Co-vG], one can give examples of non-hyperelliptic curves X of genus 3 where Δ_e has infinite order in the Griffiths group Gr^2 of $Y = X^3$.

5. One-cycles on the Jacobian.

In this section, X is a curve of genus $g \geq 1$. Let $J = \mathrm{Pic}^o(X)$ be the Jacobian of X, and let

$$(5.1) \qquad \begin{array}{rcl} i : X & \hookrightarrow & J \\ x & \longmapsto & \mathcal{O}_X(x - e) \end{array}$$

be the standard inclusion. Let $[\mathfrak{m}] : J \to J$ be the isogeny "multiplication by $\mathfrak{m}$".

The map i of (5.1) induces a map

$$(5.2) \qquad \begin{array}{rcl} i^n : S^nX & \longrightarrow & J \\ x_1 + x_2 + \cdots + x_n & \longmapsto & \mathcal{O}_X(x_1 + x_2 + \cdots + x_n - n \cdot e) \end{array}$$

and it is reasonable to study the class $(i^n)_*\Gamma_e$ in $CH_1(J)$. The following is obvious from the definitions.

PROPOSITION 5.3. — *Let $i(X) = i_*X$ be the one-cycle on J given by the image of (5.1). Then for $n \geq 1$, the direct image $(i^n \circ f)_*\Delta_e = (i^n)_*\Gamma_e$ of the modified diagonal cycle in $CH_1(J)$ is equivalent to the sum*

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} [n-k]_* i(X).$$

This sum is homologically trivial if $n \geq 3$, and, when $k = \mathbb{C}$, it is Abel-Jacobi trivial once $n \geq 4$.

Note 5.4. — The one cycle $\sum c(m)[m]_*i(X)$ is homologically trivial if and only if $\sum c(m)m^2 = 0$; if $k = \mathbb{C}$ and $\sum c(m)m^3 = 0$, it also has trivial class in the intermediate Jacobian. The subgroup spanned by these cycles in the Griffiths group $\mathrm{Gr}_1(J)$ has been studied in [CoVG].

6. Good models for triple products of curves.

Let R be a discrete valuation ring with fraction field k and residue field k_0. Let U be a smooth, proper, geometrically connected k-variety. Write $\pi \in R$ for a uniformizing parameter and $\bar{k}_0$ for a separable closure of k_0. One conjectures that there exists a regular scheme $\mathcal{U}$, proper and flat over Spec R, with general fiber U. This is known to hold when the dimension of U is 1. We show in this section that if $\mathcal{U}$ does exist and if its special fiber is sufficiently nice, then there is an explicit proceedure for constructing regular models over Spec R for powers of U. One application of this result will be the construction of a regular model for the triple product of a curve.

Let $n-1$ denote the dimension of U over k. For any local ring $\mathcal{O}$, we write $\check{\mathcal{O}}$ for a strict henselization.

DEFINITION 6.1. — *A scheme $\mathcal{U}$ over R is said to be a good model for U provided*

(1) *$\mathcal{U}$ is proper and flat over* Spec R *with generic fiber U.*

(2) *The special fibre $U_0 = \mathcal{U} \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ k_0$ is geometrically connected and every irreducible component is a non-singular variety.*

(3) *For each closed point $u_0 \in \mathcal{U}$ there is an integer r satisfying $1 \leq r \leq n$ and an isomorphism of $\check{R}$-algebras*

$$(R[x_1, ..., x_n]_{(\pi, x_1, ..., x_n)}/(x_1 ... x_r - \pi))\check{\ } \to \check{\mathcal{O}}_{\mathcal{U}, u_0}.$$

In particular $\mathcal{U}$ is a regular scheme [Mi] 3.17.

Example 6.2. — Let X be a smooth, projective, geometrically connected curve of genus $g \geq 1$ over a number field k. Let $\mathfrak{o}_k$ denote the integers of k. The theory of semi-stable reduction (cf. [Si] VII 5.4 and [Des]) shows that there is a finite extension k' of k with the property that the base change X/k' has a regular semi-stable model $\mathfrak{X}'$ over $\mathfrak{o}_{k'}$. The notions of good model and regular semi-stable model are closely related. The main differences are :

(1) The irreducible components of the special fiber are allowed to have ordinary double point singularities in a semi-stable model but not in a good model.

(2) An irreducible component of the special fiber which has genus 0 must meet the remaining components in at least two points in a semi-stable model. There is no similar requirement for good models.

Given a good model one obtains a model which is both good and semi-stable by contracting genus 0 curves which meet the other fiber components at only one point. Given a regular semi-stable model one eliminates ordinary double points in fiber components by adjoining a square root of the uniformising parameter to the base and blowing up. If all components of the special fiber are defined over the residue field, then this gives a model which is both good and semi-stable. In particular, X has a such model over the integers in some finite extension field k'' of k.

Let U_1 and U_2 be two smooth proper geometrically connected varieties over k with good models $\mathcal{U}_1$ and $\mathcal{U}_2$ over R. The fiber product $\mathcal{W}_1 := \mathcal{U}_1 \times_{\mathrm{Spec}\ R} \mathcal{U}_2$ fails to be regular precisely at the points (u_1, u_2) where each u_i is a singular point of the closed fiber $(U_i)_0 = \mathcal{U}_i \times_{\mathrm{Spec}\ R}$ Spec k_0. Thus $\mathcal{W}_1$ is not a good model for $U_1 \times U_2$ unless $\mathcal{U}_1$ or $\mathcal{U}_2$ is smooth over Spec R.

In order to describe a proceedure for desingularizing $\mathcal{W}_1$ to obtain a good model, we begin by fixing an (arbitrary) ordering of the components of the special fiber, $\mathcal{W}_1 \times_{\mathrm{Spec}\ R}$ Spec k_0. The total number of components will be denoted by $\mathbf{j}$. For $0 < j \leq \mathbf{j}$ define inductively $\gamma_j : \mathcal{W}_{j+1} \to \mathcal{W}_j$ to be the blow up centered at the strict transform of the j-th component of the special fiber. Define

$$\sigma_j : \mathcal{W}_j \to \mathcal{W}_1, \qquad \sigma_j = \gamma_{j-1} \circ \ldots \circ \gamma_1.$$

When $j = \mathbf{j} + \mathbf{1}$ we write simply

$$\sigma : \mathcal{W} \to \mathcal{W}_1.$$

Proposition 6.3. — *$\mathcal{W}$ is a good model for $U_1 \times U_2$.*

Proof. — Define $n_i = \dim.(U_i) + 1$. Given integers q, r, s, satisfying

$$1 \leq q \leq r \leq n_1 \qquad \text{and} \qquad 1 \leq s \leq n_2, \tag{6.4}$$

define the ring

$$(6.5)\quad \begin{array}{l} A_{q,r,s} = \check{R}[x_1, ..., x_{n_1+n_2}]/(x_1...x_q - \pi,\ x_1...x_r - x_{n_1+1}...x_{n_1+s}), \\ \text{and the ideal } \mathfrak{m} = (\pi, x_1, ..., x_{n_1+n_2}) \subset A_{q,r,s}. \end{array}$$

Fix a closed point $w \in \mathcal{W}_j$. We claim that there are integers q, r, s, satisfying (6.4) and an isomorphism of $\check{R}$-algebras

$$(6.6)\qquad \check{\mathcal{O}}_{\mathcal{W}_j,w} \simeq (A_{q,r,s})_{\mathfrak{m}}\check{}\ .$$

A neighborhood of $w \in \mathcal{W}_1$ is isomorphic in the étale topology to a neighborhood of the origin in Spec of

$$\check{R}[x_1, ..., x_{n_1}]/(x_1...x_r - \pi) \otimes_{\check{R}} \check{R}[x_{n_1+1}, ..., x_{n_1+n_2}]/(x_{n_1+1}...x_{n_1+s} - \pi).$$

Thus (6.6) holds when $j = 1$. (Take $r = q$.) Assume now that (6.6) holds for j and deduce that it holds for $j + 1$. To simplify the notation we write A for $A_{q,r,s}$. The components of the special fiber of $\mathcal{W}_j$ which contain w correspond to the components of

$$(6.7)\quad \operatorname{Spec} A/\pi \simeq \operatorname{Spec} \bar{k}_0[x_1, ..., x_{n_1+n_2}]/(x_1...x_q, x_1...x_r - x_{n_1+1}...x_{n_1+s}).$$

These are defined by the ideals

$$(x_i, x_{n_1+l})A, \quad 1 \le i \le r, 1 \le l \le s \qquad \text{and} \qquad x_iA, \quad r < i \le q.$$

In order to analyze the local effect of blowing up components of the special fiber of $\mathcal{W}_j$ we may work with the strict henselization or with the local ring $A_{\mathfrak{m}}$. Blowing up a component defined by a principal ideal will have no effect. Note that $(x_i, x_{n_1+l})A_{\mathfrak{m}}$ is a principal ideal if either $r = 1$ or $s = 1$. If $r > 1$ and $s > 1$, then the blow up along the ideal $(x_i, x_{n_1+l})A_{\mathfrak{m}}$ is covered by two charts given by the spectra of the following subrings of the total ring of fractions of $A_{\mathfrak{m}}$:

6.8.A

$$S^{-1}\check{R}[x_1, ..., x_i', ..., x_{n_1+n_2}]/(x_1...x_i'...x_qx_{n_1+l} - \pi,\\ x_1...x_i'...x_r - x_{n_1+1}...\widehat{x}_{n_1+l}...x_{n_1+s})$$

where $x_i' = x_i/x_{n_1+l}$ and

6.8.B

$$S^{-1}\check{R}[x_1, ..., x_{n_1+l}', ..., x_{n_1+n_2}]/(x_1...x_q - \pi,\\ x_1...\widehat{x}_i...x_r - x_{n_1+1}...x_{n_1+l}'...x_{n_1+s})$$

where $x'_{n_1+l} = x_{n_1+l}/x_i$ and $S = A - \mathfrak{m}$. The closed points of (6.8.A) are in bijective correspondence with the elements of $\bar{k}_0$. The corresponding maximal ideals have the form

$$\mathfrak{n}_a = (\pi, x_1, ..., x_{i-1}, x'_i - a, x_{i+1}, ..., x_{n_1+n_2}), \qquad a \in \check{R}.$$

In case $a \in \pi\check{R}$ we may take $a = 0$ and the henselization is isomorphic to $(A_{q+1,r,s-1})_{\mathfrak{m}}\check{}$. If a is a unit in $\check{R}$, then x'_i is a unit in the local ring at $\mathfrak{n}_a$. A change of variables in (6.8.A) such as $x'_1 = x_1 x'_i$ then gives that the henselization is isomorphic to $(A_{q,r-1,s-1})_{\mathfrak{m}}\check{}$. The situation with (6.8.B) is similar. It follows by induction that (6.6) holds for all j.

We now show that $\mathcal{W}$ satisfies (6.1) (3). It suffices to show that when $j = \mathbf{j} + 1$ then either $r = 1$ or $s = 1$ in (6.6). If this were not the case then $(x_1, x_{n_1+1})A_{\mathfrak{m}}$ would be a non-principal ideal which defines a component of the closed fiber of Spec$A_{\mathfrak{m}} \to$ Spec$\check{R}$. This component would correspond either to the strict transform of a component $\mathbf{C_j}$ of the closed fiber of $\mathcal{W}_1$ or to a new component which was introduced in the process of blowing up. We may rule out the second possiblity since the local description of the blow-ups shows that these do not introduce new components in the closed fiber. Now the strict transform of $\mathbf{C_j}$ in $\mathcal{W}_j$ is the center for the blow-up $\gamma_j : \mathcal{W}_{j+1} \to \mathcal{W}_j$. The inverse image of $\mathbf{C_j}$ in $\mathcal{W}_{j+1}$ is defined by a locally principal sheaf of ideals. The same holds for the strict transform in every $\mathcal{W}_{j'}$ with $j' > j + 1$. This contradicts $(x_1, x_{n_1+1})A_{\mathfrak{m}}$ being non-principal.

Finally we check that the components of the special fiber of $\mathcal{W}_j$ are non-singular varieties. From the explicit description of Spec A/π (6.7), this holds locally in the étale topology. It remains to check that there are no singular irreducible components of $\mathcal{W}_j \times_{\text{Spec } R}$ Spec k_0 which become unions of non-singular varieties when the base field is extended from k_0 to $\bar{k}_0$. In fact this pathology will not occur, because of the tautological bijection between components of $\mathcal{W}_j \times_{\text{Spec } R}$ Spec k_0 and components of $\mathcal{W}_1 \times_{\text{Spec } R}$ Spec k_0 and the non-singularity of the latter.

Remark 6.9. — The resolution of singularities described above is not canonical, since it depends on a choice of ordering of the components of the special fiber. Experience shows that resolutions which are canonical (eg. [Sch], [De3], 5.5) introduce components with multiplicity greater than one in the special fiber. Thus they do not yield good models.

COROLLARY 6.10. — *Let $U_1, ..., U_l$ be smooth, proper, geometrically connected k-varieties with good models $\mathcal{U}_1, ..., \mathcal{U}_l$ over R. Then $U_1 \times ... \times U_l$ has a good model over R.*

Proof. — Use the previous proposition and induction on l.

Now let X be a curve over k with a good model $\mathfrak{X}$ over R. The corollary gives us one construction of a good model for X^3 over R. A slightly different proceedure, which turns out to be more convenient in certain circumstances, is given in the next proposition.

Write $X_0 = \mathfrak{X} \times_{\mathrm{Spec} R} \mathrm{Spec}\, k_0$ for the special fiber of $\mathfrak{X}$ and $\mathfrak{X}^3$ for the 3-fold fiber product of $\mathfrak{X}$ over Spec R.

PROPOSITION 6.11 — *Fix an ordering of the components of the special fiber $X_0^3 \subset \mathfrak{X}^3$. Blow up $\mathfrak{X}^3$ along the ideal sheaf of the first component. Then blow up the resulting scheme along the ideal sheaf of the strict transform of the second component. Continue, proceeding one component at a time in increasing order. When the strict transform of the last component has been blow up, the result is a good model for $\mathfrak{X}^3$.*

Proof. — The argument is similar to the proof of (6.3) and will only be sketched. For x a closed point of $\mathfrak{X}$,

$$\check{\mathcal{O}}_{\mathfrak{X},x} \simeq \left(R[x_1, x_2] / \prod_{i=1}^{j} x_i - \pi \right)^{\check{}} ,$$

where $j = 1$ or 2, depending on whether or not X_0 is regular at x. Thus the strictly local ring at a non-regular point of $\mathfrak{X}^3$ has the form

$$(6.12.A) \qquad (R[x_1, x_2, x_3, x_4, x_5]/(x_1x_2 - \pi, x_1x_2 - x_3x_4))^{\check{}}$$

or

$$(6.12.B) \quad (R[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1x_2 - \pi, x_1x_2 - x_3x_4, x_1x_2 - x_5x_6))^{\check{}} .$$

The former situation has been treated in the proof of (6.3). The main step remaining is to analyze the blow up of the spectrum of the second ring along the component of the closed fiber defined by the ideal (x_1, x_3, x_5). This is left to the reader.

Write $\mathcal{Y}$ for the good model of X^3 constructed in (6.11) and write Y_0 for the special fiber. Some facts about the structure of Y_0 will be helpful in our construction of the local height pairing for the modified diagonal cycle on X^3. Write $\{C_a\}_{a\in A_0}$ for the components of X_0 and let S_0 be the set of singular points of X_0. Recall that the good model $\mathcal{Y}$ depends on a choice of ordering of the set A_0^3 which indexes the components of X_0^3. For $\underline{a} = (a, a', a'') \in A_0^3$ let $V_{\underline{a}}$ denote the component of Y_0 which maps to the component $Z_{\underline{a}} := C_a \times C_{a'} \times C_{a''}$ of X_0^3.

The following result will be useful in the construction of the height pairing.

LEMMA 6.13. — *The scheme Y_0 over k_0 depends only on X_0. It is independent of the choice of discrete valuation ring R and good model $\mathfrak{X}$ with special fiber X_0.*

Proof (Sketch). — Y_0 is a global normal crossing divisor on $\mathcal{Y}$ whose components are in bijective correspondence to those of X_0^3. We will describe how the components of Y_0 are obtained from those of X_0^3. Once this is done, it is not difficult to describe how the components intersect.

The main point is to show that the effect of each blow-up in the resolution $\mathcal{Y} \to \mathfrak{X}^3$ on the components of the special fiber may be described explicitly in terms of the special fiber itself without reference to the ambient scheme. Consider first $\sigma_{\underline{a}} : \hat{\mathfrak{X}}^3 \to \mathfrak{X}^3$, the blow-up of $\mathfrak{X}^3$ along the ideal sheaf of a component $Z_{\underline{a}}$ of X_0^3. It is straightforward to check that the strict transform of $Z_{\underline{b}}$, $\underline{b} \neq \underline{a}$ in $\hat{\mathfrak{X}}^3$ is isomorphic to $Z_{\underline{b}}$ blown up along $Z_{\underline{b}} \cap Z_{\underline{a}}$. With more effort we can describe the component $\hat{Z}_{\underline{a}}$ of the special fiber of $\hat{\mathfrak{X}}^3$ corresponding to $Z_{\underline{a}}$. Let $Z_{\underline{a}}^\circ$ denote the closed subscheme of X_0^3 consisting of all components except $Z_{\underline{a}}$. Write J for the ideal sheaf of the (reduced) singular locus of the divisor $Z_{\underline{a}}^\circ \cap Z_{\underline{a}}$ on $Z_{\underline{a}}$. One checks using the explicit coordinates (6.12) that the inverse image ideal sheaf [Ha],II.7.12.2, of J on $\hat{Z}_{\underline{a}}$ is an invertible sheaf of ideals. It is then not difficult to deduce from the universal property of blow-ups and the explicit geometry of the map $\hat{Z}_{\underline{a}} \to Z_{\underline{a}}$ that $\hat{Z}_{\underline{a}}$ is isomorphic to the blow-up of $Z_{\underline{a}}$ along J. Thus all the components of the special fiber of $\hat{\mathfrak{X}}^3$ may be explicitly described in terms of X_0^3. The same holds for how the components meet. Later blow-ups in the resolution process (6.11) are similar.

We give an explicit description of the components $V_{\underline{a}}$ of Y_0, leaving the details of the verification to the reader : The natural map

$$V_{\underline{a}} \to Z_{\underline{a}} \tag{6.14}$$

is a birational morphism which is biregular except possibly above points $\underline{c} = (c_1, c_2, c_3)$ with at least two of the c_i's contained in S_0. Let $\underline{b}$ be the minimal element in the ordered set A_0^3 such that the component $Z_{\underline{b}}$ contains $\underline{c}$. The nature of the fiber over $\underline{c}$ depends upon the dimension of $Z_{\underline{a}} \cap Z_{\underline{b}}$ at $\underline{c}$. If this dimension is 2, then (6.14) is locally an isomorphism. If the dimension is 1, then the fiber is isomorphic to $\mathbb{P}^1$. If $Z_{\underline{a}} = Z_{\underline{b}}$, then the fiber is either $\mathbb{P}^1$ or $\mathbb{P}^2$ blown up at 3 non-colinear points, depending on whether exactly two or all three of the c_i's are contained in S_0. Finally, the dimension of $Z_{\underline{a}} \cap Z_{\underline{b}}$ at $\underline{c}$ can be 0 only when all c_i's lie in S_0, in which case the fiber of (6.14) is again isomorphic to $\mathbb{P}^2$ blown up at 3 non-colinear points.

Example 6.15. — Suppose that X_0 consists of only two components C_a and $C_{a'}$. Then a good model $\mathcal{Y}$ can be constructed from $\mathfrak{X}^3$ in two steps. First blow up the component $Z_{(a,a,a)}$ then blow up the strict transform of $Z_{(a',a',a')}$. Of course the recipe (6.11) calls for us to continue to blow up the remaining six components of the special fiber. However these are now all Cartier divisors, so the last six blow ups have no effect.

We end this section with a lemma which gives information needed later about the desingularization process in (6.3). We recall the notations of (6.3) and write $\mathrm{pr}_i : \mathcal{W}_j \to \mathcal{U}_i, i \in \{1,2\}$, for the projection on the i'th factor.

LEMMA 6.16. — *$\mathrm{pr}_i \circ \sigma_j : \mathcal{W}_j \to \mathcal{U}_i$ is proper and flat.*

Proof. — The properness is clear. For the flatness it is useful to know that $\mathcal{W}_j$ is Cohen-Macaulay. It suffices to check that each $A_{q,r,s}$ is Cohen-Macaulay. This is the case because $A_{q,r,s}$ is the quotient of a regular ring by an ideal generated by a regular sequence (6.5).

Since $\mathcal{U}_i$ is regular and $\mathcal{W}_j$ is Cohen-Macaulay, flatness of $\mathrm{pr}_i \circ \sigma_j$ will follow if all fibers have the same dimension [Al Kl],V.3.5. To verify this we begin with a closed point $(u_1, u_2) \in \mathcal{U}_1 \times_{\mathrm{Spec} R} \mathcal{U}_2$. Suppose r components of $(U_1)_0$ pass through u_1 and s components of $(U_2)_0$ pass through u_2. Locally in the étale topology $\mathcal{U}_1 \times_{\mathrm{Spec} R} \mathcal{U}_2$ is given by $\mathrm{Spec}(A_{r,r,s})$ (6.5) and each blow up by (6.8). Choose $w \in \sigma_j^{-1}((u_1, u_2))$ so that at any stage in the sequence of blow ups, $\sigma_j : \mathcal{W}_j \to \mathcal{U}_1 \times_{\mathrm{Spec} R} \mathcal{U}_2$, the image of w in any chart of the form (6.8.A) (respectively (6.8.B)) is the maximal ideal $(\pi, x_1, ..., x_i', ..., x_{n_1+n_2})$ (respectively $(\pi, x_1, ..., x_{n+l}', ..., x_{n_1+n_2})$). We divide those components of the special fiber of $\mathcal{U}_1 \times_{\mathrm{Spec} R} \mathcal{U}_2$ which are centers of non-trivial blow-ups dominated by $\mathcal{O}_{\mathcal{W}_j,w}$ into two classes depending upon whether the image of w lies in a chart of type (6.8.A) or (6.8.B). If l (respectively i) is fixed, then there is at most one i (respectively l) such that the ideal (x_i, x_{n_1+l}) defines a component in the first (respectively second) class. We may thus reindex the x_j's so that there are natural numbers r' and s' satisfying $r' \leq r$ and $s' \leq s$ and functions

$$p : \{s'+1, ..., s\} \to \{1, ..., r\} \qquad \text{and} \qquad m : \{r'+1, ..., r\} \to \{1, ..., s\}$$

such that the ideals in the first (respectively second) class are

$$\{(x_{p(l)}, x_{n_1+l}) : s'+1 \leq l \leq s\},$$

respectively $\{(x_i, x_{n_1+m(i)}) : r'+1 \leq i \leq r\}$.

Set $\Omega_i = p^{-1}(i)$. By applying (6.8.A) $s - s'$ times and (6.8.B) $r - r'$ times, we find that the map $\mathrm{pr}_1 \circ \sigma_j$ corresponds to the map of $\check{R}$-algebras :

$$\begin{aligned}&\check{R}[x_1, ..., x_{n_1}]/(x_1...x_r - \pi)\\ &\to \check{R}[x_1, ..., x_{n_1+n_2}]/(x_1...x_r x_{n_1+s'+1}...x_{n_1+s} - \pi, x_1...x_{r'} - x_{n_1+1}...x_{n_1+s'}),\end{aligned}$$

$$x_i \to x_i \quad \text{when} \quad r < i \leq n_1; \quad x_i \to x_i \prod_{l\in\Omega_i} x_{n_1+l}, \quad \text{when} \quad 1 \leq i \leq r.$$

Here $\prod_{l\in\Omega_i} x_{n_1+l} = 1$ if Ω_i is empty. The fiber over the maximal ideal $(\pi, x_1, ..., x_{n_1})$ is Spec of

$$\begin{aligned}\bar{k}_0[x_1, ..., x_{n_1+n_2}]/(&x_1\left(\prod_{l\in\Omega_1} x_{n_1+l}\right), \ \dots, x_r\left(\prod_{l\in\Omega_r} x_{n_1+l}\right), \ x_{r+1}, \ \dots, x_{n_1},\\ &x_1...x_r x_{n_1+s'+1}...x_{n_1+s}, \ x_1...x_{r'} - x_{n_1+1}...x_{n_1+s'}),\end{aligned}$$

which clearly has dimension $n_2 - 1$, independent of what partition Ω of $\{s'+1, ..., s\}$ occurs and what the values of r, r', s, or s' may be. Since any component of the fiber $(\mathrm{pr}_1 \circ \sigma_j)^{-1}(u_1)$ includes a point w as above, all fibers have dimension $n_2 - 1$.

7. Extending the modified diagonal cycle to the regular model.

We retain the notation of the previous section : X is a smooth, projective, geometrically connected curve over the discretely valued field

k, and $\mathfrak{X}$ is a good, semi-stable model (assumed to exist) for X over the valuation ring R. Recall that A_0 indexes the components of X_0 and that an ordering of A_0^3 has been fixed. Let $\mathcal{Y} \to \mathfrak{X}^3$ be the good model of X^3 over R constructed in (6.11) and let Y_0 be the special fibre of $\mathcal{Y}$ over the residue field k_0. Let V_0 be the normalization of Y_0 over k_0 and write

$$\lambda : V_0 \longrightarrow Y_0 \longrightarrow \mathcal{Y} \tag{7.1}$$

for the composition of the natural maps over R.

We assume that $\mathfrak{X}$ has a section $\underline{e}$ over R. Write e for its generic point on X and e_0 for its specialization to X_0. Then e_0 is not a singular point, so the component C_0 of X_0 containing e_0 is uniquely defined. Let Δ_e be the modified diagonal cycle on X^3 constructed in §3, and extend Δ_e to a cycle $\underline{\Delta_{\underline{e}}}$ of codimension 2 on $\mathcal{Y}$ by summing with appropriate signs the closures of the irreducible components $\Delta_{123}, \Delta_{ij}, \Delta_i$ of Δ_e in $\mathcal{Y}$.

When X_0 is not smooth, this naive extension of Δ_e is not sufficient to construct a local height pairing. We must find an extension which pulls back to a numerically trivial cycle on V_0. In this section, we will prove that such a modification of $\underline{\Delta_{\underline{e}}}$ exists in many cases, such as when k is a global field.

Proposition 7.2. — *Assume that the residue field k_0 is finitely generated over the prime field. Then there is a rational divisor $z \in Z^1(V_0) \otimes \mathbb{Q}$ such that $\lambda^*(\underline{\Delta_{\underline{e}}} - \lambda_*(z))$ is numerically equivalent to zero in $Z^2(V_0) \otimes \mathbb{Q}$.*

Remark 7.3. — The class $\underline{\Delta_{\underline{e}}} - \lambda_*(z)$ in $Z^2(\mathcal{Y}) \otimes \mathbb{Q}$ also extends Δ_e on the general fibre. The construction of the divisor z in Proposition 7.2 is indirect, through cohomology, via the truth of the Tate conjectures for V_0. It would be useful to have a more explicit construction.

To begin the proof of 7.2, we describe the cycle

$$\underline{\Delta_{\underline{e}}} = \sum_T (-1)^{|T|+1} \underline{\Delta_T},$$

where the sum is over non-empty subsets $T \subset \{1,2,3\}$. The closure of each $\Delta_T \subset X^3$ in $\mathfrak{X}^3$ may be identified with the image of a closed immersion $\delta_T : \mathfrak{X} \to \mathfrak{X}^3$. For example, when $T = \{1,2\}$, δ_T is the diagonal on the first two factors of $\mathfrak{X}^3$ and projection onto the section $\underline{e}$ on the third factor.

Define S_T to be the set of all $s \in S_0$ such that

(1) $\delta_T(s)$ is a singular point of $\mathfrak{X}^3$ and

(2) If $\underline{a}$ is the minimal element in the ordered set A_0^3 for which $s \in Z_{\underline{a}}$, then $Z_{\underline{a}} \cap \delta_T(\mathfrak{X})$ is zero dimensional at s.

LEMMA 7.4. — *Let $\sigma_T : \mathfrak{X}_T \to \mathfrak{X}$ be the blow-up of $\mathfrak{X}$ along S_T. Then $\delta_T \circ \sigma_T$ lifts to a closed immersion*

$$\delta'_T : \mathfrak{X}_T \xrightarrow{\sim} \underline{\Delta}_T \subset \mathcal{Y}.$$

Proof. — If $\delta_T(x) \in \mathfrak{X}^3$ is a regular point, then the resolution $\mathcal{Y} \to \mathfrak{X}^3$ does not change $\delta_T(\mathfrak{X})$ in a neighborhood of $\delta(x)$. If $\delta_T(x) \in \mathfrak{X}^3$ is not regular, then $x \in S_0$. We have an explicit description of the strict henselization of $\mathfrak{X}$ and hence of $\mathfrak{X}^3$ (6.12). Using these coordinates we check that the inverse image ideal sheaf $\delta_T^{-1} I_{Z_{\underline{a}}}$ [Ha], II.7.12.2, is the ideal sheaf of the reduced scheme $Z_{\underline{a}} \cap \delta_T(\mathfrak{X})$. Thus if $Z_{\underline{a}} \cap \delta_T(\mathfrak{X})$ is locally one dimensional at x, then $\delta_T^{-1} I_{Z_{\underline{a}}}$ is locally an invertible sheaf and the blow-up of the ambient space along the strict transform of $Z_{\underline{a}}$ does not modify $\delta_T(\mathfrak{X})$ near $\delta_T(x)$. On the other hand, if $Z_{\underline{a}} \cap \delta_T(\mathfrak{X})$ is zero dimensional at x, then blowing up the ambient space along the strict transform of $Z_{\underline{a}}$ causes $\delta_T(\mathfrak{X})$ to be blown-up at $\delta_T(x)$. In either case, the fiber over $\delta_T(x)$ in the strict transform of $\delta_T(\mathfrak{X})$ is disjoint from the singular locus of the ambient scheme once the strict transform of $Z_{\underline{a}}$ has been blown up. Thus later blow-ups in the resolution $\mathcal{Y} \to \mathfrak{X}^3$ do not cause further modifications of the strict transform of $\delta_T(\mathfrak{X})$ above $\delta_T(x)$.

We may now write

$$\lambda^*(\underline{\Delta}_{\underline{e}}) = \sum_{\underline{a} \in A_0^3} \sum_{T} (-1)^{|T|+1}\, \delta'_T(\mathfrak{X}_T) \cdot V_{\underline{a}}. \tag{7.5}$$

Each exceptional $\mathbb{P}^1$ in the special fiber of $\mathfrak{X}_T$ has multiplicity two, since it meets the other fiber components in two points. Such a $\mathbb{P}^1$ is contained in exactly two $V_{\underline{a}}$'s and thus appears twice in (7.5), each time with multiplicity $(-1)^{|T|+1}$. The non-exceptional components of the special fiber of $\mathfrak{X}_T$ are contained in a unique $V_{\underline{a}}$ as indicated in the following table. (Abbreviations : s.t.=strict transform, d.=diagonal.)

T	$\underline{a}$	multiplicity	non-exceptional component in $V_{\underline{a}}$
$\{1,2,3\}$	(a,a,a)	1	s.t. of small d. in C_a^3
$\{1,2\}$	$(a,a,0)$	-1	s.t. of d. in $C_a^2 \times e_0$
$\{1\}$	$(a,0,0)$	1	s.t. of $C_a \times e_0^2$ in $Z_{\underline{a}}$.

When $T = \{1,3\}$ or $\{2,3\}$ the situation is analogous to the case $T = \{1,2\}$. Similarly when $T = \{2\}$ or $\{3\}$ the situation is analogous to the case $T = \{1\}$.

LEMMA 7.6. — 1) *The cycle $\lambda^*(\underline{\Delta}_{\underline{e}})$ in $Z^2(V_0)$ depends only on X_0 and e_0, not on $R, \mathfrak{X}$ or $\underline{e}$.*

2) *For $z \in Z^1(V_0)$, the class $\lambda^*\lambda_* z$ in $CH^2(V_0)$ depends only on X_0, not on R or $\mathfrak{X}$.*

Proof. — 1) We have seen in (6.13) that V_0 depends only on X_0 and is independent of R and $\mathfrak{X}$. It is clear that the components of $\lambda^*(\underline{\Delta}_{\underline{e}})$ listed in the table depend only on X_0 and e_0. It follows from (7.4) that the other components of $\lambda^*(\underline{\Delta}_{\underline{e}})$ depend only on X_0, e_0, and the choice of ordering on A_0^3.

2) We have $\lambda^*\lambda_* z = c_1(\mathcal{L}) \cdot z$, where $\mathcal{L}$ is the invertible sheaf on V_0 whose restriction to each component $V_{\underline{a}}$ is the normal sheaf $N_{V_{\underline{a}}/\mathcal{Y}}$. Since $(Y_0 \cdot V_{\underline{a}}) = 0$, we have $(V_{\underline{a}} \cdot V_{\underline{a}}) = -\sum_{\underline{b} \neq \underline{a}} (V_{\underline{b}} \cdot V_{\underline{a}})$. Hence $N_{V_{\underline{a}}/\mathcal{Y}} \simeq \mathcal{O}_{V_{\underline{a}}}\left(-\sum_{\underline{b}\neq\underline{a}} (V_{\underline{b}} \cdot V_{\underline{a}})\right)$. This line bundle is independent of R and $\mathfrak{X}$, as the intersections $(V_{\underline{b}} \cdot V_{\underline{a}})$ in $\mathcal{Y}$ depend only on X_0.

LEMMA 7.7. — *The Galois group $\mathrm{Gal}(\overline{k}_0/k_0)$ acts semi-simply on the ℓ-adic cohomology groups of $V_0/\overline{k}_0$, and the Tate conjecture is true for V_0 : the subspace $H^{2i}(V_0/\overline{k}_0, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}_0/k_0)}$ is the $\mathbb{Q}_\ell$-span of the classes of algebraic cycles of codimension i on V_0/k_0.*

Proof. — It suffices to check this for each component $V_{\underline{a}}$ of V_0. In each case, the cohomology is generated by the classes of algebraic cycles and H^1 of curves. Hence the Galois group acts semi-simply [Ta2].

The Tate conjecture for divisors is birationally invariant, and is true for products of curves [Ta1], [Ta2], Thm. 5.2. Hence it is true for divisors on the components $V_{\underline{a}}$ of V_0. It is then true for 1-cycles on these components, by the hard Lefshetz theorem.

We let $N^i(V_0)$ be the $\mathbb{Q}$-vector space in $H^{2i}(V_0/\overline{k}_0, \mathbb{Q}_\ell(i))$ spanned by the classes of algebraic cycles of codimension i on V_0/k_0. By Lemma 7.7 we have

$$N^i(V_0) \otimes \mathbb{Q}_\ell = H^{2i}(V_0/\overline{k}_0, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{k}_0/k_0)}. \tag{7.8}$$

The following cohomological argument, due to Beilinson [Be2], 1.1.2, gives a proof of Proposition 7.2 in the equicharacteristic case.

LEMMA 7.9. — *Assume that R is the henselization of the local ring of a k_0-rational point on a smooth curve over k_0. Let $\check{R}$ be the strict henselization of R, so* $\mathrm{Gal}(\check{R}/R) = \mathrm{Gal}(\overline{k}_0/k_0) = G$. *Assume that Δ is any cycle in $Z^2(Y)_{\mathrm{hom}}$, and let $\underline{\Delta}$ be its closure in $\mathcal{Y}/R$. Then there is a divisor $z \in Z^1(V_0) \otimes \mathbb{Q}$ such that*

$$\lambda^*(\underline{\Delta} - \lambda_*(z)) = 0 \qquad \text{in} \quad N^2(V_0).$$

Proof. — Let L be the fraction field of $\check{R}$ and let $\overline{L}$ be a separable closure of L. Put $\mathcal{G} = \mathrm{Gal}(\overline{L}/L)$ we then have the exact sequence in étale cohomology

$$0 \to H^3(Y/\overline{L}, \mathbb{Q}_\ell(2))_{\mathcal{G}} \longrightarrow H^4(Y/L, \mathbb{Q}_\ell(2)) \longrightarrow H^4(Y/\overline{L}, \mathbb{Q}_\ell(2))^{\mathcal{G}} \longrightarrow 0.$$

We also have an exact sequence in étale cohomology with supports :

$$\begin{aligned} &H^4_{Y_0}(\mathcal{Y}/\check{R}, \mathbb{Q}_\ell(2)) \longrightarrow H^4(\mathcal{Y}/\check{R}, \mathbb{Q}_\ell(2)) \\ &\qquad\qquad \longrightarrow H^4(Y/L, \mathbb{Q}_\ell(2)) \longrightarrow H^5_{Y_0}(\mathcal{Y}/\check{R}, \mathbb{Q}_\ell(2)). \end{aligned}$$

Deligne has defined a weight filtration $W_\bullet$ on these groups, and has shown [De2], 1.8, 3.6.3, that

$$\begin{aligned} W_0 H^3(Y/\overline{L}, \mathbb{Q}_\ell(2))_{\mathcal{G}} &= 0 \\ W_0 H^5_{Y_0}(\mathcal{Y}/\check{R}, \mathbb{Q}_\ell(2)) &= 0. \end{aligned}$$

Hence, there is an exact sequence

$$\begin{aligned} &W_0 H^4_{Y_0}(\mathcal{Y}/\check{R}, \mathbb{Q}_\ell(2)) \longrightarrow W_0 H^4(\mathcal{Y}/\check{R}, \mathbb{Q}_\ell(2)) \\ &\qquad\qquad \longrightarrow W_0 H^4(Y/\overline{L}, \mathbb{Q}_\ell(2))^{\mathcal{G}} \longrightarrow 0. \end{aligned}$$

The class of $\underline{\Delta}$ lies in $W_0H^4(\mathcal{Y}/\check{R},\mathbb{Q}_\ell(2))$, and maps to zero in $W_0H^4(Y/\overline{L},\mathbb{Q}_\ell(2))^{\mathcal{G}}$, as Δ is homologically trivial over $\overline{L}$. Hence $c\ell(\underline{\Delta})$ is in the image of an element of $W_0H^4_{Y_0}(\mathcal{Y}/\check{R},\mathbb{Q}_\ell(2))$. But we have an isomorphism of weighted vector spaces

$$H^4_{Y_0}(\mathcal{Y}/\check{R},\mathbb{Q}_\ell(2)) \simeq H_4(Y_0/\overline{k}_0,\mathbb{Q}_\ell(-2)).$$

Since Y_0 is a normal crossing divisor, the map

$$H^2(V_0/\overline{k}_0,\mathbb{Q}_\ell(1)) \simeq H_4(V_0/\overline{k}_0,\mathbb{Q}_\ell(-2)) \longrightarrow H_4(Y_0/\overline{k}_0,\mathbb{Q}_\ell(-2))$$

is surjective. Hence there is a class ξ in $H^2(V_0/\overline{k}_0,\mathbb{Q}_\ell(1))$ such that $\lambda_*(\xi) = c\ell(\underline{\Delta})$ in $W_0H_4(\mathcal{Y}/\check{R},\mathbb{Q}_\ell(2))$.

We may further assume that ξ is fixed by $G = \mathrm{Gal}(\overline{k}_0/k_0)$, as $c\ell(\underline{\Delta})$ is fixed and G acts semi-simply on $H^2(V_0/\overline{k}_0,\mathbb{Q}_\ell(1))$ by Lemma 7.7. The same lemma then shows that ξ lies in the subspace $N^1(V_0)\otimes\mathbb{Q}_\ell$.

But $\lambda^*\lambda_*(\xi) = \lambda^*c\ell(\underline{\Delta})$ lies in the rational vector space $N^2(V_0)$. Hence, we may modify ξ by an element of $\ker \lambda^*\lambda_*$ to obtain a class $z \in N^1(V_0)$ with $\lambda^*(\underline{\Delta} - \lambda_*(z)) = 0$ in $N^2(V_0)$. This completes the proof.

To prove Proposition 7.2 in the case when R has mixed characteristic, we use Lemma 7.6 which shows that the class of $\lambda^*(\underline{\Delta_e})$ and $\lambda^*\lambda_*(z)$ depends only on the special fibre Y_0, not on the choice of R or $\mathcal{Y}$. We construct an equicharacteristic deformation with the same special fibre, using Lemma 7.11 below, and use Lemma 7.9 (with $\Delta = \Delta_e$) to complete the proof.

In fact, we have the following result, which is slightly stronger than Proposition 7.2. Let $\check{R}$ denote the strict henselization of R.

LEMMA 7.10. — *The cohomology class of the cycle $\underline{\Delta_e} - \lambda_*(z)$ is zero in $H^4(\mathcal{Y}/\check{R},\mathbb{Q}_\ell(2))$.*

Proof. — The image of this class under the isomorphism [Mi], VI.2.7

$$i^* : H^4(\mathcal{Y}/\check{R},\mathbb{Q}_\ell(2)) \xrightarrow{\sim} H^4(Y_0/\overline{k}_0,\mathbb{Q}_\ell(2))$$

is $G = \mathrm{Gal}(\overline{k}_0/k_0)$-invariant.
It is therefore contained in $W_0H^4(Y_0/\overline{k}_0,\mathbb{Q}_\ell\ (2))$, and lies in the subspace W_{-1} if and only if it is zero. Since the pullback map

$$\kappa^* : \mathrm{Gr}^W_0 H^4(Y_0/\overline{k}_0,\mathbb{Q}_\ell(2)) \longrightarrow H^4(V_0/\overline{k}_0,\mathbb{Q}_\ell(2))$$

is injective, and $\lambda^*(c\ell(\underline{\Delta_e}) - \lambda_*(z)) = 0$, the lemma follows.

LEMMA 7.11. — *There is a finite, separable extension L_0 of k_0, and a good, semi-stable model $\mathfrak{X}$ over the henselization R of $L_0[t]$ at the maximal ideal (t), whose closed fibre $\mathfrak{X} \times_R L_0$ is isomorphic to $X_0 \times_{k_0} L_0$.*

Proof. — This follows from the moduli theory of stable curves, due to Deligne and Mumford [DeMu]. When $g = 1$ it is well known from Kodaira's theory of singular fibres for elliptic surfaces, so we will assume that $g \geq 2$.

We first contract the maximal chains of genus zero components in X_0 which meet other components in exactly 2 points to singular points in the *stable* curve $\tilde{X}_0$. Label the singular points of $\tilde{X}_0$ which arise in this way $x_i, 1 \leq i \leq r$. Let n_i be the number of curves contracted above the point x_i.

Let H denote the Hilbert scheme of genus g, tricanonically embedded, stable curves over k_0. Choosing a basis for $H^0(\tilde{X}_0, \omega_{\tilde{X}_0}^{\otimes 3})$ allows us to identify $\tilde{X}_0$ with the fibre over $h \in H$ of the universal family of stable curves over H

$$\begin{array}{ccc} \tilde{X}_0 & \longrightarrow & Z \\ \Big\downarrow & & \Big\downarrow \\ h & \longrightarrow & H \end{array} \quad .$$

But H is smooth of dimension $d = 5(g-1)$ over k_0. Moreover, we may choose an isomorphism over $\overline{k}_0$

$$\widehat{\mathcal{O}}_{H,h} \simeq \overline{k}_0[[t_1, \ldots, t_d]]$$

so that

$$\widehat{\mathcal{O}}_{Z,x_i} \simeq \overline{k}_0[[t_1, \ldots, t_d, u, v]]/(uv - t_i)$$

as $\widehat{\mathcal{O}}_{H,h}$-algebras [DeMu], 1.6, at each singular point $x_i, 1 \leq i \leq r$.

Let $\overline{R}$ be the henselization of $\overline{k}_0[t]$ at the maximal ideal (t), and choose a local homomorphism of $\overline{k}_0$-algebras $\phi : \mathcal{O}_{H,h} \to \overline{R}$ such that the induced map $\widehat{\phi}$ on completions satisfies $\operatorname{ord}(\widehat{\phi}(t_i)) = n_i$ for $1 \leq i \leq r$.

Using ϕ, we define the fibre product over $\overline{R}$

$$\overline{\mathfrak{X}} = \operatorname{Spec} \overline{R} \times_H Z.$$

Since

$$\widehat{\mathcal{O}}_{\overline{\mathfrak{X}},x_i} \simeq \overline{k}_0[[t, u, v]]/(uv - t^{n_i})$$

we may recover a good, semi-stable model $\mathfrak{X}$ by $[n_i/2]$ successive blow-ups at each point x_i [DeMu], p. 85. Clearly $\mathfrak{X}$ can be defined over the henselization R of $L_0[t]$, where L_0 is a finite extension of k_0 contained in $\overline{k}_0$. We have $\mathfrak{X} \times_R L_0 \simeq X_0 \times_{k_0} L_0$, as the blow ups at x_i give rise to the chain of genus 0 curves which were contracted to obtain $\tilde{X}_0$ from X_0.

The final result of this section extends Proposition 7.2. Let $\tau \in Z^3(Y \times Y) \otimes \mathbb{Q}$ be a self-correspondence. We show that Proposition 7.2 remains valid when $\tau_*\Delta_e$ is substituted for Δ_e.

PROPOSITION 7.12. — *Suppose given $\Theta \in Z^2(\mathcal{Y}) \otimes \mathbb{Q}$ such that $\lambda^*\Theta \equiv 0$. Let $\theta \in Z^2(Y) \otimes \mathbb{Q}$ denote the restriction of Θ to the generic fiber. Then there is a cycle class $\underline{\tau}_*\Theta \in CH^2(\mathcal{Y}) \otimes \mathbb{Q}$ satisfying $\lambda^*\underline{\tau}_*\Theta \equiv 0$ whose restriction to the generic fiber is $\tau_*\theta \in CH^2(Y) \otimes \mathbb{Q}$.*

Proof. — The proof involves extending the correspondence τ from $Y \times Y$ to a good model which we call $\mathcal{W}$. $\mathcal{W}$ is constructed from $\mathcal{Y} \times_{\text{Spec } R} \mathcal{Y}$ as in (6.3). Write f_i for the composition

$$\mathcal{W} \xrightarrow[\sigma]{} \mathcal{Y} \times_{\text{Spec } R} \mathcal{Y} \xrightarrow[\text{pr}_i]{} \mathcal{Y},$$

where $\text{pr}_i, i \in \{1, 2\}$ is projection on the i'th factor. Recall that f_i is proper and flat (6.16). Write $\underline{\tau} \in Z^3(\mathcal{W}) \otimes \mathbb{Q}$ for the closure of $\tau \in Z^3(Y \times Y) \otimes \mathbb{Q}$. Define

$$\underline{\tau}_*\Theta = f_{2*}(\underline{\tau} \cdot f_1^*(\Theta)) \in CH^2(\mathcal{Y}) \otimes \mathbb{Q},$$

where the intersection product takes place in $CH^\bullet(\mathcal{W}) \otimes \mathbb{Q}$ [GiSo], §8.

To compute $\lambda^*(\underline{\tau}_*\Theta)$ we use the following commutative diagram

$$\begin{array}{ccccccc}
V_0 & \overset{g_1}{\leftarrow} & W_0 & \overset{\nu}{\rightarrow} & \bar{W}_0 & \overset{f_2'}{\rightarrow} & V_0 \\
\downarrow \lambda & & \downarrow \varphi & & \downarrow \bar{\varphi} & & \downarrow \lambda \\
\mathcal{Y} & \underset{f_1}{\leftarrow} & \mathcal{W} & = & \mathcal{W} & \underset{f_2}{\rightarrow} & \mathcal{Y},
\end{array}$$

where $\varphi : W_0 \to \mathcal{W}$ is the normalization of the special fiber and the right hand square is Cartesian. Thus f_2' is flat and the restrictions of λ, φ and $\bar{\varphi}$ to connected components of their domains are regular embeddings. Furthermore, ν is the normalization of $\bar{W}_0$ and g_1 is a morphism between non-singular projective varieties. We are thus in a position to apply intersection theory (cf. [Fu], p. 395) to obtain :

$$\begin{aligned}
\lambda^*(\underline{\tau}_*\Theta) &= \lambda^* f_{2*}(\underline{\tau}\cdot f_1^*\Theta) & \\
&= f'_{2*}\bar{\varphi}^*(\underline{\tau}\cdot f_1^*\Theta) & \text{[Fu]}, 1.7\\
&= f'_{2*}\nu_*\varphi^*(\underline{\tau}\cdot f_1^*\Theta) & \text{[Fu]}, 2.3(b)\\
&= f'_{2*}\nu_*(\varphi^*\underline{\tau}\cdot \varphi^* f_1^*\Theta) & \text{[GiSo]}, 8.3\\
&= f'_{2*}\nu_*(\varphi^*\underline{\tau}\cdot g_1^*\lambda^*\Theta) & \text{[Fu]}, 6.5, 6.6(c)\\
&\equiv 0, & \text{since } \lambda^*\Theta\equiv 0.
\end{aligned}$$

8. The height pairing.

We now assume X is defined over a number field k, and that S is the spectrum of the ring of integers of k. We assume that X has a model $\mathfrak{X}$ over S which is good in the sense of (6.1) at each finite place of k (cf. Example 6.2). We may then construct a regular model $\mathcal{Y}$ for $Y = X^3$ over S, following the desingularization procedure (6.11). By Proposition 7.2, the modified diagonal cycle Δ_e on Y has an extension to a rational class on $\mathcal{Y}$ which is numerically trivial in each fibral component.

Bloch [Bl] and Beilinson [Be1], [Be2] have conditionally constructed, for any 3-fold Y over k, a symmetric height pairing

$$\langle\ ,\ \rangle : CH^1(Y)_{\mathrm{hom}}\otimes CH^1(Y)_{\mathrm{hom}} \longrightarrow \mathbb{R} \tag{8.1}$$

which promises to be an interesting tool in the investigation of cycle classes. The definition of the pairing $\langle a,b\rangle$ is made under the hypotheses that Y has a regular model $\mathcal{Y}$ over S, and that at least one of the cycles a, b has a rational extension to $\mathcal{Y}$ which is numerically trivial in every fibral component.

By (7.12) and the remarks above, the pairing

$$\langle \tau_*(\Delta_e), \tau'_*(\Delta_e)\rangle \tag{8.2}$$

is well-defined, for any self-correspondences $\tau, \tau' \in CH^3(Y\times Y)$ of Y. In [GrKu], §13, we conjectured the value of this pairing, for X a Shimura curve over $\mathbb{Q}$, in terms of the first derivative of triple product L-functions. The following result is useful.

PROPOSITION 8.3. — *Let $T = CH^1(X\times X)$ be the ring of correspondences on X, and assume that the correspondence $\tau' \in T^{\otimes 3} \subseteq$*

$CH^3(Y \times Y)$ of Y annihilates the module $H^0(Y, \Omega^3_Y)$. Then for any $\tau \in T^{\otimes 3}$, $\langle \tau_*(\Delta_e), \tau'_*(\Delta_e)\rangle = 0$.

Proof. — We have $H^0(Y, \Omega^3_Y) = H^0(X, \Omega^1_X)^{\otimes 3}$ by the Künneth decomposition. If $I \subset T$ is the ideal which annihilates $H^0(X, \Omega^1_X)$, then the ideal of $T^{\otimes 3}$ annihilating $H^0(Y, \Omega^3_Y)$ is the sum

$$I \otimes T \otimes T + T \otimes I \otimes T + T \otimes T \otimes I.$$

It therefore suffices to treat the case $\tau' = t \times 1 \times 1$ with $t \in I$.

Since $\mathrm{char}(k) = 0$, t annihilates $H^0(X, \Omega^1_X)$ if and only if it induces the zero endomorphism of the Jacobian J of X. This implies that t is rationally equivalent to a sum of correspondences of the form $X \times d$ or $d \times X$, where d is a point of X [Fu], 16.1.2.

If $t = X \times d$, then $\tau' = t \times 1 \times 1$ is the graph of the morphism

$$\begin{aligned} f : Y &\longrightarrow Y \\ (x_1, x_2, x_3) &\longmapsto (d, x_2, x_3). \end{aligned}$$

Hence $\tau'_*(\Delta_e) = f_*(\Delta_e)$. But

$$\begin{aligned} f_*(\Delta_{123}) = f_*(\Delta_{23}) &= \{(d, x, x)\} \\ f_*(\Delta_{12}) = f_*(\Delta_2) &= \{(d, x, e)\} \\ f_*(\Delta_{13}) = f_*(\Delta_3) &= \{(d, e, x)\} \\ f_*(\Delta_1) &= 0. \end{aligned}$$

Therefore $\tau'_*(\Delta_e) = 0$ in $CH^2(Y)_{\mathrm{hom}}$, so we clearly have $\langle \tau_*(\Delta_e), \tau'_*(\Delta_e)\rangle = 0$.

The case when $t = d \times X$ is more interesting. Then τ'_* is the transpose of f_*, so $\tau'_*(\Delta_e) = f^*(\Delta_e)$ in the sense of intersection theory. But

$$\begin{aligned} f^*(\Delta_{123}) &= \{(x, d, d)\} \\ f^*(\Delta_{12}) &= \{(x, d, e)\} \\ f^*(\Delta_{13}) &= \{(x, e, d)\} \\ f^*(\Delta_1) &= \{(x, e, e)\} \\ f^*(\Delta_{23}) = f^*(\Delta_2) = f^*(\Delta_3) &= 0. \end{aligned}$$

Therefore

$$\begin{aligned}\tau'_*(\Delta_e) &= \{(x,d,d)\} - \{(x,d,e)\} - \{(x,e,d)\} + \{(x,e,e)\}\\ &= p_{23}^*(\mathfrak{a})\end{aligned}$$

where $\mathfrak{a}$ is the zero cycle

$$\mathfrak{a} = (d,d) - (d,e) - (e,d) + (e,e) \tag{8.4}$$

on the surface X^2.

Here it is not at all obvious that $\tau'_*(\Delta_e)$ is zero in $CH^2(Y)$, or that $\mathfrak{a}$ is zero in $CH^2(X^2)$. But $\mathfrak{a}$ does have degree zero, and trivial class in the Albanese variety of X^2, so has trivial height pairing against any class in $\mathrm{Pic}^0(X^2)$ by the Néron-Tate theory [Né]. But this implies that

$$\begin{aligned}\langle \tau_*(\Delta_e), \tau'_*(\Delta_e)\rangle &= \langle \tau_*(\Delta_e), p_{23}^*(\mathfrak{a})\rangle_Y\\ &= \langle (p_{23})_*\tau_*(\Delta_e), \mathfrak{a}\rangle_{X^2}\\ &= 0\end{aligned}$$

which completes the proof.

Note 8.5. — The conjectures of Beilinson and Bloch predict that the class of the zero cycle $\mathfrak{a}$ defined in (8.4) has finite order in $CH_0(X^2)$ when k is a number field and d and e are rational points on X over k. This need *not* be true when $k = \mathbb{C}$ [Mu].

Note 8.6. — Let $X = X_0(N)$ be the modular curve over $\mathbb{Q}$ and let e be the cusp $i\infty$ of X. Let T be the commutative ring generated by the Hecke correspondences T_m of X, for all m prime to N. If $F = f * g * h$ is a triple product of newforms of weight 2 for $\Gamma_0(N)$, write $(H^0(Y,\Omega^3)\otimes\mathbb{R})^F$ for the F-isotypic component of the $(T^{\otimes 3}\otimes\mathbb{R})$-module $H^0(X,\Omega^1_X)^{\otimes 3}\otimes\mathbb{R}$. This has dimension one over $\mathbb{R}$. Let t_F be any $\mathbb{R}$-linear combination of Hecke correspondences $T_{m_1}\otimes T_{m_2}\otimes T_{m_3}$ of Y which projects to this eigencomponent, and put $\Delta_F = t_F(\Delta_e)$. Then by Proposition 8.3 the pairing $\langle\Delta_F,\Delta_F\rangle$ is well-defined. Indeed, if t'_F is another projector, the difference $t_F - t'_F$ annihilates $H^0(Y,\Omega^3)\otimes\mathbb{R}$. The precise value of $\langle\Delta_F,\Delta_F\rangle$ is conjectured in [GrKu], §13, when N is square-free : it should be zero unless $a_p(f)*a_p(g)*a_p(h) = -1$ for all primes p dividing N, in which case it should be given by a simple (non-zero) multiple of the central critical derivative $L'(F,2)$.

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