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GROWTH ORDERS OCCURRING IN EXPANSIONS OF HARDY-FIELD SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

by John SHACKELL

1. Introduction.

The reasons for seeking asymptotic-series solutions of differential equations are well known, [2], and need not be repeated here. One point that should be made however, is that there is often special interest in the behaviour of solutions in the neighbourhood of singularities and here powers of the variable may not be appropriate to describe the asymptotics. One can consider asymptotic series over other base functions, such as $\log x$ for example, but for non-linear equations that rather begs the question as to what the possible asymptotic behaviour of a solution can be. On the other hand it is not sensible to ask such a question without placing *some* restriction on the class of solutions considered. This follows from the main result of [11], where a fourth-order algebraic differential equation is given which has solutions arbitrarily close to *any* pre-assigned continuous function. One natural restriction is to look for solutions which lie in some *Hardy field*, that is to say in a differential field of germs, at $+\infty$, of real-valued $\mathcal{C}^\infty$ functions. This is the approach we shall take in this paper.

Hardy fields each carry a natural order which reflects the asymptotic growth of their elements, and they therefore provide a very natural setting for studying many questions on asymptotics. Moreover one can get some

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useful information regarding the asymptotic growth of Hardy-field solutions of algebraic differential equations over $\mathbb{R}$ (and indeed over other Hardy fields). In [14] it was shown firstly that the asymptotic behaviour of such a function could be represented by a *nested form*, which is a particular type of expression constructed using exponentials and logarithms. Secondly, it was shown that only a restricted number of these are possible for solutions of an equation of given order. For small orders the set of possibilities is quite manageable, and one can substitute its members into a given equation and check whether they correspond to possible solutions of it.

In [16], it was proved that the number of possible nested forms which can occur as Hardy-field solutions of an algebraic differential equation over $\mathbb{R}$ of order n, has growth bounded by $\exp(\mu n)$, where μ is a constant. Since one is usually interested in equations of small order, this exponential growth would not matter very much if all one ever wanted to know about was nested forms. However the nested form of a function corresponds only to the first term of an asymptotic expansion. One can ask for further asymptotic information, and this will be given by the successive parts of a *nested expansion.* A method is known for computing these, but unfortunately it requires consideration of a differential equation whose order might at worst effectively double each time a new part of the expansion is calculated. This gives an upper bound of the order of $\exp(\mu n 2^m)$ on the number of possibilities to be considered for the m-th part of the nested expansion. If it is really necessary to consider that many cases, it will not be practical to compute many terms of the expansion!

The results proved here allow one to eliminate, a priori, a substantial fraction of the possibilities for each value of m. Alas the bound on the rate of growth of the number of cases is essentially untouched by this reduction, but it is perhaps a start. In fact the possibilities that we eliminate relate to terms which tend very rapidly to zero. The exact analogue of our theorem for terms of slow growth (or to be more precise, slow diminution) is known to be false, but a weaker result along these lines might be true. In fact such a result is known for first-order equations, [17], [18].

In Section 2, we give details of material from elsewhere which we shall require. This includes further discussion of Hardy fields, definitions of "nested form" and "nested expansion" and a more detailed description of the key results from [14] and [16]. In addition, we prove a lemma, which will be needed later, concerning the level of an element of a Hardy field. The section concludes with a statement of Theorem 5, our main theorem.

Sections 3, 4 and 5 are devoted to various parts of the proof of this.

In Section 6 we take a more classical approach to the problem of describing asymptotic growth, by looking at series expansions. The drawback of using series is that Hardy-field solutions of differential equations may fail to have series expansions. The advantages are firstly that series are more familiar objects, and secondly that when series expansions do exist they generally give more precise asymptotic information than nested expansions. For example $f(x) = e^{-x}x(1 + o(1))$ is more informative than $f(x) = \exp\{-x + \log x(1 + o(1))\}$, since the latter would be equally true for $f(x) = e^{-x}x \log x$. We initially consider series with a wide range of possible base functions in order to obviate as far as possible the difficulties over the choice of these. Thus we examine series expansions in base functions given by nested expansions subject to a finiteness requirement. We prove a version of Theorem 5 which is applicable to series, under an additional hypothesis. Then in two corollaries, we illustrate how the result may be applied in particular cases. In the final section, we consider terms in an expansion which tend more slowly to zero, and make a tentative conjecture.

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2. Hardy fields and nested expansions.

Let $\mathcal{X}$ denote the ring of germs of real-valued $\mathcal{C}^\infty$ functions defined on intervals of the form $(a, +\infty) \subset \mathbb{R}$.

DEFINITION 1. — *A Hardy field is defined to be a subfield of $\mathcal{X}$ which is closed under differentiation, [1].*

Using a frequent, and very convenient, abuse of terminology, we shall often treat elements of Hardy fields as functions rather than germs of functions. In practice no problems are likely to arise from this. If $\mathcal{F}$ is a Hardy field and $f \in \mathcal{X}$, we write $\mathcal{F}\langle f\rangle$ for the field generated by $\mathcal{F}$ and all the derivatives of f.

A non-zero element of a Hardy field has to have an inverse in the field, and so cannot have arbitrarily large zeros. It must therefore be ultimately positive or ultimately negative, and hence an order can be defined on any Hardy field by setting $f_1 > f_2$ when this is true for sufficiently large values of the argument. We shall use this order on any Hardy fields which we introduce without further notification. Many key facts about Hardy fields are to be found in the papers of Maxwell Rosenlicht, [7], [8], [9], [10]. In particular the important concepts of comparability class and rank were introduced in [8].

The following theorem first appeared in [6] and can also be found in [7].

THEOREM 1. — *Let $\mathcal{F}$ be a Hardy field. Then the real algebraic closure of $\mathcal{F}$ is (isomorphic to) a Hardy field.*

The next result is due to Michael Singer in its present form, and seems to have been first published in [7].

THEOREM 2. — *Let $\mathcal{F}$ be a Hardy field, and let F and G be polynomials in $\mathcal{F}[x]$. Let y be a function with $G(y) \neq 0$ satisfying the differential equation $y' = F(y)/G(y)$. Then $\mathcal{F}(y)$ is a Hardy field.*

An important consequence of this is that integrals, exponentials and logarithms of elements can be added to Hardy fields. In particular, this applies to the variable x $(= \int 1)$, $\exp x$, $\log x$, etc., and so if f is an element of a Hardy field, comparisons like $f < \exp(x^2) + \log x$ make sense in the Hardy-field ordering.

Now let f_1 and f_2 be two elements of a Hardy field $\mathcal{F}$ with $f_1, f_2 \to \infty$. Following [8], we say that f_1 and f_2 are *comparable* if there exist positive integers m and n such that $f_1 < f_2^n$ and $f_2 < f_1^m$. This definition may be extended to the whole of $\mathcal{F} \setminus \{0\}$ by specifying firstly that $\pm f$ and $\pm f^{-1}$ are all comparable to each other, and secondly that any two elements which tend to a non-zero finite limit are comparable. Comparability is then an equivalence relation on $\mathcal{F} \setminus \{0\}$. We refer to the equivalence classes as *comparability classes* and use $\gamma(f)$ to denote the comparability class of f. The number of comparability classes, excluding $\gamma(1)$, is called the *rank* of $\mathcal{F}$.

Now with f_1 and f_2 tending to infinity, we write $\gamma(f_1) > \gamma(f_2)$ if $f_1 > f_2^n$ for all $n \in \mathbb{N}$. It is easily seen that this relation depends only on the comparability classes. If we also specify that $\gamma(1)$ be the smallest class, we obtain a total order on the set of comparability classes.

We use the following, more-or-less standard, notation for iterated exponentials and logarithms. We define $e_0(x) = l_0(x) = x$ and for $n \geq 1$,

$$e_n(x) = \exp(e_{n-1}(x)) \quad \text{and} \quad l_n(x) = \log(l_{n-1}(x)).$$

Since iterated logarithms will almost invariable take the variable x itself as argument, we shall generally abbreviate $l_n(x)$ to l_n. Thus, for example, $l_3(2+e^{-x})$ will stand for $l_3(x)\cdot(2+e^{-x})$. On the other hand, exponentials will frequently be applied to a wide range of arguments, and so for example, $e_1(x(1+e^{-x}))$ stands for $\exp(x(1+e^{-x}))$. The following lemma, which formed part of Lemma 3 of [14], will be required.

LEMMA 1. — *$\gamma(e_s(l_m^d\phi_1)) > \gamma(e_t(l_n^c\phi_2))$ if either (i) $s-m > t-n$, or (ii) $s-m = t-n$, $s > t$ and either $d > 1$, or $d = 1$ and $\phi_1 \to \infty$.*

Proof of Lemma 1. — We begin by noting that for functions f_1 and f_2 which tend to infinity, $\gamma(f_1) > \gamma(f_2)$ precisely when $\log(f_2)/\log(f_1) \to 0$. We consider first the case when $s = t$ and $s - m > t - n$; *i.e.* $n > m$. Then

$$\log(l_n^c\phi_2) \sim cl_{n+1} = o(\log(l_m^d\phi_1)),$$

which gives $\gamma(e_s(l_m^d\phi_1)) > \gamma(e_t(l_n^c\phi_2))$ in the case $s = t = 0$. Now suppose that $s = t > 0$ and that the conclusion holds when s and t are replaced by $s-1$ and $t-1$. It is then immediate that $e_{t-1}(l_n^c\phi_2)/e_{s-1}(l_m^d\phi_1) \to 0$, which gives the desired conclusion.

Now suppose that $s > t$. We have

$$e_t(l_n^c\phi_2) = e_s(l_{s-t} \circ (l_n^c\phi_1)) = e_s(l_{s-t+n}\cdot(K+o(1))). \tag{1}$$

Then if $s - m > t - n$, we have $s - t + n > m$ and the result is obtained from the previous cases. If $s-m = t-n$, then $l_{s-t+n}(K+o(1)) = l_m(K+o(1)) = o(l_m^d\phi_1)$ if either $d > 1$ or if $d = 1$ and $\phi_1 \to \infty$. Hence $\gamma(e_s(l_m^d\phi_1)) > \gamma(e_s(l_{s-t+n}(K+o(1)))$, when $s = 1$. A simple induction, as above, gives the same conclusion for general values of s, and the result follows from (1). This completes the proof of Lemma 1.

In [14] and [16] a generalisation of the concept of an asymptotic expansion was developed. Let $\mathcal{F}$ be a Hardy field and let ϕ be a positive element of $\mathcal{F}$ which tends either to zero or infinity.

DEFINITION 2. — *A nested form for ϕ will be a finite sequence $\{(\epsilon_i, s_i, m_i, d_i, \phi_i), i = 1, \dots, k\}$ with the following properties :*

(a) *For each* i, $\epsilon_i \in \{+1, -1\}$, s_i *and* m_i *are non-negative integers,* d_i *is a positive real number and* ϕ_i *is an element of a Hardy field.*

(b) $\phi = e_{s_1}^{\epsilon_1}(l_{m_1}^{d_1}\phi_1)$ *and for* $i = 2, \ldots, k$,

$$\phi_{i-1} = e_{s_i}^{\epsilon_i}(l_{m_i}^{d_i}\phi_i).$$

(c) *For* $i = 1, \ldots, k$, ϕ_i *is positive and* $\gamma(\phi_i) < \gamma(l_{m_i})$.

(d) ϕ_k *tends to a positive constant.*

(e) $d_k \neq 1$ *unless* $s_k = 0$ *or* $m_k = 0$.

Note that (c) implies that $m_{i+1} > m_i$ for $i = 1, \ldots, k-1$. Condition (e) disallows expressions like $\exp(\log x(K+o(1)))$, $K \in \mathbb{R}\backslash\{0\}$, which would instead be written in the form $x^K\phi(x)$ with $\gamma(\phi) < \gamma(x)$. The corresponding forms for elements which tend to non-zero limits and for negative elements tending to 0 or $-\infty$ are respectively $A \pm e_{s_1}^{-1}(l_{m_1}^{d_1}\phi_1)$ and $-e_{s_1}^{\epsilon_1}(l_{m_1}^{d_1}\phi_1)$. If ϕ_k is actually equal to a constant, the nested form is called *precise.* Examples of nested forms follow shortly.

The nested form may be regarded as merely another way of writing the function. (So in particular, if $\mathbf{n}$ is the nested form of f, we may write $\gamma(\mathbf{n})$ in place of $\gamma(f)$.) However, in terms of asymptotic information, the nested form corresponds to just the first term of an asymptotic series. Thus in the case when the function has an asymptotic series in a base function such as x or $\log x$, the nested form will be the first term of the series plus an error term. Our next definition concerns the higher-order terms.

DEFINITION 3. — *A nested expansion for a function* $h \in \mathcal{X}$ *is a sequence of nested forms* $\{\mathbf{n}_j\}$ *such that* $\mathbf{n}_1$ *is a nested form for* h *and if* $\mathbf{n}_j = \{(\epsilon_{j,i}, s_{j,i}, m_{j,i}, d_{j,i}, h_{j,i}),\ i = 1, \ldots, k_j\}$, $j \geq 1$, *then* $\mathbf{n}_{j+1}$ *is a nested form for* $h_{j,k_j} - \lim h_{j,k_j}$.

The sequence $\{\mathbf{n}_j\}$ could be infinite. However, because the error term is built into the nested form, it is not a restriction here to consider only finite nested expansions, and this we shall do.

As an example of a nested expansion, suppose that

$$h = e_2\big(l_1 e_1(l_3^2(7 + e_2^{-1}(l_1^3 e_2(2\sqrt{l_4}))))\big). \tag{2}$$

Then a nested expansion for h is $\{\mathbf{n}_1, \mathbf{n}_2\}$, where

$$\mathbf{n}_1 = \{(+1, 2, 1, 1, h_{1,1}), (+1, 1, 3, 2, h_{1,2})\},$$

with $\lim h_{1,2} = 7$, and $\mathbf{n}_2$ is the precise nested form

$$\{(-1,2,1,3,h_{2,1}),(+1,2,4,1/2,2)\}.$$

It is not hard to see that a series may be written as a nested expansion. For example

$$e_r(a)+b = e_r(a)(1+be_r^{-1}(a)) = \exp\{e_{r-1}(a)+\log(1+be_r^{-1}(a))\}$$
$$= \exp\{e_{r-1}(a)+be_r^{-1}(a)-b^2e_r^{-2}(a)/2+\cdots\} = \cdots.$$

However the converse is false. An example is given by $e_2(e^x/(1-x^{-1})) = e_2(e^x+e^x/x+\cdots)$.

Suppose that h has a nested expansion $\{\mathbf{n}_1,\ldots,\mathbf{n}_{J+1}\}$ and that $1 < r \le J+1$. For any real number $K \ne -\lim h_{r-1,k_{r-1}}$, we write $h[\mathbf{n}_r \curvearrowright K]$ for the function whose nested expansion agrees with that of h as far as $\mathbf{n}_{r-1}$ but with $\mathbf{n}_r$ replaced by K. Thus if each $\mathbf{n}_j$, $j=1,\ldots,J+1$, is as above, and $1 < r \le J+1$, we define $\mathbf{m}_{r-1}$ to be the precise nested form $\{(\epsilon_{r-1,i},s_{r-1,i},m_{r-1,i},d_{r-1,i},\xi_i), i=1,\ldots,k_{r-1}\}$, where $\xi_i = h_{r-1,i}$ for $i=1,\ldots,k_{r-1}-1$ and $\xi_{k_{r-1}} = \lim h_{r-1,k_{r-1}}+K$. Then $h[\mathbf{n}_r \curvearrowright K]$ has a nested expansion $\{\mathbf{n}_1,\ldots,\mathbf{n}_{r-2},\mathbf{m}_{r-1}\}$.

The finite partial expansions $h[\mathbf{n}_r \curvearrowright 0]$ for $r=2,3,\ldots$ give successively finer estimates of the asymptotic growth of h in the same way as the partial sums of an asymptotic expansion do. Note however that h and $h[\mathbf{n}_r \curvearrowright 0]$ can have different comparability classes. For example, it is easy to see that $\gamma(e_2(x)) > \gamma(e_2(x(1-l_1^{-1})))$, and similarly $\gamma(e_4(x)) < \gamma(e_4(x+e^{-x}))$.

In [16], building on the earlier work in [14], it was shown that if f belongs to a Hardy field of finite rank which contains the real constants and constant powers of positive elements then f has a nested expansion. Hardy fields which have these properties (*i.e.* which are of finite rank and contain the real constants and constant powers of positive elements) are called *Rosenlicht fields.* It follows from the results in [9] that if f belongs to a Hardy field and satisfies an algebraic differential equation of order r_1 over a Rosenlicht field of rank r_2, then f is itself contained in a Rosenlicht field of rank r_1+r_2. In particular, a Hardy-field element which satisfies a differential equation of order r_1 over $\mathbb{R}$ must belong to a Rosenlicht field of rank r_1.

The following is the main result of [14].

THEOREM 3. — *Let $\mathcal{F}$ be a Rosenlicht field of rank r. Let ϕ be a positive element of $\mathcal{F}$ which tends either to zero or to infinity. Then ϕ has* *a nested form* $\{(\epsilon_i, s_i, m_i, d_i, \phi_i),\ 1 \le i \le k\}$ *such that*

$$\sum_{j=1}^{k} s_j + \delta_k + m_k \le r, \tag{3}$$

where δ_k *is* 0 *if* $d_k = 1$ *and* $m_k = 0$ *and otherwise* $\delta_k = 1$. *Moreover there exist elements* $\Gamma_0, \ldots, \Gamma_\Sigma$ *of* $\mathcal{F}$ *such that the following properties hold :*

(i) $\Sigma = \sum_{j=1}^{k} s_j + \delta_k + m_k$.

(ii) *Each* Γ_i *may be expressed as a rational function of a finite set of real powers of* ϕ, $\phi', \ldots, \phi^{(i)}$ *with real coefficients. Conversely,* $\phi^{(i)}$ *may be similarly expressed as a rational function of a finite set of real powers of* $\Gamma_0, \Gamma_1, \ldots, \Gamma_i$. *Both of these rational functions are computable.*

(iii) *For* $i = 0, \ldots, \Sigma - 1$, *we have* $\gamma(\Gamma_i) > \gamma(\Gamma_{i+1})$. *Also* $\gamma(\Gamma_\Sigma) = \gamma(1)$.

(iv) *The set of comparability classes of the* Γ_i*'s is equal to*

$$\gamma(1) \cup \bigcup_{i=0}^{m_k+\delta_k-1} \gamma(l_i) \cup \bigcup_{j=1}^{k} \bigcup_{t=1}^{s_j} \gamma(e_t(l_{m_j}^{d_j} \phi_j)).$$

Of course if ϕ is negative, we may apply Theorem 3 to $-\phi$, while if ϕ is not a constant but tends to a non-zero constant, then Theorem 3 may be applied to $|\phi - \lim \phi|$.

Suppose that we wish to apply Theorem 3 to a Rosenlicht field of rank r. This might be of the form $\mathbb{R}\langle f \rangle$ where f satisfies a differential equation of order r over $\mathbb{R}$. We have to consider all the nested forms $\{(\epsilon_i, s_i, m_i, d_i, \phi_i),\ i = 1, \ldots, k\}$ with the d_i's remaining as parameters, for which the inequality (3) holds. These may then be substituted into the differential equation defining f in order to obtain possible values for the parameters. Having done this, one can go on to calculate the next "term" of the nested expansion for f by writing $\phi = \phi_k - \lim \phi_k$ and substituting into the differential equation in order to obtain a differential equation for ϕ. Of course the process may be repeated. However the rank will increase with each repetition. The following is from [16].

PROPOSITION 1. — *Suppose* f *belongs to a Rosenlicht field,* $\mathcal{F}$, *of rank* r *and has the nested form* $\{(\epsilon_i, s_i, m_i, d_i, \phi_i),\ i = 1, \ldots, k\}$. *Then* $\phi = \phi_k - \lim \phi_k$ *belongs to a Rosenlicht field of rank at most*

$$\sum_{i=1}^{k} s_i + \operatorname{rank}(\mathcal{F}\langle l_{m_k} \rangle),$$

and this is no greater than $m_k + 1 + \sum s_i + r$.

Inequality (3) gives a bound of $2r + 1$ for the last quantity. This has a serious computational effect in view of the next result, which is also from [16].

THEOREM 4. — *Let* $N(r)$ *denote the the number of nested forms (with the 'd's as parameters) which satisfy (3). As* $r \to \infty$, $N(r) \sim A \cdot B^r$, *where* A *and* $B > 1$ *are non-zero constants.*

Precise values are given for A and B in [16]. Here the main point is that the number of cases to be considered grows exponentially with the rank. Since the rank might double with each iteration to calculate the next "term" in the nested expansion, the number of cases to be considered could at worst grow in a doubly-exponential fashion.

A function's comparability class gives a measure of its growth. A somewhat finer measure is given by the *valuation*. Let f and g be any two non-zero elements of a Hardy field, $\mathcal{F}$. We obtain an equivalence relation on $\mathcal{F} \setminus \{0\}$ by setting $f \sim g$ whenever f/g tends to a finite non-zero limit. Under the inherited multiplication, the equivalence classes form a group called the *valuation group*. We write $\nu(f)$ for the equivalence class of f and set $\nu(f) > \nu(g)$ whenever $f/g \to 0$. The role of the valuation group was extensively discussed in the book by Lightstone and Robinson [4] and its importance re-emphasised in the work of Rosenlicht, [7].

The valuation and the comparability class may be regarded as the first two members of a sequence of growth measures of increasing coarseness. Let i be any positive integer, and let f and g be two elements of $\mathcal{F}$ both of which tend to infinity. We define $f \sim_i g$ to mean that $l_i(f)/l_i(g)$ tends to a non-zero finite limit. We write $\gamma_i(f)$ for the equivalence class of f. As with comparability classes, we can extend this notion to the whole of $\mathcal{F} \setminus \{0\}$ by decreeing that $\pm f$ and $\pm f^{-1}$ be equivalent to each other and that any two elements tending to a non-zero finite limit be equivalent. We write $\gamma_i(f) > \gamma_i(g)$ if $l_i(g)/l_i(f) \to 0$ (where $f, g \to \infty$). We may take γ_0 to be the valuation (with reversed ordering), and of course, $\gamma_1(f)$ is the comparability class of f. Note that $f \sim_i g \Rightarrow f \sim_{i+1} g$.

Now let $f, g \in \mathcal{F} \setminus \{0\}$. We define $f \sim_\infty g$ whenever there exists an integer $i \geq 0$ such that $f \sim_i g$. Once again this is an equivalence relation on $\mathcal{F} \setminus \{0\}$. We call the equivalence class of f the *level* of f and write $\gamma_\infty(f)$ for this; *cf.* [10]. We order the set of levels by writing $\gamma_\infty(f) > \gamma_\infty(g)$ whenever

$\gamma_i(f) > \gamma_i(g)$ for all i.

LEMMA 2. — *Let f and g be two elements of a Rosenlicht field $\mathcal{F}$ and suppose that the nested forms of f and g are respectively $\{(\epsilon_i, s_i, m_i, d_i, \phi_i), i = 1, \ldots, k\}$ and $\{(\delta_i, t_i, n_i, c_i, \psi_i), i = 1, \ldots, l\}$. Then $f \sim_\infty g$ if and only if $s_1 - m_1 = t_1 - n_1$.*

Thus the level of f is given by $s_1 - m_1$.

Proof of Lemma 2. — Note first that the order relation between the quantities $s_1 - m_1$ and $t_1 - n_1$ is preserved under the taking of logarithms. So if $s_1 - m_1 > t_1 - n_1$, a similar relation will hold between the nested forms of $l_i(f)$ and $l_i(g)$ for each $i = 1, \ldots$. Hence, from Lemma 1, $\gamma(l_i(f)) > \gamma(l_i(g))$ for $i = 0, \ldots$. Thus $\gamma_{i+1}(f) > \gamma_{i+1}(g)$ for $i = 0, \ldots$, and so $\gamma_\infty(f) > \gamma_\infty(g)$. The condition $s_1 - m_1 = t_1 - n_1$ is therefore necessary.

Now let this condition hold and suppose without loss of generality that $s_1 \geq t_1$. Then $l_{s_1+1}(f) \sim d_1 l_{m_1+1}$ and $l_{s_1+1}(g) \sim l_{s_1-t_1+1}(l_{n_1}^{c_1})$ which is asymptotic to $c_1 l_{m_1+1}$ if $s_1 = t_1$ and to l_{m_1+1} if $s_1 > t_1$. In either case, $l_{s_1+1}(f)/l_{s_1+1}(g)$ tends to a finite, non-zero limit, and so $f \sim_{s_1+1} g$. It follows that $f \sim_\infty g$ as required, and Lemma 2 is therefore proved.

Before commencing our main section, we give the z-function notation from [12]. Let t be an element of a Hardy field with $t \to 0$. We write

$$\operatorname{zexp}(t) = \exp(t) - 1, \qquad \operatorname{zlog}(t) = \log(1+t)$$

and for any $r \in \mathbb{R} \setminus \mathbb{N}$, we write

$$\operatorname{zpow}(r, t) = (1+t)^r - 1.$$

We have $\operatorname{zexp}(t) \sim t$, $\operatorname{zlog}(t) \sim t$ and $\operatorname{zpow}(r,t) \sim rt$. Also for $n > 0$, we make the following definitions :

$$\operatorname{zexp}_n(t) = t^{-n}\left\{\operatorname{zexp}(t) - \left(t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}\right)\right\},$$

$$\operatorname{zlog}_n(t) = t^{-n}\left\{\operatorname{zlog}(t) - \left(t - \frac{t^2}{2} + \cdots + (-1)^{n-1}\frac{t^n}{n}\right)\right\},$$

$$\operatorname{zpow}_n(r,t) = t^{-n}\left\{\operatorname{zpow}(r,t) - \left(rt + \frac{r(r-1)}{2}t^2 + \cdots + \frac{\Gamma(r+1)}{\Gamma(r-n+1)\Gamma(n+1)}t^n\right)\right\},$$

$r \in \mathbb{R} \setminus \mathbb{N}$. We take $\operatorname{zexp}_0 = \operatorname{zexp}$, $\operatorname{zlog}_0 = \operatorname{zlog}$ and $\operatorname{zpow}_0 = \operatorname{zpow}$. The functions zexp_n, zlog_n and $\operatorname{zpow}_n(r, \ )$, $n \geq 0$ are referred to collectively as

z-functions. Later, we shall want to apply z-functions to complex arguments as well as real ones, but in all cases the arguments will tend to zero. The z-functions themselves then tend to zero, and moreover they are analytic at the origin.

The main result of the paper is as follows.

THEOREM 5. — *Let g be an element of a Hardy field with nested expansion $\{\mathbf{n}_1, \ldots, \mathbf{n}_{J+1}\}$ and suppose that $\gamma(\mathbf{n}_{J+1}) > \gamma(g[\mathbf{n}_{J+1} \curvearrowright 0])$. Then $\mathbb{R}\langle g\rangle$ contains an element of comparability class at least $\gamma(\mathbf{n}_{J+1})$.*

Thus, as and when larger comparability classes arise in the nested expansion, they must give rise to larger comparability classes in $\mathbb{R}\langle g\rangle$. However, in the cases of interest, the rank of $\mathbb{R}\langle g\rangle$ will be bounded according to Theorem 3, and then Theorem 5 puts a restriction on the possible comparability classes of the $\mathbf{n}_j$s. Note that we are not assuming that $\mathbf{n}_{J+1}$ is a precise nested form.

The thrust of the proof of Theorem 5 is as follows. If the conclusion is false there will be a smallest value of J such that $\mathbb{R}\langle g\rangle$ contains no elements of comparability class as great as $\gamma(\mathbf{n}_{J+1})$. Then, of course $\gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_j)$ for every $j = 2, \ldots, J$, and moreover $\gamma(\mathbf{n}_{J+1}) > \gamma(x)$. We start Section 3 by showing that under this assumption, we can rewrite g in the form $g = f + \eta$ where $f = g[\mathbf{n}_{J+1} \curvearrowright 0]$, $\eta \to 0$ and $\gamma(\eta) = \gamma(\mathbf{n}_{J+1})$. We then construct two towers of differential algebras. The first, $\mathbb{R} = \mathcal{R}_0 \subset \mathcal{R}_1 \subset \cdots \subset \mathcal{R}_M$, contains formally defined polynomial algebras, while the second, $\mathbb{C} = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_M$, contains subalgebras of $\mathcal{X} \otimes_{\mathbb{R}} \mathbb{C}$ whose elements include certain subexpressions of the nested expansion of f. For each α with $0 \leq \alpha \leq M$, there is a set of homomorphisms, $\mathbf{G}_\alpha$, from $\mathcal{R}_\alpha$ to $\mathcal{S}_\alpha$ such that the subexpressions of the nested expansion of f are in the images of certain elements, τ_α, of $\mathbf{G}_\alpha$ for some α. The $\mathbf{G}_\alpha$ are somewhat analagous to differential Galois groups.

In Section 4, we show that if w_f is an element of $\mathcal{R}_M$ such that $\tau_M(w_f) = f$ and ρ is any element of $\mathbf{G}_M$, then as $x \to \infty$, $\rho(w_f)(x) - f(x)$ cannot tend to zero as rapidly as a power of $\eta(x)$ unless $\rho(w_f)$ is identically equal to f. Then in Section 5, by working down the tower $\mathcal{R}_M \supset \cdots \supset \mathcal{R}_0$, we prove that there is a differential polynomial, P, over $\mathbb{R}$ such that $P\langle f\rangle = 0$ but $P\langle g\rangle \neq 0$. However we can then establish that $\gamma(P\langle g\rangle) \geq \gamma(\mathbf{n}_{J+1})$, and so $\mathbb{R}\langle g\rangle$ contains an element of comparability class at least $\gamma(\mathbf{n}_{J+1})$. This will then prove the theorem.

3. Building the towers.

We begin with more notation. Suppose that h has nested expansion $\{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_N\}$, where for $j = 1, \ldots, N$, $\mathbf{m}_j = \{(\delta_{j,i}, t_{j,i}, p_{j,i}, c_{j,i}, \psi_{j,i}),\ i = 1, \ldots, k_j\}$. A *subnest* of h is either h itself, or equal to $\psi_{j,i}$ for some (j,i) with $1 \le j \le N$ and $1 \le i \le k_j$, or of the form $e_s(l_{p_{j,i}}^{c_{j,i}} \psi_{j,i})$ with $s < t_{j,i}$; in this last case the subnest will be an iterated logarithm of a subnest of one of the first two types, modulo a possible change of sign. For example if $h = e_2(l_1 e_1(l_3^2(7 + e_2^{-1}(l_1^3 e_2(2\sqrt{l_4})))))$, as in (2), then subnests of h include h, $e_1(l_1 e_1(l_3^2(7+e_2^{-1}(l_1^3 e_2(2\sqrt{l_4})))))$, $l_1 e_1(l_3^2(7+e_2^{-1}(l_1^3 e_2(2\sqrt{l_4}))))$, $e_1(l_3^2(7 + e_2^{-1}(l_1^3 e_2(2\sqrt{l_4}))))$, $l_3^2(7 + e_2^{-1}(l_1^3 e_2(2\sqrt{l_4})))$, $e_2(l_1^3 e_2(2\sqrt{l_4}))$ and $e_1(l_1^3 e_2(2\sqrt{l_4}))$, *etc.* If u_1 and u_2 are subnests of f, we define $u_2 \precsim u_1$ to mean that u_2 is a subnest of u_1, and $u_2 \prec u_1$ to mean that u_2 is a subnest of u_1 different from u_1 itself.

Our first concern now is to show that g can be written as a sum, in the required fashion. The following lemma shows that this can be done under strengthened hypotheses; later we shall show that these are not required.

LEMMA 3. — *Let g have nested expansion $\{\mathbf{n}_1, \ldots, \mathbf{n}_{J+1}\}$ and let $f = g[\mathbf{n}_{J+1} \curvearrowright 0]$. Suppose that*

$$\gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_j[\mathbf{n}_{J+1} \curvearrowright 0])$$

for every $j = 1, \ldots, J$. Then g may be written in the form $g = f + \eta$ where $\eta \to 0$ and $\gamma(\eta) = \gamma(\mathbf{n}_{J+1})$.

Proof of Lemma 3. — Suppose that the subnests of g greater than or equal to $\mathbf{n}_{J+1}$ are $\mathbf{n}_{J+1} = g_0 \prec g_1 \prec \cdots \prec g_I = g$. Then for $1 \le i \le I$, g_i will be of one of the forms $l_m^d g_{i-1}$, $A + l_m^d g_{i-1}$, $\exp(\pm g_{i-1})$, $A + \exp(-g_{i-1})$, with A constant. We show by induction on i that g_i can be written in the form $g_i = f_i + \eta_i$ with $f_i = g_i[\mathbf{n}_{J+1} \curvearrowright 0]$, $\eta_i \to 0$ and $\gamma(\eta_i) = \gamma(\mathbf{n}_{J+1})$. For the case $i = 0$ this is a triviality, so suppose it holds for g_{i-1}.

We consider first the case when $g_i = l_m^d g_{i-1}$. Then

$$g_i = l_m^d f_{i-1} + l_m^d \eta_{i-1}.$$

This is of the required form since, by the hypotheses of the lemma, $\gamma(\eta_{i-1}) = \gamma(\mathbf{n}_{J+1}) > \gamma(l_m^d)$, and therefore $l_m^d \eta_{i-1} \to 0$ and $\gamma(l_m^d \eta_{i-1}) = \gamma(\eta_{i-1})$. A similar conclusion holds if $g_i = A + l_m^d g_{i-1}$. Next suppose that $g_i = \exp(g_{i-1})$. Then

$$g_i = \exp(f_{i-1} + \eta_{i-1}) = \exp(f_{i-1}) + \exp(f_{i-1})\,\mathrm{zexp}(\eta_{i-1}). \tag{4}$$

Since $\operatorname{zexp}(\eta_{i-1}) \sim \eta_{i-1}$, we have $\gamma(\operatorname{zexp}(\eta_{i-1})) = \gamma(\eta_{i-1}) > \gamma(\exp(f_{i-1}))$ and moreover $\exp(f_{i-1})\operatorname{zexp}(\eta_{i-1}) \to 0$. Thus (4) expresses g_i as required. The corresponding form for $\exp(-g_{i-1})$ is obtained by multiplying through by -1 before exponentiation, and trivially, if A is constant $A+\exp(-g_{i-1})$ can be similarly expressed. Thus g_i may be written in the form $g_i = f_i + \eta_i$ with $f_i = g_i[\mathbf{n}_{J+1} \curvearrowright 0]$, $\eta_i \to 0$ and $\gamma(\eta_i) = \gamma(\mathbf{n}_{J+1})$, and Lemma 3 follows by induction on i.

If the conclusion of Theorem 5 is false, then for a suitable J, $\mathbb{R}\langle g\rangle$ contains an element of comparability class at least $\max\{\gamma(\mathbf{n}_1), \dots, \gamma(\mathbf{n}_J)\}$, but none as large as $\gamma(\mathbf{n}_{J+1})$, and so $\gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_j)$ for $j = 2, \dots, J$. In fact we can assume that $\gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_j[\mathbf{n}_{J+1} \curvearrowright 0])$, as the next lemma shows.

LEMMA 4. — *Suppose that $1 \le K \le J$. If $\gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_j)$, for $j = K, \dots, J$, then likewise $\gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_j[\mathbf{n}_{J+1} \curvearrowright 0])$, for $j = K, \dots, J$.*

Proof of Lemma 4. — If not, let j be the largest value, $K \le j \le J$, such that $\gamma(\mathbf{n}_{J+1}) \le \gamma(\mathbf{n}_j[\mathbf{n}_{J+1} \curvearrowright 0])$. Then $\gamma(\mathbf{n}_j[\mathbf{n}_{J+1} \curvearrowright 0]) > \gamma(\mathbf{n}_j)$. Now $\mathbf{n}_j$ cannot be of the form $L(A+\varepsilon)$ with $A \in \mathbb{R}$, L a product of real powers of logarithms and $\varepsilon \to 0$, since then we would have $\gamma(\mathbf{n}_j) = \gamma(L) = \gamma(\mathbf{n}_j[\mathbf{n}_{J+1} \curvearrowright 0])$. Thus $\mathbf{n}_j$ must be an exponential. Now $\gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_j) \Rightarrow \gamma_\infty(\mathbf{n}_{J+1}) \ge \gamma_\infty(\mathbf{n}_j)$ and so, using Lemma 2, we have

$$(5) \quad \gamma_\infty(\mathbf{n}_{J+1}) \ge \gamma_\infty(\mathbf{n}_j) = \gamma_\infty(\mathbf{n}_j[\mathbf{n}_{J+1} \curvearrowright 0]) > \gamma_\infty(\log \mathbf{n}_j[\mathbf{n}_{J+1} \curvearrowright 0]).$$

By our choice of j, $\gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_k[\mathbf{n}_{J+1} \curvearrowright 0])$ for $j < k \le J$. Since, from (5), we also have $\gamma(\mathbf{n}_{J+1}) > \gamma(\log \mathbf{n}_j[\mathbf{n}_{J+1} \curvearrowright 0])$, we may apply Lemma 3 with g replaced by the subnest $\log \mathbf{n}_j$, and we see that we may write $\log \mathbf{n}_j = H + \psi$ where $H = \log \mathbf{n}_j[\mathbf{n}_{J+1} \curvearrowright 0]$, $\psi \to 0$ and $\gamma(\psi) = \gamma(\mathbf{n}_{J+1})$. But then $\mathbf{n}_j = \exp H \cdot (1 + \operatorname{zexp} \psi)$ and so $\gamma(\mathbf{n}_j) = \gamma(\exp H) = \gamma(\mathbf{n}_j[\mathbf{n}_{J+1} \curvearrowright 0])$. This contradiction establishes the lemma.

COROLLARY 1. — *Let g be as in Theorem 5, and assume, as we may, that $\gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_j)$ for $2 \le j \le J$. Then we may write $g = f + \eta$, where $f = g[\mathbf{n}_{J+1} \curvearrowright 0]$, $\eta \to 0$ and $\gamma(\eta) = \gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_j[\mathbf{n}_{J+1} \curvearrowright 0])$, for every $j = 1, \dots, J$.*

Proof of Corollary 1. — The hypotheses of Theorem 5 give us that $\gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_1[\mathbf{n}_{J+1} \curvearrowright 0])$. By Lemma 4, $\gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_j[\mathbf{n}_{J+1} \curvearrowright 0])$, for $j = 2, \dots, J$. The corollary now follows from Lemma 3.

COROLLARY 2. — *Under the assumption, that $\gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_j)$, for $2 \le j \le J$, we have*

$$\gamma(\mathbf{n}_{J+1}) > \gamma(g[\mathbf{n}_{J+1} \curvearrowright 0]) \Leftrightarrow \gamma(\mathbf{n}_{J+1}) > \gamma(g).$$

Proof of Corollary 2. — That the left-hand inequality implies the right, follows immediately from Corollary 1. The reverse implication is obtained by taking $K = 1$ in Lemma 4.

Our next construction bears some resemblance to that used in [15]. We build two towers of differential algebras, $\mathcal{R}_0 \subset \mathcal{R}_1 \subset \cdots \subset \mathcal{R}_M$ and $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_M$. The $\mathcal{R}_\alpha$s will be defined formally as polynomial $\mathbb{R}$-algebras in indeterminates $v_1, \ldots, v_M$. For $1 \le \alpha \le M$, each $\mathcal{R}_\alpha$ will be an extension of $\mathcal{R}_{\alpha-1}$ either by v_α or by v_α and v_α^{-1}. The $\mathcal{S}_\alpha$s will be $\mathbb{C}$-algebras of function germs, each $\mathcal{S}_\alpha$ being a subalgebra of $\mathcal{X} \otimes_{\mathbb{R}} \mathbb{C}$. In addition we will define, for each $\alpha = 0, \ldots, M$, a set $\mathbf{G}_\alpha$, of differential $\mathbb{R}$-algebra homomorphisms of $\mathcal{R}_\alpha$ into $\mathcal{S}_\alpha$. In effect, the elements of $\mathbf{G}_\alpha$ will be the representations of $\mathcal{R}_\alpha$ in $\mathcal{S}_\alpha$. Furthermore, for each α, we will designate a special element, τ_α, of $\mathbf{G}_\alpha$, with the property that $\tau_\alpha(v_\alpha)$ is part of the nested expansion of f, *e.g.* a subnest. The τ_αs thus allow us to recover f and its sub-expressions from the elements of the first tower. For the lowest fields of this tower, the v_α will be equal to l_m for values of m which increase as we go further up the tower. For the higher fields, they will either be of the form l_m^d for d a non-integral real number, or else (increasing) subnests of f.

We are now ready to give the formal definitions of the various $\mathcal{R}_\alpha$, $\mathcal{S}_\alpha$, $\mathbf{G}_\alpha$ etc. The definitions are split into different cases for different ranges of α, and different $\tau_\alpha(v_\alpha)$. In each case, $\mathcal{R}_{\alpha-1}$ will be a differential subalgebra of $\mathcal{R}_\alpha$, so we use the same symbol, D, for the derivation, and with D already defined on $\mathcal{R}_{\alpha-1}$, we need only specify $D(v_\alpha)$ in order to define the derivation on $\mathcal{R}_\alpha$. In each case, the derivation on $\mathcal{S}_\alpha$ will be d/dx.

DEFINITION 4 ($\alpha = 0$). — *We define $\mathcal{R}_0 = \mathbb{R}$ and $\mathcal{S}_0 = \mathbb{C}$. The derivation, D, is defined to act trivially on $\mathcal{R}_0$, i.e. $D(r) = 0$ for all $r \in \mathbb{R}$. Let τ_0 be the natural embedding of $\mathbb{R}$ in $\mathbb{C}$, and let $\mathbf{G}_0 = \{\tau_0\}$.*

DEFINITION 4 ($\alpha = 1$). — *Let $\mathcal{R}_1 = \mathbb{R}(v_1)$ and $\mathcal{S}_1 = \mathbb{C}(x)$. The derivation on $\mathcal{R}_1$ is defined by setting $D(v_1) = 1$. Then we let $\mathbf{G}_1$ be the set of $\mathbb{R}$-algebra homomorphisms of $\mathbb{R}(v_1)$ into $\mathcal{S}_1$ whose restrictions to $\mathcal{R}_0$ belong to $\mathbf{G}_0$ and which map v_1 into $x + K$ for some value of $K \in \mathbb{C}$. The special homomorphism, τ_1, maps v_1 to x.*

Before giving the next part of Definition 4, we introduce two pieces of notation which we shall employ throughout the rest of the paper. Suppose that for $1 \le j \le J$, the nested form $\mathbf{n}_j$ is $\{(\epsilon_{j,i}, s_{j,i}, m_{j,i}, d_{j,i}, \phi_{j,i}), i = 1, \ldots, k_j\}$. Let $\beta = \max_{1 \le j \le J}\{m_{j,k_j}\}$. Then the expression for f as a nested expansion will contain l_β but not l_m for any $m > \beta$. The "lowest fields" mentioned above will be the lowest $\beta + 2$. Secondly, if ρ is a mapping from $\mathcal{R}_\alpha$ to $\mathcal{S}_\alpha$, $\rho_\lrcorner$ denotes the restriction of ρ to the algebra below in the tower, i.e. $\mathcal{R}_{\alpha-1}$.

Definition 4 $(2 \le \alpha \le \beta + 1)$. — *For $2 \le \alpha \le \beta+1$, we set*

$$\mathcal{R}_\alpha = \mathbb{R}(v_1, \ldots, v_{\alpha-1}, v_\alpha) \simeq \mathcal{R}_{\alpha-1}(v_\alpha).$$

Then the derivation is extended to $\mathcal{R}_\alpha$ by defining $D(v_\alpha) = v_1^{-1}v_2^{-1}\cdots v_{\alpha-1}^{-1}$.

We then define $\mathcal{S}_\alpha$ to be the $\mathbb{C}$-algebra generated by $\mathcal{S}_{\alpha-1}$ and all elements of the forms $\log(\sigma(v_{\alpha-1}))$, $(\log\sigma(v_{\alpha-1}) + K)^{-1}$ for $\sigma \in \mathbf{G}_{\alpha-1}$ and $K \in \mathbb{C}$. Then we set

$$\mathbf{G}_\alpha = \{\rho : \mathcal{R}_\alpha \to \mathcal{S}_\alpha;\ \rho_\lrcorner \in \mathbf{G}_{\alpha-1}\ \&\ \rho(v_\alpha) = \log\rho_\lrcorner(v_{\alpha-1}) + K_\rho\ ,\ K_\rho \in \mathbb{C}\}.$$

Here the special homomorphism, τ_α, of $\mathbf{G}_\alpha$ is defined by specifying that $\tau_{\alpha\lrcorner} = \tau_{\alpha-1}$ and that $\tau_\alpha(v_\alpha) = l_{\alpha-1}$.

We now continue building the towers so that the $\mathcal{S}_\alpha$s will contain in turn the nested forms ϕ_{J,k_J-1}, $\phi_{J,k_J-2}, \ldots$, $\phi_{J,1}$, $\phi_{J-1,k_{J-1}-1}$, $\ldots$, $\phi_{J-1,1}, \ldots$, $\phi_{1,1}$. In fact these will be the images under the various τ_αs of certain elements in the $\mathcal{R}_\alpha$s. At each stage, we have to add either an element of the form l_m^d, where d is a non-integral real number, or an exponential of an existing element. Suppose then that the appropriate definitions have been made for $\alpha - 1$ that we wish to add l_m^d. In this case we do not introduce an inverse for v_α, and this will be the pattern henceforth.

Definition 4 $(\alpha > \beta + 1,\ \tau_\alpha(v_\alpha) = l_m^d)$. — *We take $\mathcal{R}_\alpha = \mathcal{R}_{\alpha-1}[v_\alpha]$. We recall that since $m \le \beta$, $v_{m+1}^{-1} \in \mathcal{R}_\alpha$ and $\tau_\alpha(v_{m+1}) = l_m$. We may thus define D on $\mathcal{R}_\alpha$ by setting*

$$D(v_\alpha) = dv_\alpha D(v_{m+1})/v_{m+1}.$$

Now we take $\mathcal{S}_\alpha$ to be the $\mathbb{C}$-algebra generated by $\mathcal{S}_{\alpha-1}$ and the complex roots $(\rho_\lrcorner(l_m))^d$. To add a rational power of l_m we may take d to be of the form $1/q$ with q a positive integer; for example, $l_m^{-1/q} = (l_m^{1/q})^{q-1}l_m^{-1}$. In this case we let $\mathbf{G}_\alpha$ be the set of homomorphisms, ρ, such that $\rho_\lrcorner \in \mathbf{G}_{\alpha-1}$ and $\rho(v_\alpha)$ is one of the complex q-th roots of $\rho_\lrcorner(l_m)$. In the irrational case, we take

$$\mathbf{G}_\alpha = \{\rho;\ \rho_\lrcorner \in \mathbf{G}_{\alpha-1}\ \&\ \rho(v_\alpha) = K_\rho(\rho_\lrcorner(v_{m-1}))^d,\ K_\rho \in \mathbb{C}\}. \tag{6}$$

In both the rational and irrational cases, we define τ_α *by setting* $\tau_{\alpha\lrcorner} = \tau_{\alpha-1}$ *and* $\tau_\alpha(v_\alpha)$ *equal to the positive root* l_m^d.

Note that here we allow K_ρ to be zero. For this reason, if l_m^{-d} for example, occurs later in the nested expansion of f, it will be represented in the $\mathcal{R}$-tower by a different v_i. More generally we will ignore any $\mathbb{Q}$-linear relations between the numbers d_j which occur as exponents of the same l_m in the nested expansion of f, and represent each $l_m^{d_j}$ by a different v_i.

Now consider the case of an expression of the form $\exp(h)$ occurring as a subnest in the expansion of f. Because of the way in which the τ_is are defined, h will be of the form $h = \tau_{\alpha-1}(w_h)$ where $w_h \in \mathcal{R}_{\alpha-1}$. If there are several such w_h, we select one arbitrarily.

DEFINITION 4 ($\alpha > \beta + 1$, $\tau_\alpha(v_\alpha) = \exp(h)$). — *As before,* $\mathcal{R}_\alpha = \mathcal{R}_{\alpha-1}[v_\alpha]$. *We define the derivation,* D_α, *on* $\mathcal{R}_\alpha$ *by* $D_\alpha(v_\alpha) = v_\alpha \cdot D(w_h)$, *where* $h = \tau_{\alpha-1}(w_h)$. *We let* $\mathcal{S}_\alpha$ *be the* $\mathbb{C}$-*algebra generated by* $\mathcal{S}_{\alpha-1}$ *and all elements of the form* $\exp(\sigma(w_h))$ *for* $\sigma \in \mathbf{G}_{\alpha-1}$, *and we take*

$$\mathbf{G}_\alpha = \{\rho;\ \rho_\lrcorner \in \mathbf{G}_{\alpha-1}\ \&\ \rho(v_\alpha) = K_\rho \exp(\rho_\lrcorner(w_h)),\ K_\rho \in \mathbb{C}\}. \tag{7}$$

We define τ_α *by* $\tau_{\alpha\lrcorner} = \tau_{\alpha-1}$ *and* $\tau_\alpha(v_\alpha) = \exp(\tau_{\alpha-1}(w_h)) = \exp(h)$.

Here again, K_ρ is allowed to be zero. We note that if h is of the form $h = l_m^d f_i$ where f_i is a subnest of f, then there are elements $w_{l_m^d}$ and w_{f_i} of $\mathcal{R}_{\alpha-1}$ such that $w_h = w_{l_m^d} w_{f_i}$ and $\tau_{\alpha-1}(w_{l_m^d}) = l_m^d$ and $\tau_{\alpha-1}(w_{f_i}) = f_i$. We can choose $w_{l_m^d}$ to be one of the v_is, or the d-th power of one of them if d is an integer.

The following proposition sums up the properties of the $\mathcal{R}_\alpha$, $\mathcal{S}_\alpha$ and $\mathbf{G}_\alpha$.

PROPOSITION 2. — *There exists an integer* M, *and for each* $\alpha = 0, \ldots, M$ *there exists a differential* $\mathbb{R}$-*algebra,* $\mathcal{R}_\alpha$, *a differential* $\mathbb{C}$-*subalgebra,* $\mathcal{S}_\alpha$, *of* $\mathcal{X} \otimes_{\mathbb{R}} \mathbb{C}$ *and a set,* $\mathbf{G}_\alpha$ *of differential* $\mathbb{R}$-*algebra homomorphisms of* $\mathcal{R}_\alpha$ *into* $\mathcal{S}_\alpha$ *with the following properties :*

(i) $\mathcal{R}_0 = \mathbb{R}$, $\mathcal{S}_0 = \mathbb{C}$.

(ii) *For* $\alpha = 1, \ldots, M$, *we have either* $\mathcal{R}_\alpha = \mathcal{R}_{\alpha-1}(v_\alpha)$ *or* $\mathcal{R}_\alpha = \mathcal{R}_{\alpha-1}[v_\alpha]$. *Furthermore there exists a* $\tau_\alpha \in \mathbf{G}_\alpha$ *such that* $\tau_\alpha(v_\alpha)$ *is of one of the forms* $l_{\alpha-1}$, l_m^d *where* $m \in \mathbb{N}$ *and* d *is a non-integral real number, or* $\exp(h)$ *where* $h \in \tau_{\alpha\lrcorner}(\mathcal{R}_{\alpha-1})$. *In the second case,* d *will either be of the form* $1/q$, *where* q *is an integer greater than 1, or else irrational.*

(iii) *For any subnest, f_i, of f, there exists an $\alpha \le M$ and an element w_{f_i} of $\mathcal{R}_\alpha$ such that $\tau_\alpha(w_{f_i}) = f_i$. In particular, $f = \tau_M(w_f)$.*

(iv) *For each $\alpha = 1, \ldots, M$, the set $\mathbf{G}_\alpha$ is parameterised over a space of the form*

$$\mathbb{C}^{a(\alpha)} \times \prod_{b=1}^{b(\alpha)} \{1, e^{2\pi i/q_b}, \ldots e^{2\pi i(q_b-1)/q_b}\},$$

where $a(\alpha) + b(\alpha) = \alpha$, and each q_b is an integer greater than one.

(v) *If $\tau_\alpha(v_\alpha) = l_m^{1/q}$ and $\rho \in \mathbf{G}_\alpha$, then $(\rho((l_m)^{1/q}))^q = \rho_\lrcorner(l_m)$. Moreover given any $\sigma \in \mathbf{G}_{\alpha-1}$ and any complex q-th root of $\sigma(l_m)$, there is a $\rho \in \mathbf{G}_\alpha$ which agrees with σ on $\mathcal{R}_{\alpha-1}$ and takes v_α to the given q-th root.*

(vi) *Similarly, if $\tau_\alpha(v_\alpha)$ is of one of the forms $l_{\alpha-1}$, l_m^d with d irrational, or $\exp(h)$ with $h \in \tau_{\alpha-1}(\mathcal{R}_{\alpha-1})$, let $D(y) = \Lambda_\alpha(y)/\Omega_\alpha(y)$ denote the differential equation satisfied by v_α over $\mathcal{R}_{\alpha-1}$. That is to say, the differential equation $D(y) = D(v_{\alpha-1})/v_{\alpha-1}$, $D(y) = ydD(v_{m+1})/v_{m+1}$ or $D(y) = yD(w_h)$, where $\tau_{\alpha-1}(w_h) = h$. Let $\sigma \in \mathbf{G}_{\alpha-1}$ and let ζ be any solution of the differential equation $y' = \tilde{\sigma}(\Lambda_\alpha(y))/\tilde{\sigma}(\Omega_\alpha(y))$, where we have written $\tilde{\sigma}(\Lambda_\alpha)$ and $\tilde{\sigma}(\Omega_\alpha)$ for the polynomials obtained by applying σ to the coefficients of Λ_α and Ω_α respectively. Then there exists a $\rho \in \mathbf{G}_\alpha$ such that $\rho_\lrcorner = \sigma$ and $\rho(v_\alpha) = \zeta$.*

Note that the last sentence of (vi) only holds for the case $\zeta = 0$ because we have allowed $K_\rho = 0$ in (6) and (7). A final point in this section is that because we have ignored any algebraic relationships that may exist between the various l_m^d for different d, the τ_αs are not necessarily monomorphisms and so the elements of $\mathbf{G}_\alpha$ cannot be thought of as acting on $\tau_\alpha(\mathcal{R}_\alpha)$.

4. $\rho(w_f) - f$ cannot be too small.

Our goal in this section is to prove the following result.

PROPOSITION 3. — *Let g be as in Theorem 5, let $f = g[\mathbf{n}_{J+1} \curvearrowright 0]$ and let $\rho \in \mathbf{G}_M$. Then $g \neq \rho(w_f)$.*

We may regard f as an element of $\mathcal{X} \otimes_{\mathbb{R}} \mathbb{C}$. We establish the proposition by showing that $\rho(w_f)(x) - f(x)$ cannot tend to zero as rapidly as a positive power of $\eta(x)$ unless $\rho(w_f) = f$. Note that $\rho(w_f)(x)$ itself *can* tend to zero more rapidly then any power of a subnest of f. For example, we might have $f = e_3^{-1}(x)$, $\rho(w_f) = e_2^{-1}(2e^x)$ or $f = e_1^{-1}(x(1+e_2^{-1}(x)))$, $\rho(w_f) = e_1^{-1}(x(1+e_2^{+1}(x)))$.

We require yet more notation. For any $j \geq 0$, let $\mathcal{L}_j$ denote the field generated over $\mathbb{C}$ by all real powers of $l_0, l_1, \dots, l_j$. We define the *chain* of f, written $\mathrm{Ch}(f)$, to be the set consisting of f and its subnests down as far as the greatest subnest (in the ordering $\prec$) which is contained in $\mathcal{L}_\beta$. So if the elements of $\mathrm{Ch}(f)$ are $f_0 \prec f_1 \prec \cdots \prec f_p = f$, then the expressions for $f_1, \dots, f_p$ contain exponentials, but that for f_0 does not. For example if $f = e_1(l_1 l_2(2+e_1^{-1}(l_1^2 l_2 + 2)))$ then $\mathrm{Ch}(f)$ consists of the subnests

$$l_1^2 l_2 + 2 \prec e_1(l_1^2 l_2 + 2) \prec l_2(2+e_1^{-1}(l_1^2 l_2+2)) \prec l_1 l_2(2+e_1^{-1}(l_1^2 l_2+2)) \prec f.$$

Next we define a set $\mathcal{U}(f_i)$ for each element $f_i \in \mathrm{Ch}(f)$ by inducting down the chain. We shall eventually induct *up* the chain to show that if $u \in \mathcal{U}(f_i)$, $\rho \in \mathbf{G}_i$ and $\tau_M(w_f) = f$, then $\rho(w_f) - u$ cannot tend to zero more rapidly than every non-zero element of $\tau_M(\mathcal{R}_M)$. We start by taking $\mathcal{U}(f) = \{f\}$. Now suppose that $\mathcal{U}(f_i)$ has been defined. If $f_i = \exp(\pm f_{i-1})$ with $f_{i-1} \in \mathrm{Ch}(f)$, we set

$$\mathcal{U}(f_{i-1}) = \{K \pm \log u;\ K \in \mathbb{C},\ u \in \mathcal{U}(f_i) \setminus \{0\}\}.$$

Similarly if $f_i = l_m^d f_{i-1}$ where $d \in \mathbb{R} \setminus \{0\}$, and $f_{i-1} \in \mathrm{Ch}(f)$, we select $w_{l_m^d} \in \mathcal{R}_\alpha$ such that $\tau_\alpha(w_{l_m^d}) = l_m^d$. We then define

$$\mathcal{U}(f_{i-1}) = \{u/\rho(w_{l_m^d});\ u \in \mathcal{U}(f_i)\ ,\ \rho \in \mathbf{G}_M\ ,\ \rho(w_{l_m^d}) \neq 0\}.$$

Likewise if $f_i = A + f_{i-1}$ where $A \in \mathbb{C}$ and $f_{i-1} \in \mathrm{Ch}(f)$, we set

$$\mathcal{U}(f_{i-1}) = \{u - A;\ u \in \mathcal{U}(f_i)\}.$$

We note that any f_i in $\mathrm{Ch}(f)$ apart from the least element must be of one of these three forms, and so $\mathcal{U}(f_i)$ is defined for every f_i in the chain. We note also that $f_i \in \mathcal{U}(f_i)$ in all cases. We recall that by Proposition 2(iii), for any $f_i \in \mathrm{Ch}(f)$, there is a $w_{f_i} \in \mathcal{R}_\alpha$ such that $\tau_\alpha(w_{f_i}) = f_i$. We define

$$\Gamma = \max\{\gamma(x), \gamma(f_0), \gamma(f_1), \dots, \gamma(f_p)\}.$$

Clearly $\Gamma = \max\{\gamma(x), \gamma(\mathbf{n}_1), \dots, \gamma(\mathbf{n}_J)\}$, and so by the choice of J, $\mathbb{R}\langle g\rangle$ contains an element of comparability class at least Γ.

We prove Proposition 3 by establishing the following Lemma.

LEMMA 5. — *Let $f_i \in \mathrm{Ch}(f)$ and let $w_{f_i} \in \mathcal{R}_\alpha$ be such that $\tau_\alpha(w_{f_i}) = f_i$. Let $\rho \in \mathbf{G}_\alpha$, $u \in \mathcal{U}(f_i)$ and suppose that $\rho(w_{f_i}) \neq u$. Then* *there exist a sequence of real numbers* $x_n \to \infty$, *an* $h \in \mathrm{Ch}(f) \cup \{x\}$ *and an* $N \in \mathbb{Z}$ *such that* $h^N \to \infty$ *and for* $n = 1, 2, \ldots$,

$$|\rho(w_{f_i})(x_n) - u(x_n)| > h^{-N}(x_n).$$

On taking the case $f_p = f$, we obtain that $|\rho(w_f)(x_n) - f(x_n)| > h^{-N}(x_n)$. But if $\rho(w_f) - f \to 0$, this is inconsistent with $\gamma(\rho(w_f) - f) > \Gamma$. Hence $\rho(w_f) - f \neq \eta$, and so Proposition 3 will follow from Lemma 5.

As a first stage in proving Lemma 5, we obtain the following.

LEMMA 6. — *For* $f_i \in \mathrm{Ch}(f)$, *let* $Y(f_i)$ *be the smallest field containing* $\mathrm{Ch}(f_i) \cup \mathcal{L}_{\beta+i}$ *which is closed under the application of the* z*-functions* zexp_n, zlog_n, *and* $\mathrm{zpow}_n(r, \quad)$, $r \in \mathbb{R}$, $n \geq 0$. *Then for any* $\xi \in Y(f_i) \setminus \{0\}$, *there exists a real number* x_0, *an* $h \in \mathrm{Ch}(f) \cup \{x\}$, *and an* $N \in \mathbb{Z}$ *such that* $h^N \to \infty$ *and for* $x \geq x_0$,

$$h^{-N}(x) < |\xi(x)| < h^N(x).$$

The proof of the lemma uses arguments similar to those in [12]. However matters are simpler here because we are dealing with elements given as nested forms.

Proof of Lemma 6. — For each $Y(f_i)$, we define a finite set, $\mho(f_i)$, of nested forms all of whose comparability classes are distinct and no greater than Γ, with the elements of $\mho(f_i)$ being either equal to l_n for some $n \geq 0$, or else of the form $\exp(v)$ with $v \in Y(f_{i-1})$. The real powers of the elements of $\mho(f_i)$ will generate $Y(f_i)$ under field operations and the action of the z-functions. An element of $Y(f_i)$ will then be a product of powers of elements of $\mho(f_i)$ times a function which is analytic in powers of elements of $\mho(f_i)$ which tend to zero, and its asymptotic behaviour will be given by the first non-zero term in the power series expansion of the analytic function. The lemma will thus follow once the $\mho(f_i)$s are defined.

We use induction on the ordering of the chain. In the initial case, $Y(f_0)$ is generated from $\mathcal{L}_\beta$ using z-functions, and we take $\mho(f_0) = \{x, l_1, \ldots, l_\beta\}$. At a particular stage in the induction, the next subnest in the ordering will be obtained by either adding a constant possibly after having inverted, multiplying by a power of some l_α for $\alpha \leq \beta$ or exponentiating. Of these, the first two clearly leave the field Y unchanged apart from the addition of the appropriate $l_{\beta+i}$, which we include in $\mho(f_i)$; thus we have only to prove that the conclusion of the lemma is preserved under exponentiation. Suppose then that $Y(f_i)$ is generated by $\mho(f_i)$ and that $f_{i+1} = e^v$, where $v \in Y(f_i)$. If f_{i+1} is of different comparability class from every element of $Y(f_i)$, we take $\mho(f_{i+1}) = \mho(f_i) \cup \{f_{i+1}, l_{\beta+i+1}\}$. Otherwise there is a $\varphi_0 \in \mho(f_i)$ which has the same comparability class as f_{i+1}. Then there exists $r_0 \in \mathbb{R} \setminus \{0\}$ and ξ_1 with $\gamma(\xi_1) < \gamma(f_{i+1})$ such that $f_{i+1} = \varphi_0^{r_0}\xi_1$. Now if $\gamma(\xi_1)$ is not among the comparability classes of $\mho(f_i)$, we set $\mho(f_{i+1}) = \mho(f_i) \cup \{\xi_1, l_{\beta+i+1}\}$. Note that by induction, φ_0 must be of the form $\varphi_0 = \exp(\psi_0)$ with $\psi_0 \in Y(f_{i-1})$. Hence $\xi_1 = \exp(v - r_0\psi_0)$ and $v - r_0\psi_0 \in Y(f_i)$ as required. If on the other hand, ξ_1 has the same comparability class as an element, $\varphi_1 \in \mho(f_i)$, there must exist $r_1 \in \mathbb{R}\setminus\{0\}$ and ξ_2 with $\gamma(\xi_2) < \gamma(\xi_1)$ such that $\xi_1 = \varphi_1^{r_1}\xi_2$. This process may be continued. If at some stage we obtain a ξ_k of comparability class distinct from all those in $\mho(f_i)$, we define $\mho(f_{i+1}) = \mho(f_i) \cup \{\xi_k, l_{\beta+i+1}\}$. We will then have $f_{i+1} = \varphi_0^{r_0}\varphi_1^{r_1}\cdots\varphi_{k-1}^{r_{k-1}}\xi_k$, where $\varphi_0, \ldots, \varphi_{k-1} \in \mho(f_i)$, and so $\mho(f_{i+1})$ will indeed generate $Y(f_{i+1})$ as required. Otherwise, since $\mho(f_i)$ is finite, we must eventually obtain $f_{i+1} \sim K\varphi_0^{r_0}\varphi_1^{r_1}\cdots\varphi_k^{r_k}$, with $K \in \mathbb{C}\setminus\{0\}$, $r_0, \ldots, r_k \in \mathbb{R}$ and $\varphi_0, \ldots, \varphi_k \in \mho(f_i)$. Now each φ_j, $j \leq k$, is either of the form l_n, $n \leq \beta + i$ or of the form $\exp(t)$, $t \in Y(f_{i-1})$. We take $\mho(f_{i+1}) = \mho(f_i) \cup \{l_{\beta+i+1}\}$. Then for each $j \leq k$, $\log\varphi_j$ belongs to the field generated by $\mho(f_{i+1})$. Now recall that $f_{i+1} = e^v$ and write $\varphi = v - \log K - r_0\log\varphi_0 - \cdots - r_k\log\varphi_k$. Then φ belongs to the field generated by $\mho(f_{i+1})$, ϕ tends to zero, and $f_{i+1} = e^v = K\varphi_0^{r_0}\cdots\varphi_k^{r_k}(1+\mathrm{zexp}\,\varphi)$. Thus we see that $\mho(f_{i+1})$ generates $Y(f_{i+1})$ as required. Clearly the comparability classes in the various $\mho(f_i)$ do not exceed those in $\mathrm{Ch}(f)$, and so are no greater than Γ, as was asserted. We have therefore proved Lemma 6.

The following is an immediate consequence, which will be of relevance later in the proof.

COROLLARY.

$$\max\{\gamma(h);\ h \in \tau_M(\mathcal{R}_M)\} \leq \Gamma.$$

Proof. — $\tau_M(\mathcal{R}_M)$ is generated by $\mho(f_M)$ as above.

LEMMA 7. — *For any $f_i \in \mathrm{Ch}(f)$, we have $\mathcal{U}(f_i) \subset Y(f)$.*

Proof of Lemma 7. — For each $i = 0, \ldots, p$, we define X_i to be the smallest field containing $\mathrm{Ch}(f) \cup \mathcal{L}_{\beta+p-i}$ which is closed under application of the z-functions. Note that X_i is required to contain $\mathrm{Ch}(f)$, and not merely $\mathrm{Ch}(f_i)$. We prove by inducting down from $\mathcal{U}(f_p)$ that $\mathcal{U}(f_i) \subset X_i$. Since $X_i \subset Y(f)$ for each i, this will suffice to prove the lemma.

For the initial case, $\mathcal{U}(f_p) = \{f\} \subset X_p$. Now suppose that $i \leq p$, $\rho \in \mathbf{G}_i$ and $l_m^d \in \mathcal{S}_i$ and let $w_{l_m^d}$ be as above. Then there exist constants $K, K_0, \ldots, K_m$, such that

$$\begin{aligned}\rho(w_{l_m^d}) &= K(K_0+\log(K_1+\log(\cdots+\log(K_{m-1}+\log(K_m+x)\cdots))))^d \\ &= K(K_0 + \log(K_1 + \log(\cdots \log(\log x + K_{m-1} \\ &\qquad + \operatorname{zlog}(K_m/x)))\cdots)))^d \\ &= K(K_0 + \log(K_1 + \log(\cdots \log(l_2 + \operatorname{zlog}(l_1^{-1}(K_{m-1} \\ &\qquad + \operatorname{zlog}(K_m/x))))\cdots)))^d,\end{aligned} \tag{8}$$

etc. It follows that $\rho(w_{l_m^d}) \in X_i$ for any i and any $\rho \in \mathbf{G}_i$. Now suppose that $\mathcal{U}(f_i) \subset X_i$ and $f_i = l_m^d f_{i-1}$, then

$$\mathcal{U}(f_{i-1}) = \{u/\rho(w_{l_m^d});\ u \in \mathcal{U}(f_i)\ ,\ \ \rho \in \mathbf{G}_M\ ,\ \ \rho(w_{l_m^d}) \neq 0\} \subset X_{i-1}.$$

Similarly if $f_i = A + f_{i-1}$ with $A \in \mathbb{C}$,

$$\mathcal{U}(f_i) \subset X_i \Rightarrow \mathcal{U}(f_{i-1}) \subset X_{i-1}.$$

It only remains to consider the case when $f_i = \exp(f_{i-1})$, and here it is enough to show that $u \in X_i \setminus \{0\} \Rightarrow \log u \in X_{i-1}$. Now any element u of $X_i \setminus \{0\}$ can be written in the form

$$u = K l_{\alpha_1}^{d_1} \cdots l_{\alpha_j}^{d_j} \phi_1 \cdots \phi_k Z\{1+F\},$$

where $\phi_1, \ldots, \phi_k$ are exponentials in $\mathrm{Ch}(f)$, Z is a product of z-functions, K belongs to $\mathbb{C} \setminus \{0\}$ and F belongs to X_i and tends to zero. Clearly $\log \phi_1, \ldots, \log \phi_k \in X_{i-1}$, and in addition, we may write $\log(1+F) = \operatorname{zlog}(F)$. Moreover $l_{\alpha_1}, \ldots, l_{\alpha_j} \in X_i \Rightarrow l_{\alpha_1+1}, \ldots, l_{\alpha_j+1} \in X_{i-1}$. So to establish the lemma, it suffices to consider logarithms of the z-functions. We take $\operatorname{zlog}_s(h)$ as a typical factor of Z. From the definition, we have

$$\begin{aligned}\log(\operatorname{zlog}_s(h)) &= \log\{(-1)^s h/(s+1) + h\ \operatorname{zlog}_{s+1}(h))\} \\ &= \log\{(-1)^s h/(s+1)\} + \operatorname{zlog}\{(-1)^s(s+1)\operatorname{zlog}_{s+1}(h)\}.\end{aligned}$$

By structural induction, we may assume that $\log h \in X_{i-1}$ and hence $\log(\operatorname{zlog}_s(h)) \in X_{i-1}$. Similarly $\log(\operatorname{zexp}_s(h))$ and $\log(\operatorname{zpow}_s(c,h))$, $c \in \mathbb{R} \setminus \{0\}$, belong to X_{i-1} and Lemma 7 follows.

LEMMA 8. — *For every $f_i \in \mathrm{Ch}(f)$, the set $\mathcal{U}(f_i)$ does not contain the zero function.*

Proof of Lemma 8. — This is reasonably straightforward when one looks at the way in which the various $\mathcal{U}(f_i)$ are built. Suppose that

$f = e_{r_1}(\psi_1)$, with $r_1 > 0$. Then $\mathcal{U}(e_{r_1-1}(\psi_1)) = \{K_0 + e_{r_1-1}(\psi_1);\ K_0 \in \mathbb{C}\}$. Assuming that $r_1 - 1 > 0$, we have

$$\mathcal{U}(e_{r_1-2}(\psi_1)) = \{K_1 + \log(K_0 + e_{r_1-1}(\psi_1)));\ K_0, K_1 \in \mathbb{C}\}.$$

If $K_0 \neq 0$, the element $K_1 + \log(K_0 + e_{r_1-1}(\psi_1))$ has transcendence degree over $\mathbb{C}$ equal to r_1 plus the transcendence degree of ψ_1, and it is clear that no succession of operations $u \to K + \log u$, $u \to u/\rho(w_{l_m^d})$ and $u \to u - A$ can reduce this transcendence degree. So zero cannot be obtained from one of these elements. When $K_0 = 0$, the element under consideration is $K_1 + e_{r_1-2}(\psi_1)$. A similar argument now applies to this, with r_1 replaced by $r_1 - 1$. We can continue until we reach the element ψ_1, which we now assume has the form $L(A_1 + e_{r_2}^{-1}(\psi_2))$, with $r_2 \geq 1$, L a product of real powers of logarithms and $A_1 \in \mathbb{C}$. The operations $u \to u - A$ and $u \to u/\rho(w_{l_m^d})$, do not reduce the transcendence degree of $e_{r_2}^{-1}(\psi_2)$, and so we may apply the same argument to this element as we applied to f. At the last stage, we have $f_1 = \exp(L(A + f_0))$, where $A \in \mathbb{C} \setminus \{0\}$, L and f_0 belong to $\mathcal{L}_\beta$, $L \to \infty$ and $f_0 \to 0$. The elements of $\mathcal{U}(f_0)$ under consideration are of the form $K + \log f_1$, which has modulus tending to infinity and therefore cannot be zero! Hence zero does not belong to any $\mathcal{U}(f_i)$. Lemma 8 has thus been proved.

The following is now an immediate consequence of Lemmas 6, 7 and 8.

COROLLARY. — *For any $f_i \in \mathrm{Ch}(f)$ and any $u \in \mathcal{U}(f_i)$, there exists a real number x_0, an element h of* Ch(f) *and an integer N, such that for $x \geq x_0$,*

$$h^{-N}(x) < |u(x)| < h^N(x).$$

Proof of Lemma 5. — We induct up the chain of f. Recall that f_0, the minimal element of Ch(f), belongs to $\mathcal{L}_\beta$ and hence $\rho(f_0) - u \in Y(f)$ for all u in $\mathcal{U}(f_0)$. So in this case, the Lemma follows from Lemma 6.

Now suppose the result holds for some f_{i-1} in the chain, and consider first the case when $f_i = l_m^d f_{i-1}$, with $d \in \mathbb{R}\setminus\{0\}$. There exist $w_{l_m^d}$, $w_{f_{i-1}}$ and w_{f_i} in $\mathcal{R}_\alpha$ such that $w_{f_i} = w_{l_m^d} w_{f_{i-1}}$, $\tau_\alpha(w_{l_m^d}) = l_m^d$ and $\tau_\alpha(w_{f_{i-1}}) = f_{i-1}$. Let $u \in \mathcal{U}(f_i)$. If $\rho(w_{l_m^d}) = 0$, then $\rho(w_{f_i}) = 0$ and so $\rho(w_{f_i}) - u = -u$. The result then follows from the corollary to Lemma 8. Otherwise, we have

$$\rho(w_{f_i}) - u = \rho(w_{l_m^d})(\rho(w_{f_{i-1}}) - u/\rho(w_{l_m^d})).$$

But $u/\rho(w_{l_m^d}) \in \mathcal{U}(f_{i-1})$ and $\rho(w_{f_i}) \neq u \Rightarrow \rho(w_{f_{i-1}}) \neq u/\rho(w_{l_m^d})$. So by the induction hypothesis,

$$|\rho(w_{f_{i-1}})(x_n) - u(x_n)/\rho(w_{l_m^d})(x_n)| \geq h^{-N}(x_n),$$

for $n \geq 1$, where $\{x_n\}$, h and N are as in the statement of the Lemma. Since $\rho(w_{l_m^d}) \in Y(f) \setminus \{0\}$, the conclusion in this case now follows from Lemma 6. Similarly, if $f_i = A + f_{i-1}$ with $A \in \mathbb{C}$, let $u \in \mathcal{U}(f_i)$ and $\rho \in \mathbf{G}_M$. Then $\rho(w_{f_i}) - u = \rho(w_{f_{i-1}}) - (u - A)$ and by the induction hypothesis, $|\rho(w_{f_{i-1}})(x_n) - (u(x_n) - A)| \geq h^{-N}(x_n)$, for suitable $\{x_n\}$, h and N. Thus the desired conclusion is obtained here also.

Now consider the case when $f_i = \exp(\pm f_{i-1})$, and let $u \in \mathcal{U}(f_i)$. There exist $w_{f_{i-1}}$ and w_{f_i} in $\mathcal{R}_\alpha$ such that $\tau_\alpha(w_{f_{i-1}}) = f_{i-1}$, and $\tau_\alpha(w_{f_i}) = f_i$. If $\rho(w_{f_i})/u \nrightarrow 1$ as $x \to \infty$, then there is a sequence $\{x_n\} \to \infty$ on which $|\rho(w_{f_i}) - u| > k|u|$ for some real $k > 0$, and the result then follows from the corollary to Lemma 8. Otherwise $\rho(w_{f_i}) \sim u$ and we have $\rho(w_{f_i}) = \exp\{\pm(\rho(w_{f_{i-1}}) - K)\}$ for some $K \in \mathbb{C}$. Therefore

$$|\rho(w_{f_i})(x) - u(x)| = |u(x)||\exp\{\pm(\rho(w_{f_{i-1}})(x) - K) - \log u(x)\} - 1|.$$

Moreover $\rho(w_{f_i})/u \to 1$ implies that $\rho(w_{f_{i-1}}) - (K \pm \log|u|)$ must tend to zero, and hence for x sufficiently large,

$$|\rho(w_{f_i})(x) - u(x)| \geq \frac{1}{2}|u(x)||\rho(w_{f_{i-1}})(x) - (K \pm \log u(x))|.$$

But $K \pm \log u \in \mathcal{U}(f_{i-1})$ and so $\{x_n\}$, h and N exist as in the statement of the lemma, such that $|\rho(w_{f_{i-1}})(x_n) - (K \pm \log u(x_n))| \geq h^{-N}(x_n)$. The conclusion for this case now follows from the corollary to Lemma 8, since $u \in \mathcal{U}(f_i)$. The lemma is thus proved by induction, and Proposition 3 follows as previously established.

5. Obtaining the differential polynomial.

In this section, our objective is to obtain a differential polynomial over $\mathbb{R}$ which vanishes at the function f but not at g. In fact the polynomial we construct will have all its zeros in the set $\{\rho(w_f);\ \rho \in \mathbf{G}_M\}$, and the only property of g which we use here is that g does not belong to this set. Our construction is obtained by working down the tower $\mathcal{R}_M \supset \cdots \supset \mathcal{R}_0$. In the algebraic case, that is to say when $\tau_\alpha(v_\alpha) = l_m^{1/q}$ for some $m, q \in \mathbb{N}$, the transition from $\mathcal{R}_\alpha$ to $\mathcal{R}_{\alpha-1}$ is relatively straightforward. We recall that if $\mathcal{F}$ is a Hardy field and $f \in \mathcal{X}$, $\mathcal{F}\langle f\rangle$ denotes the field generated by $\mathcal{F}$ and all the derivatives of f. Now we note that an element, σ of $\mathbf{G}_\alpha$ induces a differential homomorphism, $\tilde{\sigma} : \mathcal{R}_\alpha\langle y\rangle \longrightarrow \mathcal{S}_\alpha\langle y\rangle$. We define $\mathcal{T}_\alpha = \tau_\alpha(\mathcal{R}_\alpha)$, for $1 \leq \alpha \leq M$.

PROPOSITION 4. — *Suspect that* $1 \leq \alpha \leq M$, *and that* $\tau_\alpha(v_\alpha)$ *is algebraic over* $\mathcal{T}_{\alpha-1}$ *(i.e.* $\tau_\alpha(v_\alpha) = l_m^{1/q}$ *for some* m *and* q*). Let* P_α *be a differential polynomial over* $\mathcal{R}_\alpha$ *with the property that* $\tilde{\tau}_\alpha(P_\alpha\langle f\rangle) = 0$ *but for all* $\sigma \in \mathbf{G}_\alpha$, $\tilde{\sigma}(P_\alpha)\langle g\rangle \neq 0$. *Then there exists a differential polynomial,* $P_{\alpha-1}$ *over* $\mathcal{R}_{\alpha-1}$ *with similar properties to* P_α. *i.e.* $\tilde{\tau}_{\alpha-1}(P_{\alpha-1}\langle f\rangle) = 0$ *but for all* $\sigma \in \mathbf{G}_{\alpha-1}$, $\tilde{\sigma}(P_{\alpha-1})\langle g\rangle \neq 0$.

Proof of Proposition 4. — Suppose that $\tau_\alpha(v_\alpha) = l_m^{1/q}$. In the polynomial $P_\alpha\langle y\rangle$, we replace v_α by a new indeterminate, z. This is not strictly necessary, but will hopefully shed light on subsequent arguments. Then we take

$$P_{\alpha-1}\langle y\rangle = \mathrm{res}_z\{P_\alpha(z)\langle y\rangle, z^q - l_m\}. \tag{9}$$

The standard properties of the resultant now show that $P_{\alpha-1}$ satisfies the conditions required by the lemma. Firstly $\tilde{\tau}_{\alpha-1}(P_{\alpha-1}\langle f\rangle) = 0$ because $P_\alpha(z)\langle f\rangle$ becomes zero when $l_m^{1/q}$ is substituted for z. Suppose that $\tilde{\sigma}(P_{\alpha-1}\langle g\rangle) = 0$, for some $\sigma \in \mathbf{G}_{\alpha-1}$. By extending our notation slightly, we may regard $\tilde{\sigma}$ as acting on $\mathcal{R}_{\alpha-1}[z]\langle y\rangle$ by applying σ to the coefficients in $\mathcal{R}_{\alpha-1}$. Now, since the resultant is given by the Sylvester determinant and σ is algebra homomorphism,

$$\tilde{\sigma}(P_{\alpha-1}\langle g\rangle) = \mathrm{res}_z\{\tilde{\sigma}(P_\alpha(z)\langle g\rangle), z^q - \sigma(l_m)\}.$$

Thus $\tilde{\sigma}(P_\alpha(z)\langle g\rangle)$ must become zero when z is replaced by some root, call it z_0, of $z^q = \sigma(l_m)$. But by part (v) of Proposition 2, there is a $\rho \in \mathbf{G}_\alpha$ which agrees with σ on $\mathcal{R}_{\alpha-1}$ and takes v_α to z_0. But then $\tilde{\rho}(P_\alpha(v_\alpha)\langle g\rangle) = \tilde{\sigma}(P_\alpha(z_0)\langle g\rangle) = 0$, contrary to hypothesis. This completes the proof of Proposition 4.

The case when $\tau_\alpha(v_\alpha)$ is transcendental over $\mathcal{S}_{\alpha-1}$ is substantially more difficult. We recall that $\mathcal{R}_\alpha = \mathcal{R}_{\alpha-1}[v_\alpha]$ and that v_α satisfies a differential equation

$$D(z) = \Lambda_\alpha(z)/\Omega_\alpha(z), \tag{10}$$

where Λ_α and Ω_α are polynomials over $\mathcal{R}_{\alpha-1}$. Of course (10) will be either $D(y) = D(v_{\alpha-1})/v_{\alpha-1}$, $D(y) = ydD(v_m)/v_m$ or $D(y) = yD(w_h)$, where $\tau_{\alpha-1}(w_h) = h$. We regard P_α as an element of $\mathcal{R}_{\alpha-1}[v_\alpha]\langle y\rangle$, and as above, we replace v_α in P_α, by a new indeterminate z to form a polynomial which we now denote by P; thus $P \in \mathcal{R}_{\alpha-1}[z]\langle y\rangle$. On $\mathcal{R}_{\alpha-1}[z]\langle y\rangle$ we use the differential operator, D_α^*, defined by

$$D_\alpha^*(P) = \tilde{D}(P)\Omega_\alpha(z) + \frac{\partial P}{\partial z}\Lambda_\alpha(z) + \Omega_\alpha(z)\sum_{i=0} y^{(i+1)}\frac{\partial P}{\partial y^{(i)}},$$

where $\tilde{D}(P)$ is obtained by applying D to the coefficients of P. Note that $D^*_\alpha(P)|_{z\to v_\alpha} = D(P(v_\alpha))\Omega_\alpha(v_\alpha)$. The idea is to eliminate z between P and $D^*_\alpha(P)$ to obtain $P_{\alpha-1}$. However we will need to first ensure that $\tilde{\sigma}(P(z)\langle g\rangle)$ is square free (as a polynomial in z) for every $\sigma \in \mathbf{G}_{\alpha-1}$. We begin with the following proposition.

PROPOSITION 5. — *Suppose that m and α are non-negative integers with $\alpha \leq M$, and let $r_1, \ldots, r_m$ be complex-valued parameters. Write $\mathbf{r} = (r_1, \ldots, r_m)$ and let $P(\mathbf{r})$ be a polynomial in $r_1, \ldots, r_m$ with coefficients belonging to $\mathcal{R}_\alpha\langle y\rangle$. Now let $\chi : \mathbb{C}^{a(\alpha)} \times \mathcal{B}_\alpha \to \mathbf{G}_\alpha$ be the parameterisation of $\mathbf{G}_\alpha$, as in Proposition 2, where*

$$\mathcal{B}_\alpha = \prod_{b=1}^{b(\alpha)} \{1, e^{2\pi i/q_b}, \ldots e^{2\pi i(q_b-1)/q_b}\}.$$

Let $\mathbf{q}_0$ be any element of $\mathcal{B}_\alpha$. Then the set

$$V = \{(\mathbf{p}, \mathbf{r}) \in \mathbb{C}^{a(\alpha)+m};\ \tilde{\chi}(\mathbf{p}, \mathbf{q}_0)(P(\mathbf{r})\langle g\rangle) = 0\}$$

is an analytic subvariety of $\mathbb{C}^{a(\alpha)+m}$ (1).

Appropriate definitions and results concerning analytic varieties may be found in [3] or [20] for example. We will require the following facts :

(i) Let U be a domain in $\mathbb{C}^n$. A subset V is a *subvariety* of U if for every z in U there is a neighbourhood U_z and functions $f_1, \ldots, f_t$ analytic in U_z such that

$$V \cap U_z = \{x \in U_z;\ f_1(x) = \cdots = f_t(x) = 0\}.$$

(ii) A point p of V is said to be a *regular* point of V if there is a neighbourhood, U_p of p such that $U_p \cap V$ is a complex submanifold of U_p. Otherwise p is called a *singular* point of V.

(iii) Finite unions and intersections of subvarieties are subvarieties.

(iv) If V is a subvariety at 0 of dimension ς, then the regular points of V form a submanifold of dimension ς which is dense in V, and the singular points form a subvariety of dimension less than ς.

We also require the following lemma, which is adapted from Lemma $\bar{\text{V}}$.D.1 in [3].

(1) Note that for $(\mathbf{p}, \mathbf{r})$ to belong to V we require $\tilde{\chi}(\mathbf{p}, \mathbf{q}_0)(P(\mathbf{r})\langle g\rangle)$ to be the *zero function* (of x).

Lemma 9. — *Let V be an analytic variety of dimension ς. Let F be a holomorphic mapping of V into $\mathbb{C}^m$, where $m > \varsigma$, and let ω be the natural embedding of $\mathbb{R}^m \to \mathbb{C}^m$. Then $F(V) \cap \omega(\mathbb{R}^m)$ is of first category in $\omega(\mathbb{R}^m)$.*

Proof of Lemma 9. — We use induction on ς. If $\varsigma = 0$, then V is countable, and so $F(V)$ is countable. For the case $\varsigma > 0$, let $\Re(V)$ denote the set of regular point of V, and let $V_0 = \{x \in \Re(V);\ \operatorname{rank}_x F = n\}$, where n is the maximal rank of F. Then $n \leq \varsigma < m$, and if $x \in V_0$, there is a neighbourhood, U_x, of x in V such that $F(U_x)$ is a complex submanifold of $\mathbb{C}^m$ of dimension n. Then certainly $F(U_x) \cap \omega(\mathbb{R}^m)$ is of first category in $\omega(\mathbb{R}^m)$. We can cover V_0 by countably many such U_x and so $F(V_0) \cap \omega(\mathbb{R}^m)$ is of first category. But $V \setminus V_0$ is of dimension less than ς, and so by induction $F(V \setminus V_0) \cap \omega(\mathbb{R}^m)$ is of first category. Thus

$$F(V) \cap \omega(\mathbb{R}^m) = \{F(V_0) \cap \omega(\mathbb{R}^m)\} \cup \{F(V \setminus V_0) \cap \omega(\mathbb{R}^m)\}$$

is of first category also, and the lemma is proved.

Proof of Proposition 5. — We temporarily reinterprete elements of $\mathcal{R}_\alpha(\mathbf{r})\langle y\rangle$ as functions on the set $\mathbb{C}^{a(\alpha)+m} \times \mathcal{B}_\alpha$. The idea is to 'differentiate out' the 'arbitrary constants', i.e $p_1, \ldots, p_{a(\alpha)}, r_1, \ldots, r_m$, to form a differential equation satisfied by every $\tilde{\chi}(\mathbf{p}, \mathbf{q}_0)(P(\mathbf{r})\langle g\rangle)$. For particular values of $\mathbf{p}$ and $\mathbf{r}$, $\tilde{\chi}(\mathbf{p}, \mathbf{q}_0)(P(\mathbf{r})\langle g\rangle)$ will then be functionally equivalent to zero precisely when it and enough of its derivatives vanish at a suitably chosen evaluation point x_0.

For any $Q \in \mathcal{R}_\alpha(\mathbf{r})\langle y\rangle$, we denote by $\Xi(Q)$ the function from $\mathbb{C}^{a(\alpha)+m} \times \mathcal{B}_\alpha$ to $\mathcal{S}_\alpha\langle g\rangle$ given by

$$(p_1, \ldots, p_{a(\alpha)}, r_1, \ldots, r_m, q_1, \ldots, q_{b(\alpha)}) \longrightarrow \tilde{\chi}(\mathbf{p}, \mathbf{q})(Q(\mathbf{r})\langle g\rangle).$$

Next we define a tower of differential algebras containing the various $\Xi(Q)$. We take $\Upsilon_0 = \mathbb{C}\langle g\rangle(\mathbf{r})$ and suppose that Υ_i has been defined for $0 \leq i < j$. We then set $\Upsilon_j = \Upsilon_{j-1}(\Xi(v_j))$, where $\Xi(v_j)$ is the function on $\mathbb{C}^{a(j)+m} \times \mathcal{B}_j$ given by

$$(p_1, \ldots, p_{a(j)}, r_1, \ldots, r_m, q_1, \ldots, q_{b(j)}) \longrightarrow \chi(\mathbf{p}, \mathbf{q})(v_j).$$

We extend the differentiation from Υ_{j-1} to Υ_j by declaring the derivative of $\Xi(v_j)$ to be $\Xi(\Lambda_j(v_j)/\Omega_j(v_j))$; *i.e.* as given by the differential equation of v_j as an exponential, logarithm or real power, as the case may be.

The transcendence degree of Υ_j over Υ_{j-1} is at most one. By induction the transcendence degree of each Υ_α over Υ_0 is finite and hence so is the transcendence degree over $\mathbb{C}\langle g\rangle$. Suppose first that g satisfies an algebraic differential equation over $\mathbb{R}$. Then every element of Υ_α satisfies an algebraic differential equation over $\mathbb{C}$, and in particular, this is true of $\Xi(P)$, where P is the element of $\mathcal{R}_\alpha(\mathbf{r})\langle y\rangle$ in the statement of the proposition. Let μ be the order of the differential equation satisfied by $\Xi(P)$. Then for particular values of $p_1,\dots,p_{a(\alpha)},r_1,\dots,r_m,q_1,\dots,q_{b(\alpha)}$, we have that $\Xi(P)(\mathbf{p},\mathbf{q})$ will be the zero element of $\mathcal{S}_\alpha\langle g\rangle$ if and only if it satisfies initial conditions of the form $y(x_0)=y'(x_0)=\cdots=y^{(\mu-1)}(x_0)=0$. Let $(\mathbf{p_0},\mathbf{r_0})$ be an element of $\mathbb{C}^{a(\alpha)+m}$ and $\mathbf{q}_0$ an element of $\mathcal{B}_\alpha$. Let U_0 be any relatively compact neighbourhood of $(\mathbf{p_0},\mathbf{r_0})$. There exists a sufficiently large $x_0\in\mathbb{R}$ such that all the logarithms occurring in $\Xi(P)$ with $(\mathbf{p},\mathbf{r})\in U_0$ are defined at x_0. Then, for each $i=0,\dots,\mu-1$, $d^i(\Xi(P))/dx^i(\mathbf{p},\mathbf{r},\mathbf{q}_0)|_{x=x_0}$ will be an analytic function of $p_1,\dots,p_{a(\alpha)},r_1,\dots,r_m$ in $U_0\times\mathbb{C}^m$. Therefore the set of $(\mathbf{p},\mathbf{r})\in U_0$ for which $\tilde{\chi}(\mathbf{p},\mathbf{q}_0)(P(\mathbf{r}))\langle g\rangle=0$ is given by the vanishing of a finite set of analytic functions. So V is an an analytic subvariety of $\mathbb{C}^{a(\alpha)+m}$ as asserted.

On the other hand, if g satisfies no algebraic differential equation over $\mathbb{R}$, then it cannot satisfy one over $\mathcal{S}_\alpha$. For the transcendence degree of $\mathcal{S}_\alpha\langle g\rangle$ over $\mathcal{S}_\alpha$ would then be finite, and hence so would its transcendence degree over $\mathbb{R}$. It follows that in this case an element of $\mathcal{S}_\alpha\langle g\rangle$ is zero if and only if all the coefficients of the monomials in g and its derivatives are zero. So the condition for $\tilde{\chi}(\mathbf{p},\mathbf{q}_0)(P(\mathbf{r}))\langle g\rangle$ to be zero is given by the vanishing of a finite set of elements of $\mathcal{S}_\alpha$ and thus is an analytic variety by the above argument. Proposition 5 is therefore proved.

We need information about the dimension of varieties such as V above. This is given by the following result.

LEMMA 10. — *Let $\mathcal{F}$ be an infinite field and let $P,Q\in\mathcal{F}[x]$, where x is an indeterminate. Let $r\geq 2$ be a fixed integer and suppose that P and Q have no common root of multiplicity greater than or equal to r. Then there are only a finite number of $h\in\mathcal{F}$ such that $P+hQ$ has a root of multiplicity at least r.*

Proof. — The following elegant little proof of this result is due to Chris Woodcock.

Suppose on the contrary that there are infinitely-many $h\in\mathcal{F}$ for which a root, $\alpha(h)$, of multiplicity at least r exists in the algebraic closure of $\mathcal{F}$. Then for such h

$$P(\alpha(h)) + hQ(\alpha(h)) = 0,$$
$$P'(\alpha(h)) + hQ'(\alpha(h)) = 0.$$

Therefore $(PQ' - P'Q)(\alpha(h)) = 0$ for all such h. Suppose first that there are infinitely-many distinct $\alpha(h)$. Then $PQ' - P'Q$ is the zero polynomial. So P is a constant multiple of Q, and by the hypotheses of the lemma, Q can have no root of multiplicity greater than or equal to r. Therefore there is just one value of h for which $P + hQ$ has a root of multiplicity at least r, namely that for which $P + hQ \equiv 0$. This contradicts our earlier assumption and therefore this case cannot occur.

On the other hand, if there are only finitely-many distinct $\alpha(h)$ then there must exist $h_1 \neq h_2$ such that $\alpha(h_1) = \alpha(h_2)$. But then $(x - \alpha(h_1))^r$ divides both $P + h_1Q$ and $P + h_2Q$. Hence it divides both P and Q, contrary to hypothesis. The proof of the lemma is thus complete.

Before stating our next proposition, we give the following lemma.

LEMMA 11. — *Let $Q \in \mathcal{R}_{\alpha-1}[z]\langle y\rangle$ and $\sigma \in \mathbf{G}_\alpha$. Then*

$$\tilde{\sigma}(D_\alpha^*(Q)) = \Theta_{\alpha,\sigma}^*(\tilde{\sigma}(Q)),$$

where $\Theta_{\alpha,\sigma}^*$ is defined analagously to D_α^*. i.e. for $S \in \mathcal{S}_\alpha\langle y\rangle$,

$$\Theta_{\alpha,\sigma}^*(S) = \frac{\partial S}{\partial x}\tilde{\sigma}(\Omega_\alpha(z)) + \frac{\partial S}{\partial z}\tilde{\sigma}(\Lambda_\alpha(z)) + \tilde{\sigma}(\Omega_\alpha(z))\sum_{i=0} y^{(i+1)}\frac{\partial S}{\partial y^{(i)}}.$$

Proof of Lemma 11. — $\tilde{\sigma}(D_\alpha^*(Q)(z))$ is equal to

$$\tilde{\sigma}\left(\tilde{D}(Q)\Omega_\alpha(z) + \frac{\partial Q}{\partial z}\Lambda_\alpha(z) + \Omega_\alpha(z)\sum_{i=0} y^{(i+1)}\frac{\partial Q}{\partial y^{(i)}}\right)$$
$$= \tilde{\sigma}(\tilde{D}(Q))\tilde{\sigma}(\Omega_\alpha(z)) + \tilde{\sigma}\left(\frac{\partial Q}{\partial z}\right)\tilde{\sigma}(\Lambda_\alpha(z)) + \tilde{\sigma}(\Omega_\alpha(z))\sum_{i=0} y^{(i+1)}\tilde{\sigma}\left(\frac{\partial Q}{\partial y^{(i)}}\right)$$
$$= \frac{\partial(\tilde{\sigma}(Q))}{\partial x}\tilde{\sigma}(\Omega_\alpha(z)) + \frac{\partial\tilde{\sigma}(Q)}{\partial z}\tilde{\sigma}(\Lambda_\alpha(z)) + \tilde{\sigma}(\Omega_\alpha(z))\sum_{i=0} y^{(i+1)}\frac{\partial\tilde{\sigma}(Q)}{\partial y^{(i)}},$$

since σ commutes with the derivations. But this last expression is just $\Theta_{\alpha,\sigma}^*(\tilde{\sigma}(Q))$, which establishes the result.

Our method of obtaining a P such that $\tilde{\sigma}(P)\langle g\rangle(z)$ is square free is as follows. We reduce the maximal multiplicity of a root of a $\tilde{\sigma}(P)$ by adding a term to P of the form $\lambda D_\alpha^*(P)$ where λ is a polynomial in x. We use a dimension argument to show that suitable λ exist.

PROPOSITION 6. — *Suspect that $\tau_\alpha(v_\alpha)$ is transcendental over $\mathcal{T}_{\alpha-1}$. Let P be an element of $\mathcal{R}_{\alpha-1}[z]\langle y\rangle$ such that $\tilde{\tau}_\alpha(P(v_\alpha)\langle f\rangle)=0$ but $\tilde{\rho}(P(v_\alpha)\langle g\rangle)$ is not zero for any $\rho\in\mathbf{G}_\alpha$. Then we can find a polynomial, $Q\in\mathcal{R}_{\alpha-1}[z]\langle y\rangle$ with the same properties as P, and such that $\tilde{\rho}(Q(z)\langle g\rangle)$ is a square-free polynomial in z for every $\rho\in\mathbf{G}_{\alpha-1}$.*

Proof of Proposition 6. — Let $\sigma\in\mathbf{G}_{\alpha-1}$ be such that the polynomial $\tilde{\sigma}(P(z)\langle g\rangle)$ has a root, $z=\zeta$, in the algebraic closure of $\mathcal{R}_{\alpha-1}\langle g\rangle$, of maximal multiplicity r, where $r\geq 2$. If ζ were to satisfy the differential equation for $\sigma(v_\alpha)$, *i.e.*

$$z'=\tilde{\sigma}(\Lambda_\alpha(z))/\tilde{\sigma}(\Omega_\alpha(z)), \tag{11}$$

then by Proposition 2(vi), there would be a $\rho\in\mathbf{G}_\alpha$ whose restriction to $\mathcal{R}_{\alpha-1}$ is equal to σ and such that $\rho(v_\alpha)=\zeta$. But then $\tilde{\rho}(P(v_\alpha)\langle g\rangle)=\tilde{\sigma}(P(\zeta)\langle g\rangle)=0$, contrary to our assumptions. So ζ does not satisfy (11).

$\Theta^*_{\alpha,\sigma}(\tilde{\sigma}(P))$, will be a sum of terms of which all but one contains a factor $(z-\zeta)^r$. The remaining term will instead contain the factor $r(z-\zeta)^{r-1}\{\tilde{\sigma}(\Lambda_\alpha(z))-\zeta'\tilde{\sigma}(\Omega_\alpha(z))\}$ but no higher power of $z-\zeta$. Now $z=\zeta$ is not a zero of $\tilde{\sigma}(\Lambda_\alpha(z))-\zeta'\tilde{\sigma}(\Omega_\alpha(z))$ since ζ does not satisfy (11). So ζ is a zero of order $r-1$ of $\Theta^*_{\alpha,\sigma}(\tilde{\sigma}(P))$, which by Lemma 11, is equal to $\tilde{\sigma}(D^*_\alpha(P))$.

It now follows from Lemma 10 that for each $\sigma\in\mathbf{G}_{\alpha-1}$ there are only a finite number of $h\in\mathcal{R}_{\alpha-1}$ for which $\tilde{\sigma}(P)(z)\langle g\rangle+h\tilde{\sigma}(D^*_\alpha(P))(z)\langle g\rangle$ has a root in z of multiplicity greater than $r-1$. Let $m\in\mathbb{N}$ and $t_0,t_1,\ldots,t_{m-1}\in\mathbb{C}$. We shall write $\mathbf{t}=(t_0,t_1,\ldots,t_{m-1})$ and $\mathbf{x}=(1,x,x^2,\ldots,x^{m-1})$, so that $\mathbf{t}\cdot\mathbf{x}=t_0+t_1x+\cdots+t_{m-1}x^{m-1}$. Recall that $\mathbf{G}_\alpha$ is parameterised by the map $\chi:\mathbb{C}^{a(\alpha)}\times\mathcal{B}_\alpha\to\mathbf{G}_\alpha$ and write $Q_\mathbf{t}=(P+(\mathbf{t}\cdot\mathbf{x})D^*_\alpha(P))\langle g\rangle$. Let $V_1\subset\mathbb{C}^{a(\alpha)+m}$ be the set

$$V_1=\{(\mathbf{p},\mathbf{t})\in\mathbb{C}^{a(\alpha)+m};\ \exists\mathbf{q}\in\mathcal{B}_\alpha\ /\ \tilde{\chi}(\mathbf{p},\mathbf{q})(Q_\mathbf{t}(v_\alpha))=0\}.$$

Then, by Proposition 5, V_1 is an analytic variety. Moreover for a given set of values of $\mathbf{p}$ and $\mathbf{q}$, there is at most one $\mathbf{t}=\{t_0,t_1,\ldots,t_{m-1}\}$ such that $\tilde{\chi}(\mathbf{p},\mathbf{q})(Q_\mathbf{t}(v_\alpha))=0$, since $\tilde{\chi}(\mathbf{p},\mathbf{q})((P(v_\alpha)\langle g\rangle)\neq 0$. Here we have used the fact that the elements of $\mathbf{G}_1$ are injective, and so if $\mathbf{t}_1\neq\mathbf{t}_2$ then $\tilde{\chi}(\mathbf{p},\mathbf{q})(\mathbf{t}_1\cdot\mathbf{x})\neq\tilde{\chi}(\mathbf{p},\mathbf{q})(\mathbf{t}_2\cdot\mathbf{x})$. Thus V_1 is a subset of an analytic cover over $\mathbb{C}^{a(\alpha)}$ and so is of dimension at most $a(\alpha)$. Similarly, let

$$V_2=\{(\mathbf{p},\mathbf{t})\in\mathbb{C}^{a(\alpha)+m};\exists\mathbf{q}\in\mathcal{B}_\alpha\ /\ \tilde{\chi}(\mathbf{p},\mathbf{q})(Q_\mathbf{t}(z))\ \text{has a root of multiplicity }\geq r\}.$$

The condition for $(\mathbf{p}, \mathbf{t})$ to belong to V_2 can be expressed in terms of the vanishing of certain polynomials in the coefficients of $\tilde{\chi}(\mathbf{p}, \mathbf{q})(Q_{\mathbf{t}})$. This follows on taking F_i, to be the i-th derivative of F, $i = 1, \ldots, r$, in the following theorem of elimination theory, [19].

THEOREM. — *Let $F_0, \ldots, F_{r-1}$ be r polynomials in a single variable of given degree with indeterminate coefficients. Then there exists a system, $D_1, \ldots, D_k$ of integral polynomials in these coefficients with the property that if those coefficients are assigned values from a field, $\mathcal{K}$, the conditions $D_1 = 0, \ldots, D_k = 0$ are necessary and sufficient in order that either the equations $F_0 = 0, \ldots, F_{r-1} = 0$ have a solution in a suitable extension field or that the formal leading coefficients of all the polynomials $F_0, \ldots, F_{r-1}$ vanish.*

Thus V_2 will be given by a finite set of equations of the form $\tilde{\chi}(\mathbf{p}, \mathbf{q})(S\langle g\rangle) = 0$, where $S \in \mathcal{R}_{\alpha-1}(\mathbf{t})\langle y\rangle$, and so will also be an analytic variety, by Proposition 5. However Lemma 10 shows that for each fixed $\mathbf{p}$, there is only a finite set of $\mathbf{t}$ for which $(\mathbf{p}, \mathbf{t}) \in V_2$. So V_2, like V_1, and so is of dimension $a(\alpha)$ at most. Now if we take $m > a(\alpha)$ and apply Lemma 9 with $V = V_1 \cup V_2$ and F equal to the projection onto $\mathbb{C}^m$, we obtain that the projection intersects the image of $\mathbb{R}^m$ in a subset of the first category. Hence there exist real values of $t_1, \ldots, t_m$ such that for any $\rho \in \mathbf{G}_{\alpha-1}$, $\tilde{\rho}(Q_{\mathbf{t}})(v_\alpha) \neq 0$ and moreover $\tilde{\rho}(Q_{\mathbf{t}})$ has no root of multiplicity greater than $r-1$.

On the other hand $\tilde{\tau}_\alpha(Q_{\mathbf{t}}(v_\alpha)\langle f\rangle) = 0$; for $\tilde{\tau}_\alpha(P(z)\langle f\rangle)$ must contain a factor $z - \tau_\alpha(v_\alpha)$, since $\tilde{\tau}_\alpha(P(v_\alpha)\langle f\rangle) = 0$. Therefore we can write $\tilde{\tau}_\alpha(P(z)\langle f\rangle) = (z - \tau_\alpha(v_\alpha))\Phi(z)$. Then

$$\begin{aligned}\Theta^*_{\alpha,\tau_\alpha}(\tilde{\tau}_\alpha(P(z)\langle f\rangle)) = (z - \tau_\alpha(v_\alpha))&\Theta^*_{\alpha,\tau_\alpha}(\Phi(z)) \\ &+\{\tilde{\tau}_\alpha(\Lambda_\alpha(z)) - \tau_\alpha(v_\alpha)'\tilde{\tau}_\alpha(\Omega_\alpha(z))\}\Phi(z).\end{aligned}$$

However, $(\tau_\alpha(v_\alpha))' = \tau_\alpha(\Lambda_\alpha(v_\alpha))/\tau_\alpha(\Omega_\alpha(v_\alpha))$ and so $z - \tau_\alpha(v_\alpha)$ is a factor of $\Theta^*_{\alpha,\tau_\alpha}(P(z)\langle f\rangle)$. *i.e.* $\Theta^*_{\alpha,\tau_\alpha}(P(v_\alpha)\langle f\rangle) = 0$. But for any $\mathbf{t}$,

$$\begin{aligned}\tau_\alpha(Q_{\mathbf{t}}(v_\alpha)\langle f\rangle) &= \tilde{\tau}_\alpha(P(v_\alpha)\langle f\rangle) + \tau_\alpha(\mathbf{t}\cdot\mathbf{x})\tilde{\tau}_\alpha(D^*_\alpha(P)(z)\langle f\rangle|_{z\to\tau_\alpha(v_\alpha)} \\ &= \tilde{\tau}_\alpha(P(v_\alpha)\langle f\rangle) + \tau_\alpha(\mathbf{t}\cdot\mathbf{x})\Theta^*_{\alpha,\tau_\alpha}(\tilde{\tau}_\alpha(P(z)\langle f\rangle))|_{z\to\tau_\alpha(v_\alpha)},\end{aligned}$$

by Lemma 11. Hence $\tilde{\tau}_\alpha(Q_{\mathbf{t}}(v_\alpha)\langle f\rangle) = 0$ and so $Q_{\mathbf{t}}$ has similar properties to P but none of its images under elements of $\mathbf{G}_\alpha$ have roots of multiplicity exceeding $r-1$. By induction we can then find a square-free polynomial as required, and this completes the proof of Proposition 6.

We can now show how $P_{\alpha-1}$ is obtained from P_α. This is given by the following proposition, which uses similar arguments to those above.

PROPOSITION 7. — *Suppose that $\tau_\alpha(v_\alpha)$ is transcendental over $\mathcal{R}_{\alpha-1}$. Let P_α be an element of $\mathcal{R}_{\alpha-1}(z)\langle y\rangle$, such that $\tilde{\tau}_\alpha(P_\alpha(v_\alpha)\langle f\rangle) = 0$ and $\tilde{\rho}(P_\alpha(v_\alpha))\langle g\rangle$ is non-zero for every $\rho \in \mathbf{G}_\alpha$. Then we can find a polynomial $P_{\alpha-1} \in \mathcal{R}_{\alpha-1}\langle y\rangle$ which satisfies conditions analagous to those satisfied by $P_\alpha(v_\alpha)$; i.e. $\tilde{\tau}_{\alpha-1}(P_{\alpha-1}\langle f\rangle) = 0$ but for all $\sigma \in \mathbf{G}_{\alpha-1}$, $\tilde{\sigma}(P_{\alpha-1}))\langle g\rangle \neq 0$.*

Proof of Proposition 7. — After Proposition 6, we can assume that every $\tilde{\rho}(P_\alpha(z)\langle g\rangle)$ is a square-free polynomial in z. Let $P_{\alpha-1}$ be the resultant, $\mathrm{res}_z(P_\alpha, D^*_\alpha(P_\alpha))$; we show that $P_{\alpha-1}$ has the required properties.

Firstly, $\tilde{\tau}_\alpha(P_\alpha(z)\langle f\rangle)$ and $\Theta^*_{\alpha,\tau_\alpha}(\tilde{\tau}_\alpha(P_\alpha(z)\langle f\rangle)$ contain a common factor, $z-\tau_\alpha(v_\alpha)$, as in the proof of Proposition 6, and so $\tilde{\tau}_{\alpha-1}(P_{\alpha-1}\langle f\rangle) = 0$.

Now let $\sigma \in \mathbf{G}_{\alpha-1}$ and suppose that $\tilde{\sigma}(P_{\alpha-1})\langle g\rangle = 0$. We shall show that this contradicts the hypotheses of the proposition. As before, $\tilde{\sigma}$ acts on coefficients and is a algebra homomorphism, and so we have

$$\begin{aligned}\tilde{\sigma}(P_{\alpha-1}\langle g\rangle) &= \tilde{\sigma}(\mathrm{res}_z\{P_\alpha\langle g\rangle, D^*_\alpha(P_\alpha\langle g\rangle)\})\\ &= \mathrm{res}_z\{\tilde{\sigma}(P_\alpha\langle g\rangle), \tilde{\sigma}(D^*_\alpha(P_\alpha\langle g\rangle))\}\\ &= \mathrm{res}_z\{\tilde{\sigma}(P_\alpha\langle g\rangle(z)), \Theta^*_{\alpha,\sigma}(\tilde{\sigma}(P_\alpha\langle g\rangle(z)))\},\end{aligned}$$

by Lemma 11. So if $\tilde{\sigma}(P_{\alpha-1})\langle g\rangle = 0$, then $\tilde{\sigma}(P_\alpha(z))\langle g\rangle$ and $\Theta^*_{\alpha,\sigma}(\tilde{\sigma}(P_\alpha(z))\langle g\rangle)$ must have a factor $z-\zeta$ in common, where ζ belongs to the algebraic closure of $\mathcal{S}_{\alpha-1}\langle g\rangle$. Let $\tilde{\sigma}(P_\alpha(z))\langle g\rangle = (z-\zeta)Y(z)$. The domain of $\Theta^*_{\alpha,\sigma}$ can obviously be extended to the algebraic closure of $\mathcal{S}_{\alpha-1}\langle g\rangle$. Then

$$\Theta^*_{\alpha,\sigma}(\tilde{\sigma}(P_\alpha(z))\langle g\rangle) = (z-\zeta)\Theta^*_{\alpha,\sigma}(Y(z)) + \{\tilde{\sigma}(\Lambda_\alpha)(z) - \zeta'\tilde{\sigma}(\Omega_\alpha)(z)\}Y(z).$$

However $z-\zeta$ is not a factor of Y, since $\tilde{\sigma}(P_\alpha(z)\langle g\rangle)$ is square free. Therefore $\tilde{\sigma}(\Lambda_\alpha)(\zeta) - \zeta'\tilde{\sigma}(\Omega_\alpha)(\zeta) = 0$. Then, by Proposition 2(vi), there is a $\rho \in \mathbf{G}_\alpha$ which agrees with σ on $\mathcal{R}_{\alpha-1}$ and takes v_α to ζ. So $\tilde{\rho}(P_\alpha(v_\alpha)\langle g\rangle) = \tilde{\sigma}(P_\alpha(\zeta)\langle g\rangle) = 0$, contrary to hypothesis. Thus $\tilde{\sigma}(P_{\alpha-1}\langle g\rangle) \neq 0$ for all $\sigma \in \mathbf{G}_{\alpha-1}$ and Proposition 7 is proved.

We shall use Proposition 7 to obtain a differential polynomial, P, over $\mathbb{R}$ which vanishes at f but not at g. Our next Lemma shows that $P\langle g\rangle$ will have the growth properties we require.

LEMMA 12. — *Let g, f and η be as above, with $g = f+\eta$ and $\gamma(\eta) > \Gamma$. Then if P is any differential polynomial over $\mathbb{R}$ such that $P\langle g\rangle \neq P\langle f\rangle$, we have $\gamma(P\langle g\rangle - P\langle f\rangle) \geq \gamma(\eta)$.*

Proof of Lemma 12. — Clearly we can write $P\langle f+\eta\rangle - P\langle f\rangle$ as a differential polynomial in η over $\mathcal{T}_M$ with every term involving η or one of its derivatives. Now any derivative of η must tend to zero. Moreover by part (ii) of the following result, which appears as Lemma 2 of [13], it has comparability class equal to $\gamma(\eta)$.

LEMMA 13. — *Let h be an element of a Hardy field.*

(i) *If $\gamma(h)=\gamma(x)$ and $\log|h| \not\sim \log x$ then $\gamma(h')=\gamma(x)=\gamma(h)$.*

(ii) *If $\gamma(h)>\gamma(x)$ then $\gamma(h')=\gamma(h)$.*

(iii) *If $\gamma(h) < \gamma(x)$ and $\nu(h)\neq 0$, then $\log h' \sim -\log x$ (and in particular $\gamma(h')=\gamma(x)$).*

Thus every term of $P\langle f+\eta\rangle - P\langle f\rangle$ contains a product of derivatives of η, each of which tends to zero and has comparability class at least $\gamma(\eta)$. But the coefficients of these products of derivatives belong to $\mathcal{T}_M$, and therefore have comparability class no greater than Γ by the corollary to Lemma 6. So $P\langle g\rangle = P\langle f+\eta\rangle - P\langle f\rangle$ itself must have comparability class at least $\gamma(\eta)$, which is the conclusion of Lemma 12.

For completeness we give a short proof of Lemma 13.

Proof of Lemma 13. — We use two key facts. The first is that for elements h_1, h_2 of a Hardy field, $\gamma(h_1) > \gamma(h_2)$ if and only if $\log|h_2|/\log|h_1| \to 0$; this is a direct consequence of the definitions involving γ. The other is that if $h_2/h_1 \to 0$ and $\nu(h_1)\neq 0$, then $h_2'/h_1' \to 0$; this follows from L'Hôpital's rule (see [7]).

To establish (i) we note that its hypotheses imply that $\log|h| \sim k\log x$ for some real constant $k\neq 1$. Let ε be any positive real number which is sufficiently small to ensure that $k-1-\varepsilon$ and $k-1+\varepsilon$ have the same sign. Then $x^{k-\varepsilon} < |h| < x^{k+\varepsilon}$, and hence $x^{k-1-\varepsilon} < |h'| < x^{k-1+\varepsilon}$. So $\gamma(h)=\gamma(x)$ as required.

To prove (ii) we may suppose that $|h|\to\infty$; for otherwise we may replace h by h^{-1}. Then $\log|h| > K\log x$ for every $K\in\mathbb{R}$. On differentiating, we obtain that $h'/h > K/x$, and it follows that $\gamma(h')\geq \gamma(h)$. On the other hand, $h^{-1}\to 0$ and hence $h'/h^2 \to 0$. Since the hypotheses ensure that $|h'|\to\infty$, this implies that $\gamma(h')\leq\gamma(h)$. Thus $\gamma(h')=\gamma(h)$.

As regards (iii), we have $x^{-\delta} < |h| < x^{\delta}$, for every $\delta\in\mathbb{R}^+$. Therefore $x^{-\delta-1} < |h'| < x^{\delta-1}$, and so $\log|h'| \sim -\log x$. This completes the proof of

Lemma 13.

Completion of the proof of Theorem 5. — If the conclusion is false, let J be the smallest value such that $\mathbb{R}\langle g\rangle$ contains no elements of comparability class as great as $\gamma(\mathbf{n}_{J+1})$. Then we may assume that $\gamma(\mathbf{n}_{J+1}) > \gamma(\mathbf{n}_j)$ for $j = 1, \ldots, J$. Corollary 1 to Lemma 4 then allows us to write g in the form $g = f+\eta$. Proposition 3 then shows that g is never equal to $\rho(w_f)$ with $\rho \in \mathbf{G}_M$. Now if we take $P_M\langle y\rangle$ to be the polynomial $y - w_f \in \mathcal{R}_M\langle y\rangle$, then $\tilde{\tau}_M(P_M\langle f\rangle) = 0$ but for every $\rho \in \mathbf{G}_M$, $\tilde{\rho}(P_M)\langle g\rangle \neq 0$. By Propositions 4 and 7 we can find a polynomial $P_0 \in \mathbb{R}\langle y\rangle$ with similar properties; *i.e.* such that $\tilde{\tau}_0(P_0\langle f\rangle) = 0$ but $\tilde{\tau}_0(P_0\langle g\rangle) \neq 0$. By Lemma 12, $\gamma(\tilde{\tau}_0(P_0\langle g\rangle)) \geq \gamma(\eta)$ and that suffices to establish Theorem 5 since $\tilde{\tau}_0(P_0\langle g\rangle) \in \mathbb{R}\langle g\rangle$.

If we apply Corollary 2 of Lemma 4, we obtain a slightly different version of Theorem 5, which has a somewhat simpler statement but is more difficult to use in practice.

COROLLARY (to Theorem 5). — *Let g be an element of a Hardy field with nested expansion $\{\mathbf{n}_1, \ldots, \mathbf{n}_{J+1}\}$ and suppose that $\gamma(\mathbf{n}_{J+1}) > \gamma(g)$. Then $\mathbb{R}\langle g\rangle$ contains an element of comparability class at least $\gamma(\mathbf{n}_{J+1})$.*

6. The series case.

The first thing to say is that this is not just a matter of applying Theorem 5. Certainly a series of nested forms can always be rewritten as a nested expansion, but there may be terms in the series which do not occur in the nested expansion. An example is

$$\begin{aligned} e^x + e^x x^{-1} + e^{x-x^2} &= \exp\{x + \log(1 + x^{-1} + e^{-x^2})\} \\ &= \exp\{x + x^{-1} - x^{-2}/2 + x^{-3}/3 + \cdots\}, \end{aligned}$$

and of course, one never reaches a term involving e^{-x^2} in the nested expansion. In this sort of example, the series gives a more accurate picture of the asymptotic behaviour than the nested expansion, which is one reason for using a series expansion where possible. Another is that a series is easier to understand.

We shall consider base elements $X_1, X_2, \ldots, X_k$ with $\gamma(X_1) > \gamma(X_2) > \cdots > \gamma(X_k)$. We restrict $X_2, \ldots, X_k$ to functions which have *precise nested expansions*; that is to say nested expansions $\{\mathbf{n}_1, \ldots, \mathbf{n}_\lambda\}$, $\lambda \geq 1$, with $\mathbf{n}_\lambda$ a precise nested form. We do not need to restrict X_1 to have a precise nested expansion, and this will be convenient since it will allow X_1 to act as the tail of the series.

We shall say that a function $f \in \mathcal{X}$ has *asymptotic series*

$$\sum a_n X_1^{r_{1,n}} X_2^{r_{2,n}} \cdots X_k^{r_{k,n}}$$

if for each $N \geq 0$,

$$f - \sum_{n=0}^{N} a_n X_1^{r_{1,n}} X_2^{r_{2,n}} \cdots X_k^{r_{k,n}} = o(X_1^{r_{1,N}} X_2^{r_{2,N}} \cdots X_k^{r_{k,N}}).$$

It would be nice to frame our treatment of series in terms of formal expansions only, and thus dispense with the need to restrict attention to solutions in a Hardy field. The author is of the opinion that this could be done. However there are a number of difficulties with formal series in several base elements; see [5], pages 24-27. One that is particularly relevant here concerns the meaning to be assigned to the statement *"the formal series $\sum a_n X_1^{r_{1,n}} X_2^{r_{2,n}} \cdots X_k^{r_{k,n}}$ satisfies the differential equation $P\langle y \rangle = 0$"*. If one requires merely that for every $N = 0, 1, \ldots$,

$$P\left\langle \sum_{n=0}^{N} a_n X_1^{r_{1,n}} \cdots X_k^{r_{k,n}} \right\rangle = o(X_1^{r_{1,N}} \cdots X_k^{r_{k,N}}),$$

then one has to face the fact that $\sum x(\log x)^{-n}$ satisfies $y' = 0$. Here we avoid such issues by considering only expansions of Hardy-field elements, and probably little is lost in practice by this.

In addition to our previous assumptions concerning $X_2, \ldots, X_k$, we shall also require that for each $i = 2, \ldots, k$, $\gamma(X_i)$ is strictly greater than the comparability class of any of its proper subnests. This is a reasonable restriction. For suppose that X_i is an exponential and that X_i contains a subnest $\mathbf{n}$ with $\gamma(\mathbf{n}) \geq \gamma(X_i)$. We may apply Corollary 1 of Lemma 4 to $\log X_i$, and thus rewrite X_i in the form $X_i = e^f e^\eta = e^f(1 + \operatorname{zexp} \eta)$ with $\eta \to 0$. It would then be more logical to expand in terms of e^f, η and the other X_js. Of course if $X_i = l_m^d \phi$ one would expand in terms of l_m, ϕ and the other X_js. In fact the series analogue of Theorem 5 can fail in a rather silly way without such a restriction on $X_2, \ldots, X_k$. For example, suppose that we perversely decide to seek a series-expansion solution of the equation $y' = y$ in terms of a set of base functions which includes $\exp(x + \phi)$, where ϕ tends to zero and $\gamma(\phi) > \gamma(e^x)$. We will obtain an expansion

$$y = \exp(x + \phi)\{1 - \phi + \phi^2/2 - \cdots\},$$

which contains terms of comparability class $\gamma(\phi)$. Here of course is one of the big disadvantages of series; one has to specify the right base-functions in advance!

Essentially, the techniques of Sections 3 and 5 can be applied to the series case without great difficulty. However Section 4 is another matter. The proofs there rely heavily on the ability to "unwind" the nested expansion, and it does not seem to be entirely straightforward to adapt this to series expansions. For this reason, we give our main result of this section, Theorem 6, with the analogue of Proposition 3 assumed as a hypothesis. We then look at some special cases in which it is easy to show that this is satisfied.

Firstly we note that the methods of Section 3 can be used, almost without change, to build towers of functions with the top tower containing $X_2, \ldots, X_k$. Thus, we have the following.

PROPOSITION 8. — *There exist towers $\mathcal{R}_\alpha$ and $\mathcal{S}_\alpha$, of differential algebras, and $\mathbf{G}_\alpha$, of homomorphisms from $\mathcal{R}_\alpha$ to $\mathcal{S}_\alpha$ ($1 \leq \alpha \leq M$), satisfying conditions (i), (ii), (iv), (v) and (vi) of Proposition 2, and such that for each $j = 2, \ldots, k$ and each subnest, $\mathbf{n}$, of X_j there exists a $w_\mathbf{n}$ in some $\mathcal{R}_\alpha$ with $\tau_\alpha(w_\mathbf{n}) = \mathbf{n}$.*

Henceforth we shall identify $\mathcal{R}_\alpha$ with its image under τ_α. Our main result of Section 6 is then the following.

THEOREM 6. — *Let g be an element of a Hardy field such that $g = \sum_1^{n_0} a_n X_2^{r_{2,n}} \cdots X_k^{r_{k,n}} + X_1$, where*

1. *$a_1, \ldots, a_{n_0}$ and $r_{2,1}, \ldots, r_{2,n_0}, \ldots, r_{k,1}, \ldots, r_{k,n_0}$ are real numbers.*
2. *$X_2, \ldots, X_k$ are given by precise nested expansions.*
3. *For $i = 2, \ldots, k$, $\gamma(X_i)$ is strictly greater than the comparability class of any proper subnest of X_i.*
4. *$\gamma(X_1) > \cdots > \gamma(X_k)$.*
5. *If $\mathcal{R}_\alpha$, $\mathcal{S}_\alpha$ and $\mathbf{G}_\alpha$, $\alpha = 1, \ldots, M$ are as in Proposition 8, then for every $\rho \in \mathbf{G}_M$,*

$$\rho\Big(\sum_1^{n_0} a_n X_2^{r_{2,n}} \cdots X_k^{r_{k,n}}\Big) - \sum_1^{n_0} a_n X_2^{r_{2,n}} \cdots X_k^{r_{k,n}} \neq X_1.$$

Then $\mathbb{R}\langle g\rangle$ contains an element of comparability class at least $\gamma(X_1)$.

Proof of Theorem 6. — We merely have to observe that Propositions 4 and 7 are applicable here, and so we can find a differential polynomial,

P, over $\mathbb{R}$ such that $P\langle \sum_{1}^{n_0} a_n X_2^{r_{2,n}} \cdots X_k^{r_{k,n}} \rangle = 0$ but $P\langle g \rangle \neq 0$. Lemma 12 now gives the conclusion.

We now show how Theorem 6 may be applied to yield results about the non-appearance of certain types of term in series expansions of solutions of algebraic differential equations of a given order. Our intention is to illustrate the kind of result that can be obtained rather than to seek maximum generality.

COROLLARY 1. — *Let g be an element of a Hardy field which has an asymptotic series expansion in x, e^x and λ where $\gamma(\lambda) \geq \gamma(e_2(x))$. If λ actually occurs* (2) *in the expansion, then g cannot satisfy a first-order algebraic differential equation over $\mathbb{R}(x)$.*

Proof of Corollary 1. — The main part of the proof consists of showing that $\mathbb{R}(x)\langle g \rangle$ contains an element of comparability class at least $\gamma(e_2(x))$.

If λ occurs in a product which tends to infinity, then $\gamma(g) \geq \gamma(e_2(x))$. Otherwise we may write g in the form $g = S_{n_0} + X_1$ where S_{n_0} is a real-power polynomial in x and e^x, and $X_1 \to 0$ with $\gamma(X_1) \geq \gamma(e_2(x))$. Now we take $\mathcal{R}_0 = \mathbb{R}$, $\mathcal{R}_1 = \mathbb{R}(v_1)$ where $v_1' = 1$, and $\mathcal{R}_2 = \mathbb{R}(v_1, v_2)$ where $v_2' = v_2$; thus $\mathbb{R}(v_1, v_2) \simeq \mathbb{R}(x, e^x)$. We further extend, say by $\mathcal{R}_3, \ldots, \mathcal{R}_M$ in order to incorporate any non-integral powers of x and e^x occurring in S_{n_0}. Let $\mathcal{S}_0, \ldots, \mathcal{S}_M$ and $\mathbf{G}_0, \ldots, \mathbf{G}_M$ be as in Proposition 2. It is easy to see that for any $\rho \in \mathbf{G}_M$, $\rho(S_{n_0})$ must be a complex polynomial in elements of the form $(x+K)^r$, e^{sx} with $r, s \in \mathbb{R}$. Hence $\rho(S_{n_0}) - S_{n_0}$ is asymptotic to $x^r e^{sx}$ for some $r, s \in \mathbb{R}$ and so cannot be equal to X_1. Therefore by Theorem 6, $\mathbb{R}(x)\langle g \rangle$ contains an element, h say, of comparability class at least $\gamma(e_2(x))$.

Now by Lemma 1 of [14], $\gamma(h) \geq \gamma(e_2(x))$ implies that $\nu(h'/h) \leq \nu(e_2(x)'/e_2(x)) = \nu(e^x)$, where ν is the valuation. Hence $\gamma(h'/h) \geq \gamma(e^x)$. We show that $\gamma(h'/h) \neq \gamma(h)$. We may suppose that $h \to \infty$. Then for every $r \in \mathbb{R}^+$, $h^{-r} \to 0$, and hence $h'h^{-r-1} \to 0$. Thus $h'/h < h^r$ for every $r \in \mathbb{R}^+$, and so $\gamma(h'/h) < \gamma(h)$.

Thus $\mathbb{R}(x)\langle g \rangle$ contains comparability classes $\gamma(x) < \gamma(h'/h) < \gamma(h)$ and therefore has rank at least three. In particular, it has transcendence

(2) We take this to exclude appearances such as $e_2^{-1}(x)e_2(x + \log x)$.

degree at least two over $\mathbb{R}(x)$. But this is not possible if g satisfies a first-order algebraic differential equation over $\mathbb{R}(x)$, since then g' and all higher derivatives of g would depend algebraically on x and g. This completes the proof of Corollary 1.

COROLLARY 2. — *Let g be an element of a Hardy field with $g \sim Kl_n^r(x)$, where $K, r \in \mathbb{R}\setminus\{0\}$ and $n \in \mathbb{N}$. Suppose that g has an asymptotic series expansion in $x, l_1(x), \ldots, l_n(x), \lambda$, where $\gamma(\lambda) \geq \gamma(e^x)$. Then if λ actually occurs in the expansion, g cannot satisfy an algebraic differential equation of order $n+1$ over $\mathbb{R}(x)$.*

Proof of Corollary 2. — By hypothesis, g has nested form $g = l_n^r(x)(K+\mu)$, where $\mu \to 0$. By Theorem 3, $\mathbb{R}\langle g\rangle$ contains comparability classes $\gamma(x), \gamma(l_1(x)), \ldots, \gamma(l_n(x))$. Suppose that $g = S_{n_0} + X_1$ where S_{n_0} is a real-power polynomial in $x, l_1(x), \ldots, l_n(x)$ and $\gamma(X_1) \geq \gamma(e^x)$. It is easy to see that condition 5 of Theorem 6 will be satisfied in this case, and so $\mathbb{R}\langle g\rangle$ will contain an element of comparability class at least $\gamma(e^x)$ also. But this gives $\mathbb{R}\langle g\rangle$ a rank of at least $n+2$, and g cannot then satisfy an algebraic differential equation of order $n+1$. So Corollary 2 is established.

Smaller comparability classes.

Our main theorem still allows the possibility of a combinatorial explosion in the number of cases to be considered when computing a nested expansion. But can this explosion really happen? Certainly the precise analogue of Theorem 6 for smaller comparability classes does not hold. This is shown by the following example, adapted from one in [17]. Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{(1+y)x}. \tag{12}$$

It follows from Theorem 2 that any solution of this equation belongs to a Rosenlicht field of rank two. However inspection shows that (12) possesses a solution with asymptotic expansion

$$y(x) \sim \log x - \log\log x + \ldots.$$

But a Rosenlicht field of rank two cannot contain an element of comparability class $\gamma(l_2)$.

An example of a slightly different sort is given by the equation $y' = (2x+1)y$, which has $e^{x^2}e^x$ as a solution, but $\mathbb{R}\langle x, e^{x^2}e^x\rangle$ contains no term of comparability class $\gamma(e^x)$.

As regards a positive result, we may note that Propositions 4 and 7 are still available for dealing with smaller comparability classes. However the rest of the proof of Theorem 5 is not applicable, and one might expect that it would be harder to get results for smaller classes, since the phenomenum of slow diminution is destroyed by differentiation and may also be destroyed by cancellation. Nonetheless one such result is known. The following theorem, which was based on work in [18], appears in [17].

THEOREM 7 (Strodt). — *Suppose that $y(x)$ satisfies a first-order algebraic differential equation over $\mathbb{R}(x)$ of degree d. Then the asymptotic expansion of $y(x)$ cannot contain l_q for any $q > 2d$.*

Examples would suggest that the appearance of new comparability classes in later terms of expansions is a rare phenomenum. However, this evidence is biased by the fact that research here has usually been conducted by looking for series expansions in terms of particular base elements, and almost invariably those for which the associated Hardy field is of small rank. It seems somewhat rash to make a conjecture on the basis of such knowledge, but the following might at least serve as a focus for attention.

CONJECTURE. — *Let g be a Hardy-field solution of an algebraic differential equation of order $r-p$ over a Rosenlicht field, $\mathcal{H}$, of rank p. Suppose that g has a series expansion,*

$$S = \sum a_n X_1^{r_{1,n}} \cdots X_k^{r_{k,n}}.$$

as above. Then the rank of the Hardy field $\mathcal{F}\langle X_1, \ldots, X_k\rangle$ is at most $2r$.

Less ambitiously, one might replace two by a larger constant.

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