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RATIONAL EQUIVALENCE ON SOME FAMILIES OF PLANE CURVES (*)

by J.M. MIRET and S. XAMBÓ DESCAMPS

Introduction.

For a given irreducible n-dimensional family $\mathcal{C}$ of plane curves of degree d, there is interest in enumerative geometry to know the number of curves in $\mathcal{C}$ that satisfy n simple conditions. Geometrically this often can be reduced to finding the number of intersection points of n hypersurfaces of $\mathcal{C}$, and the solution of problems of this type usually involves a complete knowledge of $\mathrm{Pic}(\mathcal{C})$.

For the family $\mathcal{C}_d$ of all *smooth* plane curves of degree d, $\mathrm{Pic}(\mathcal{C}_d)$ is a cyclic group of order $3(d-1)^2$, for plane curves of degree d are parameterized by a projective space of dimension $d(d+3)/2$ and the singular curves fill a hypersurface of degree $3(d-1)^2$ in that space (the discriminant hypersurface).

Another interesting example is the family $V_{d,\delta}$ of irreducible plane curves of degree d with exactly δ nodes as singularities. This variety is irreducible (Severi [8], Harris [4], Ran [7]). Unfortunately, the group $\mathrm{Pic}(V_{d,\delta})$ is not known. However, it has been conjectured by Diaz and Harris [1] (see also Diaz and Harris [2]), that $\mathrm{Pic}(V_{d,\delta})$ is a torsion group.

The main goal of this note is to study rational equivalence on some families of plane curves, or closely related families, and to determine, as an application (see Theorem (22)) the group $\mathrm{Pic}(V_{d,1})$, obtaining, in particular, that it is a finite group of order $6(d-2)(d^2-3d+1)$ (cf. Miret–Xambó [6], where it is proved that $\mathrm{Pic}(V_{3,1})$ is a cyclic group of order 6).

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To be more specific, we determine, among others, the intersection ring of natural compactifications of the following families of plane curves of degree d : curves with an ordinary multiple point of multiplicity $m > 1$ and no other singularity (section 1; the compactification is denoted $X_{d,m}$); curves with an ordinary cusp and no other singularity (section 2; the compatification is X_d^{cusp}); and curves with two ordinary nodes and no other singularity (section 3; the compactification is X_d^{binod}). The compactifications we consider are projective bundles over varieties naturally related to the corresponding problem (for example, the point-line incidence correspondence for the cuspidal curves) and the determination of the intersection ring relies on the construction and geometric interpretation of explicit resolutions of the vector bundles involved (incidentally, in a remark at the end of section 2 we point out a bug concerning $\text{Pic}(V_3^{\text{cusp}})$ that slipped into Miret–Xambó [6]). The baring of these varieties on the Severi variety $V_{d,1}$ is, roughly speaking, that $X_{d,2}$ is a compactification of $V_{d,1}$ with boundary divisors X_d^{cusp} and X_d^{binod}. This fact, and the knowledge of the rational equivalence of all the objects involved, allows us to calculate $\text{Pic}(V_{d,1})$.

Notations. — The term variety will be used to mean an algebraic variety defined over a fixed algebraically closed ground field $\boldsymbol{k}$ and a morphism of varieties is a $\boldsymbol{k}$-morphism.

If $f: X \to X'$ is a morphism of varieties and E' is a vector bundle on X', we shall write f^*E' to denote the pull-back of E' with respect to the map f. If f is clear from the context, we shall also write $E'|X$. In particular, if X is a variety and W is a vector space (which we will regard as a vector bundle on $\text{Spec}\,\boldsymbol{k}$), the trivial vector bundle on X whose fiber is W is the pull-back $W|X$ to X of W with respect to the constant map $X \to \text{Spec}(\boldsymbol{k})$.

In the sequel we take V to denote a vector space of dimension 3 over an algebraically closed field $\boldsymbol{k}$, $\mathbb{P}^2 = \mathbb{P}(V)$ the projective plane associated to V and $\mathbb{P}^{2*} = \mathbb{P}(V^*)$ the dual projective plane. If $v \in V - \{0\}$, we shall write $[v]$ to denote the point in $\mathbb{P}^2$ defined by v. As usual, we shall identify the points of $\mathbb{P}^{2*}$ with the lines of $\mathbb{P}^2$: if $w \in V^*$, $w \neq 0$, $[w]$ is identified with the line whose points $[v]$ satisfy $w(v) = 0$. Dually, given a point $[v]$, the lines $[w]$ such that $w(v) = 0$ are just the pencil of lines through $[v]$.

The *tautological line subbundle* of the trivial rank three bundle $V|\mathbb{P}^2$ over $\mathbb{P}^2$ will be denoted by L. Thus the fiber $L_{[v]}$ of L over the point $[v] \in \mathbb{P}^2$ defined by the non-zero vector $v \in V$ is the vector line $\langle v \rangle \subseteq V$ spanned by v. The quotient of $V|\mathbb{P}^2$ by L is, by definition, the *tautological quotient* *bundle* Q. Thus Q is a rank 2 bundle and we have an exact sequence

$$(1) \qquad 0 \to L \to V|\mathbb{P}^2 \to Q \to 0.$$

Dualizing we get an exact sequence

$$(2) \qquad 0 \to Q^* \to V^*|\mathbb{P}^2 \to L^* \to 0.$$

$Q^*_{[v]}$ is the space of linear forms $w \in V^*$ that vanish on $\langle v \rangle$, that is, such that $w(v) = 0$. Thus we see that $\mathbb{P}(Q^*)$ is embedded in $\mathbb{P}(V^*|\mathbb{P}^2) = \mathbb{P}^2 \times \mathbb{P}^{2*}$ in such a way that the fiber of $\mathbb{P}(Q^*)$ over $[v] \in \mathbb{P}^2$ is identified with the pencil of lines through $[v]$.

In what follows we shall write $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1)) = -c_1(L)$. The exact sequences above give us that $c(Q) = (1-h)^{-1} = 1 + h + h^2$ and $c(Q^*) = 1 - h + h^2$.

In order to simplify the appearance of some expressions that will be needed often, for any positive integer m we set

$$c_1(m) = \binom{m+1}{2} \quad \text{and} \quad c_2(m) = \binom{c_1(m)+1}{2} = c_1(c_1(m)).$$

1. Plane curves with a distinguished multiple point.

We study the variety whose closed points parameterize (point) plane curves of degree $d \geq 3$ with a distinguished multiple point of multiplicity $m > 0$. Here we consider any m, even though in later Sections we will be concerned only with the case $m = 2$, both because the case $m = 3$ is needed, in part, in the proof of 9, and also for future reference.

Let S^dV^* denote the d-th symmetric power of V^*. Then S^dV^* has dimension $N+1$ over $\boldsymbol{k}$, where $N = d(d+3)/2$, and the closed points of the projective space $\mathbb{P}^N = \mathbb{P}(S^dV^*)$ parameterize (point) plane curves of degree d.

We shall write $E_{d,m}$ to denote the subbundle of $S^dV^*|\mathbb{P}^2$ whose fiber over $[v] \in \mathbb{P}^2$ is the linear subspace of S^dV^* the elements of which are the forms φ that have multiplicity at least m at $[v]$. It is clear that $E_{d,m}$ has rank $N + 1 - c_1(m)$.

Define $X_{d,m} = \mathbb{P}(E_{d,m})$. It is clear that $X_{d,m}$ has dimension $N - c_1(m) + 2$. Since $\mathbb{P}(S^dV^*|\mathbb{P}^2) = \mathbb{P}^N \times \mathbb{P}^2$, $X_{d,m}$ is the *incidence subvariety* of $\mathbb{P}^N \times \mathbb{P}^2$ whose closed points are pairs (f, a) such that $a \in \mathbb{P}^2$ is a point of multiplicity at least m for the plane point curve $f \in \mathbb{P}^N$. Thus the fiber over a of the projection $\pi : X_{d,m} \to \mathbb{P}^2$ (that is, the restriction to $X_{d,m}$ of the projection $\mathbb{P}^N \times \mathbb{P}^2 \to \mathbb{P}^2$) is naturally isomorphic to the $(N - c_1(m))$-dimensional linear system of plane curves of degree d for which a is a multiple point of multiplicity at least m.

Let

$$S^{d-m-1}V^* \otimes \Lambda^2 Q^* \otimes S^{m-1}Q^* \xrightarrow{\alpha_{d,m}} S^{d-m}V^* \otimes S^m Q^*$$

be the unique map such that

$$\varphi \otimes (\xi_1 \wedge \xi_2) \otimes \psi \longmapsto (\varphi\xi_1) \otimes (\xi_2\psi) - (\varphi\xi_2) \otimes (\xi_1\psi)$$

and

$$S^{d-m}V^* \otimes S^m Q^* \xrightarrow{\beta_{d,m}} S^d V^* | \mathbb{P}^2$$

be the unique map such that $\varphi \otimes \xi \mapsto \varphi\xi$. If there is no danger of confusion we shall write simply α to denote any of the "alternating maps" $\alpha_{d,m}$ and β to denote any of the "product maps" $\beta_{d,m}$. Notice that from the definitions it follows that α maps $S^{d-m-1}Q^* \otimes \Lambda^2 Q^* \otimes S^{m-1}Q^*$ into $S^{d-m}Q^* \otimes S^m Q^*$ and that β maps $S^{d-m}V^* \otimes S^m Q^*$ onto $E_{d,m}$ and $S^{d-m}Q^* \otimes S^m Q^*$ onto $S^d Q^*$.

Next proposition gives a resolution of $E_{d,m}$ in terms of the maps α and β :

(3) PROPOSITION. — *The sequence*

$$0 \to S^{d-m-1}V^* \otimes \Lambda^2 Q^* \otimes S^{m-1}Q^* \xrightarrow{\alpha} S^{d-m}V^* \otimes S^m Q^* \xrightarrow{\beta} E_{d,m} \to 0$$

is exact.

Proof. — From the definitions of α and β it is clear that $\beta\alpha = 0$. To see that α is injective, it is enough to establish this property for the fiber map over an arbitrary point a. Given the point a, choose projective coordinates $[x_0, x_1, x_2]$ so that $a = [1, 0, 0]$. Thus x_0, x_1, x_2 is a basis of V^* and x_1, x_2 a basis of Q_a^*. Take an element

$$\xi = \sum_{i=0}^{m-1} \xi_i \otimes (x_1 \wedge x_2) \otimes x_1^i x_2^{m-1-i}$$

in $S^{d-m-1}V^* \otimes \Lambda^2 Q^* \otimes S^{m-1}Q^*$. Then

$$\alpha(\xi) = (\xi_0 x_1) \otimes x_2^m + \sum_{i=1}^{m-1} (\xi_i x_1 - \xi_{i-1} x_2) \otimes x_1^i x_2^{m-i} - (\xi_{m-1} x_2) \otimes x_1^m.$$

If $\alpha(\xi) = 0$, then we must have $\xi_0 x_1 = 0$ (hence $\xi_0 = 0$), $\xi_1 x_1 - \xi_0 x_2 = 0$ (hence $\xi_1 = 0$), and so on. This shows that $\ker(\alpha) = 0$.

To end the proof it is now enough to show that the difference of the ranks of the middle and left bundles is equal to $N + 1 - c_1(m)$. But this can be checked with a straightforward computation.

The two lemmas that come next about Chern classes will be used below to determine the total Chern class of $E_{d,m}$.

(4) LEMMA.

$$c(S^m Q^*) = 1 - c_1(m)h + c_2(m)h^2.$$

Proof. — Let s and t be the roots of Q^*. From the exact sequence (2) it follows that $c(Q^*) = (1 + c_1 L^*)^{-1} = 1 - h + h^2$ and therefore $s + t = -h$, $st = h^2$. The roots of $S^m Q^*$ are $is + (m-i)t$, $0 \le i \le m$ (Fulton [3], Ex. 3.2.6), and so

$$c_1 S^m Q^* = \sum_{i=0}^{m} is + (m-i)t = \binom{m+1}{2}(s+t) = -\binom{m+1}{2}h.$$

Similarly we have that

$$\begin{aligned} c_2 S^m Q^* &= \sum_{0 \le i < j \le m} \big(is + (m-i)t\big)\big(js + (m-j)t\big) \\ &= \sum_{0 \le i < j \le m} \big(ijs^2 + (m-i)(m-j)t^2 + i(m-j)st + j(m-i)st\big) \\ &= \sum_{0 \le i < j \le m} ij(s^2 + t^2 - 2st) + m \sum_{0 \le i < j \le m} (i+j)st \\ &= p_m\big((s+t)^2 - 4st\big) + m q_m st = (-3p_m + m q_m)h^2, \end{aligned}$$

where we set $p_m = \sum_{0 \le i < j \le m} ij$ and $q_m = \sum_{0 \le i < j \le m} (i+j)$. Now it is easy to see that

$$p_m = p_{m-1} + m\binom{m}{2} \quad \text{and} \quad q_m = q_{m-1} + m^2 + \binom{m}{2}$$

and hence we see that p_m and q_m are polynomials of degree 4 in m (for non-negative integers m). Since the proposed coefficient of h^2 (let it be k_m) is also a polynomial of degree 4 in m, to end the proof it is enough to see that k_m and $-3p_m + mq_m$ have the same value for $0 \leq m \leq 4$, which is indeed the case and easy to check. Alternatively, it is straightforward to obtain the expression $mq_m - 3p_m = m + 4\binom{m}{2} + 6\binom{m}{3} + 3\binom{m}{4}$ and to check that this coincides with k_m.

(5) Lemma.

$$c(\Lambda^2 Q^* \otimes S^{m-1}Q^*)^{-1} = 1 + c_1(m)h + c_2(m)h^2.$$

Proof. — We first calculate the Chern classes of $\Lambda^2 Q^* \otimes S^{m-1}Q^*$:

$$\begin{aligned} c_1(\Lambda^2 Q^* \otimes S^{m-1}Q^*) &= \operatorname{rank} S^{m-1}Q^* c_1(\Lambda^2 Q^*) + c_1(S^{m-1}Q^*) \\ &= m(-h) - c_1(m-1)h \\ &= -c_1(m)h, \end{aligned}$$

and

$$\begin{aligned} c_2(\Lambda^2 Q^* \otimes S^{m-1}Q^*) &= c_2(S^{m-1}Q^*) + (m-1)c_1(S^{m-1}Q^*)c_1(\Lambda^2 Q^*) \\ &\quad + \binom{m}{2} c_1(\Lambda^2 Q^*)^2 \\ &= \big(c_2(m-1) + mc_1(m-1)\big)h^2. \end{aligned}$$

From this calculation it follows that

$$\begin{aligned} c(\Lambda^2 Q^* \otimes S^{m-1}Q^*)^{-1} &= 1 + c_1(m)h + \big(c_1(m)^2 - c_2(m-1) - mc_1(m-1)\big)h^2 \\ &= 1 + c_1(m)h + c_2(m)h^2. \end{aligned}$$

(6) Proposition. — *The total Chern class of $E_{d,m}$ is given by*

$$c(E_{d,m}) = 1 - (d-m+1)c_1(m)h + (d-m+1)^2 c_2(m)h^2 .$$

Proof. — According to the exact sequence (3),

$$c(E_{d,m}) = (1 - c_1h + c_2h^2)^{\binom{d-m+2}{2}} (1 + c_1h + c_2h^2)^{\binom{d-m+1}{2}},$$

where for simplicity we have written c_1 and c_2 instead of $c_1(m)$ and $c_2(m)$. Since $(1 - c_1h + c_2h^2)(1 + c_1h + c_2h^2) = \big(1 + (2c_2 - c_1^2)h^2\big)$,

$$\begin{aligned} c(E_{d,m}) &= (1 + (2c_2 - c_1^2)h^2)^{\binom{d-m+1}{2}} (1 - c_1h + c_2h^2)^{d-m+1} \\ &= 1 - (d-m+1)c_1h + (d-m+1)^2 c_2h^2. \end{aligned}$$

The intersection ring of $X_{d,m}$. — The previous corollary allows us to describe explicitly the Chow ring of $X_{d,m}$.

(7) THEOREM. — *The ring $A^*(X_{d,m})$ is isomorphic to the quotient of the polynomial ring $Z[\mu, h]$ by the ideal*

$$(h^3, \mu^{N+1-c_1(m)} - (d-m+1)c_1(m)h\mu^{N-c_1(m)} \\ +(d-m+1)^2c_2(m)h^2\mu^{N-c_1(m)-1}).$$

In particular $\operatorname{Pic}(X_{d,m})$ *is a rank 2 free group generated by μ, h. Moreover, the following relations hold :*

$$\begin{aligned} h^2\mu^{N-c_1(m)} &= 1, \\ h\mu^{N+1-c_1(m)} &= (d-m+1)c_1(m), \\ \mu^{N+2-c_1(m)} &= (d-m+1)^2\big(c_1(m)^2 - c_2(m)\big). \end{aligned}$$

Proof. — Given a line u in $\mathbb{P}^2$, the pullback of $h = [u] \in \operatorname{Pic}(\mathbb{P}^2)$ to $X_{d,m}$ by the projection $X_{d,m} \to \mathbb{P}^2$ will be still denoted h. Since this projection is flat, h is the class of the pullback of u, that is, the class of the hypersurface whose points (f, a) satisfy $a \in u$. Similarly, the pullback to $X_{d,m}$ of the hyperplane class $\mu = c_1(\mathcal{O}_{\mathbb{P}^N}(1))$ of $\mathbb{P}^N$ under the projection $X_{d,m} \to \mathbb{P}^N$ will still be denoted μ. Since the condition of going through a point is linear, μ is the class of the hypersurface whose points (f, a) satisfy that f goes through a given point. Moreover, μ coincides with the class $c_1(\mathcal{O}_{X_{d,m}}(1))$, by the functoriality of the hyperplane class.

Now we know (see, for example, Fulton [3], Ex. 8.3.4) that $A^*(X_{d,m})$ is isomorphic to

$$A^*(\mathbb{P}^2)[\mu]/\mu^{N+1-c_1(m)} - (d-m+1)c_1(m)h\mu^{N-c_1(m)} \\ +(d-m+1)^2c_2(m)h^2\mu^{N-c_1(m)-1}.$$

Since $A^*(\mathbb{P}^2) = \mathbb{Z}/(h^3)$, the result follows.

We define $X_d^{\text{nod}} = X_{d,2}$. This variety contains the open set $V_{d,1}$ whose points parameterize plane curves of degree d with one node and no other singularities. In the next two sections we introduce and study varieties X_d^{cusp} and X_d^{binod}. These will be used in section 4 to show that $X_d^{\text{nod}} - V_{d,1}$ is the union of two codimension one irreducible subvarieties X'_d and X''_d whose generic points are plane curves of degree d that have, respectively, a cusp and two nodes as only singularities. Furthermore, X'_d and X''_d will in turn play a key role in the determination of $\operatorname{Pic}(V_{d,1})$.

2. Cuspidal curves of degree d.

Let $F = \mathbb{P}(Q^*) \subset \mathbb{P}^2 \times \mathbb{P}^{2^*}$ be the flag variety, that is, the variety whose closed points are pairs $(a, \ell) \in \mathbb{P}^2 \times \mathbb{P}^{2^*}$ with $a \in \ell$. Define E_d^{cusp} to be the subvector bundle of $E_{d,2}|F \subset S^dV^*|F$ whose fiber over $(a,\ell) \in F$ is the linear subspace of S^dV^* consisting of those forms φ for which a is a cusp and ℓ is the cuspidal tangent. The rank of E_d^{cusp} is $N-4$. Then the fiber of $X_d^{\text{cusp}} := \mathbb{P}(E_d^{\text{cusp}})$ over (a,ℓ) is canonically isomorphic to the linear system of curves of degree d that have a cusp at a whose cuspidal tangent is ℓ (that is, a is a double point for f and ℓ is a double tangent to f at a).

By the definition of E_d^{cusp}, there exists a map of vector bundles

$$S^{d-2}V^* \otimes \mathcal{O}_F(-2) \to E_d^{\text{cusp}}$$

given by composing the map $1 \otimes i$ with the product map $S^{d-2}V^* \otimes S^2Q^*|F \to S^dV^*|F$. Adding this map and the product map

$$S^{d-3}V^* \otimes S^3Q^*|F \to S^dV^*|F$$

yields a surjective map

$$\left(S^{d-3}V^* \otimes S^3Q^*|F\right) \oplus \left(S^{d-2}V^* \otimes \mathcal{O}_F(-2)\right) \xrightarrow{\pi} E_d^{\text{cusp}}.$$

(8) Proposition. — *There exists a resolution of E_d^{cusp} of the following form :*

$$\begin{aligned}
0 \to\, & S^{d-4}V^* \otimes \Lambda^2Q^* \otimes \mathcal{O}_F(-2) \\
& \xrightarrow{\delta} \left(S^{d-4}V^* \otimes \Lambda^2Q^* \otimes S^2Q^*|F\right) \oplus \left(S^{d-3}V^* \otimes Q^* \otimes \mathcal{O}_F(-2)\right) \\
& \quad \xrightarrow{\rho} \left(S^{d-3}V^* \otimes S^3Q^*|F\right) \oplus \left(S^{d-2}V^* \otimes \mathcal{O}_F(-2)\right) \xrightarrow{\pi} E_d^{\text{cusp}} \to 0,
\end{aligned}$$

where $\mathcal{O}_F(-2) \stackrel{i}{\hookrightarrow} S^2Q^*|F$ is the natural inclusion map,

$$\rho = \begin{pmatrix} \alpha & 0 \\ 1 \otimes \bar{\beta} & -\beta \otimes 1 \end{pmatrix}$$

with $Q^* \otimes \mathcal{O}_F(-2) \xrightarrow{\bar{\beta}} S^3Q^*$ the map obtained by composing $1 \otimes i$ with the product map β, and $\delta = (-1 \otimes 1 \otimes i, \alpha \otimes 1)$.

Proof. — As explained before, π is surjective, and from the definition it is clear that δ is injective. Furthermore, it is straightforward to check that the sequence is a complex.

Let us now check that $\ker(\rho) \subseteq \mathrm{Im}(\delta)$. Given $(a, \ell) \in F$, take projective coordinates so that $a = [1, 0, 0]$ and $\ell = \{x_2 = 0\}$. Assume that $\xi \in \ker(\rho)$. Then we can write

$$\xi = \big(\xi_1 \otimes x_1 \wedge x_2 \otimes x_1^2 + \xi_{12} \otimes x_1 \wedge x_2 \otimes x_1 x_2 \\ + \xi_2 \otimes x_1 \wedge x_2 \otimes x_2^2, \xi_1' \otimes x_1 \otimes x_2^2 + \xi_2' \otimes x_2 \otimes x_2^2\big)$$

and by definition of ρ we must have

$$\xi_1 x_1 \otimes x_2 x_1^2 - \xi_1 x_2 \otimes x_1^3 + \xi_{12} x_1 \otimes x_1 x_2^2 - \xi_{12} x_2 \otimes x_1^2 x_2 \\ + \xi_2 x_1 \otimes x_2^3 - \xi_2 x_2 \otimes x_1 x_2^2 + \xi_1' \otimes x_1 x_2^2 + \xi_2' \otimes x_2^3 = 0$$

and

$$(\xi_1' x_1 + \xi_2' x_2) \otimes x_2^2 = 0.$$

The second relation yields that there exists $\eta \in S^{d-4}V^*$ such that

$$\xi_2' = \eta x_1, \quad \xi_1' = -\eta x_2.$$

If we now perform these substitutions in the first relation we get that

$$(\xi_1 x_1 - \xi_{12} x_2) \otimes x_1^2 x_2 - \xi_1 x_2 \otimes x_1^3 \\ + (\xi_{12} x_1 - \eta x_2 - \xi_2 x_2) \otimes x_1 x_2^2 + (\xi_2 + \eta) x_1 \otimes x_2^3 = 0.$$

From these we get immediately that $\xi_1 = 0$, $\xi_{12} = 0$ and $\xi_2 = -\eta$ and so

$$\xi = \delta(\eta \otimes x_1 \wedge x_2 \otimes x_2^2).$$

Notice that the argument actually shows that $\ker(\rho) = \mathrm{Im}(\delta)$. Finally it is easy to check that the alternating sum of the ranks of the bundles of the sequence in question is zero, which is enough to complete the proof.

In next proposition we determine the total Chern class of E_d^{cusp}. Notice that (7) yields, since $Q^* = E_{1,1}$ and so $F = X_{1,1}$, that $A^1F = \mathrm{Pic}(F)$ is rank 2 free group generated by c and q, where now c and q are the pullbacks to F of the hyperplane classes of $\mathbb{P}^2$ and $\mathbb{P}^{2^*}$, respectively.

(9) Proposition. — *The total Chern class of E_d^{cusp} is given by the following formula :*

$$c(E_d^{\mathrm{cusp}}) = 1 - 5(d-2)c - 2q + 15(d-2)^2c^2 + 12(d-2)cq - 42(d-2)^2c^2q.$$

From the preceding proposition we get that $c(E_d^{\text{cusp}})$ can be written as follows :

$$\frac{c(S^{d-3}V^* \otimes S^3Q^* | F)}{c(S^{d-4}V^* \otimes \Lambda^2 Q^* \otimes S^2Q^* | F)} \cdot \frac{c\big(S^{d-2}V^* \otimes \mathcal{O}_F(-2)\big)\, c\big(S^{d-4}V^* \otimes \Lambda^2Q^* \otimes \mathcal{O}_F(-2)\big)}{c\big(S^{d-3}V^* \otimes Q^* \otimes \mathcal{O}_F(-2)\big)}.$$

Now the first fraction in this product is equal to

$$c(E_{d,3}|F) = 1 - 6(d-2)c + 21(d-2)^2c^2.$$

To find the value of the second fraction, define $G = S^{d-2}V^*/E_{d-2,1}$, so that we have the exact sequence

$$0 \to S^{d-4}V^* \otimes \Lambda^2Q^* \to S^{d-3}V^* \otimes Q^* \to S^{d-2}V^* \to G \to 0.$$

Then it is clear that the second fraction is equal to

$$c(G \otimes \mathcal{O}_F(-2)) = c(G) - 2q = c(E_{d-2,1})^{-1} - 2q = 1 + (d-2)c - 2q.$$

From this our formula follows readily.

In the following we shall write $X_d^{\text{cusp}} = \mathbb{P}(E_d^{\text{cusp}})$. Thus the points of X_d^{cusp} may be thought as triples (f, a, ℓ) such that f is a plane curve of degree d having a cusp at the point a with cuspidal tangent ℓ.

The intersection ring of X_d^{cusp}. — The pullbacks of the classes c and q on F to X_d^{cusp} will be denoted c and q, respectively. Thus c and q are the classes of the hypersurfaces of triples (f, a, ℓ) such that, respectively, a lies on a line and q goes through a point. On the other hand, the hyperplane class on X_d^{cusp} will be denoted μ. It is easy to see that μ is the class of the hypersurface of triples (f, a, ℓ) such that the curve f goes through a given point.

(10) THEOREM. — *The ring $A^*(X_d^{\text{cusp}})$ is isomorphic to the quotient of the polynomial ring $Z[\mu, c, q]$ by the ideal*

$$\big(c^3, c^2 - cq + q^2, \mu^{N-4} - (5(d-2)c + 2q)\mu^{N-5} \\ +3(d-2)\big(5(d-2)c^2 + 4cq)\mu^{N-6}\big) - 42(d-2)^2c^2q\mu^{N-7}\big).$$

In particular, $\mathrm{Pic}(X_d^{\text{cusp}})$ *is a rank 3 free group generated by* μ, c, q.

Proof. — Let μ be the hyperplane class on X_d^{cusp}. Then

$$A^*(X_d^{\text{cusp}}) = A^*(F)[\mu]/\sum \pi^* c_i(E_d^{\text{cusp}})\mu^{N-i-4}$$

where $\pi : X_d^{\text{cusp}} \to F$ is the natural projection and the statement follows from the preceding proposition in a straighforward manner.

(11) COROLLARY.

$$\begin{aligned} c^2q\mu^{N-5} &= 1 & c^2\mu^{N-4} &= 2 \\ cq\mu^{N-4} &= 2 + 5(d-2) & q^2\mu^{N-4} &= 5(d-2) \\ c\mu^{N-3} &= 4 + 8(d-2) & q\mu^{N-3} &= 8(d-2) + 10(d-2)^2. \\ \mu^{N-2} &= 12(d-2) + 12(d-2)^2 \end{aligned}$$

Proof. — The first relation is obtained from the fact that the curves of degree d that satisfy the condition c^2q form a linear system. The remaining follow from the basic relation

$$\mu^{N-4} - \big(5(d-2)c + 2q\big)\mu^{N-5} \\ +3(d-2)\big(5(d-2)c^2 + 4cq\big)\mu^{N-6} - 42(d-2)^2c^2q\mu^{N-7} = 0$$

and straighforward calculations.

Remark. — Let V_3^{cusp} be the variety of non-degenerate plane cuspidal cubics. Then we claim that $\operatorname{Pic}(V_3^{\text{cusp}})$ *is an infinite cyclic group generated by* $c-q$. This statement corrects the assertion in Miret–Xambó [5], Theorem 1.3 : the last sentence "and from this the claim follows" of the proof of that theorem is incorrect, for the argument up to that point only allows to conclude that there exists an exact sequence $\mathbb{Z} \to \operatorname{Pic}(V_3^{\text{cusp}}) \to \mathbb{Z}/(5) \to 0$, but not that this sequence is split. It is also to be remarked that this error affects no other statement in the quoted paper, for there Theorem 1.3 is used only in the last sentence of Remark 10.1.1, as a cross-checking, and this sentence is in fact correct, for it relies only on the correct part of the proof of Theorem 1.3.

Now let us prove our claim. Let E' be the subbundle of $S^2V^*|F$ whose fiber over a point (a,ℓ) is the vector subspace of S^2V^* consisting of forms that are tangent to ℓ at a. Let $E'' = S^3Q^*|F \subseteq S^3V^*|F$. Then $\sigma = \mathbb{P}(E' \otimes \mathcal{O}_F(-1)) \subset X_3^{\text{cusp}}$ is the hypersurface whose generic point consists of a non-degenerate conic with a distinguished tangent, the contact point being the cusp and the tangent the cuspidal tangent, and $\tau = \mathbb{P}(E'')$

is the hypersurface of X_3^{cusp} whose generic point is a triple of lines through a. With these notations it is easy to see that

$$X_3^{\mathrm{cusp}} - V_3^{\mathrm{cusp}} = \sigma \cup \tau.$$

Now by (10) we can express σ and τ as an integral linear combination of μ, c and q. In fact it is not hard to see that

$$\begin{aligned} \sigma &= \mu - 3c + 3q \\ \tau &= \mu + c - 2q \end{aligned}$$

(independently, these relations are a consequence of the first and fifth relations of Theorem 10.1 in Miret–Xambó [5]). These relations imply the relations

$$\begin{aligned} \mu &= 3(c-q) + \sigma \\ q &= 4(c-q) + \sigma - \tau \\ c &= 5(c-q) + \sigma - \tau \end{aligned}$$

and since there exists an exact sequence

$$A^0(\sigma \cup \tau) \to \mathrm{Pic}\,(X_3^{\mathrm{cusp}}) \to \mathrm{Pic}\,(V_3^{\mathrm{cusp}}) \to 0$$

we see that $\mathrm{Pic}\,(V_3^{\mathrm{cusp}})$ is generated by $c-q$. To end the proof of the claim we only have to see that $\mathrm{Pic}\,(V_3^{\mathrm{cusp}})$ is not a torsion group. Let us argue by contradiction. Assume that n is a non-zero integer such that $n(c-q)=0$. By the above exact sequence, this implies that there are integers r, s such that

$$n(c-q) = r\sigma + s\tau.$$

Substituting σ and τ in this relation by their expressions above in terms of μ, c and q, we would get a relation

$$(r+s)\mu - (3r - s + n)c + (3r - 2s + n)q = 0,$$

which is impossible if $n \neq 0$.

3. Binodal curves of degree d.

In this section we introduce a variety X_d^{binod} that parameterizes plane curves of degree d with an ordered pair of double points. In fact, X_d^{binod} will be defined as a projective bundle over the variety G whose closed points are triples $(a, b, m) \in (\mathbb{P}^2)^2 \times \mathbb{P}^{2^*}$ such that $a, b \in m$. We write M to denote the line bundle on G that is the pullback of $\mathcal{O}_{\mathbb{P}^{2*}}(-1)$ under the projection map $G \to \mathbb{P}^{2^*}$.

The bundle E_d^{bisec}. — In order to study the vector bundle E_d^{binod} such that $\mathbb{P}(E_d^{\text{binod}}) = X_d^{\text{binod}}$ we introduce first an auxiliary vector bundle E_d^{bisec} over G : it is the subbundle of $S^d V^* | G$ whose fiber over $(a, b, m) \in G$ is the linear subspace of $S^d V^*$ whose elements are forms φ of degree d whose restriction to m vanishes on $a + b$ (as a divisor on the line m). It is clear that E_d^{bisec} has rank $N + 1 - 2 = N - 1$.

Next statement provides a resolution of E_d^{bisec}. We use the following notations :

- Q_a^* (resp. Q_b^*) for the pullback of Q^* to G under the projection map $G \to \mathbb{P}^2$, $(a, b, m) \mapsto a$ (resp. $(a, b, m) \mapsto b$);
- $S_{ab} = Q_a^* \oplus Q_b^*$ and $P_{ab} = Q_a^* \otimes Q_b^*$;
- $p : M \otimes S^k V^* \to S^{k+1} V^*$ (product map) for the inclusion map given by $m \otimes \omega \mapsto m\omega$;
- $i_a : M \to Q_a^*$ and $i_b : M \to Q_b^*$ for the natural inclusions and $\Delta : M \to S_{ab}$ for the map (i_a, i_b);
- $\tau : M \otimes S_{ab} \to P_{ab}$ for the map such that $m \otimes (\alpha, \beta) \mapsto \alpha \otimes i_b(m) - i_a(m) \otimes \beta$;
- $\delta : S_{ab} \otimes S^k V^* \to S^{k+1} V^*$ for the map such that $(\alpha, \beta) \otimes \omega \mapsto \alpha\omega - \beta\omega$; and
- $q : P_{ab} \otimes S^k V^* \to S^{k+2} V^*$ the map such that $\alpha \otimes \beta \otimes \omega \mapsto \alpha\beta\omega$.

To simplify notations we also make the convention to denote with the same symbol than any of the above maps the maps that it induces after tensoring by any bundle on the left and any other bundle on the right. For example, the map

$$p : M^2 \otimes S_{ab} \otimes S^{d-3} V^* \to M \otimes S_{ab} \otimes S^{d-2} V^*$$

is to be interpreted as the composition of the transposition isomorphism

$$M^2 \otimes S_{ab} \otimes S^{d-3} V^* \simeq M \otimes S_{ab} \otimes M \otimes S^{d-3} V^*$$

with the map $1_M \otimes 1_{S_{ab}} \otimes p$.

(12) Proposition. — *The following sequence is exact :*

$$0 \to M^3 \otimes S^{d-3}V^* \xrightarrow{\gamma_1} \left(M^2 \otimes S^{d-2}V^*\right) \oplus \left(S_{ab} \otimes M^2 \otimes S^{d-3}V^*\right)$$
$$\xrightarrow{\gamma_2} \left(S_{ab} \otimes M \otimes S^{d-2}V^*\right) \oplus \left(P_{ab} \otimes M \otimes S^{d-3}V^*\right)$$
$$\xrightarrow{\gamma_3} \left(M \otimes S^{d-1}V^*\right) \oplus \left(P_{ab} \otimes S^{d-2}V^*\right) \xrightarrow{\gamma_4} E_d^{\mathrm{bisec}} \to 0,$$

where

$$\gamma_1 = \begin{pmatrix} p \\ \Delta \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} -\Delta & p \\ 0 & \tau \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} -\delta & q \\ \tau & -p \end{pmatrix}, \quad \gamma_4 = (p, q).$$

Proof. — Since p is injective, γ_1 is injective. Now

$$\gamma_2\gamma_1 = (-\Delta p + p\Delta, \tau\Delta) = (0,0)$$

as follows directly from the definitions. Similarly,

$$\gamma_3\gamma_2 = \begin{pmatrix} -\delta\Delta & -\delta p + q\tau \\ -\tau\Delta & \tau p - p\tau \end{pmatrix} \quad \text{and} \quad \gamma_4\gamma_3 = (p\delta + q\tau, pq - qp),$$

which are both zero, as it is easy to check. So the stated sequence is a complex. On the other hand a straighforward calculation shows that the alternate sum of the ranks of the bundles is 0, so that to prove exactness it is enough to see that γ_4 is surjective, that $\ker(\gamma_2) \subseteq \mathrm{Im}\,(\gamma_1)$ and that $\ker(\gamma_4) \subseteq \mathrm{Im}\,(\gamma_3)$.

That γ_4 is surjective follows from the definition of E_d^{bisec} : a form of degree d whose restriction to m vanishes at the divisor $a+b$ of m, for a given $(a,b,m) \in G$, either vanishes identically on m, in which case it is in $\gamma_4(M \otimes S^{d-1}V^*)$, or else it is, up to a form that vanishes identically on m, in $\gamma_4(P_{ab} \otimes S^{d-2}V^*)$.

The other two relations are established by a straightforward calculation. To illustrate we shall prove that $\ker(\gamma_2) \subseteq \mathrm{Im}\,(\gamma_1)$. Let $\omega \in \ker(\gamma_2)$ and $(a,b,m) \in G$ the point over which ω lies. We shall assume that $a \neq b$. The case $a = b$ can be done in a similar way. Fix projective coordinates so that $a = [0,0,1]$, $b = [0,1,0]$ and $m = \{x_0 = 0\}$. Then

$$\omega = (x_0^2 \otimes \omega_0, x_0^2 \otimes x_0 \otimes \omega_1' + x_0^2 \otimes x_1 \otimes \omega_1'', x_0^2 \otimes x_0 \otimes \omega_2' + x_0^2 \otimes x_2 \otimes \omega_2''),$$

where $\omega_0 \in S^{d-2}V^*$ and $\omega_1', \omega_1'', \omega_2', \omega_2'' \in S^{d-3}V^*$. Since

$$\begin{aligned}\gamma_2(\omega) = (&x_0 \otimes x_0 \otimes \omega_0 - x_0 \otimes x_0 \otimes x_0\omega_1' - x_0 \otimes x_1 \otimes x_0\omega_1'', \\ &-x_0 \otimes x_0 \otimes \omega_0 + x_0 \otimes x_0 \otimes x_0\omega_2' - x_0 \otimes x_2 \otimes x_0\omega_2'', \\ &x_0 \otimes x_0 \otimes x_0\omega_1' + x_0 \otimes x_1 \otimes x_0\omega_1'' - x_0 \otimes x_0 \otimes x_0\omega_2' - x_0 \otimes x_2 \otimes x_0\omega_2'')\end{aligned}$$

the relation

$$\gamma_2(\omega) = 0$$

is easily seen to be equivalent to the relations

$$\omega_1'' = \omega_2'' = 0\ , \quad \text{and} \quad \omega_0 = x_0\omega_1' = x_0\omega_2'.$$

Therefore $\omega_1' = \omega_2'$ (say $= \omega'$) and

$$\omega = (x_0^2 \otimes x_0\omega', x_0^2 \otimes x_0 \otimes \omega', x_0^2 \otimes x_0 \otimes \omega'),$$

and from this it is clear that

$$\omega = \gamma_1(x_0^3 \otimes \omega').$$

Next we apply the resolution above to calculate the total Chern class $c(E_d^{\mathrm{bisec}})$.

(13) Proposition. — *The total Chern class of $c(E_d^{\mathrm{bisec}})$ is given by the following expression :*

$$1-m-(d-1)(a+b)-(d^2-1)m^2+\frac{(d-1)(d+4)}{2}m(a+b)+\frac{(d-1)(d-2)}{2}(a+b)^2$$
$$+(d+1)(d-1)(d-3)m^2(a+b) - \frac{(d-1)(d-2)(d+3)}{2}m(a+b)^2$$
$$-\frac{(d-1)(d-2)(d-3)}{6}(a+b)^3 + (d-1)(d-2)(d^2+3d-3)m^2ab.$$

Proof. — From the resolution in the previous proposition we obtain :

$$c(E_d^{\mathrm{bisec}}) = \frac{c(M^2 \otimes S^{d-2}V^*)c(P_{ab} \otimes S^{d-2}V^*)}{c(M \otimes S_{ab} \otimes S^{d-2}V^*)} \cdot \frac{c(M^2 \otimes S_{ab} \otimes S^{d-3}V^*)c(M \otimes S^{d-1}V^*)}{c(M \otimes P_{ab} \otimes S^{d-3}V^*)c(M^3 \otimes S^{d-3}V^*)}.$$

Define $R_a = Q_a^*/M$, $R_b = Q_b^*/M$ and $R_{ab} = R_a \otimes R_b$. From the exact sequence

$$(*) \qquad 0 \to M^2 \to M \otimes S_{ab} \to P_{ab} \to R_{ab} \to 0$$

(obtained tensoring the exact sequences $0 \to M \to Q_a^* \to R_a \to 0$ and $0 \to M \to Q_b^* \to R_b \to 0$), we see that the first fraction of the expression above is $c(R_{ab} \otimes S^{d-2}V^*)$. Therefore we see that

$$c(E_d^{\mathrm{bisec}}) = \frac{c(R_{ab} \otimes S^{d-2}V^*)c(M^2 \otimes S_{ab} \otimes S^{d-3}V^*)}{c(M \otimes P_{ab} \otimes S^{d-3}V^*)c(M^3 \otimes S^{d-3}V^*)}c(M \otimes S^{d-1}V^*).$$

To evaluate the fraction in this expression, consider the obvious exact sequence

$$0 \to M \otimes S^{d-3}V^* \to S^{d-2}V^* \to S^{d-2}V^*/M \otimes S^{d-3}V^* \to 0$$

and tensor it with R_{ab}. Since the first term of the resulting exact sequence is $M \otimes R_{ab} \otimes S^{d-3}V^*$, which is the last of the result of tensoring $(*)$ with $M \otimes S^{d-3}V^*$, we get the exact sequence

$$\begin{aligned}0 \to M^3 \otimes S^{d-3}V^* \to M^2 \otimes S_{ab} \otimes S^{d-3}V^* \to M \otimes P_{ab} \otimes S^{d-3}V^* \\ \to R_{ab} \otimes S^{d-2}V^* \to R_{ab} \otimes S^{d-2}V^*/M \otimes S^{d-3}V^* \to 0.\end{aligned}$$

Now with this sequence the fraction of the expression of $c(E_d^{\text{bisec}})$ is $c\big(R_{ab} \otimes \big(S^{d-2}V^*/M \otimes S^{d-3}V^*\big)\big)$ and so

$$c(E_d^{\text{bisec}}) = c(M \otimes S^{d-1}V^*)c\big(R_{ab} \otimes \big(S^{d-2}V^*/M \otimes S^{d-3}V^*\big)\big).$$

From this the result follows by a straightforward computation of which we indicate the main steps. It is clear that $c(M) = 1 - m$. Using this and the relation $c(Q_a^*) = 1 - a + a^2$, we obtain that $c(R_a) = 1 - a + m$. Hence $c(R_{ab}) = 1 - a - b + 2m$. On the other hand,

$$\begin{aligned}c(M \otimes S^kV^*) = c(M)^{\binom{k+2}{2}} = 1 - \frac{(k+1)(k+2)}{2}m \\ + \frac{k(k+1)(k+2)(k+3)}{8}m^2\end{aligned}$$

and

$$\begin{aligned}c(S^kV^*/M \otimes S^{k-1}V^*) = c(M)^{-\binom{k+1}{2}} = 1 + \frac{k(k+1)}{2}m \\ + \frac{k(k+1)(k^2+k+2)}{8}m^2.\end{aligned}$$

Finally $c(M \otimes S^{d-1}V^*)$ and $c\big(R_{ab} \otimes \big(S^{d-2}V^*/M \otimes S^{d-3}V^*\big)\big)$ can be evaluated, the latter using the fact that R_{ab} is a line bundle (see Fulton [3], Example 3.2.2). Putting everything together yields the stated formula.

The intersection ring of X_d^{bisec}. — We are going to study on the variety $X_d^{\text{bisec}} = \mathbb{P}(E_d^{\text{bisec}})$ the following conditions : the characteristic condition μ and the conditions a, b and m (for example, and as usual, a denotes the condition that the point a of a quadruple $(f, a, b, m) \in X_d^{\text{bisec}}$ lies on a line).

(14) THEOREM. — *The ring $A^*(X_d^{\mathrm{bisec}})$ is isomorphic to the quotient of the polynomial ring $Z[\mu, a, b, m]$ by the ideal*

$$\begin{aligned}&\big(a^3, a^2 - am + m^2, b^2 - bm + m^2, \mu^{N-1} - \big(m + (d-1)(a+b)\big)\mu^{N-2}\\ &-\big((d^2-1)m^2 - \frac{(d-1)(d+4)}{2}m(a+b) - \frac{(d-1)(d-2)}{2}(a+b)^2\big)\mu^{N-3}\\ &+\big((d+1)(d-1)(d-3)m^2(a+b) - \frac{(d-1)(d-2)(d+3)}{2}m(a+b)^2\\ &-\frac{(d-1)(d-2)(d-3)}{6}(a+b)^3\big)\mu^{N-4} + (d-1)(d-2)(d^2+3d-3)m^2ab\mu^{N-5}\big).\end{aligned}$$

In particular, $\mathrm{Pic}(X_d^{\mathrm{bisec}})$ *is a rank 4 free group generated by* μ, a, b, m.

Proof. — Let μ be the hyperplane class on X_d^{bisec}. Then

$$A^*(X_d^{\mathrm{bisec}})) = A^*(G)[\mu]/\sum \pi^* c_i(E_d^{\mathrm{bisec}})\mu^{N-i-1},$$

where $\pi : X_d^{\mathrm{bisec}} \to G$ is the natural projection. Using the resolution of the previous proposition the theorem follows in a straightforward manner.

(15) COROLLARY. — *In $A^*(X_d^{\mathrm{bisec}})$ the following relations hold :*

$$\begin{array}{llll} \mu^{N+2} & = 0 & a\mu^{N+1} & = 0 \\ m\mu^{N+1} & = 0 & ab\mu^N & = d^2 \\ a^2\mu^N & = 0 & ma\mu^N & = d(d-1) \\ m^2\mu^N & = d(d-1) & a^2b\mu^{N-1} & = d \\ mab\mu^{N-1} & = 2d-1 & m^2a\mu^{N-1} & = d-1 \\ m^2ab\mu^{N-2} & = 1. & & \end{array}$$

Proof. — Use (14) and the fact that $a^2b^2\mu^{N-2} = 1$.

The bundle E_d^{binod}. — Now we define E_d^{binod} as the subbundle of the trivial bundle $S^dV^*|G$ whose fiber over $(a, b, m) \in G$ is the linear subspace of S^dV^* whose elements are forms φ of degree d whose restriction to m vanishes on $2a + 2b$ (as a divisor on the line m). It is clear that E_d^{binod} has rank $N + 1 - 6 = N - 5$. Define also $T_{ab} = \big(Q_a^* \otimes S^2Q_b^*\big) \oplus \big(S^2Q_a^* \otimes Q_b^*\big)$ and $\Pi_{ab} = S^2Q_a^* \otimes S^2Q_b^*$.

Next statement provides a resolution of E_d^{binod}. We use the following notations :

- $k : M \otimes P_{ab} \to T_{ab}$, the map $m \otimes \alpha \otimes \beta \mapsto (\alpha \otimes m\beta, m\alpha \otimes \beta)$;

- $w : M \otimes T_{ab} \to \Pi_{ab}$, the map $m \otimes (\alpha_1 \otimes \beta_2, \alpha_2 \otimes \beta_1) \mapsto m\alpha_1 \otimes \beta_2 - \alpha_2 \otimes m\beta_1$;
- $i : T_{ab} \otimes S^k V^* \to E_{k+3}^{\text{bisec}}$, the map $(\alpha_1 \otimes \beta_2, \alpha_2 \otimes \beta_1) \otimes \omega \mapsto (\alpha_1\beta_2 - \alpha_2\beta_1)\omega$;
- $j : \Pi_{ab} \otimes S^k V^* \to E_{k+4}^{\text{bisec}}$, the map $\alpha \otimes \beta \otimes \omega \mapsto \alpha\beta\omega$;
- recall also that $p : M \otimes S^k V^* \to S^{k+1} V^*$ (product map) denotes the inclusion map given by $m \otimes \omega \mapsto m\omega$.

Moreover, we use similar conventions to those explained before Proposition (12) to simplify notations.

(16) Proposition. — *The following sequence is exact :*

$$\begin{aligned}0 \to M^3 \otimes P_{ab} \otimes S^{d-5}V^* \xrightarrow{\beta_1} \big(M^2 \otimes T_{ab} \otimes S^{d-5}V^*\big) \oplus \big(M^2 \otimes P_{ab} \otimes S^{d-4}V^*\big) \\ \xrightarrow{\beta_2} \big(M \otimes T_{ab} \otimes S^{d-4}V^*\big) \oplus \big(M \otimes \Pi_{ab} \otimes S^{d-5}V^*\big) \\ \xrightarrow{\beta_3} \big(M \otimes E_{d-1}^{bisec}\big) \oplus \big(\Pi_{ab} \otimes S^{d-4}V^*\big) \xrightarrow{\beta_4} E_d^{\text{binod}} \to 0\end{aligned}$$

where

$$\beta_1 = \begin{pmatrix} k \\ p \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} p & -k \\ w & 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} -i & j \\ w & -p \end{pmatrix}, \quad \beta_4 = (p, j).$$

[In the definition of β_4, note that the image of the map $j : \Pi_{ab} \otimes S^{d-4}V^* \to E_d^{\text{bisec}}$ is contained in E_d^{binod} and that we denote the corresponding map $j : \Pi_{ab} \otimes S^{d-4}V^* \to E_d^{\text{binod}}$ by the same symbol j.]

Proof. — It is similar to that of Proposition (12) and we omit it.

(17) Proposition. — *The total Chern class $c(E_d^{\text{binod}})$ is given by the following expression :*

$$\begin{aligned}&1 - \big(5m + (3d-8)(a+b)\big) \\ &-\Big(3(d^2-2d-5)m^2 - \frac{1}{2}(3d^2+29d-102)m(a+b) - \frac{1}{2}(d-3)(9d-26)(a+b)^2\Big) \\ &+\Big(3(3d^3-19d^2+8d+62)m^2(a+b) - \frac{1}{2}(d-3)(9d^2+40d-204)m(a+b)^2 \\ &\qquad - \frac{1}{2}(d-3)(3d-10)^2(a+b)^3\Big) + 36(d-2)(d-3)(d^2+d-13)m^2ab.\end{aligned}$$

Proof. — From the resolution given in the last proposition we have that

$$\frac{c(M^2\otimes P_{ab}\otimes S^{d-4}V^*)c(\Pi_{ab}\otimes S^{d-4}V^*)}{c(M\otimes T_{ab}\otimes S^{d-4}V^*)} \cdot \frac{c(M^2\otimes T_{ab}\otimes S^{d-5}V^*)c(M\otimes E_{d-1}^{\mathrm{bisec}})}{c(M\otimes \Pi_{ab}\otimes S^{d-5}V^*)c(M^3\otimes P_{ab}\otimes S^{d-5}V^*)}.$$

Define $U_a = S^2Q_a^*/M\otimes Q_a^*$, $U_b = S^2Q_b^*/M\otimes Q_b^*$ and $U_{ab} = U_a\otimes U_b$.

There exists an exact sequence

$$(*)\qquad 0\to M^2\otimes P_{ab}\to M\otimes T_{ab}\to \Pi_{ab}\to U_{ab}\to 0$$

which is obtained tensoring

$$0\to M\otimes Q_a^*\to S^2Q_a^*\to U_a\to 0$$

with the analogous sequence for U_b. So the expression above is

$$c(M\otimes E_{d-1}^{\mathrm{bisec}})\frac{c(M^2\otimes T_{ab}\otimes S^{d-5}V^*)c(U_{ab}\otimes S^{d-4}V^*)}{c(M\otimes \Pi_{ab}\otimes S^{d-5}V^*)c(M^3\otimes P_{ab}\otimes S^{d-5}V^*)}.$$

Now we are going to see that there is an exact sequence

$$0\to M^3\otimes P_{ab}\otimes S^{d-5}V^*\to M^2\otimes T_{ab}\otimes S^{d-5}V^*\to$$
$$M\otimes\Pi_{ab}\otimes S^{d-5}V^*\to U_{ab}\otimes S^{d-4}V^*\xrightarrow{\phi} U_{ab}\otimes\big(S^{d-4}V^*/M\otimes S^{d-5}V^*\big)\to 0.$$

Indeed, the kernel of ϕ is $M\otimes U_{ab}\otimes S^{d-5}V^*$ and this bundle coincides with the rightmost non-zero term of $M\otimes(*)\otimes S^{d-5}V^*$, so that our sequence is just the result of splicing together this last sequence with the sequence

$$0\to M\otimes U_{ab}\otimes S^{d-5}V^*\to U_{ab}\otimes S^{d-4}V^*\to U_{ab}\otimes\big(S^{d-4}V^*/M\otimes S^{d-5}V^*\big)\to 0.$$

Finally, the last displayed exact sequence allows us to evaluate the fraction in the expression of $c(E_d^{\mathrm{binod}})$ and hence we obtain

$$c(E_d^{\mathrm{binod}}) = c(M\otimes E_{d-1}^{\mathrm{bisec}})c(U_{ab}\otimes S^{d-4}V^*/M\otimes S^{d-5}V^*).$$

By definition, $U_a = S^2Q_a^*/M\otimes Q_a^*$, and so, using previous formulas, $c(U_a) = 1+2m-2a$ and $c(U_{ab}) = 1+4m-2a-2b$. Using this, and the fact that $c(E_{d-1}^{\mathrm{bisec}})$ is already known, the proposition follows by a staightforward computation.

The intersection ring of X_d^{binod}. — We shall set X_d^{binod} to denote the projective bundle $\mathbb{P}(E_d^{\mathrm{binod}})$. Thus X_d^{binod} is smooth and has dimension

$N-6+4=N-2$. We are going to calculate the intersection ring of X_d^{binod} using the knowledge of the Chern classes of E_d^{binod}, in terms of the following classes : the characteristic condition μ and the conditions a, b and m (for example, and as usual, a denotes the condition that the point a of a quadruple $(f,a,b,m)\in X_d^{\text{binod}}$ lies on a line).

(18) THEOREM. — *The ring $A^*(X_d^{\text{binod}})$ is isomorphic to the quotient of the polynomial ring $Z[\mu,a,b,m]$ by the ideal*

$$\Big(a^3, a^2-am+m^2, b^2-bm+m^2, \mu^{N-5}-(5m+3(d-8)(a+b))\mu^{N-6}$$
$$-\big(3(d^2-2d-5)m^2-\frac{1}{2}(3d^2+29d-102)m(a+b)-\frac{1}{2}(d-3)(9d-26)(a+b)^2\big)\mu^{N-7}$$
$$+\big(3(3d^3-19d^2+8d+62)m^2(a+b)-\frac{1}{2}(d-3)(9d^2+40d-204)m(a+b)^2$$
$$-\frac{1}{2}(d-3)(3d-10)^2(a+b)^3\big)\mu^{N-8}+36(d-2)(d-3)(d^2+d-13)m^2ab\mu^{N-9}\Big).$$

In particular, $\mathrm{Pic}(X_d^{\text{binod}})$ *is a rank 4 free group generated by μ,a,b,m.*

Proof. — Let μ be the hyperplane class on X_d^{binod}. Then

$$A^*(X_d^{\text{binod}})) = A^*(G)[\mu]/\sum \pi^* c_i(E_d^{\text{binod}})\mu^{N-i-5},$$

where $\pi : X_d^{\text{binod}} \to G$ is the natural projection. Using the resolution of the previous proposition the statement follows straightforwardly.

(19) COROLLARY. — *In $A^*(X_d^{\text{binod}})$ the following relations hold :*

$$\begin{aligned}
\mu^{N-2} &= 3(d-1)(3d^3-9d^2-5d+22) & a\mu^{N-3} &= 9d^3-27d^2-d+30\\
m\mu^{N-3} &= 2(d-2)(9d^2-34d+27) & ab\mu^{N-4} &= 9d^2-18d+2\\
a^2\mu^{N-4} &= 3d^2-6d-4 & ma\mu^{N-4} &= 12d^2-49d+46\\
m^2\mu^{N-4} &= (d-2)(9d-25) & a^2b\mu^{N-5} &= 3(d-1)\\
mab\mu^{N-5} &= 6d-11 & m^2a\mu^{N-5} &= 3d-8\\
m^2ab\mu^{N-6} &= 1.
\end{aligned}$$

Proof. — Use (18) and the fact that $a^2b^2\mu^{N-6}=1$.

4. The group Pic$(V_{d,1})$.

Recall that we set X_d^{nod} to denote be the variety $X_{d,2}$. Now there exist natural maps $X_d^{\text{cusp}} \to X_d^{\text{nod}}$ and $X_d^{\text{binod}} \to X_d^{\text{nod}}$ whose images will be denoted X'_d and X''_d, respectively. From our analysis in the previous sections it follows that X'_d and X''_d are the closures of the sets $V^c_{d,1}$ and $V^b_{d,2}$ of plane curves of degree d that have a cusp or an ordered pair of nodes as only singularities, respectively, so that in particular this yields the irreducibility of $V^c_{d,1}$ and $V^b_{d,2}$ (the irreducibility of $V^b_{d,2}$ implies the irreducibility of $V_{d,2}$, for the former is a double cover of the latter; cf. Harris [4], Ran [7]). It follows that $X^{\text{nod}}_d - V_{d,1} = X'_d \cup X''_d$. We set $\gamma = [X'_d]$ and $\xi = [X''_d]$, so that $\gamma, \xi \in \operatorname{Pic}(X^{\text{nod}}_d)$.

(20) PROPOSITION. — *The group* $\operatorname{Pic}(X^{\text{nod}}_d)$ *is a rank two free group generated by* μ, h. *Moreover, the following relations hold in* $\operatorname{Pic}(X^{\text{nod}}_d)$ *:*

$$\begin{aligned}\gamma =& 2\mu + 2(d-3)h \\ \xi =& (3d^2 - 6d - 4)\mu + (-7d + 18)h.\end{aligned}$$

Proof. — By the second part of Theorem (7), there exist integers r, s such that $\gamma = r\mu + sh$. If we multiply this relation by $h^2\mu^{N-4}$ we get $r = h^2\mu^{N-4}\gamma$. To evaluate this, let φ be the map from X^{cusp}_d to X^{nod}_d. The image of this map is X'_d and so $\varphi_*(1) = m\gamma$, where m is the degree of φ. So, by the projection formula and $mr = (\varphi^* h)^2(\varphi^*\mu)^{N-4} = c^2\mu^{N-4} = 2$ (Corollary (11)). On the other hand, since φ is generically bijective, m has to be a power of the characteristic of the ground field, which we have assumed to be not 2. Thus we conclude that $m = 1$ (this means that φ is a birational morphism onto its image) and $r = 2$. If we now multiply γ by $h\mu^{N-3}$ we get (Corollary (11) and Theorem (7))

$$s = h\mu^{N-3}\gamma - 2h\mu^{N-2} = 2(d-3).$$

Similarly, there are integers r', s' such that $\xi = r'\mu + s'h$. Multiplying by $h^2\mu^{N-4}$ and using Corollary (19) we get $r' = 3d^2 - 6d - 4$, and multiplying by $h\mu^{N-3}$ we get $s' = -7d + 18$, this time using Corollary (14) and Theorem (7).

(21) COROLLARY.

$$\begin{aligned}6(d-2)(d^2 - 3d + 1)\mu =& (7d - 18)\gamma + 2(d-3)\xi \\ 6(d-2)(d^2 - 3d + 1)h =& (3d^2 - 6d - 4)\gamma - 2\xi.\end{aligned}$$

Proof. — If follows immediately from Cramer's rule.

(22) THEOREM. — *If d is odd, then*

$$\operatorname{Pic}(V_{d,1}) = \mathbb{Z}/6(d-2)(d^2 - 3d + 1)$$

and if d is even then

$$\mathrm{Pic}(V_{d,1}) = \mathbb{Z}_2 \times \mathbb{Z}/3(d-2)(d^2-3d+1).$$

Proof. — The two relations of (20) above yield $2\mu = 2(3-d)h$ and $(3d^2-6d-4)\mu = (7d-18)h$. If d is odd, then $3d^2-6d-4$ is odd, say $3d^2-6d-4 = 2n+1$, and in this case the two relations imply that $\mu = (3d^3-15d^2+22d-9)h$. Thus $\mathrm{Pic} = \mathrm{Pic}(V_{d,1})$ is cyclic generated by h and the second relation in (21) implies that the order of Pic is at most $6(d-2)(d^2-3d+1)$. Now let r be the order of h (or of Pic), and let s be such that $rs = 6(d-2)(d^2-3d+1)$. Then, because of the exact sequence

$$A^0(\gamma \cup \xi) \to \mathrm{Pic}(X_d^{\mathrm{nod}}) \to \mathrm{Pic}(V_{d,1}) \to 0,$$

there exist integers p, q such that $rh = p\gamma + q\xi$. Therefore $6(d-2)(d^2-3d+1)h = srh = sp\gamma + sq\xi$. It follows that $sp = 3d^2-6d-4$, $sq = 2$, and hence that $s = 1$. Notice that γ and ξ are linearly independent over $\mathbb{Z}$.

Assume now that d is even, $d = 2k$. Cancelling the factor 2 in the first relation above, which is possible by (20), we get the relation

$$3(d-2)(d^2-3d+1)h = (6k^2-6k-2)\gamma - \xi.$$

In particular, $3(d-2)(d^2-3d+1)h = 0$ in Pic. As before, it is easy to see that $3(d-2)(d^2-3d+1)$ is the order of h.

In Pic we also have $2\mu = 2(3-d)h$ (the difference of both expressions is γ). Hence if we set $\mu' = \mu + (d-3)h$, then $2\mu' = 0$ in Pic. Since μ' and h generate Pic, there are two possibilities : either $\mu' \in \mathbb{Z}h$, in which case Pic would be cyclic of order $3(d-2)(d^2-3d+1)$, or else $\mu' \notin \mathbb{Z}h$, in which case Pic would be the product of two cyclic groups of orders 2 and $3(d-2)(d^2-3d+1)$. To complete the proof it is enough to see that the first possibility cannot occur.

Since $\mu' = \mu + (d-3)h$ if $\mu' \in \mathbb{Z}h$ there would exist integers p, q and r such that

$$\mu + ph = q\gamma + r\xi.$$

Substituting γ and ξ as expressed in (20) we would obtain

$$\mu + ph = \big(2q + (3d^2-6d-4)r\big)\mu + \big(2(d-3)q + (-7d+18)r\big)h.$$

But this is a contradiction, for the coefficient of μ on the right hand side is even.

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