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THE RECIPROCAL OF THE BETA FUNCTION AND $GL(n,\mathrm{R})$ WHITTAKER FUNCTIONS

by Eric STADE

1. Definitions; statement of results.

Our first result will concern the classical beta function

$$\mathrm{B}(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

In §2 below we will prove:

THEOREM 1. — *If* $\mathrm{Re}\,(x+y-1)>0$, *then*

$$\frac{1}{(x+y-1)\mathrm{B}(x,y)}=\frac{1}{2\pi i}\int_{|u|=1}(1+1/u)^{x-1}\,(1+u)^{y-1}\,\frac{du}{u},$$

the integral taken counterclockwise in the complex plane.

We note that, by the functional equation of the gamma function, the left–hand side of the above expression equals $\Gamma(x+y-1)/\{\Gamma(x)\Gamma(y)\}$. We also remark that the result of Theorem 1 is similar in form to the classical formula

$$\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,$$

and even more so to the expression

$$\mathrm{B}(x,y)=\int_0^\infty (1+1/u)^{-x}(1+u)^{-y}\,\frac{du}{u}$$

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(which follows from the first integral for $\mathrm{B}(x,y)$ by the substitution $t = (1+1/u)^{-1}$).

The remainder of the work in this paper will concern the application of Theorem 1 to the study of $GL(n,\mathrm{R})$ Whittaker functions. We begin by recalling some basic facts regarding these functions.

Whittaker functions for arbitrary reductive groups were first studied by Jacquet [6]; we will consider in this paper only the groups $GL(n,\mathrm{R})$. In fact, the results we obtain will pertain specifically to $GL(3,\mathrm{R})$ and $GL(4,\mathrm{R})$ Whittaker functions; even so, some terminology in a general setting will be helpful. We introduce the following notation. Let $X \subset GL(n,\mathrm{R})$ be the group of upper triangular, unipotent matrices: if $x \in X$, we write $x = (x_{ij})$. Also let $Y \subset GL(n,\mathrm{R})$ be the group of diagonal matrices y of the form

$$y = \{\mathrm{diag}(y_1y_2\cdots y_{n-1}, y_2y_3\cdots y_{n-1}, \ldots, y_{n-1}, 1) \mid y_i > 0 \text{ for all } i\}.$$

We will be interested in functions on the "generalized upper half–plane" $\mathcal{H}^n$, which is a homogeneous space for $GL(n,\mathrm{R})$:

$$\mathcal{H}^n = GL(n,\mathrm{R})/(KZ),$$

where K is the orthogonal group and Z the group of scalar matrices. By the Iwasawa decomposition, every $z \in \mathcal{H}^n$ has a unique representation

$$z \equiv xy \pmod{KZ}$$

with $x \in X$, $y \in Y$. In what follows, functions on $\mathcal{H}^n$ will be defined in terms of these coordinates.

In particular, we will want to study eigenfunctions of the algebra D of $GL(n,\mathrm{R})$–invariant differential operators on $\mathcal{H}^n$. One such eigenfunction is given (*cf.* [11]), for $\nu = (\nu_1, \nu_2, \ldots, \nu_{n-1}) \in \mathrm{C}^{n-1}$, by

$$H_\nu(z) \equiv H_\nu(y) = \prod_{i=1}^{n-1}\prod_{j=1}^{n-1} {y_i}^{b_{ij}\nu_j}$$

where $b_{ij} = \min\{ij,\ (n-i)(n-j)\}$. If we denote the eigenvalues of H_ν by $\lambda_\nu(d)$—that is,

$$dH_\nu = \lambda_\nu(d)H_\nu$$

for all $d \in D$—then these eigenvalues are known to be invariant under a certain group action. Namely, let $\mathcal{W}$ be the Weyl group (identified with the group of $n \times n$ permutation matrices) of $GL(n,\mathrm{R})$. Also, for an arbitrary $t = (t_1, t_2, \ldots, t_{n-1}) \in \mathrm{C}^{n-1}$, let $t - \dfrac{1}{n}$ denote the element

$\left(t_1 - \frac{1}{n}, t_2 - \frac{1}{n}, \ldots, t_{n-1} - \frac{1}{n}\right)$. If we define an action $\omega(\nu)$ of $\mathcal{W}$ on $\mathbb{C}^{n-1}$ by requiring, for each $\omega \in \mathcal{W}$, that

$$H_{\nu-\frac{1}{n}}(y) = H_{\omega(\nu)-\frac{1}{n}}(\omega y \omega^{-1})$$

for all $y \in Y$, then $\lambda_{\omega(\nu)}(d) = \lambda_\nu(d)$ for all $\omega \in \mathcal{W}$, $d \in D$.

Now Whittaker functions arise as eigenfunctions of D that transform under the "superdiagonal" character Θ of X. Namely, if

$$\Theta(x) = \mathrm{e}(x_{1,2} + x_{2,3} + \cdots + x_{n-1,n}),$$

where $\mathrm{e}(t)$ denotes $\mathrm{e}^{2\pi i t}$, then we make the following

DEFINITION 1.1. — *A $GL(n,\mathbb{R})$ Whittaker function is a function $f_\nu(z)$, smooth on $\mathcal{H}^n$ and meromorphic on $\mathbb{C}^{n-1}$, such that*

(a) $\quad df_\nu = \lambda_\nu(d) f_\nu \qquad \forall d \in D;$

(b) $\quad f_\nu(x_1 z) = \Theta(x_1) f_\nu(z) \qquad \forall x_1 \in X,\ z \in \mathcal{H}^n.$

Casselman and Zuckerman (in unpublished work), and independently Kostant [7], have shown that the space $S_{n,\nu}$ of $GL(n,\mathbb{R})$ Whittaker functions with fixed eigenvalues $\lambda_\nu(d)$ has, for almost all values of ν, dimension $n! = |\mathcal{W}|$. Further, Hashizume [5] has demonstrated the existence of a "fundamental" Whittaker function $M_{n,\nu}(z)$ that may be used to generate $S_{n,\nu}$. Specifically, let $K = (k_1, k_2, \ldots, k_{n-1})$ and, for $y \in Y$ as above, let πy denote the matrix product of $\mathrm{diag}(\pi^{n-1}, \pi^{n-2}, \ldots, \pi, 1)$ with y. Then there exist complex coefficients $G_K(n;\nu)$ and a $GL(n,\mathbb{R})$ Whittaker function

(1.1)

$$M_{n,\nu}(z) = \Theta(x) H_\nu(\pi y) \sum_{k_1,k_2,\ldots,k_{n-1}=0}^{\infty} G_K(n;\nu)\,(\pi y_1)^{2k_1}(\pi y_2)^{2k_2}\ldots(\pi y_{n-1})^{2k_{n-1}}$$

such that the set $\{M_{n,\omega(\nu)}(z) \,|\, \omega \in \mathcal{W}\}$ spans $S_{n,\nu}$. (Of course, the quantities $\Theta(x)$ and $H_\nu(\pi y)$ in equation (1.1) depend on n as well, but we suppress this dependence in our notation.)

Before stating our main results (*cf.* Theorems 2 and 3 below) on Whittaker functions, we recall their relevance to the theory of automorphic forms. By a $GL(n,\mathbb{R})$ *automorphic form of type* ν, we mean a smooth function $\varphi(z)$ on $\mathcal{H}^n$ such that, if $\Gamma = GL(n,\mathbb{Z})$:

(i) $\varphi(\gamma z) = \varphi(z)$ for all $\gamma \in \Gamma$, $z \in \mathcal{H}^n$;

(ii) $d\varphi = \lambda_\nu(d)\varphi$ for all $d \in D$;

(iii) φ is of at most polynomial growth in each y_i.

Such a function φ has a "Fourier expansion" (*cf.* [10], [12]) whose Fourier coefficients are given by a certain Whittaker function in $S_{n,\nu}$ (in addition, some of the Fourier coefficients of φ may involve "degenerate" Whittaker functions, which are eigenfunctions of D that do not satisfy condition (b) of Definition 1.1).

The Whittaker function arising in the above Fourier expansion is not in fact the fundamental Whittaker function $M_{n,\nu}(z)$, but rather the "class one principal series" Whittaker function $W_{n,\nu}(z)$. This function corresponds to a certain principal series representation of $GL(n,\mathrm{R})$ induced from the subgroup XY (*cf.* [4]). (That $W_{n,\nu}(z)$ should be the unique Whittaker function occuring in the Fourier development of an automorphic form follows from work of Shalika [12] and Wallach [17].) Of course, $W_{n,\nu}(z)$ may be expressed as a linear combination of the $M_{n,\omega(\nu)}(z)$'s.

Automorphic forms of particular interest are the *cusp forms*, whose expansions contain *no* degenerate terms. These eigenfunctions give rise to the discrete spectrum of D acting on $\mathcal{L}^2(\Gamma\backslash\mathcal{H}^n)$. ($\mathcal{L}^2(\Gamma\backslash\mathcal{H}^n)$ denotes the space of automorphic functions that are square–integrable, with respect to the $GL(n,\mathrm{R})$–invariant measure, over a fundamental domain for Γ in $\mathcal{H}^n$.) Thus the cusp forms (and consequently the $GL(n,\mathrm{R})$ Whittaker functions) play a central role in the spectral theory of $\mathcal{L}^2(\Gamma\backslash\mathcal{H}^n)$.

One may relate $M_{n,\nu}(z)$ more directly to the study of automorphic forms, by considering the Poincaré series

$$P_n(z;\nu) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} M_{n,\nu}(\gamma z)$$

(where $\Gamma_\infty = \Gamma\cap X$). This series has been studied when $n=2$ (*cf.* [8] and [9]) and $n=3$ (*cf.* [14]); there is found in either case a correspondence between the poles of $P_n(z;\nu)$ and the eigenvalues of cusp forms in $\mathcal{L}^2(\Gamma\backslash\mathcal{H}^n)$. (When $n=3$, the series does not actually converge, but makes sense as a linear functional on spaces of cusp forms.) Moreover, a certain linear combination (over $\omega\in\mathcal{W}$) of the $P(z;\omega(\nu))$'s yields an "Eisenstein series;" the latter is an automorphic form (of type ν) whose eigenvalues belong to the *continuous* spectrum of D acting on $\mathcal{L}^2(\Gamma\backslash\mathcal{H}^n)$.

Let us now turn to the statement of our results concerning Whittaker functions. If we write $M_{n,\nu}(z) = \Theta(x)\, M_{(\nu_1,\ldots,\nu_{n-1})}(y_1,\ldots,y_{n-1})$ then, in §3 and 4 below, we will prove:

THEOREM 2. — *If* $\operatorname{Re}(\nu_1+\nu_2)>0$, *then*

$$M_{(\nu_1,\nu_2)}(y_1,y_2) = \pi^2 y_1^{1-(\nu_1-\nu_2)/2} y_2^{1+(\nu_1-\nu_2)/2} \prod_{j=1}^{2}\prod_{k=1}^{3-j}\Gamma\Big(\frac{3\nu_k+\cdots+3\nu_{k+j-1}-(j-2)}{2}\Big) \cdot \frac{1}{2\pi i}\int_{|u|=1} I_{(3\nu_1+3\nu_2-2)/2}\big(2\pi y_1\sqrt{1+1/u}\big) \cdot I_{(3\nu_1+3\nu_2-2)/2}\big(2\pi y_2\sqrt{1+u}\big)\, u^{(3\nu_1-3\nu_2)/4}\,\frac{du}{u},$$

the path of integration being the same as in Theorem 1.

We also have:

THEOREM 3. — *If* $\rho=2\nu_1+2\nu_2+2\nu_3-3/2$ *and* $\operatorname{Re}\rho+1>0$, *then*

$$M_{(\nu_1,\nu_2,\nu_3)}(y_1,y_2,y_3) = \pi^5 y_1^{(3/2)-\nu_1+\nu_3} y_2^2 y_3^{(3/2)+\nu_1-\nu_3} \prod_{j=1}^{3}\prod_{k=1}^{4-j}\Gamma\Big(\frac{4\nu_k+\cdots+4\nu_{k+j-1}-(j-2)}{2}\Big) \cdot \frac{1}{(2\pi i)^2}\int_{|u_1|=1}\int_{|u_2|=1} I_\rho\big(2\pi y_1\sqrt{1+1/u_1}\big) \cdot I_\rho\big(2\pi y_2\sqrt{(1+u_1)(1+1/u_2)}\big)\, I_\rho\big(2\pi y_3\sqrt{1+u_2}\big) \cdot I_{2\nu_2-1/2}\big(2\pi y_2\sqrt{u_1/u_2}\big)\, u_1^{\nu_1-\nu_3} u_2^{\nu_1-\nu_3}\,\frac{du_1du_2}{u_1u_2},$$

the path of integration in each variable being as in Theorems 1 and 2.

Theorem 2 (a form of which we have previously stated, with only a brief allusion to the proof, in [14]) has already been applied to the study of the Poincaré series $P_3(z;\nu)$ discussed above. Moreover, Bump and Huntley [3] have used the result of this theorem to deduce asymptotics, as y_1 or $y_2\to\infty$, for $M_{(\nu_1,\nu_2)}(y_1,y_2)$. It is hoped that the expression for $M_{4,\nu}(z)$ embodied by Theorem 3 will prove equally useful.

Regarding possible generalizations of Theorems 2 and 3 to $GL(n,\mathbb{R})$, we expect that there should be an integral formula giving us $M_{n,\nu}(z)$ in terms of $M_{n-2,\nu}(z)$. To state our conjecture precisely, it is convenient to divide out by some powers of the y_i's, and by some gamma factors. Specifically, let $\nu\in\mathbb{C}^{n-1}$ as above. We define

$$r_{j,k}=(n\nu_k+\cdots+n\nu_{k+j-1}-j)/2$$

for $1 \le j \le n-1$ and $1 \le k \le n-j$, and $\mu = r_{n-1,1}$. If we write

$$Q_{n,\nu}(y) = \pi^{(n^3-n)/12} H_\nu(y) \prod_{j=1}^{n-1} \prod_{k=1}^{n-j} \{ y_j^{-r_{j,k}} \Gamma(1+r_{j,k}) \}$$

and

$$M_{n,\nu}^*(y) = \frac{M_{(\nu_1,\ldots,\nu_{n-1})}(y_1,\ldots,y_{n-1})}{Q_{n,\nu}(y)},$$

then we have

CONJECTURE. — *Let $n \ge 2$. If $\nu \in \mathbb{C}^{n-1}$, put $\lambda_{j-1} = n\nu_j/(n-2)$ for $2 \le j \le n-2$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-3})$. Also, formally define $u_0 = 1/u_{n-1} = 0$ and $(u_{n-1})^0 = 1$. Then, if* $\mathrm{Re}\,(\mu+1) > 0$,

$$\begin{aligned} &M_{n,\nu}^*(y) \\ &= \frac{1}{(2\pi i)^{n-2}} \int_{|u_1|=1} \cdots \\ &\cdots \int_{|u_{n-2}|=1} \prod_{i=1}^{n-1} \{ u_i^{(r_{i,1}-r_{i,n-i})/2} I_\mu \big(2\pi y_i \sqrt{(1+u_{i-1})(1+1/u_i)} \big) \} \\ &\cdot M_{n-2,\lambda}^* \left(y_2 \sqrt{\frac{u_1}{u_2}}, y_3 \sqrt{\frac{u_2}{u_3}}, \ldots, y_{n-2} \sqrt{\frac{u_{n-3}}{u_{n-2}}} \right) \frac{du_1}{u_1} \cdots \frac{du_{n-2}}{u_{n-2}} \end{aligned}$$

(each integral taken counterclockwise), where I denotes the I–Bessel function (1.4).

We remark that the $GL(n-2,\mathbb{R})$–Whittaker function appearing on the right–hand side of this conjecture is understood to equal the constant function one when $n = 2$ or $n = 3$; we also note that, when $n = 2$, the "zero–fold" integral arising denotes the integrand itself. Our conjecture is then supported by two observations: first, that it does indeed hold for $n \le 4$ (as Theorems 2 and 3 above, and equation (1.3) below, attest). Moreover, we have shown [15] that an analogous statement concerning the *class one* Whittaker function $W_{n,\nu}(z)$, as described above, in fact holds *for all n*. Namely, if one replaces the fundamental Whittaker functions $M_{n,\nu}$ and $M_{n-2,\lambda}$ appearing in our conjecture by the corresponding functions $W_{n,\nu}$ and $W_{n-2,\lambda}$, while simultaneously replacing each I–Bessel function by the corresponding K–Bessel function, and finally replacing each integration around the unit circle by one on the positive real axis, then one indeed gets a true statement. This statement is embodied by Theorem 2.1 in [15] (which takes a slightly different form than just described, due to different normalizations and the replacement of each u_i by u_i^2).

Our proofs of Theorems 2 and 3 will rely on knowledge of the coefficients $G_K(3;\nu)$ and $G_K(4;\nu)$ of the Whittaker functions in question. It is, in fact, only for $n \le 4$ that the $G_K(n;\nu)$'s have been determined explicitly. For this reason, we are unable at present to supply a proof of the above conjecture that is valid for general values of n.

We close this section by recalling the relevant information concerning $G_K(n;\nu)$ for $n = 2, 3, 4$: we first define, for $e \in \mathbb{C}$,

(1.2) $$(e)_k = (k+e-1)(k+e-2)\cdots(1+e)e \quad (k \in \mathbb{Z}^+); \quad (e)_0 = 1.$$

It then follows from work of Whittaker and Watson [18] that

$$G_K(2;\nu) = \frac{1}{k_1!\,(\nu_1+\frac{1}{2})_{k_1}}.$$

That is, writing x for x_1, y for y_1, and ν for ν_1 (and recalling (1.1)):

(1.3) $$M_{2,\nu}(z) = e(x)\Gamma(\nu+\tfrac{1}{2})\sqrt{\pi y}\, I_{\nu-1/2}(2\pi y)$$

where

(1.4) $$I_s(y) = \sum_{k=0}^{\infty} \frac{(y/2)^s}{k!\,\Gamma(k+s+1)}$$

is the classical I–Bessel function. Next, Bump [2] has shown that

(1.5) $$G_K(3;\nu) = \frac{(\frac{3\nu_1+3\nu_2}{2})_{k_1+k_2}}{k_1!\,k_2!\,(\frac{3\nu_2+1}{2})_{k_1}(\frac{3\nu_1+3\nu_2}{2})_{k_1}(\frac{3\nu_1+1}{2})_{k_2}(\frac{3\nu_1+3\nu_2}{2})_{k_2}}.$$

Since $\Gamma(s+1) = s\Gamma(s)$, and thus

(1.6) $$(e)_k = \frac{\Gamma(e+k)}{\Gamma(e)}$$

(also $k! = \Gamma(k+1)$), we find that $G_K(2;\nu)$ and $G_K(3;\nu)$ are in fact ratios of gamma functions. (The gamma function actually has simple poles at the nonpositive integers; if e is such a number then the right–hand side of (1.6) is defined by taking the appropriate limit.)

The expression for $G_K(4;\nu)$ is somewhat more complicated. Surprisingly, $G_K(4;\nu)$ is *not* expressible as a ratio of Gamma functions, but rather takes the form of a *finite sum* of ratios of Gamma functions. Namely, we have shown [16] that

(1.7) $$G_K(4;\nu)$$

$$= \frac{(-1)^{k_1+k_3}(a)_{k_2}\,(b)_{k_2}\,{}_4F_3(a',b',1-c-k_2,-k_2;c',1-a-k_2,\,1-b-k_2;1)}{k_1!\,k_2!\,k_3!\,(a)_{k_1}\,(b)_{k_3}\,(c)_{k_2}\,(a+a')_{k_2}\,(b+b')_{k_2}}$$

where

(1.8)

$$a = -k_1 - 2\nu_3 + 1/2; \qquad b = -k_3 - 2\nu_1 + 1/2; \qquad c = 2\nu_2 + 1/2;$$
$$a' = k_1 + \rho + 1; \qquad b' = k_3 + \rho + 1; \qquad c' = \rho + 1$$

(ρ as in Theorem 3), and ${}_4F_3$ denotes a generalized hypergeometric series (*cf.* [13]). This series terminates because of the $-k_2$ appearing as a "numerator" parameter, and in fact we may write

$$(1.9) \qquad \begin{aligned} &{}_4F_3(a', b', 1-c-k_2,\ -k_2;\ c', 1-a-k_2,\ 1-b-k_2;\ 1) \\ &= \sum_{\ell=0}^{k_2} \binom{k_2}{\ell} \frac{(a')_\ell\,(b')_\ell\,(1-c-k_2)_\ell}{(c')_\ell\,(1-a-k_2)_\ell\,(1-b-k_2)_\ell} (-1)^\ell. \end{aligned}$$

We now proceed to the proofs of our theorems.

2. Proof of Theorem 1.

Let us rewrite the statement of Theorem 1 according to the remark immediately following that theorem: we get

$$(2.1) \qquad \frac{1}{2\pi i} \int_{|u|=1} (1+1/u)^{x-1}(1+u)^{y-1}\,\frac{du}{u} = \frac{\Gamma(x+y-1)}{\Gamma(x)\Gamma(y)}$$

for $\mathrm{Re}(x+y-1) > 0$. We will prove (2.1) initially under the assumption that $\mathrm{Re}(x-1),\ \mathrm{Re}(y-1) > 0$. The principle of analytic continuation will yield the complete result because, as

$$(1+1/u)^{x-1}(1+u)^{y-1} = u^{y-1}(1+1/u)^{x+y-2},$$

both sides of equation (2.1) are seen to be holomorphic in x, y for $\mathrm{Re}\,(x+y-1) > 0$.

To prove equation (2.1), we recall the binomial theorem

$$(1-r)^a = \sum_{k=0}^{\infty} \frac{(-a)_k}{k!}\, r^k.$$

The radius of convergence of this series, for an arbitrary complex number a, equals one. Moreover one checks (using, for example, Stirling's formula [18] for the gamma function or Raabe's test [1]),that $\mathrm{Re}\,(a) > 0$ is sufficient for absolute convergence on the boundary $|r| = 1$.

Assuming that $\operatorname{Re}(x-1)$, $\operatorname{Re}(y-1>0$, we may then apply the binomial theorem to each of the factors $(1+1/u)^{x-1}$ and $(1+u)^{y-1}$ in our integrand. We find

$$\frac{1}{2\pi i}\int_{|u|=1}(1+1/u)^{x-1}(1+u)^{y-1}\,\frac{du}{u}$$

$$=\frac{1}{2\pi i}\int_{|u|=1}\left[\sum_{k_1=0}^{\infty}\frac{(1-x)_{k_1}}{k_1!}\,(-u)^{-k_1}\right]\left[\sum_{k_2=0}^{\infty}\frac{(1-y)_{k_2}}{k_2!}\,(-u)^{k_2}\right]\frac{du}{u}$$

$$=\sum_{k_1,k_2=0}^{\infty}\frac{(1-x)_{k_1}\,(1-y)_{k_2}}{k_1!\,k_2!}\left\{\frac{1}{2\pi i}\int_{|u|=1}(-u)^{-k_1+k_2}\,\frac{du}{u}\right\}.$$

The interchange of integration and summation is justified by the fact that each integral in braces is bounded by 1, and by the absolute convergence (again, by Stirling's formula or Raabe's test) of

$$\sum_{k_1,k_2=0}^{\infty}\frac{(1-x)_{k_1}\,(1-y)_{k_2}}{k_1!\,k_2!}.$$

Since, for integers k_1,k_2,

$$\frac{1}{2\pi i}\int_{|u|=1}(-u)^{-k_1+k_2}\,\frac{du}{u}=\delta_{k_1,k_2},$$

we have

$$\frac{1}{2\pi i}\int_{|u|=1}(1+1/u)^{x-1}(1+u)^{y-1}\,\frac{du}{u}$$

$$=\sum_{k=0}^{\infty}\frac{(1-x)_k\,(1-y)_k}{k!\,k!}\;=\;{}_2F_1(1-x,1-y;1;1).$$

Here

$${}_2F_1(\alpha,\beta;\gamma;z)=\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k}{k!\,(\gamma)_k}\,z^k$$

is the hypergeometric series or Gauss function. But the Gauss summation theorem (*cf.* [13]) gives

$${}_2F_1(\alpha,\beta;\gamma;1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\beta-\alpha)},$$

provided the parameters of the series satisfy $\operatorname{Re}(\gamma-\alpha-\beta)>0$. If $\operatorname{Re}(x-1),\operatorname{Re}(y-1)>0$, then $\operatorname{Re}(1-(1-x)-(1-y))=\operatorname{Re}(x+y-1)>0$, so by (2.2)

$$\frac{1}{2\pi i}\int_{|u|=1}(1+1/u)^{x-1}(1+u)^{y-1}\,\frac{du}{u}$$

$$=\frac{\Gamma(1)\Gamma(1-(1-x)-(1-y))}{\Gamma(1-(1-x))\Gamma(1-(1-y))}=\frac{\Gamma(x+y-1)}{\Gamma(x)\Gamma(y)}.$$

This proves equation (2.1) (under the assumption $\operatorname{Re}(x-1)>0$, $\operatorname{Re}(y-1>0)$; as already noted, Theorem 1 follows.

3. Proof of Theorem 2.

By equations (1.1), (1.5), and (1.6), we have

$$M_{(\nu_1,\nu_2)}(y_1,y_2) = \Gamma((3\nu_1+1)/2)\,\Gamma((3\nu_2+1)/2)\,\Gamma((3\nu_1+3\nu_2)/2)$$

$$\cdot \sum_{k_1,k_2=0}^{\infty}\left[\frac{(\pi y_1)^{2k_1+\nu_1+2\nu_2}}{k_1!\,\Gamma(k_1+\frac{3\nu_1+3\nu_2}{2})}\right]\left[\frac{(\pi y_2)^{2k_2+2\nu_1+\nu_2}}{k_2!\,\Gamma(k_2+\frac{3\nu_1+3\nu_2}{2})}\right]$$

$$\cdot\left[\frac{\Gamma(k_1+k_2+\frac{3\nu_1+3\nu_2}{2})}{\Gamma(k_1+\frac{3\nu_2+1}{2})\,\Gamma(k_2+\frac{3\nu_1+1}{2})}\right].$$

To the third expression in brackets, we may apply Theorem 1. Putting $x=k_1+(3\nu_2+1)/2$ and $y=k_2+(3\nu_1+1)/2$, our theorem tells us that this bracketed expression is equal to

$$\frac{1}{2\pi i}\int_{|u|=1}(1+1/u)^{k_1+(3\nu_2-1)/2}\,(1+u)^{k_2+(3\nu_1-1)/2}\,\frac{du}{u}.$$

Since

$$(\pi y_1)^{2k_1+\nu_1+2\nu_2}(\pi y_2)^{2k_2+2\nu_1+\nu_2}$$

$$=\pi^2\,y_1^{1-(\nu_1-\nu_2)/2}\,y_2^{1+(\nu_1-\nu_2)/2}(\pi y_1)^{2k_1+(3\nu_1+3\nu_2-2)/2}(\pi y_2)^{2k_2+(3\nu_1+3\nu_2-2)/2},$$

our most recent expression for $M_{(\nu_1,\nu_2)}(y_1,y_2)$ becomes

$$M_{(\nu_1,\nu_2)}(y_1,y_2)$$

$$=\pi^2\,y_1^{1-(\nu_1-\nu_2)/2}\,y_2^{1+(\nu_1-\nu_2)/2}\prod_{j=1}^{2}\prod_{k=1}^{3-j}\Gamma\left(\frac{3\nu_k+\cdots+3\nu_{k+j-1}-(j-2)}{2}\right)$$

$$\cdot\sum_{k_1,k_2=0}^{\infty}\left[\frac{(\pi y_1)^{2k_1+(3\nu_1+3\nu_2-2)/2}}{k_1!\,\Gamma(k_1+\frac{3\nu_1+3\nu_2}{2})}\right]\left[\frac{(\pi y_2)^{2k_2+(3\nu_1+3\nu_2-2)/2}}{k_2!\,\Gamma(k_2+\frac{3\nu_1+3\nu_2}{2})}\right]$$

$$\cdot\frac{1}{2\pi i}\int_{|u|=1}(1+1/u)^{k_1+(3\nu_2-1)/2}\,(1+u)^{k_2+(3\nu_1-1)/2}\,\frac{du}{u}.$$

We now wish to interchange integration and summation. To justify this, note the following:

$$\left|\int_{|u|=1}(1+1/u)^{k_1+(3\nu_2-1)/2}\,(1+u)^{k_2+(3\nu_1-1)/2}\,\frac{du}{u}\right|$$

$$\leq 2^{k_1+k_2}\int_{|u|=1}\left|(1+1/u)^{(3\nu_2-1)/2}\,(1+u)^{(3\nu_1-2)/2}\,\frac{du}{u}\right|,$$

since $|1+1/u|, |1+u| \le 2$. The arguments of $1+1/u$ and $1+u$ are bounded, so the latter integral is

$$\ll \int_{|u|=1} |1+1/u|^{\operatorname{Re}((3\nu_2-1)/2)} |1+u|^{\operatorname{Re}((3\nu_1-1)/2)} \left|\frac{du}{u}\right|$$

(the implied constant depending on ν_1, ν_2). Writing $u = e^{i\theta}$, the integral directly above becomes

$$\int_0^{2\pi} |2+2\cos\theta|^{\operatorname{Re}((3\nu_1+3\nu_2-2)/4)}\, d\theta = \int_0^{2\pi} |2\cos(\theta/2)|^{\operatorname{Re}((3\nu_1+3\nu_2-2)/2)}.$$

The last integral converges for $\operatorname{Re}(\nu_1+\nu_2) > 0$, so all told we find

$$\left|\int_{|u|=1} (1+1/u)^{k_1+(3\nu_2-1)/2} (1+u)^{k_2+(3\nu_1-1)/2} \frac{du}{u}\right| \ll 2^{k_1+k_2}.$$

Substituting this estimate into the above expression for $M_{(\nu_1,\nu_2)}(y_1, y_2)$, we are left with a series that is easily seen (e.g. by the ratio test) to converge absolutely. Thus our interchange is allowed.

We therefore have

$$\begin{aligned}
&M_{(\nu_1,\nu_2)}(y_1,y_2) \\
&= \pi^2\, y_1^{1-(\nu_1-\nu_2)/2}\, y_2^{1+(\nu_1-\nu_2)/2} \prod_{j=1}^{2}\prod_{k=1}^{3-j} \Gamma\left(\frac{3\nu_k+\cdots+3\nu_{k+j-1}-(j-2)}{2}\right) \\
&\quad\cdot \frac{1}{2\pi i}\int_{|u|=1} \sum_{k_1,k_2=0}^{\infty} \left[\frac{(\pi y_1)^{2k_1+(3\nu_1+3\nu_2-2)/2}}{k_1!\,\Gamma(k_1+\frac{3\nu_1+3\nu_2}{2})}\right] \left[\frac{(\pi y_2)^{2k_2+(3\nu_1+3\nu_2-2)/2}}{k_2!\,\Gamma(k_2+\frac{3\nu_1+3\nu_2}{2})}\right] \\
&\quad\cdot \left(\sqrt{1+1/u}\right)^{2k_1+3\nu_2-1} \left(\sqrt{1+u}\right)^{2k_2+3\nu_1-1} \frac{du}{u} \\
&= \pi^2\, y_1^{1-(\nu_1-\nu_2)/2}\, y_2^{1+(\nu_1-\nu_2)/2} \prod_{j=1}^{2}\prod_{k=1}^{3-j} \Gamma\left(\frac{3\nu_k+\cdots+3\nu_{k+j-1}-(j-2)}{2}\right) \\
&\quad\cdot \frac{1}{2\pi i}\int_{|u|=1} \sum_{k_1,k_2=0}^{\infty} \left[\frac{\left(\pi y_1\sqrt{1+1/u}\right)^{2k_1+(3\nu_1+3\nu_2-2)/2}}{k_1!\,\Gamma(k_1+\frac{3\nu_1+3\nu_2}{2})}\right] \\
&\quad\cdot \left[\frac{\left(\pi y_2\sqrt{1+u}\right)^{2k_2+(3\nu_1+3\nu_2-2)/2}}{k_2!\,\Gamma(k_2+\frac{3\nu_1+3\nu_2}{2})}\right] (1+1/u)^{(3\nu_2-3\nu_1)/4}(1+u)^{(3\nu_1-3\nu_2)/4}\frac{du}{u} \\
&= \pi^2\, y_1^{1-(\nu_1-\nu_2)/2}\, y_2^{1+(\nu_1-\nu_2)/2} \prod_{j=1}^{2}\prod_{k=1}^{3-j} \Gamma\left(\frac{3\nu_k+\cdots+3\nu_{k+j-1}-(j-2)}{2}\right) \\
&\quad\cdot \frac{1}{2\pi i}\int_{|u|=1} I_{(3\nu_1+3\nu_2-2)/2}\left(2\pi y_1\sqrt{1+1/u}\right) \\
&\quad\cdot I_{(3\nu_1+3\nu_2-2)/2}\left(2\pi y_2\sqrt{1+u}\right) u^{(3\nu_1-3\nu_2)/4}\,\frac{du}{u},
\end{aligned}$$

by the definition of the I–Bessel function (*cf.* (1.4)) and the fact that

$$(1+u)\,(1+1/u)^{-1} = u.$$

This concludes the proof of Theorem 2.

4. Proof of Theorem 3.

From equations (1.1) and (1.7), we see that

$$M_{(\nu_1,\nu_2,\nu_3)}(y_1,y_2,y_3) \tag{4.1}$$
$$= \sum_{k_1,k_2,k_3=0}^{\infty} \left[\frac{(-1)^{k_1+k_3}\,(a)_{k_2}\,(b)_{k_2}}{k_1!\,k_2!\,k_3!\,(a)_{k_1}\,(b)_{k_3}\,(c)_{k_2}\,(a+a')_{k_2}\,(b+b')_{k_2}} \right]$$
$$\cdot {}_4F_3(a',b',1-c-k_2,\,-k_2;\,c',1-a-k_2,\,1-b-k_2;\,1)$$
$$\cdot (\pi y_1)^{2k_1+\nu_1+2\nu_2+3\nu_3}(\pi y_2)^{2k_2+2\nu_1+4\nu_2+2\nu_3}(\pi y_3)^{2k_3+3\nu_1+2\nu_2+\nu_3}.$$

We wish to examine the sum in k_2. We note that, in this section, all interchanges of sums and integrals with other sums and integrals may be justified using arguments similar to those employed in §3.

So let us write, according to equations (4.1) and (1.9),

$$M_{(\nu_1,\nu_2,\nu_3)}(y_1,y_2,y_3) \tag{4.2}$$
$$= \sum_{k_1,k_3=0}^{\infty} \frac{(-1)^{k_1+k_3}\,(\pi y_1)^{2k_1+\nu_1+2\nu_2+3\nu_3}(\pi y_3)^{2k_3+3\nu_1+2\nu_2+\nu_3}}{k_1!\,k_3!\,(a)_{k_1}\,(b)_{k_3}}$$
$$\cdot \sum_{k_2=0}^{\infty} \frac{(a)_{k_2}\,(b)_{k_2}\,(\pi y_2)^{2k_2+2\nu_1+4\nu_2+2\nu_3}}{k_2!\,(c)_{k_2}\,(a+a')_{k_2}\,(b+b')_{k_2}}$$
$$\cdot \sum_{\ell=0}^{k_2} \binom{k_2}{\ell} \frac{(a')_\ell\,(b')_\ell\,(1-c-k_2)_\ell}{(c')_\ell\,(1-a-k_2)_\ell\,(1-b-k_2)_\ell}\,(-1)^\ell.$$

We recall that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \tag{4.3}$$

so that, for e any complex number and $0 \le \ell \le k_2$,

$$\frac{(1-e-k_2)_\ell}{(e)_{k_2}} = \frac{\Gamma(1-e-k_2+\ell)\,\Gamma(e)}{\Gamma(1-e-k_2)\Gamma(e+k_2)} = \frac{\sin \pi(e+k_2)\,\Gamma(e)}{\Gamma(k_2-\ell+e)\,\sin \pi(k_2-\ell+e)}$$
$$= \frac{(-1)^{k_2}\sin \pi(e)\,\Gamma(e)}{\Gamma(k_2-\ell+e)\,(-1)^{k_2-\ell}\sin \pi(e)} = \frac{(-1)^\ell\Gamma(e)}{\Gamma(k_2-\ell+e)} = \frac{(-1)^\ell}{(e)_{k_2-\ell}}.$$

(The above calculations are valid for e not a negative integer or zero; by analytic continuation, the result is true for all e.) Let us apply this equation, with $e = a; b; c$ respectively, to the inner double sum (in $k_2; \ell$) that appears in (4.2). Writing $m = k_2 - \ell$, this sum becomes

$$\sum_{\ell,m=0}^{\infty} \frac{(a')_\ell\,(b')_\ell\,(a)_m\,(b)_m\,(\pi y_2)^{2\ell+2m+2\nu_1+4\nu_2+2\nu_3}}{\ell!\,m!\,(c')_\ell\,(c)_m\,(a+a')_{\ell+m}\,(b+b')_{\ell+m}},$$

so that (4.2) may now be written

$$M_{(\nu_1,\nu_2,\nu_3)}(y_1,y_2,y_3) \tag{4.4}$$

$$= \sum_{k_1,k_3,\ell,m=0}^{\infty} \left[\frac{(-1)^{k_1}(a')_\ell\,(a)_m}{(a)_{k_1}\,(a+a')_{\ell+m}}\right] \left[\frac{(-1)^{k_3}(b')_\ell\,(b)_m}{(b)_{k_3}\,(b+b')_{\ell+m}}\right] \cdot \left[\frac{(\pi y_1)^{2k_1+\nu_1+2\nu_2+3\nu_3}\,(\pi y_2)^{2\ell+2m+2\nu_1+4\nu_2+2\nu_3}\,(\pi y_3)^{2k_3+3\nu_1+2\nu_2+\nu_3}}{k_1!\,k_3!\,\ell!\,m!\,(c')_\ell\,(c)_m}\right].$$

But from (4.3), we find that

$$\frac{(-1)^{k_1}(a')_\ell\,(a)_m}{(a)_{k_1}\,(a+a')_{\ell+m}} = \frac{(-1)^{k_1}\Gamma(\ell+a')\,\Gamma(m+a)\,\Gamma(a+a')}{\Gamma(a')\,\Gamma(k_1+a)\,\Gamma(\ell+m+a+a')}$$

$$= \frac{(-1)^{k_1}\Gamma(a+a')\pi}{\Gamma(a')\,\Gamma(k_1+a)\sin\pi(m+a)} \left[\frac{\Gamma(\ell+a')}{\Gamma(1-a-m)\,\Gamma(\ell+m+a+a')}\right]$$

$$= \frac{(-1)^{m}\Gamma(a+a')\Gamma(1-a-k_1)}{\Gamma(a')} \left[\frac{\Gamma(\ell+a')}{\Gamma(1-a-m)\,\Gamma(\ell+m+a+a')}\right]$$

$$= \frac{(-1)^{m}\Gamma(a+a')\Gamma(1-a-k_1)}{2\pi i\,\Gamma(a')} \int_{|u_1|=1} (1+1/u_1)^{-a-m}(1+u_1)^{\ell+m+a+a'-1}\frac{du_1}{u_1},$$

the last equality following from Theorem 1 and the remark immediately below it. (The application of Theorem 1 is valid because our assumption $\operatorname{Re}(\rho+1) > 0$ implies $\operatorname{Re}(\ell+a') > 0$.) Similarly,

$$\frac{(-1)^{k_3}(b')_\ell\,(b)_m}{(b)_{k_3}\,(b+b')_{\ell+m}}$$

$$= \frac{(-1)^{m}\Gamma(b+b')\Gamma(1-b-k_3)}{2\pi i\,\Gamma(b')} \int_{|u_2|=1} (1+1/u_2)^{-b-m}(1+u_2)^{\ell+m+b+b'-1}\frac{du_2}{u_2},$$

so that (4.4) reads

$$M_{(\nu_1,\nu_2,\nu_3)}(y_1,y_2,y_3)=\frac{1}{(2\pi i)^2}\sum_{k_1,k_3,\ell,m=0}^{\infty} \tag{4.5}$$

$$\cdot\left[\frac{(\pi y_1)^{2k_1+\nu_1+2\nu_2+3\nu_3}(\pi y_2)^{2\ell+2m+2\nu_1+4\nu_2+2\nu_3}(\pi y_3)^{2k_3+3\nu_1+2\nu_2+\nu_3}}{k_1!\,k_3!\,\ell!\,m!\,(c')_\ell\,(c)_m}\right]$$

$$\cdot\left\{\frac{\Gamma(a+a')\Gamma(1-a-k_1)}{\Gamma(a')}\int_{|u_1|=1}(1+1/u_1)^{-a-m}(1+u_1)^{\ell+m+a+a'-1}\frac{du_1}{u_1}\right\}$$

$$\cdot\left\{\frac{\Gamma(b+b')\Gamma(1-b-k_3)}{\Gamma(b')}\int_{|u_2|=1}(1+1/u_2)^{-b-m}(1+u_2)^{\ell+m+b+b'-1}\frac{du_2}{u_2}\right\}$$

or, by (1.8) and some rearranging:

$$M_{(\nu_1,\nu_2,\nu_3)}(y_1,y_2,y_3)=(\pi y_1)^{(3/2)-\nu_1+\nu_3}(\pi y_2)^2(\pi y_3)^{(3/2)+\nu_1-\nu_3} \tag{4.6}$$

$$\cdot\Gamma(2\nu_1+1/2)\Gamma(2\nu_2+1/2)\Gamma(2\nu_3+1/2)\Gamma(2\nu_1+2\nu_2)\Gamma(2\nu_2+2\nu_3)$$

$$\cdot\Gamma(2\nu_1+2\nu_2+2\nu_3-1/2)\frac{1}{(2\pi i)^2}$$

$$\cdot\int_{|u_1|=1}\int_{|u_2|=1}\sum_{k_1,k_3,k,m=0}^{\infty}\left[\frac{(\pi y_1)^{2k_1+\rho}(\pi y_2)^{(2\ell+\rho)+(2m+2\nu_2-1/2)}(\pi y_3)^{2k_3+\rho}}{k_1!\,k_3!\,\ell!\,m!\,\Gamma(\ell+\rho+1)\,\Gamma(m+2\nu_2+1/2)}\right]$$

$$\cdot\left[\frac{(1+1/u_1)^{k_1-m+2\nu_3-1/2}(1+u_1)^{\ell+m+2\nu_1+2\nu_2-1}}{\Gamma(k_1+\rho+1)}\right]$$

$$\cdot\left[\frac{(1+1/u_2)^{k_3-m+2\nu_1-1/2}(1+u_2)^{\ell+m+2\nu_2+2\nu_3-1}}{\Gamma(k_3+\rho+1)}\right]\frac{du_1du_2}{u_1u_2}$$

$$=\pi^5y_1^{(3/2)-\nu_1+\nu_3}y_2^2y_3^{(3/2)+\nu_1-\nu_3}\prod_{j=1}^{3}\prod_{k=1}^{4-j}\Gamma\left(\frac{4\nu_k+\cdots+4\nu_{k+j-1}-(j-2)}{2}\right)$$

$$\cdot\frac{1}{(2\pi i)^2}\int_{|u_1|=1}\int_{|u_2|=1}\left[\sum_{k_1=0}^{\infty}\frac{\left(\pi y_1\sqrt{1+1/u_1}\right)^{2k_1+\rho}}{k_1!\,\Gamma(k_1+\rho+1)}\right]$$

$$\cdot\left[\sum_{\ell=0}^{\infty}\frac{\left(\pi y_2\sqrt{(1+u_1)(1+u_2)}\right)^{2\ell+\rho}}{\ell!\,\Gamma(\ell+\rho+1)}\right]$$

$$\cdot\left[\sum_{k_3=0}^{\infty}\frac{\left(\pi y_3\sqrt{1+1/u_2}\right)^{2k_3+\rho}}{k_3!\,\Gamma(k_3+\rho+1)}\right]\left[\sum_{m=0}^{\infty}\frac{\left(\pi y_2\sqrt{u_1u_2}\right)^{2m+2\nu_2-1/2}}{m!\,\Gamma(m+2\nu_2+1/2)}\right]$$

$$\cdot u_1^{\nu_1-\nu_3}\,u_2^{\nu_3-\nu_1}\frac{du_1du_2}{u_1u_2}=\pi^5y_1^{(3/2)-\nu_1+\nu_3}y_2^2y_3^{(3/2)+\nu_1-\nu_3}$$

$$\cdot \prod_{j=1}^{3} \prod_{k=1}^{4-j} \Gamma\left(\frac{4\nu_k + \cdots + 4\nu_{k+j-1} - (j-2)}{2} \right)$$

$$\cdot \frac{1}{(2\pi i)^2} \int_{|u_1|=1} \int_{|u_2|=1} I_\rho\big(2\pi y_1 \sqrt{1 + 1/u_1}\,\big) I_\rho\big(2\pi y_2 \sqrt{(1+u_1)(1+1/u_2)}\,\big)$$

$$\cdot I_\rho\big(2\pi y_3 \sqrt{1+u_2}\,\big) \cdot I_{2\nu_2 - 1/2}\big(2\pi y_2 \sqrt{u_1/u_2}\,\big)\, u_1^{\nu_1 - \nu_3}\, u_2^{\nu_1 - \nu_3}\, \frac{du_1 du_2}{u_1 u_2}.$$

(At the last step, we have applied the definition (1.4) of the I–Bessel function, and the change of variable $u_2 \to u_2^{-1}$.) This completes the proof of Theorem 3.

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