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BERNSTEIN'S INEQUALITY ON ALGEBRAIC CURVES

by Charles FEFFERMAN and Raghavan NARASIMHAN

Dedicated to Bernard Malgrange

Introduction.

In this paper, we estimate the growth of polynomials on a smooth algebraic curve Γ in the plane. Let $\Gamma = \{(x,y) \in \mathbb{R}^2 : y = \psi(x) \text{ and } |x| \leq 1\}$ where $Q(x,\psi(x)) = 0, Q(x,y)$ being a polynomial with real coefficients. To control Γ, we make the following :

Assumptions.

(I) $\quad |\psi(x)| \leq 1$ for $|x| \leq 1$

(II) $\quad Q(x,y)$ has degree at most D.

(III) $\quad |Q(x,y)| \leq C$ for $|x|, |y| \leq 1$.

(IV) $\quad \left|\frac{\partial Q}{\partial y}(x,y)\right| \geq c > 0$ for $(x,y) \in \Gamma$.

Under these assumptions, we prove :

THEOREM 1. — *Let $P(x,y)$ be a polynomial of degree d and $f(x) = P(x,\psi(x))$. Then, there exists a constant C_* depending only on d, D, C, c, such that the following hold :*

(A) $\quad \max_{|x|\leq 1} |f(x)| \leq C_* \max_{|x|\leq \frac{1}{2}} |f(x)|$;

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(B) $\max_{|x|\leq 1}|f'(x)| \leq C_* \max_{|x|\leq 1}|f(x)|$ (Bernstein's inequality);

(C) $\max_{|x|\leq 1}|f(x)| \leq C_* \int_{-1}^{1}|f(x)|dx.$

Thus, f behaves like a polynomial of one variable. The point of Theorem 1 is to control the dependence of the constant C_* on the curve Γ. For fixed Γ, estimates (A), (B), (C) are obvious consequences of the fact that any two norms on a finite dimensional vector space are equivalent.

Theorem 1 was conjectured by A. Parmeggiani; he uses it in his analysis of pseudodifferential operators $[P]$.

It is tempting to make the following conjecture from which Theorem 1 would follow easily : Let $P(x,y)$ be a polynomial of degree at most d, and let Γ satisfy (I), (II), (III), (IV). Then there exists another polynomial $\tilde{P}(x,y)$ of degree at most d_*, such that $P=\tilde{P}$ on Γ, and $\max_{|x|,|y|\leq 1}|\tilde{P}(x,y)| \leq C_* \max_{\Gamma}|P|$. Here, d_* and C_* depend only on d, D, C, c. Unfortunately, this conjecture is disproved by elementary examples. For instance, take

$$\Gamma_\varepsilon = \{(x,y)\in\mathbb{R}^2 : |x|,|y|\leq 1 \text{ and } \ y(1+x^2+y^2)-\varepsilon=0\}$$

for small nonzero ε. The polynomial $P_\varepsilon = \dfrac{y}{\varepsilon}$ satisfies

$$(*) \qquad P_\varepsilon = \frac{1}{1+x^2+y^2} \text{ on } \Gamma_\varepsilon,$$

so that $P_\varepsilon|\Gamma_\varepsilon$ is bounded uniformly in ε. Suppose we could extend $P_\varepsilon|\Gamma_\varepsilon$ to a polynomial $Q_\varepsilon(x,y)$ whose degree and coefficients are bounded uniformly in ε. Then for a sequence $\varepsilon_\nu \to 0$, we would have $Q_{\varepsilon_\nu} \to Q$ uniformly, for a polynomial $Q(x,y)$. Since Γ_ε approaches the x-axis, $(*)$ implies that $Q(x,0) = \dfrac{1}{1+x^2}$, which contradicts the fact that Q is a polynomial.

The above counterexample suggests the cure for the false conjecture : instead of extending $P|_\Gamma$ to a polynomial, we should look for a rational function F/G, with $F(x,y), G(x,y)$ of degree at most d_*, such that $F/G = P$ on Γ, $c_* < G < C_*$ on $\{|x|,|y|\leq 1\}$ and $\max_{|x|,|y|\leq 1}|F| \leq C_* \max_{\Gamma}|P|$. We give the precise formulation of our extension theorem in the next section, and then we present its proof. Finally, we derive Theorem 1 from the extension theorem. Note that we formulate the extension theorem for hypersurfaces in $\mathbb{R}^n$. The proof is no harder than the special case $n=2$ that we use for Theorem 1. We have recently proven an extension theorem in higher codimension.

We are happy to dedicate this paper to Bernard Malgrange. He has influenced us in many ways. Moreover, parts of this paper are inspired by his work.

1. The Extension theorem.

Let Q_ρ denote the cube $\{(x_1, ..., x_n) \in \mathbb{R}^n : |x_j| \leq \rho \text{ for } j = 1, 2, \ldots, n\}$.

THEOREM. — *Let p be a polynomial in n variables, having real coefficients and of degree at most D. Let $V = \{x \in \mathbb{R}^n : p(x) = 0\}$. Assume that $p(0) = 0$, $|\bigtriangledown p(0)| > c > 0$, and $\max_{Q_1} |p| \leq C$.*

Then, there exist constants $\rho_1 > 0, C' > 0$ and $D' > 0$ depending only on the constants n, c, C, D appearing in the assumptions on p such that the following holds :

Given a polynomial f on $\mathbb{R}^n$ and a number $\rho \in (0, \rho_1]$, we can find polynomials F, G of degree at most D' satisfying the following conditions:

(a) $f = F/G$ *on* $V \cap Q_{2\rho}$;

(b) $\frac{1}{2} \leq G \leq 2$ *on* $Q_{2\rho}$;

(c) $\max_{Q_{2\rho}} |F| \leq C' \max_{V \cap Q_\rho} |f|$;

2. Notation and definitions.

Let D be an integer, and let $c_1 \in (0, 1)$ be given. We fix these parameters throughout the paper. Define H^d = the vector space of all polynomials of degree at most d on $\mathbb{R}^n$, with real coefficients. Define

$$\| f \| = \left(\sum_{|\alpha| \leq d} |f_\alpha|^2 \right)^{\frac{1}{2}} \text{ for } f = \sum_{|\alpha| \leq d} f_\alpha x^\alpha \in H^d$$

$$W = \{p \in H^D : \| p \| = 1, p(0) = 0, |\bigtriangledown p(0)| \geq c_1\}$$

$$Q_\rho = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x_j| \leq \rho \text{ for } j = 1, \ldots, n\}$$

as in the extension theorem. Given $f \in W$, we can factor f as $f = pq$ with p irreducible in $\mathbb{R}[x_1, \ldots, x_n], p(0) = 0$ and $q(0) = 1$. This uniquely determines p and q in terms of f. We define mainfac(f) to be p, and we define otherfac(f) to be q for the above factorization of f. Note that mainfac(f) is irreducible in $\mathbb{C}[x_1,\dots,x_n]$, as well as in $\mathbb{R}[x_1,\dots,x_n]$. If $f \in W$ and $p = \text{mainfac}(f)$, then define

$$V(f) = \{x \in \mathbb{R}^n : p(x) = 0\}.$$

The *type* of $f \in W$ is defined as the degree of mainfac (f). Set

$$W^t = \{f \in W : f \text{ has type } t\}, \text{ and}$$
$$W^{t+} = \{f \in W : f \text{ has type } \geq t\}.$$

Suppose $p \in W, q \in H^{D'}$ and $\rho > 0$ are given. We define Norm(p,q,D',ρ) to be the least constant C' such that for any $f \in H^D$ there exists an $\tilde{f} \in H^{D'}$ that satisfies

(a) $\tilde{f} = qf$ on $V(p)$, and

(b) $\| \tilde{f} \| \leq C' \max_{Q_\rho \cap V(p)} |f|$.

If no such constant C' exists, then we define Norm$(p,q,D',\rho) = +\infty$.

Next suppose $p \in W, C', D', \rho > 0$ are given. We define Norm$_*(p,D',C',\rho)$ to be the infimum of Norm(p,q,D',ρ) over all $q \in H^{D'}$ that satisfy $q(0) = 1, \| q \| \leq C'$.

3. Preliminary observations.

LEMMA 3.1. — *Let $\varphi_1,\dots,\varphi_s \in H^D$, and let $E \subset \mathbb{R}^n$. Assume that $\varphi_1,\dots,\varphi_s$ are linearly independent as functions on E. Then we can find points $x_1,\dots,x_s \in E$ such that*

(1)

$$\sum_{1\leq j\leq s} |\xi_j| \leq C \cdot \max_{1\leq k\leq s} \left| \sum_{1\leq j\leq s} \xi_j \varphi_j(x_k) \right| \text{ for any real numbers } \xi_1,\dots,\xi_s.$$

Moreover, suppose $\tilde{\varphi}_1,\dots,\tilde{\varphi}_s \in H^D$ and $\tilde{x}_1,\dots,\tilde{x}_s \in \mathbb{R}^n$ are given, and assume that

(2) $$\|\varphi_j - \tilde{\varphi}_j\|, |x_j - \tilde{x}_j| < c.$$

Then we have the estimate

(3)

$$\sum_{1\leq j\leq s} |\xi_j| \leq C' \max_{1\leq k\leq s} \left| \sum_{1\leq j\leq s} \xi_j \tilde{\varphi}_j(\tilde{x}_k) \right| \text{ for any real numbers } \xi_1,\dots,\xi_s.$$

Here, c and C' depend only on E, $\varphi_1, \ldots, \varphi_s$, D , n. In particular, they are independent of $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_s$, $\tilde{x}_1, \ldots, \tilde{x}_s$, $\xi_1, \ldots, \xi_s$.

Proof. — By induction on $\nu (1 \leq \nu \leq s)$ we can find $x_\nu \in E$ so that the vector space $\{\varphi \in \operatorname{span}(\varphi_1, \ldots, \varphi_s) : \varphi(x_1) = \ldots = \varphi(x_\nu) = 0\}$ has dimension $s - \nu$. Hence the linear map

$$T : (\xi_1, \ldots, \xi_s) \in \mathbb{R}^s \mapsto \left(\sum_{1 \leq j \leq s} \xi_j \varphi_j(x_\nu) \right)_{1 \leq \nu \leq s} \in \mathbb{R}^s$$

is one-to-one and hence an isomorphism. This implies (1), since all norms on a finite-dimensional vector space are equivalent.

Now suppose $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_s$ and $\tilde{x}_1, \ldots, \tilde{x}_s$ are given. Introduce the linear map

$$\tilde{T} : (\xi_1, \ldots, \xi_s) \in \mathbb{R}^s \mapsto \left(\sum_{1 \leq j \leq s} \xi_j \tilde{\varphi}_j(\tilde{x}_\nu) \right)_{1 \leq \nu \leq s} \in \mathbb{R}^s.$$

If $\|\tilde{\varphi}_j - \varphi_j\|$ and $|\tilde{x}_\nu - x_\nu|$ are small enough, then $\tilde{T}$ is a small perturbation of T. Therefore, $\|\tilde{T}\| < C$ and $|\det(\tilde{T})| > c$, with C and c depending only on T. Hence $\|(\tilde{T})^{-1}\| < C'$ with C' depending only on T. In other words, C' depends only on $\varphi_1, \ldots, \varphi_s$, $x_1, \ldots, x_s$. The $x_1, \ldots, x_s$ were specified in terms of $\varphi_1, \ldots, \varphi_s, E$, therefore C' depends only on $\varphi_1, \ldots, \varphi_s, E$. We have shown that

$$\|\tilde{\varphi}_j - \varphi_j\|, |\tilde{x}_\nu - x_\nu| < c' \text{ implies } \|\tilde{T}^{-1}\| \leq C', \tag{4}$$

with C' depending only on $\varphi_1, \ldots, \varphi_s$, E; and with c' depending only on $\varphi_1, \ldots, \varphi_s$, $x_1, \ldots, x_s$, D, n. Again, since $x_1, \ldots, x_s$ were specified in terms of $\varphi_1, \ldots, \varphi_s$, E, it follows that c' depends only on $\varphi_1, \ldots, \varphi_s$, E, D,n.

Estimate (3) is immediate from (4).

LEMMA 3.2. — *Suppose $f \in W$ and F is a polynomial on $\mathbb{R}^n$. Assume that F vanishes on $Q_\rho \cap V(f)$ for some $\rho > 0$. Then F is a multiple of* $\operatorname{mainfac}(f)$. *In particular F vanishes on all of $V(f)$.*

Proof. — Let $p = \operatorname{mainfac}(f)$. Thus, $p(0) = 0$ and $\nabla p(0) \neq 0$. Without loss of generality we may assume that $\dfrac{\partial p}{\partial x_n}(0) \neq 0$. Hence in a small neighborhood of the origin, $V = \{z \in \mathbb{C}^n : p(z) = 0\}$ is the graph $\{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_n = G(z_1, \ldots, z_{n-1})\}$ of an analytic function G. The restriction of that graph to $(z_1, \ldots, z_{n-1})$ real is the intersection of $V(f)$ with a small neighborhood of the origin. Hence, $F(z_1, \dots, z_{n-1}, G(z_1, \dots, z_{n-1}))$ is an analytic function that vanishes for real $(z_1, \dots, z_{n-1})$ near the origin. It follows that $F(z_1, \dots, z_{n-1}, G(z_1, \dots, z_{n-1})) = 0$ for complex $(z_1, \dots, z_{n-1})$ near the origin. Thus, F vanishes on some neighborhood of the origin in V. Since p is irreducible, the regular points of V form a connected complex manifold. So it follows by analytic continuation that F vanishes at all the regular points of V. The regular points of V are dense in V, and F is continuous on $\mathbb{C}^n$. Hence, F vanishes on V. Since p is irreducible, this implies that F is a multiple of p, $F = p\tilde{F}$ for some polynomial $\tilde{F}$. The proof of the lemma is complete.

LEMMA 3.3. — *Let $p \in W$, $q \in H^d$ and $\rho > 0$. If D' is an integer with $D + d \leq D'$, then* Norm $(p,q, D',\rho) < \infty$.

Proof. — Let $\mathcal{H}$ be the vector space H^D/multiples of mainfac(p). If $f \in H^D$ and $f = 0$ on $Q_\rho \cap V(p)$, then, by the preceding lemma, f is a multiple of mainfac(p). Hence, the function

$$f \mapsto \max_{Q_\rho \cap V(p)} |f|$$

is a norm on $\mathcal{H}$.

On the other hand, let $f_1,\dots,f_m \in H^D$ be such that their images in $\mathcal{H}$ form a basis of $\mathcal{H}$. The function

$$f \mapsto \sum_{k=1}^{m} |A_k| \|f_k q\|, \text{ where } f = \sum_{k=1}^{m} A_k f_k \bmod (\text{mainfac}\,(p))$$

defines a norm on $\mathcal{H}$ (if $q \neq 0$). Since two norms on a finite-dimensional vector space are equivalent, there is a constant C' with

$$(+) \qquad \sum_{k=1}^{m} |A_k| \|f_k q\| \leq C' \max_{Q_\rho \cap V(p)} |f|$$

whenever $f \in H^D$ and $f = A_1 f_1 + \dots + A_m f_m \bmod (\text{mainfac}\,(p))$.

To $f \in H^D$, we associate $\tilde{f} = \sum_{k=1}^{m} A_k f_k q$ where $A_1,\dots,A_m$ are as above. We then have $\tilde{f} = qf$ on $V(p)$ and $\|\tilde{f}\| \leq C' \max_{Q_\rho \cap V(p)} |f|$ by (+). This shows that Norm $(p, q, D', \rho) < \infty$, since $\tilde{f} \in H^{D'}$ (we have $\tilde{f} \in H^{D'}, q \in H^d, f_k \in H^D$ and $d + D \leq D'$). The proof of the lemma is complete.

COROLLARY. — *If $D' \geq D$ and $C' \geq 1$, then* $\text{Norm}_*(p, D', C', \rho) < \infty$ *for $p \in W$ and all $\rho > 0$.*

Proof. — We have Norm $(p, 1, D', \rho) < \infty$ by the preceding lemma. Clearly $q \equiv 1$ satisfies $q \in H^{D'}$, $q(0) = 1$ and $\|q\| \leq C'$.

4. General properties of the main factor.

LEMMA 4.1. — *If $f \in W$, then*

$$c < \|\text{mainfac}\,(f)\| < C, \textit{ and}$$

$$c < \|\text{otherfac}\,(f)\| < C.$$

Here, $c > 0$ and C depend only on c_1, D, n.

Proof. — Let $\varphi_1, \ldots, \varphi_s$ be a list of all the monomials of degree $\leq D$, and let E be the unit cube in $\mathbb{R}^n$. According to Lemma 3.1, we can find $x_1, \ldots, x_s \in E$ and $c_2 > 0$ depending only on D, n, with the following property : If $\tilde{x}_1, \ldots, \tilde{x}_s \in \mathbb{R}^n$ and $|\tilde{x}_\nu - x_\nu| \leq c_2$, then

$$(1) \qquad \|f\| \leq C_2 \max_{1 \leq \nu \leq s} |f(\tilde{x}_\nu)| \text{ for any } f \in H^D.$$

Here, C_2 depends only on D, n.

Now suppose $f \in W$ is given. We can factor f as

$$(2) \qquad f = pq \text{ with } p \text{ irreducible, } p(0) = 0,\ \|p\| = 1.$$

Since all norms on a finite-dimensional vector space are equivalent, we have, for each ν, the estimate $\|p\| \leq C_3 \max\{|p(\tilde{x}_\nu)| : |\tilde{x}_\nu - x_\nu| \leq c_2\}$, with C_3 depending only on D, n. Since our p satisfies $\|p\| = 1$, it follows that we can find $\tilde{x}_1, \ldots, \tilde{x}_s \in \mathbb{R}^n$ with $|\tilde{x}_\nu - x_\nu| \leq c_2$ and $|p(\tilde{x}_\nu)| \geq c_3$. Here, $c_3 > 0$ depends only on D, n.

Since $\|f\| = 1$, $x_\nu \in E$ and $|\tilde{x}_\nu - x_\nu| \leq c_2$, it follows that $|f(\tilde{x}_\nu)| \leq C_4$, with C_4 depending only on D, n. Therefore, $|q(\tilde{x}_\nu)| = |f(\tilde{x}_\nu)|/|p(\tilde{x}_\nu)| \leq C_4 c_3^{-1}$. Hence (1) applied to q yields

$$(3) \qquad \|q\| \leq C_2 C_4 c_3^{-1}.$$

On the other hand, since $f \in W$, we have $c_1 \leq |\nabla f(0)| = |q(0)||\nabla p(0)| \leq |q(0)| \cdot C_5 \|p\|$, with C_5 depending only on D, n. Recall that $\|p\| = 1$; this gives

$$(4) \qquad |q(0)| \geq c_4, \text{ with } c_4 > 0 \text{ depending only on } c_1, D, n.$$

From (3) and (4) we get

$$(5) \qquad c_6 \leq \|q\| \leq C_6, \text{ with } C_6, c_6 \text{ depending only on } c_1, D, n.$$

Comparing (2) with the definitions of mainfac, otherfac, we see that

$$\mathrm{mainfac}\,(f) = q(0)\cdot p, \mathrm{otherfac}\,(f) = (q(0))^{-1}\cdot q.$$

Hence from (4), (5) and $\|p\| = 1$, we obtain the estimates $c_7 \leq \|\mathrm{mainfac}\,(f)\|$, $\|\mathrm{otherfac}\,(f)\| \leq C_7$, with C_7, $c_7 > 0$ depending only on c_1, D, n. This is the conclusion of the lemma.

LEMMA 4.2. — *W^{t+} is a relatively open subset of W.*

Proof. — If not, we could find $f_\nu \in W$ of type $< t$, with $f_\nu \to f$ in H^D and type $(f) \geq t$. Write $f_\nu = p_\nu q_\nu$ with $p_\nu = \mathrm{mainfac}\,(f_\nu)$,$q_\nu = \mathrm{otherfac}\,(f_\nu)$.The previous lemma shows that $\|p_\nu\|$, $\|q_\nu\|$ are bounded as $\nu \to \infty$. Passing to a subsequence, we may suppose $p_\nu \to p$, $q_\nu \to q$ in H^D. Since $f_\nu = p_\nu q_\nu$, we pass to the limit and conclude that $f = pq$. Since $q_\nu(0) = 1$ we have $q(0) = 1$. Comparing $f = pq$, $q(0) = 1$ with the factorization of f into irreducible factors, we conclude that $\mathrm{mainfac}\,(f) \mid p$. Each p_ν has degree $< t$, since type $(f_\nu) < t$. Taking the limit as $\nu \to \infty$, we see that p has degree $< t$. Since mainfac $(f) \mid p$, it follows that $\mathrm{mainfac}\,(f)$ has degree $< t$. This contradicts type $(f) \geq t$.

LEMMA 4.3. — *The mappings $f \mapsto \mathrm{mainfac}\,(f)$ and $f \mapsto \mathrm{otherfac}\,(f)$ are continuous when restricted to W^t.*

Proof. — Suppose $f_\nu \to f$ with f_ν, $f \in W^t$. We must show that $\mathrm{mainfac}\,(f_\nu) \to \mathrm{mainfac}\,(f)$ and $\mathrm{otherfac}\,(f_\nu) \to \mathrm{otherfac}\,(f)$. Suppose not. Lemma 4.1 shows that $\|\mathrm{mainfac}\,(f_\nu)\|$, $\|\mathrm{otherfac}\,(f_\nu)\|$ remain bounded as $\nu \to \infty$. Hence we can find p, $q \in H^D$ such that (after we pass to a subsequence) $\mathrm{mainfac}\,(f_\nu) \to p$, $\mathrm{otherfac}\,(f_\nu) \to q$, yet $(p,q) \neq (\mathrm{mainfac}\,(f), \mathrm{otherfac}\,(f))$. We have deg $(\mathrm{mainfac}\,(f_\nu)) = t$, hence $\deg p \leq t$. Also, $f_\nu = \mathrm{mainfac}\,(f_\nu)\cdot\mathrm{otherfac}\,(f_\nu)$, so $f = pq$. Moreover, $\mathrm{otherfac}\,(f_\nu) = 1$ at $x = 0$, so $q(0) = 1$. Since $f = pq$ with $q(0) = 1$, examination of the factorization of f into irreducible factors shows that $\mathrm{mainfac}\,(f) \mid p$. On the other hand, $\deg p \leq t$, while $\mathrm{mainfac}\,(f)$ has degree t exactly. Hence, $p = a\cdot\mathrm{mainfac}\,(f)$ for a constant a. Now we have $\mathrm{mainfac}\,(f)\cdot aq = pq = f = \mathrm{mainfac}\,(f)\cdot\mathrm{otherfac}\,(f)$, hence $aq = \mathrm{otherfac}\,(f)$. Since q and $\mathrm{otherfac}\,(f)$ are both equal to 1 at the origin, it follows that $a = 1$. Thus, $p = \mathrm{mainfac}\,(f)$ and $q = \mathrm{otherfac}\,(f)$, contradicting $(p,q) \neq (\mathrm{mainfac}\,(f), \mathrm{otherfac}\,(f))$.

5. Semi-algebraic sets.

Many of the results of this section may be well known to experts. We have given direct proofs since we could not find any in the literature.

The *semi-algebraic* sets $E \subset \mathbb{R}^n$ are those which can be formed from the sets $\{P > 0\}$, P being any polynomial with real coefficients, by making finitely many Boolean operations (union, intersection, complement).

A function $f : E \to \mathbb{R}^N$ defined on $E \subset \mathbb{R}^M$ is called semi-algebraic if its graph $\{(x,y) \in \mathbb{R}^M \times \mathbb{R}^N : x \in E \text{ and } y = f(x)\}$ is a semi-algebraic set. The set E is then semi-algebraic (by the Tarski-Seidenberg theorem 5.1, below).

A function $f : E \to \mathbb{R}^1 \cup \{+\infty, -\infty\}$ defined on $E \subset \mathbb{R}^M$ is called an *extended semi-algebraic function* if the sets $E_+ = f^{-1}(\infty)$, $E_- = f^{-1}(-\infty)$, $E_0 = f^{-1}(\mathbb{R}^1)$ are semi-algebraic and the restriction of f to E_0 is a semi-algebraic function.

An important fact concerning semi-algebraic sets is as follows (usually called the Tarski-Seidenberg theorem) :

THEOREM 5.1. — *The image of a semi-algebraic set under a polynomial map from $\mathbb{R}^N$ to $\mathbb{R}^M$ is again semi-algebraic.*

A beautiful proof of this result can be found in the book [R] by A. Robinson. The proof is given in more conventional mathematical language in the book [BR] of Benedetti and Risler.

The next few results are simple consequences of Theorem 5.1.

COROLLARY 1. — *If $E \subset \mathbb{R}^{N+M}$ is semi-algebraic, then $\{x \in \mathbb{R}^N : (x,y) \in E$ for some $y \in \mathbb{R}^M\}$ is semi-algebraic.*

Proof. — Apply the preceding theorem to the projection $(x,y) \mapsto x$ from $\mathbb{R}^{N+M}$ to $\mathbb{R}^N$.

COROLLARY 2. — *If $E \subset \mathbb{R}^{N+M}$ is semi-algebraic, then $\{x \in \mathbb{R}^N : (x,y) \in E$ for all $y \in \mathbb{R}^M\}$ is semi-algebraic.*

Proof. — The complement of the given set is $\{x \in \mathbb{R}^N : (x,y) \in (\mathbb{R}^{N+M} \setminus E)$ for some $y \in \mathbb{R}^M\}$, which is semi-algebraic by the preceding corollary.

COROLLARY 3. — *Let $f : E \to \mathbb{R}^1$ be a semi-algebraic function, with $E \subset \mathbb{R}^{N+M}$. For $x \in \mathbb{R}^N$, define $\tilde{f}(x) = \sup\{f(x,y) : (x,y) \in E\}$, where the sup of the empty set is defined to be $-\infty$. Then $\tilde{f}$ is an extended semi-algebraic function.*

Proof. — By hypothesis, the set

$$F = \{(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^1 : (x,y) \in E \text{ and } f(x,y) = t\}$$

is semi-algebraic. Hence, so is the set $F_1 = \{(x,y,t,u) \in \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^1 \times \mathbb{R}^1 : (x,y) \in E, f(x,y) = t, u < t\}$. Corollary 1 implies that $F_2 = \{(x,u) \in \mathbb{R}^N \times \mathbb{R}^1 : u < \sup_{(x,y)\in E} f(x,y)\}$ is semi-algebraic. By definition of $\tilde{f}$ we have $F_2 = \{(x,u) \in \mathbb{R}^N \times \mathbb{R}^1 : u < \tilde{f}(x)\}$. Applying Corollaries 1 and 2 above, we see that $\tilde{f}^{-1}(\infty)$ and $\tilde{f}^{-1}(\mathbb{R}^1 \cup \{\infty\})$ are semi-algebraic sets. Hence the sets $E_+ = \tilde{f}^{-1}(\infty), E_- = \tilde{f}^{-1}(-\infty), E_0 = \tilde{f}^{-1}(\mathbb{R}^1)$ are all semi- algebraic. Since E_0 and F_2 are semi-algebraic, the set $F_3 = \{(x,u,v) \in \mathbb{R}^N \times \mathbb{R}^1 \times \mathbb{R}^1 : x \in E_0, \text{ and } u < \tilde{f}(x) \text{ or } u \geq v\}$ is semi-algebraic. By Corollary 2, the set $F_4 = \{(x,v) \in \mathbb{R}^N \times \mathbb{R}^1 : (x,u,v) \in F_3 \text{ for all } u \in \mathbb{R}^1\}$ is semi-algebraic. However, $F_4 = \{(x,v) \in \mathbb{R}^N \times \mathbb{R}^1 : x \in E_0,\ v \leq \tilde{f}(x)\}$. Since F_2 and E_0 are semi-algebraic, the set $F_5 = \{(x,v) \in \mathbb{R}^N \times \mathbb{R}^1 : x \in E_0,\ v < \tilde{f}(x)\}$ is semi-algebraic. The graph $\{(x,v) \in \mathbb{R}^N \times \mathbb{R}^1 : x \in E_0 \text{ and } v = \tilde{f}(x)\}$ is equal to $F_4 \backslash F_5$, and is therefore semi-algebraic.

Thus, E_+, E_-, E_0 are semi-algebraic sets, and the restriction of $\tilde{f}$ to E_0 is a semi-algebraic function. That is, $\tilde{f}$ is an extended semi-algebraic function.

LEMMA 5.1. — *If $f : \mathbb{R}^1 \to \mathbb{R}^1$ is semi-algebraic, then there is a polynomial $Q(x,y)$, not identically zero, with $Q(x, f(x)) = 0$ for all $x \in \mathbb{R}^1$.*

Proof. — By hypothesis, the graph of f is semi-algebraic, hence it is a finite union of sets of the form

$$\Gamma = \{P_1 > 0, \ldots, P_m > 0, Q_1 = 0, \ldots, Q_k = 0\}. \tag{1}$$

We may assume that these Γ are non- empty. Each of these Γ must have at least one non-zero Q_j in (1), for otherwise, Γ would be a non-empty open set, so we could not have Γ contained in the graph of a function.

Thus, for each of the Γ that make up the graph of f, we have a non-zero polynomial Q that vanishes on Γ. Multiplying together the Q arising from each Γ, we obtain a polynomial $\tilde{Q} \neq 0$ that vanishes on the graph of f.

COROLLARY. — *If $f : \mathbb{R}^1 \to \mathbb{R}^1$ is semi-algebraic, then in a small enough interval* $(0,\delta)$ *we have* $|f(t)| \leq Ct^{-m}$.

Proof. — Let $Q(x,y)$ be as in the preceding lemma. Since $Q = 0$ on the graph of f, Q cannot be independent of y. Hence

$$Q(x,y) = \sum_{k=0}^{r} p_k(x)y^k \quad \text{with} \quad r \geq 1 \quad \text{and} \quad p_r(x) \not\equiv 0.$$

We now make the following remark : if w, $a_{r-k}(1 \leq k \leq r)$ are complex numbers with $w^r + a_{r-1}w^{r-1} + \ldots + a_1 w + a_0 = 0$, then $|w| < 2\cdot\max_k |a_{r-k}|^{\frac{1}{k}}$ unless all the a_i are 0. To see this, we may assume that $\alpha = \max_k |a_{r-k}|^{1/k} > 0$. Let $z = \frac{w}{\alpha}$. We have $z^r + \frac{a_{r-1}}{\alpha} z^{r-1} + \ldots + \frac{a_0}{\alpha^r} = 0$ and $\frac{|a_{r-k}|}{\alpha^k} \leq 1$. Hence $|z|^r \leq 1 + |z| + \ldots + |z|^{r-1}$, which implies that $|z| < 2$ (because, if $u \geq 2$, we have $1 + u + \ldots + u^{r-1} = \frac{u^r - 1}{u - 1} \leq u^r - 1$), proving the statement in the remark.

Now, if $y = f(x)$, we have $p_r(x)y^r + \ldots + p_0(x) = 0$; if we multiply this by $(p_r(x))^{r-1}$ and set $w = p_r(x)f(x)$, we have $w^r + p_{r-1}(x)w^{r-1} + \ldots + (p_r(x))^{r-1}p_0(x) = 0$, so that

$$|p_r(x)f(x)| \leq 2 \max_{1\leq k\leq r} |p_r(x)^{k-1}p_{r-k}(x)|^{\frac{1}{k}}.$$

If p_r has a zero of order m at $x = 0$, we have $|p_r(x)| \geq c|x|^m$ for $|x| \leq \delta_0(\leq 1)$, so that $|f(x)| \leq C|x|^{-m}$ for $0 < |x| \leq \delta_0$.

LEMMA 5.2. — *Let S be a compact semi-algebraic subset of $\mathbb{R}^N$, and let $E \subset S$ be a semi-algebraic subset of S. Let $f : E \to \mathbb{R}^1$ be a semi-algebraic function. Assume that f is locally bounded on E. Then for positive constants m, C, we have*

$$|f(x)| \leq C(\operatorname{dist}(x, S\backslash E))^{-m} \text{ for all } x \in E.$$

Proof. — Set $g(x,y) = |x - y|$ and $G = \mathbb{R}^N \times (S\backslash E)$. These are semi-algebraic. Corollary 3 to Theorem 5.1 shows that $\operatorname{dist}(x, S\backslash E) = \inf\{g(x,y) : (x,y) \in G\}$ is an extended semi-algebraic function. We may assume $S\backslash E$ to be non- empty, since otherwise Lemma 5.2 is trivial. It follows that $\operatorname{dist}(x, S\backslash E)$ always belongs to $[0, \infty)$, so that $\operatorname{dist}(x, S\backslash E)$ is a semi-algebraic function. Thus, the graph $\{(x,u) \in E \times \mathbb{R}^1 : \operatorname{dist}(x,S\backslash E) = u\}$ is semi-algebraic. Hence also $\{(x, u, t) \in E \times \mathbb{R}^1 \times \mathbb{R}^1 : \operatorname{dist}(x, S\backslash E) = u$ and $u \geq t\}$ is semi-algebraic. Corollary 1 to Theorem 5.1 now shows that

$\{(x, t) \in E \times \mathbb{R}^1 : \operatorname{dist}(x, S\backslash E) \geq t\}$ is semi-algebraic. Applying Corollary 3 to Theorem 5.1, we see that

$$\tilde{f}(t) = \sup\{|f(x)| : x \in E, \operatorname{dist}(x, S\backslash E) \geq t\} \tag{1}$$

is an extended semi-algebraic function. If $t > 0$, then $\{x \in E : \operatorname{dist}(x, S\backslash E) \geq t\} = S \cap \bigcap_{y \in S\backslash E} \{x : |x-y| \geq t\}$ is compact; and we know that f is locally bounded on E. Hence, f is bounded on $\{x \in E : \operatorname{dist}(x, S\backslash E) \geq t\}$, so that

$$\tilde{f}(t) < +\infty \text{ for } t > 0. \tag{2}$$

We may assume that $\delta_0 = \operatorname{dist}(x^0, S\backslash E) > 0$ for some $x^0 \in E$, since otherwise lemma 5.2 holds vacuously. Fix such an x_0. If $t < \delta_0$, then the sup in (1) is taken over a set of x that includes x^0. Hence,

$$\tilde{f}(t) > -\infty \text{ for } t < \delta_0. \tag{3}$$

In view of (2) and (3), the restriction of $\tilde{f}(t)$ to $(0,\delta_0)$ is a semi-algebraic function. We may redefine $\tilde{f}(t)$ to be identically zero outside $(0,\delta_0)$ to get a semi-algebraic function on all of $\mathbb{R}^1$. Applying the corollary to Lemma 5.1, we get the estimate $|\tilde{f}(t)| \leq Ct^{-m}$ for $t \in (0, \delta]$, with $0 < \delta < \delta_0$. In view of the definition (1), this means that

$$|f(x)| \leq C(\operatorname{dist}(x, S\backslash E))^{-m} \text{ if } x \in E \text{ and } 0 < \operatorname{dist}(x, S\backslash E) < \delta, \tag{4}$$

and that

$$|f(x)| \leq C\delta^{-m}, \text{ if } \operatorname{dist}(x, S\backslash E) \geq \delta. \tag{5}$$

Since also $(\operatorname{dist}(x, S\backslash E))^{-m} \geq (\operatorname{diam} S)^{-m}$ for $x \in E$, (5) implies

$$|f(x)| \leq [C\delta^{-m}(\operatorname{diam} S)^m](\operatorname{dist}(x, S\backslash E))^{-m} \text{ if } x \in E, \operatorname{dist}(x, S\backslash E) \geq \delta. \tag{6}$$

The conclusion of Lemma 5.2 follows at once from (4) and (6).

LEMMA 5.3. — *The sets W, W^t, $W^{t+} \subset H^D$ are semi-algebraic. The function $(p, q, \rho) \in W \times H^{D'} \times (0, \infty) \mapsto \operatorname{Norm}(p, q, D', \rho)$ is an extended semi- algebraic function. For $D' \geq D, C' \geq 1$, the function $(p, \rho) \in W \times (0, \infty) \mapsto \operatorname{Norm}_*(p, D', C', \rho)$ is semi-algebraic.*

Here, of course, to define semi-algebraic sets and functions on H^D, we identify H^D with an euclidean space.

Proof of Lemma 5.3. — The set $W \subset H^D$ is defined by polynomial equations and inequalities, so W is obviously semi-algebraic. The set $\{(f, p, q) \in H^D \times H^s \times H^{D-s} : f = pq, p(0) = 0\}$ is semi-algebraic. Hence so is $\{(f,p,q) \in W \times H^s \times H^{D-s} : f = pq, p(0) = 0\}$. Theorem 5.1 Corollary 1 implies that $\{f \in W : f$ has a factor p of degree $\leq s$ with $p(0) = 0\}$ is semi-algebraic. This set is equal to $W \backslash W^{(s+1)+}$. Hence W^{t+} is semi-algebraic, as is $W^t = W^{t+} \backslash W^{(t+1)+}$.

From the previous paragraph, we see that $\{(f,p,q) \in W^t \times H^t \times H^{D-t} : f = pq, p(0) = 0, q(0) = 1\}$ is semi-algebraic. Theorem 5.1 Corollary 1 implies that $\{(f,p) \in W^t \times H^D : \deg p \leq t, f = pq$, for some $q \in H^{D-t}$ with $q(0) = 1\}$ is semi-algebraic. The union of these sets over all $t (1 \leq t \leq D)$ is $\{(f, p) \in W \times H^D : p = \mathrm{mainfac}(f)\}$. Hence $f \in W \mapsto \mathrm{mainfac}(f)$ is semi- algebraic.

It follows that $\{(f, p, x) \in W \times H^D \times \mathbb{R}^n : p = \mathrm{mainfac}(f), p(x) = 0\}$ is semi-algebraic. Theorem 5.1 Corollary 1 implies that $\{(f, x) \in W \times \mathbb{R}^n : x \in V(f)\}$ is semi- algebraic. Hence $\{(f, x, p, \rho) \in H^D \times \mathbb{R}^n \times W \times (0, \infty) : x \in V(p), x \in Q_\rho\}$ is semi-algebraic. Theorem 5.1 Corollary 3 shows that $\phi : (f, p, \rho) \in H^D \times W \times (0, \infty) \mapsto \sup_{V(p) \cap Q_\rho} |f(x)|$ is an extended semi-algebraic function. Since $V(p) \cap Q_\rho$ is a compact set containing the origin, $\sup_{V(p) \cap Q_\rho} |f(x)|$ is never $\pm\infty$. Thus, ϕ is a semi-algebraic function. Also, $\{(f, x, p) \in H^{D'} \times \mathbb{R}^n \times W : x \in V(p), f(x) \neq 0\}$ is semi-algebraic. Theorem 5.1 Corollary 1 shows that $\{(f, p) \in H^{D'} \times W : f$ does not vanish on $V(p)\}$ is semi-algebraic. It follows that

(a) $$\{(\tilde{f}, f, p, q) \in H^{D'} \times H^D \times H^D \times H^{D'} : \tilde{f} = fq \text{ on } V(p)\}$$

is semi-algebraic. Since ϕ is semi-algebraic, the set $\{(\tilde{f}, f, p, C', \rho, u) \in H^{D'} \times H^D \times W \times [0, \infty) \times (0, \infty) \times [0, \infty) : \|\tilde{f}\| \leq C'u, u = \sup_{V(p) \cap Q_\rho} |f|\}$ is semi-algebraic. Theorem 5.1 Corollary 1 shows that

(b)
$$\{(\tilde{f}, f, p, C', \rho) \in H^{D'} \times H^D \times W \times (0, \infty) \times (0, \infty) : \|\tilde{f}\| \leq C' \sup_{V(p) \cap Q_\rho} |f|\}$$

is semi-algebraic. From (a), (b) it follows that the set $\{(\tilde{f}, f, p, q, C', \rho) \in H^{D'} \times H^D \times W \times H^{D'} \times [0, \infty) \times (0, \infty) : \tilde{f} = qf$ on $V(p)$ and $\|\tilde{f}\| \leq C' \sup_{V(p) \cap Q_\rho} |f|\}$ is semi-algebraic. Applying Theorem 5.1 Corollaries 1 and 2, we find that the set $E = \{(p, q, C', \rho) \in W \times H^{D'} \times [0, \infty) \times (0, \infty)$: for all $f \in H^D$ there exist $\tilde{f} \in H^{D'}$ for which $\tilde{f} = qf$ on $V(p)$ and $\|\tilde{f}\| \leq C' \sup_{V(p) \cap Q_\rho} |f|\}$ is semi-algebraic.

By definition, $\mathrm{Norm}(p, q, D', \rho) = \inf\{C' \in [0, \infty) : (p, q, C', \rho) \in E\}$. Hence Theorem 5.1 Corollary 3 shows that $(p, q, \rho) \in W \times H^{D'} \times (0,$

$\infty) \mapsto \mathrm{Norm}\,(p,\, q,\, D',\, \rho)$ is an extended semi-algebraic function. Note that $\mathrm{Norm}\,(p,q,D',\rho)$ may be $+\infty$, but it cannot be $-\infty$. Set $E_0 = \{(p,\, q,\, \rho) \in W \times H^{D'} \times (0,\, \infty) : \mathrm{Norm}\,(p,\, q,\, D',\, \rho) < \infty\}$. Thus, E_0 is semi-algebraic, and $\mathrm{Norm}\,(p,\, q,\, D',\, \rho)$ is a semi-algebraic function on E_0. The set $E_1 = \{(p,q,\rho) \in W \times H^{D'} \times (0,\infty) : (p,q,\rho) \in E_0, q(0) = 1, \|q\| \le C'\}$ is semi-algebraic, since E_0 is semi-algebraic. Since

$$\mathrm{Norm}_*(p,D',C',\rho) = \inf\{\mathrm{Norm}\,(p,q,D',\rho) : (p,q,\rho) \in E_1\},$$

it follows from Theorem 1 Corollary 3 that

$$\psi : (p,\rho) \in W \times (0,\infty) \mapsto \mathrm{Norm}_*(p,D',C',\rho)$$

is an extended semi-algebraic function. Note that $\mathrm{Norm}\,(p,q,D',\rho) \ge 0$, so $\mathrm{Norm}_*(p,D',C',\rho) \ge 0$. In particular, $\mathrm{Norm}_*(p,D',C',\rho)$ is never equal to $-\infty$. If we note that by the corollary to Lemma 3.3, $\mathrm{Norm}_*(p,\, D',\, C',\, \rho)$ is never equal to $+\infty$ provided that $C' \ge 1$ and D' is $\ge D$, we find that ψ is an extended semi-algebraic function which is never $\pm\infty$, and so ψ is a semi-algebraic function. We have proven all the assertions of Lemma 5.3.

6. The main technical lemmas.

LEMMA 6.1. — *Let $p \in W^t$ and $\rho > 0$ be given. Then there exists a neighborhood U of p in H^D such that $\tilde p \mapsto \mathrm{Norm}\,(\tilde p,\, 1,\, D,\, \rho)$ is bounded on $U \cap W^t$.*

Proof. — Let ψ_β be a list of all monomials of degree $\le D - t$, and (φ_α), a list of monomials of degree $\le D$ such that the φ_α and $\psi_\beta \mathrm{mainfac}\,(p)$ form a basis for H^D. If $\tilde p \in W^t$ and $\tilde p$ is close enough to p, then φ_α, $\psi_\beta \mathrm{mainfac}\,(\tilde p)$ belong to H^D and lie near φ_α, $\psi_\beta \mathrm{mainfac}\,(p)$. Hence (φ_α) and $(\psi_\beta \mathrm{mainfac}\,(\tilde p))$ form a basis for H^D, provided $\tilde p \in U \cap W^t$. Here U denotes a small enough neighborhood of p. Consequently, if $f \in H^D$ is given, then we can find real numbers ξ_α such that

$$f = \sum_\alpha \xi_\alpha \varphi_\alpha \quad \text{on} \quad V(\tilde p). \tag{1}$$

Note that the φ_α are linearly independent $\mathrm{mod}\, H^{D-t} \cdot \mathrm{mainfac}\,(p)$. Hence they are linearly independent as functions on $V(p) \cap Q_{\frac{\varrho}{2}}$. (This follows from Lemma 3.2.) We can now apply Lemma 3.1 to produce points $x_\beta \in V(p) \cap Q_{\frac{\varrho}{2}}$ such that whenever $\tilde x_\beta \in \mathbb{R}^n$ with $|\tilde x_\beta - x_\beta|$ small enough, then we have

$$\sum_\alpha |\xi_\alpha| \le C \max_\beta \left| \sum_\alpha \xi_\alpha \varphi_\alpha(\tilde x_\beta) \right|, \text{ with } C \text{ independent of the } \xi_\alpha, \tilde x_\beta. \tag{2}$$

If $\tilde{p} \in W^t$ lies close enough to p, then $\|\text{mainfac}(\tilde{p}) - \text{mainfac}(p)\|$ will be small, by Lemma 4.3. Also, mainfac (p) vanishes at 0 with a non- zero gradient. Hence there is $\rho_0 > 0$ depending only on p such that whenever $x \in Q_{\frac{\rho_0}{2}} \cap \{$ zeroes of mainfac $(p)\}$, then for $\tilde{p} \in W^t$ close enough to p we can find $\tilde{x} \in Q_{\rho_0} \cap \{$ zeroes of mainfac $(\tilde{p})\}$ as close as we please to x. Assuming $\rho \leq \rho_0$, we may apply the above observation to the x_β. Thus, given $\varepsilon > 0$ there is some $\delta > 0$ such that whenever $\tilde{p} \in W^t$, $\|\tilde{p} - p\| < \delta$ then we can find $\tilde{x}_\beta \in V(\tilde{p}) \cap Q_\rho$ with $|\tilde{x}_\beta - x_\beta| < \varepsilon$. If we take ε small enough, then (2) applies. Hence, we can find δ, $C > 0$ with the following property. If $\tilde{p} \in W^t$ and $\|\tilde{p} - p\| < \delta$, then

$$\text{(3)} \qquad \sum_\alpha |\xi_\alpha| \leq C \cdot \max_{x \in V(\tilde{p}) \cap Q_\rho} \left| \sum_\alpha \xi_\alpha \varphi_\alpha(x) \right|, \text{ for all } (\xi_\alpha).$$

For a given $f \in H^D$, we take $\tilde{f} = \sum_\alpha \xi_\alpha \varphi_\alpha$ with ξ_α as in (1). Then (1) and (3) yield

$$\text{(4)} \qquad \tilde{f} = f \text{ on } V(\tilde{p}), \text{ and } \|\tilde{f}\| \leq C \max_{V(\tilde{p}) \cap Q_\rho} |f|.$$

This holds whenever $\tilde{p} \in W^t$ and $\|\tilde{p} - p\| < \delta$. Since also $\tilde{f}$ has degree $\leq D$, it follows from (4) that Norm $(\tilde{p}, 1, D, \rho) \leq C$ for all $\tilde{p} \in W^t$ with $\|\tilde{p} - p\| < \delta$. This is the conclusion of Lemma 6.1. We proved it under the assumption $\rho \leq \rho_0$, but that assumption can immediately be removed, since $\rho \mapsto$ Norm $(\tilde{p}, 1, D, \rho)$ is monotone decreasing.

LEMMA 6.2. — *Let $p \in W^t$ be given, and let integers m, D' be given. Then there exist a neighborhood U of p in H^D, and continuous maps $\Phi_\alpha : U \cap W^t \to H^{D'+mD} (1 \leq \alpha \leq s)$ and a constant C_* with the following properties :*

(a) *The polynomials $\Phi_\alpha(p)$ are linearly independent as functions on $V(p)$.*

(b) *Suppose we are given $\tilde{p} \in U \cap W^t$, $g \in H^D$ with $\|g\| \leq 1$, and $\tau \in (0, 1)$. Let $\tilde{\varphi}_\alpha = \Phi_\alpha(\tilde{p})$, and let $\tilde{q} =$ otherfac $(\tilde{p})$. Then given $f \in H^{D'}$, we can find coefficients $A_\alpha (1 \leq \alpha \leq s)$ and a polynomial $f_\# \in H^{D'+mD}$ such that*

$$\text{(i)} \qquad \tilde{q}^m f = \sum_{1 \leq \alpha \leq s} A_\alpha \tilde{\varphi}_\alpha + \tau^m f_\# \bmod (\tilde{p} + \tau g),$$

and

$$\text{(ii)} \qquad |A_\alpha|, \|f_\#\| \leq C_* \|f\|.$$

Proof. — We use induction on m. If $m = 0$, the result is trivial. In fact, we take $s = 0$ (i.e. there are no Φ_α). Condition (a) holds vacuously.

To check condition (b) with $m = 0$, we take $f_{\#} = f$ and note that (i) and (ii) are obvious.

Assume Lemma 6.2 holds for a given m. We will prove it for $m+1$. Thus, we assume we are given U, Φ_α, C_* satisfying (a) and (b). Let $(\psi_\beta)_{1\le\beta\le s'}$ be a list of all monomials of degree $\le D' + mD$.

Let A be a maximal set of β's such that the polynomials $\psi_\beta(\beta \in A)$ and $\Phi_\alpha(p)(1 \le \alpha \le s)$ are linearly independent as functions on $V(p)$. After renumbering the ψ's, we may assume that $A = \{1, \dots, s''\}$, with $s'' \le s'$. Thus,

$$\psi_\beta(1 \le \beta \le s'') \text{ and } \Phi_\alpha(p)(1 \le \alpha \le s) \tag{1}$$

are linearly independent as functions on $V(p)$, and

$$\text{For } 1 \le \gamma \le s' \text{ we have } \psi_\gamma = \sum_{1\le\beta\le s''} C_\gamma^\beta \psi_\beta + \sum_{1\le\alpha\le s} E_\gamma^\alpha \Phi_\alpha(p) \tag{2}$$

as functions on $V(p)$.

Next we add new mappings Φ_α to our list, by setting

$$\Phi_{s+\beta}(\tilde{p}) = \psi_\beta \text{ for } 1 \le \beta \le s''. \tag{3}$$

Our new mappings are obviously continuous. Moreover, the new list $(\Phi_\alpha)_{1\le\alpha\le s+s''}$ still satisfies (a) and (b). In fact, (a) is equivalent to (1); while (b) for the new list follows from (b) for the original list by simply taking $A_\alpha = 0$ for $s+1 \le \alpha \le s+s''$.

Property (2) asserts that each ψ_γ is a linear combination of the $\Phi_\alpha(p)$, as functions on $V(p)$. After a trivial change of notation, we thus have found $U, (\Phi_\alpha)_{1\le\alpha\le s}, C_*$ such that (a) and (b) are satisfied, and

(c) Each ψ_β is given as $\psi_\beta = \sum\limits_{1\le\alpha\le s} C_\beta^\alpha \Phi_\alpha(p)$ as functions on $V(p)$.

Condition (c) means that as polynomials in $H^{D'+mD}$,

$$\psi_\beta = \sum_{1\le\alpha\le s} C_\beta^\alpha \Phi_\alpha(p) + h_\beta \operatorname{mainfac}(p), \text{ with} \tag{4}$$

$$h_\beta \in H^{D'+mD-t}, \text{ since mainfac}(p) \text{ has degree } t. \tag{5}$$

Now let $T_{\tilde{p}} : \mathbb{R}^s \times H^{D'+mD-t} \to H^{D'+mD}$ be the linear map defined by

$$T_{\tilde{p}}((\xi_\alpha)_{1\le\alpha\le s}, h) = \sum_{1\le\alpha\le s} \xi_\alpha \Phi_\alpha(\tilde{p}) + h \operatorname{mainfac}(\tilde{p}) \quad \text{for} \quad \tilde{p} \in U \cap W^t. \tag{6}$$

Since the Φ_α are continuous maps on $U \cap W^t$, Lemma 4.3 shows that $T_{\tilde{p}}$ depends continuously on $\tilde{p} \in U \cap W^t$. Moreover, (4) shows that T_p is onto.

We now make the following :

Observation : let $T_{\tilde{p}} : \mathbb{R}^N \to \mathbb{R}^M$ be a linear map depending continuously on a parameter $\tilde{p}$. Suppose T_p is onto, and that $T_p v_1 = w_1 \in \mathbb{R}^M - \{0\}$. For $\tilde{p}$ close enough to p, we can find $\tilde{v}_1$ as close as we please to v_1, satisfying $T_{\tilde{p}}\tilde{v}_1 = w_1$.

To prove the observation, we complete w_1 to a basis $w_1, \ldots, w_M$ of $\mathbb{R}^M$, and we extend v_1 to a list $v_1, \ldots, v_M \in \mathbb{R}^N$ such that $T_p v_j = w_j$. Let V denote the span of $v_1, \ldots, v_M$, and let $T^{\#}_{\tilde{p}} = T_{\tilde{p}} \mid_V$.

Thus, $T^{\#}_{\tilde{p}} : V \to \mathbb{R}^M$ is a linear map depending continuously on $\tilde{p}$, and $T^{\#}_p$ is an isomorphism, with $T^{\#}_p v_1 = w_1$.

The linear map $T^{\#}_{\tilde{p}}(T^{\#}_p)^{-1}$ depends continuously on $\tilde{p}$ and is equal to the identity at $\tilde{p} = p$. Hence for $\tilde{p}$ in a small neighborhood of $p, \tilde{p} \mapsto (T^{\#}_{\tilde{p}}(T^{\#}_p)^{-1})^{-1}$ is well-defined and continuous.

Set $\tilde{v}_1 = (T^{\#}_p)^{-1}(T^{\#}_{\tilde{p}}(T^{\#}_p)^{-1})^{-1} w_1$. This depends continuously on $\tilde{p}$ provided $\tilde{p}$ stays close enough to p. We have $T^{\#}_{\tilde{p}}\tilde{v}_1 = [T^{\#}_{\tilde{p}}(T^{\#}_p)^{-1}]$ $[T^{\#}_{\tilde{p}}(T^{\#}_p)^{-1}]^{-1} w_1 = w_1$. Hence also $T_{\tilde{p}}\tilde{v}_1 = w_1$. The proof of the observation is complete.

We use the observation with $T_{\tilde{p}}$ given by (6), and with w_1 taken to be ψ_β. The observation gives us functions

$$\mathcal{C}^\alpha_\beta : U_1 \cap W^t \to \mathbb{R}^1 \text{ and } \mathcal{H}_\beta : U_1 \cap W^t \to H^{D'+mD-t} \tag{7}$$

defined on $U_1 \cap W^t$ with $U_1 \subset U$ an open neighborhood of p in H^D, continuous at p, and satisfying.

$$\psi_\beta = \sum_\alpha \mathcal{C}^\alpha_\beta(\tilde{p})\Phi_\alpha(\tilde{p}) + \mathcal{H}_\beta(\tilde{p})\text{mainfac}\,(\tilde{p}) \text{ for } \tilde{p} \in U_1 \cap W^t. \tag{8}$$

Since the functions (7) are continuous at p, they are bounded on a small neighborhood of p. Thus

(9) There exist $\overline{C} > 0$, and $U_2 \subset H^D$ open, with $p \in U_2 \subset U_1$, and $|\mathcal{C}^\alpha_\beta(\tilde{p})|, \|\mathcal{H}_\beta(\tilde{p})\| < \overline{C}$ for $\tilde{p} \in U_2 \cap W^t$.

We are ready to define maps $\underline{\Phi_\alpha} : U_2 \cap W^t \to H^{D'+(m+1)D}$ such that (a) and (b) will hold with U_2 in place of U, with $(m+1)$ in place of m, and with a suitable constant $\underline{C_*}$ in place of C_*. In fact, we define

$$\underline{\Phi_\alpha}(\tilde{p}) = \Phi_\alpha(\tilde{p}) \cdot \text{otherfac}\,(\tilde{p}). \tag{10}$$

Since Φ_α is a continuous map from $U_2 \cap W^t$ into $H^{D'+mD}$, and since otherfac is a continuous map from $U_2 \cap W^t$ into H^D (see Lemma 4.3), it follows that $\underline{\Phi_\alpha}$ is a continuous map from $U_2 \cap W^t$ into $H^{D'+(m+1)D}$, as required for the inductive step.

Next, we verify that the $\underline{\Phi_\alpha}$ satisfy (a). Recall that the Φ_α satisfy (a), and that a polynomial vanishes on $V(p)$ if and only if it vanishes on some small neighborhood of the origin in $V(p)$. Hence, the $\Phi_\alpha(p)$ are linearly independent as germs at the origin of functions on $V(p)$. Since otherfac$\,(p) = 1$ at the origin, we have otherfac$\,(p) \neq 0$ in a small neighborhood of the origin. Hence, the functions otherfac$\,(p) \cdot \Phi_\alpha(p)$ are linearly independent as germs at 0 of functions on $V(p)$. So the $\underline{\Phi_\alpha}$ satisfy (a).

Next we verify that the $\underline{\Phi_\alpha}$ satisfy (b), with $(m+1)$ in place of m, and with a new constant $\underline{C_*}$ in place of C_*. In fact, let f be given, let $\tilde{p} \in U_2 \cap W^t$ be given, let $g \in H^D$ be given with $\|g\| \leq 1$, and let $\tau \in (0,1)$ be given. We apply (b) to write

$$\text{(11)} \qquad [\text{otherfac}\,(\tilde{p})]^m f = \sum_\alpha A_\alpha \Phi_\alpha(\tilde{p}) + \tau^m f_\# \bmod(\tilde{p} + \tau g), \text{ with}$$

$$\text{(12)} \qquad f_\# = \sum_\beta B_\beta \psi_\beta, \text{ and}$$

$$\text{(13)} \qquad |A_\alpha|, |B_\beta| \leq C_* \|f\|.$$

Substituting (8) into (12), and then putting the result into (11), we find that

$$[\text{otherfac}\,(\tilde{p})]^m f = \sum_\alpha A_\alpha \Phi_\alpha(\tilde{p}) + \tau^m \sum_{\beta\alpha} B_\beta C^\alpha_\beta(\tilde{p}) \Phi_\alpha(\tilde{p})$$
$$+ \sum_\beta \tau^m B_\beta \mathcal{H}_\beta(\tilde{p}) \text{ mainfac}\,(\tilde{p}) \bmod(\tilde{p} + \tau g).$$

Multiply this equation by otherfac$(\tilde{p})$, and note that mainfac$(\tilde{p})\cdot$otherfac$(\tilde{p})$ $= \tilde{p} = -\tau g \bmod(\tilde{p} + \tau g)$. Thus we obtain

$$\text{(14)} \qquad [\text{otherfac}\,(\tilde{p})]^{m+1} f = \sum_\alpha \underline{A_\alpha} \cdot \underline{\Phi_\alpha}(\tilde{p}) + \tau^{m+1} \underline{f_\#} \bmod(\tilde{p} + \tau g),$$

where we have set

$$\text{(15)} \qquad \underline{A_\alpha} = A_\alpha + \tau^m \sum_\beta B_\beta C^\beta_\alpha(\tilde{p})$$

and

$$\text{(16)} \qquad \underline{f_\#} = -g \sum_\beta B_\beta \mathcal{H}_\beta(\tilde{p}).$$

Recall that $\mathcal{H}_\beta(\tilde{p}) \in H^{D'+mD-t}$, while $g \in H^D$. Hence $\underline{f_\#} \in H^{D'+(m+1)D}$ as required for the inductive step.

Substituting (9) and (13) into (15), (16), and recalling that $\tau \in (0, 1)$ and $\|g\| \leq 1$, we obtain a constant $\underline{C}_*$ independent of $\tilde{p}, g, \tau, f$, such that

$$(17) \qquad |\underline{A_\alpha}|, \|\underline{f_\#}\| \leq \underline{C}_* \|f\|.$$

Equation (14) and estimate (17) show that $U_2, \underline{\Phi_\alpha}, \underline{C}_*$ satisfy (b), with $(m+1)$ in place of m.

Thus we have found $U_2, \underline{\Phi_\alpha}, \underline{C_*}$ that satisfy (a) and (b) with $(m+1)$ in place of m. The inductive step is complete and Lemma 6.2 is proven.

LEMMA 6.3. — *There is a small constant $\rho_0 > 0$ with the following properties :*

(a) *For all $p \in W$, $V(p) \cap Q_{\rho_0} = \{x \in Q_{\rho_0} : p(x) = 0\}$.*

(b) *For all $p \in W$ and $x \in Q_{\rho_0}$, we have $|\nabla p(x)| \geq \frac{1}{2} c_1$.*

(c) *Let $p \in W$ and $x \in V(p) \cap Q_{\frac{\rho_0}{2}}$ be given. Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $\hat{p} \in W$ and $\|\hat{p} - p\| < \delta$, then there is a point $\hat{x} \in V(\hat{p}) \cap Q_{\rho_0}$ with $|\hat{x} - x| < \varepsilon$.*

Proof. — (a) amounts to saying that otherfac$(p) \neq 0$ throughout Q_{ρ_0}. Since otherfac$(p) = 1$ at the origin, and since $\|\text{otherfac}(p)\| \leq C$ by Lemma 4.1, we can pick $\rho_0 > 0$ depending only on c_1, D, n such that otherfac$(p) \geq \frac{1}{2}$ throughout Q_{ρ_0}. (b) is obvious, since $|\nabla p(0)| \geq c_1$ and $\|p\| = 1$ for $p \in W$. (c) is proven as follows. By (b), we can pick $i_0 (1 \leq i_0 \leq n)$ so that $\left|\frac{\partial p}{\partial x_{i_0}}\right| \geq \frac{c_1}{2n}$ at x. The implicit function theorem shows that $\hat{p}(\hat{x}) = 0$ if and only if $\hat{x}_{i_0} = F(\hat{p}, (\hat{x}_j)_{j \neq i_0})$, provided $\|\hat{p} - p\|$ and $|\hat{x} - x|$ are small. Here, F denotes a smooth function. Restricting attention to those $\hat{x}$ that differs from x only in the i_0-coordinate, we obtain $\hat{p}(\hat{x}) = 0$ when $\hat{x} = G(\hat{p})$ for a smooth function G defined on a neighborhood of p. Moreover, $G(p) = x$. In view of (a), we have $\hat{x} = G(\hat{p}) \in V(\hat{p})$, provided $\hat{x} \in Q_{\rho_0}$. Since $|\hat{x} - x| = |G(\hat{p}) - G(p)| \leq C\|\hat{p} - p\|$ and $x \in Q_{\frac{\rho_0}{2}}$, it follows that $|\hat{x} - x| < \varepsilon$ and $\hat{x} \in Q_{\rho_0}$ if $\|\hat{p} - p\| < \delta$. Therefore $|\hat{x} - x| < \varepsilon$ and $\hat{x} \in Q_{\rho_0} \cap V(\hat{p})$ if $\|\hat{p} - p\| < \delta$, which proves (c).

LEMMA 6.4. — *Let $p \in W^t$ be given, and let D', m, $\rho > 0$ be given. Then there exist C, $\delta > 0$ with the following property :*

Let $\hat{p} = \tilde{p} + \tau g$, with $\tilde{p} \in W^t$, $g \in H^D$, $\tau > 0$, $\|g\| \leq 1$. Assume that $\|\tilde{p} - p\|$, $\tau < \delta$. Then given any $f \in H^{D'}$ we can find $\tilde{f} \in H^{D'+mD}$ such *that*

(a) $$\tilde{f} = [\text{otherfac}\,(\tilde{p})]^m f \textit{ on } V(\hat{p}),$$

and

(b) $$\|\tilde{f}\| \le C\{\max_{Q_\rho \cap V(\hat{p})} |f| + \tau^m \|f\|\}.$$

Proof. — Let U, Φ_α, C_* be as in Lemma 6.2. Given $f \in H^{D'}$, Lemma 6.2 produces coefficients A_α and a polynomial $f_\# \in H^{D'+mD}$ such that

(1) $$[\text{otherfac}\,(\tilde{p})]^m f = \sum_{\alpha=1}^{s} A_\alpha \Phi_\alpha(\tilde{p}) + \tau^m f_\# \bmod \hat{p},$$

and

(2) $$|A_\alpha|, \|f_\#\| \le C_* \|f\|,$$

provided we take δ small enough that $\|\tilde{p} - p\| < \delta$ implies $\tilde{p} \in U$. We define

(3) $$\tilde{f} = \sum_{1 \le \alpha \le s} A_\alpha \Phi_\alpha(\tilde{p}) + \tau^m f_\# \in H^{D'+mD}.$$

Then (a) is immediate from (1), and it remains only to check (b). Let $\rho_0 > 0$ be as in Lemma 6.3. Set $\overline{\rho} = \min\left\{\frac{1}{2}\rho_0, \frac{1}{2}\rho\right\}$. According to Lemma 6.2, the $\Phi_\alpha(p)$ are linearly independent as functions on $V(p)$. Hence, they are also linearly independent as functions on $Q_{\overline{\rho}} \cap V(p)$. According to Lemma 3.1, we can find points $x_1, \dots, x_s \in Q_{\overline{\rho}} \cap V(p)$ and constants C', $\varepsilon > 0$ such that

(4) $$\sum_{1 \le j \le s} |\xi_j| \le C' \max_{1 \le k \le s} \left| \sum_{1 \le j \le s} \xi_j \tilde{\varphi}_j(\tilde{x}_k) \right|,$$

provided

(5) $$\|\tilde{\varphi}_\alpha - \Phi_\alpha(\tilde{p})\|, |\tilde{x}_j - x_j| < \varepsilon.$$

Now suppose that $\|\tilde{p} - p\|$, $\tau < \delta$ with δ small enough. The continuity of the maps Φ_α shows that

(6) $$\|\Phi_\alpha(\tilde{p}) - \Phi_\alpha(p)\| < \varepsilon,$$

while Lemma 6.3 (c) implies that we can find $\hat{x}_1, \dots, \hat{x}_s \in V(\hat{p}) \cap Q_{2\overline{\rho}}$ such that

(7) $$|\hat{x}_j - x_j| < \varepsilon.$$

Estimates (6) and (7) show that (5) holds with $\tilde{\varphi}_\alpha = \Phi_\alpha(\tilde{p})$ and with $\tilde{x}_j = \hat{x}_j \in Q_{2\overline{\rho}} \cap V(\hat{p})$. Therefore (4) holds with these data, which means that

$$\begin{aligned}\sum_{1\le\alpha\le s} |A_\alpha| &\le C' \max_{1\le j\le s} \Big| \sum_{1\le\alpha\le s} A_\alpha \Phi_\alpha(\tilde{p})(\tilde{x}_j)\Big| \\ &\le C' \max_{Q_{2\overline{\rho}}\cap V(\hat{p})} \Big|\sum_{\alpha=1}^{s} A_\alpha \Phi_\alpha(\tilde{p})\Big|;\end{aligned} \tag{8}$$

this estimate holds under the assumption that $\|\tilde{p}-p\|, \tau < \delta$.

In view of (6), we have also the weaker estimate

$$\|\Phi_\alpha(\tilde{p})\| < C'' \text{ for } \|\tilde{p}-p\| < \delta, \tag{9}$$

with a constant C'' independent of $\tilde{p}$. Combining (8) and (9), we obtain a constant $\tilde{C}$ independent of $\tilde{p}$, τ, g, A_α such that

$$\Big\|\sum_{\alpha=1}^{s} A_\alpha\Phi_\alpha(\tilde{p})\Big\| \le \tilde{C} \max_{Q_{2\overline{\rho}}\cap V(\hat{p})} \Big|\sum_{\alpha=1}^{s} A_\alpha \Phi_\alpha(\tilde{p})\Big|. \tag{10}$$

Moreover, since $2\overline{\rho} \le \rho_0$, we have

$$\max_{Q_{2\overline{\rho}}\cap V(\hat{p})} |f_\#| \le \max_{Q_{\rho_0}} |f_\#| \le C\|f_\#\|, \tag{11}$$

with C depending only on c_1, D, n.

By (3), (11) and the equation $[\text{otherfac}\,(\tilde{p})]^m f = \tilde{f}$ on $V(\hat{p})$, we have

$$\begin{aligned}\max_{Q_{2\overline{\rho}}\cap V(\hat{p})} \Big|\sum_{\alpha=1}^{s} A_\alpha\Phi_\alpha(\tilde{p}).\Big| &\le \max_{Q_{2\overline{\rho}}\cap V(\hat{p})} |\tilde{f}| + \tau^m \max_{Q_{2\overline{\rho}}\cap V(\hat{p})} |f_\#| \\ &\le \max_{Q_{2\overline{\rho}}\cap V(\hat{p})} |[\text{otherfac}\,(\tilde{p})]^m f| + \tau^m C\|f_\#\| \\ &\le C_\#\Big\{\max_{Q_{2\overline{\rho}}\cap V(\hat{p})} |f| + \tau^m\|f\|\Big\},\end{aligned} \tag{12}$$

in view of estimate (2) and Lemma 4.1 Here, $C_\#$ is independent of f, $\tilde{p}$, τ, g. By (10), (12) and the fact that $2\overline{\rho} \le \rho$, we have

$$\Big\|\sum_{\alpha=1}^{s} A_\alpha\Phi_\alpha(\tilde{p})\Big\| \le C'_\#\Big\{\max_{Q_\rho\cap V(\hat{p})} |f| + \tau^m\|f\|\Big\} \tag{13}$$

with $C'_\#$ independent of f, $\tilde{p}$, τ, g.

Substituting (13) and (2) into (3), we obtain the estimate

$$\|\tilde{f}\| \le C''_\#\Big\{\max_{Q_\rho\cap V(\hat{p})} |f| + \tau^m\|f\|\Big\},$$

with $C''_\#$ independent of f, $\tilde{p}$, τ, g. This is precisely conclusion (b).

7. Local boundedness of the norm.

LEMMA 7. — *Fix $t(1 \leq t \leq D)$. There exist constants D', C', $(D' \geq D, C' \geq 1)$ such that the following holds : let $p \in W^t$ and $\rho > 0$ be given. Then there exist positive numbers δ, K such that* $\mathrm{Norm}_*(\hat{p}, D', C', \rho) \leq K$ *for all $\hat{p} \in W$ with $\|\hat{p} - p\| < \delta$.*

Proof. — We use backwards induction on t.

First suppose $t = D$. Lemma 4.2 shows that W^D is a relatively open subset of W. Thus, given $p \in W^D$ we can find $\delta_1 > 0$ such that

(1) $$\hat{p} \in W, \qquad \|\hat{p} - p\| < \delta_1 \text{ imply } \hat{p} \in W^D.$$

On the other hand, let $p \in W^D$ and $\rho > 0$ be given. By Lemma 6.1, we can find K, $\delta_2 > 0$ such that

(2) $$\hat{p} \in W^D, \qquad \|\hat{p} - p\| < \delta_2 \text{ imply } \mathrm{Norm}\,(\hat{p}, 1, D, \rho) \leq K.$$

Set $\delta = \min\{\delta_1, \delta_2\}$ for a given $p \in W^D$, $\rho > 0$. Then (1) and (2) show at once that $\hat{p} \in W$, $\|\hat{p} - p\| < \delta$ imply $\mathrm{Norm}\,(\hat{p}, 1, D, \rho) \leq K$.

Hence by definition of $\mathrm{Norm}_*(\hat{p}, D, C', \rho)$, we have

(3)
$$\mathrm{Norm}_*(\hat{p}, D, C', \rho) \leq K \text{ for all } \hat{p} \in W \text{ with } \|\hat{p} - p\| < \delta, \text{ provided } C' \geq 1.$$

This proves Lemma 7 for $t = D$.

Next suppose Lemma 7 holds with t replaced by any $t' > t$. $(1 \leq t \leq D - 1)$. We will prove that Lemma 7 holds for the given t. This backwards induction step will complete the proof of Lemma 7.

Thus, for suitable constants $D', C'(D' \geq D, C' \geq 1)$, we assume the following.

Let $p \in W^{(t+1)+}, \rho > 0$ be given. Then there exist δ, $K > 0$ such that $\hat{p} \in W$, $\|\hat{p} - p\| < \delta$ imply $\mathrm{Norm}_*(\hat{p}, D', C', \rho) \leq K$. In other words, for each $\rho > 0$, the map $\hat{p} \in W^{(t+1)+} \mapsto \mathrm{Norm}_*(\hat{p}, D', C', \rho)$ is locally bounded.

Since also $\mathrm{Norm}_*(\hat{p}, D', C', \rho)$ is monotone decreasing in ρ, it follows that the map

(4) $$f : (\hat{p}, \rho) \mapsto \mathrm{Norm}_*(\hat{p}, D', C', \rho)$$

is locally bounded on $E = W^{(t+1)+} \times (0, 1]$.

We apply Lemma 5.2, with f and E as above, and with $S = W \times [0, 1]$. Note that S is compact, E and S are semi-algebraic sets, and f is a semi-algebraic function, by Lemma 5.3. Thus the hypotheses of Lemma 5.2 are satisfied.

Lemma 5.2 gives us constants C'', m such that

(5) $\mathrm{Norm}_*(\hat{p}, D', C', \rho) \leq C''[\mathrm{dist}((\hat{p}, \rho), W \times [0,1] \backslash W^{(t+1)+} \times (0,1])]^{-m}$

for all $\hat{p} \in W^{(t+1)+}, \rho \in (0,1]$.

Note that $\mathrm{dist}((\hat{p}, \rho), W \times [0,1] \backslash W^{(t+1)+} \times (0,1]) \geq \min(\rho, \mathrm{dist}(\hat{p}, W \backslash W^{(t+1)+})) \geq c\,\rho \cdot \mathrm{dist}(\hat{p}, W \backslash W^{(t+1)+})$, since $W \times [0,1] \backslash W^{(t+1)+} \times (0,1] = (W \backslash W^{(t+1)+}) \times [0,1] \cup W \times \{0\}$. Putting these inequalities into (5), we get

(6) $$\mathrm{Norm}_*(\hat{p}, D', C', \rho) \leq C'' \rho^{-m} [\mathrm{dist}(\hat{p}, W \backslash W^{(t+1)+})]^{-m}$$

for $\hat{p} \in W^{(t+1)+}$.

Recalling the definitions of $\mathrm{Norm}_*(\hat{p},D',C',\rho)$ and of $\mathrm{Norm}(p,q,D',\rho)$, we can reformulate (6) as follows :

(7) Let $\hat{p} \in W^{(t+1)+}$ be given. Then we can find $\hat{q} \in H^{D'}$ with

(a) $$\hat{q}(0) = 1 \text{ and } \|\hat{q}\| \leq C',$$

such that for any $f \in H^D$ we can find an $\hat{f} \in H^{D'}$ for which we have

(b) $$\hat{f} = \hat{q} f \text{ on } V(\hat{p})$$

and

(c) $$\|\hat{f}\| \leq C'' \rho^{-m} [\mathrm{dist}(\hat{p}, W \backslash W^{(t+1)+})]^{-m} \max_{Q_\rho \cap V(\hat{p})} |f|.$$

Our task now is to find constants $\underline{D}'$, $\underline{C}'$ with $\underline{D}' \geq D$, $\underline{C}' \geq 1$, such that given $p \in W^t$ and $\rho > 0$ there exist $\underline{\delta}$, $\underline{K} > 0$ such that $\hat{p} \in W$, $\|\hat{p} - p\| < \underline{\delta}$ imply $\mathrm{Norm}_*(\hat{p}, \underline{D}', \underline{C}', \rho) \leq \underline{K}$. We begin by picking $\underline{D}'$ and $\underline{C}'$. With D', m as in (7), we set

(8) $$\underline{D}' = D' + mD.$$

Then we pick $\underline{C}'$ to satisfy the following conditions :

(9a) $$\underline{C}' \geq 1$$

and

(9b) If $\tilde{p} \in W$ and $\hat{q} \in H^{D'}$ with $\|\hat{q}\| \leq C'$, then $\|[\mathrm{otherfac}(\tilde{p})]^m \hat{q}\| \leq \underline{C}'$.

We can satisfy (9b) by virtue of Lemma 4.1.

Now suppose we are given $p \in W^t$ and $\rho > 0$. We pick $\underline{\delta} > 0$ small enough to satisfy the following conditions :

(10) If $\hat{p} \in W$ and $\|\hat{p} - p\| \leq 2\underline{\delta}$ then $\hat{p} \in W^{t+}$.

(11) There is a constant K_1 such that any $\hat{p} \in W^t$ with $\|\hat{p} - p\| \leq \underline{\delta}$ satisfies $\mathrm{Norm}(\hat{p}, 1, D', \rho) \leq K_1$.

There exists $K_2 > 0$ with the following property :

(12) Assume $\hat{p} = \tilde{p} + \tau g$ with $\tilde{p} \in W^t, g \in H^D, \tau > 0,$ $\|g\| \leq 1, \|\tilde{p} - p\| < 10\underline{\delta}, \tau < 10\underline{\delta}.$

Then given any $\hat{f} \in H^{D'}$ there exists $\tilde{f} \in H^{D'+mD}$ such that

$$\tilde{f} = [\mathrm{otherfac}\,(\tilde{p})]^m \hat{f} \text{ on } V(\hat{p}), \tag{12a}$$

and

$$\|\tilde{f}\| \leq K_2 \Big\{ \max_{Q_\rho \cap V(\hat{p})} |\hat{f}| + \tau^m \|\hat{f}\| \Big\}. \tag{12b}$$

Note that we can satisfy (10) by Lemma 4.2, we can satisfy (11) by Lemma 6.1, and we can satisfy (12) by Lemma 6.4. This completes our selection of $\underline{\delta}$. We fix constants K_1 as in (11) and K_2 as in (12). Next we pick $\underline{K}$. Let $C_\#$ be a positive constant for which we have

$$\hat{q} \in H^{D'}, \|\hat{q}\| \leq C' \text{ imply } \max_{Q_\rho} |\hat{q}| \leq C_\#. \tag{13}$$

Then we define

$$\underline{K} = K_1 + K_2\{C_\# + C'' \rho^{-m}\}. \tag{14}$$

Now let $\hat{p} \in W$ be given, with $\|\hat{p} - p\| < \underline{\delta}$. We will prove that

$$\mathrm{Norm}_*(\hat{p}, \underline{D}', \underline{C}', \rho) \leq \underline{K}. \tag{15}$$

This will complete the backwards induction on t.

In fact, (10) shows that either $\hat{p} \in W^t$ or $\hat{p} \in W^{(t+1)+}$. If $\hat{p} \in W^t$, then (11) implies (15) since $\underline{D}' \geq D', \underline{C}' \geq 1$ and $K_1 \leq \underline{K}$. Hence to prove (15), we may assume $\hat{p} \in W^{(t+1)+}$. Let $\hat{q}$ be as in (7). Define

$$\tau = \mathrm{dist}\,(\hat{p}, W \backslash W^{(t+1)+}), \tag{16}$$

and let $\tilde{p} \in W \backslash W^{(t+1)+}$ with

$$\|\hat{p} - \tilde{p}\| = \tau. \tag{17}$$

We can find such a $\tilde{p}$, by Lemma 4.2. Note that $\tau > 0$ since $\hat{p} \in W^{(t+1)+}$. Also, since $p \in W \backslash W^{(t+1)+}$, we have $\tau \leq \|\hat{p} - p\| < \underline{\delta}$. Thus, (17) shows that

$$\|\tilde{p} - p\| \leq 2\underline{\delta} \text{ and } 0 < \tau < \underline{\delta}. \tag{18}$$

Now set $g = (\hat{p} - \tilde{p})/\tau$. Then (17) implies

$$\hat{p} = \tilde{p} + \tau g \text{ with } g \in H^D, \|g\| = 1. \tag{19}$$

By (10), (18) and the property $\tilde{p} \in W \backslash W^{(t+1)+}$, we have

(20) $$\tilde{p} \in W^t.$$

Now suppose $f \in H^D$ is given. By (7) and (16), we can find an $\hat{f} \in H^{D'}$ that satisfies

(21) $$\hat{f} = \hat{q} f \quad \text{on } V(\hat{p}), \text{ and}$$

(22) $$\|\hat{f}\| \leq C'' \rho^{-m} \tau^{-m} \max_{Q_\rho \cap V(\hat{p})} |f|.$$

To this $\hat{f}$ we apply (12). Note that the asumptions of (12) are satisfied, by virtue of (18), (19), (20). Thus there exists an $\tilde{f} \in H^{D'+mD}$ that satisfies

(23) $$\tilde{f} = [\text{otherfac}\,(\tilde{p})]^m \hat{f} \text{ on } V(\hat{p}), \text{ and}$$

(24) $$\|\tilde{f}\| \leq K_2 \Big\{ \max_{Q_\rho \cap V(\hat{p})} |\hat{f}| + \tau^m \|\hat{f}\| \Big\}.$$

From (21), (23) we get

(25) $$\tilde{f} = (\hat{q}[\text{otherfac}\,(\tilde{p})]^m) f \quad \text{on } V(\hat{p}).$$

From (21) we get $\max_{Q_\rho \cap V(\hat{p})} |\hat{f}| \leq \left(\max_{Q_\rho} |\hat{q}| \right) \cdot \max_{Q_\rho \cap V(\hat{p})} |f|$. Hence, (7a) and (13) yield $\max_{Q_\rho \cap V(\hat{p})} |\hat{f}| \leq C_\# \max_{Q_\rho \cap V(\hat{p})} |f|$. Putting this and (22) into (24), we see that

(26) $$\|\tilde{f}\| \leq K_2 \{C_\# + C'' \rho^{-m}\} \max_{Q_\rho \cap V(\hat{p})} |f|.$$

Comparing (26) with (14), we get

(27) $$\|\tilde{f}\| \leq \underline{K} \max_{Q_\rho \cap V(\hat{p})} |f|.$$

Thus, given $f \in H^D$, we have found $\tilde{f} \in H^{D'+mD}$ that satisfies (25) and (27). By definition of the Norm, this means that

(28) $$\text{Norm}\,(\hat{p}, \hat{q} \cdot [\text{otherfac}\,(\tilde{p})]^m, D' + mD, \rho) \leq \underline{K}.$$

Let us examine $\hat{q} \cdot [\text{otherfac}\,(\tilde{p})]^m$. Since $\hat{q} \in H^{D'}$, we have

(29) $$\hat{q} \cdot [\text{otherfac}\,(\tilde{p})]^m \in H^{D'+mD}.$$

By (7a) and the defining property of otherfac $(\tilde{p})$, we have also

(30) $$\hat{q} \cdot [\text{otherfac}\,(\tilde{p})]^m = 1 \text{ at the origin.}$$

Moreover, (7a) and (9b) imply

(31) $$\|\hat{q} \cdot [\text{otherfac}\,(\tilde{p})]^m\| \leq \underline{C}'.$$

Since $\underline{D}' = D' + mD$, we see from (28), (29), (30), (31) and the definition of the Norm_* that $\text{Norm}_*(\hat{p}, \underline{D}', \underline{C}', \rho) \leq \underline{K}$, which is the desired estimate (15). Thus, (15) holds for all $\hat{p} \in W$ with $\|\hat{p} - p\| < \underline{\delta}$. We have shown that there exist $\underline{D}', \underline{C}'$ such that for any $p \in W^t, \rho > 0$ there exist $\underline{\delta}, \underline{K} > 0$ such that for any $\hat{p} \in W$ with $\|\hat{p} - p\| < \underline{\delta}$ we have $\text{Norm}_*(\hat{p}, \underline{D}', \underline{C}', \rho) \leq \underline{K}$. This completes our backwards induction on t, thus proving Lemma 7.

Proof of the extension theorem.

It is now trivial to prove the extension theorem. By Lemma 7 we can find constants $D', C' (D' \geq D, C' \geq 1)$ with the following property :

For any $\rho > 0$, the function $p \in W \mapsto \text{Norm}_*(p, D', C', \rho)$ is locally bounded. Since W is compact, it follows that

$$f(\rho) = \sup\{\text{Norm}_*(p, D', C', \rho) : p \in W\} \tag{1}$$

satisfies

$$0 \leq f(\rho) < \infty \text{ for all } \rho \in (0, \infty). \tag{2}$$

Theorem 5.1 Corollary 3 and Lemma 5.3 imply that $f(\rho)$ is an extended semi-algebraic function on $(0, \infty)$. Estimate (2) then shows that $f(\rho)$ is a semi-algebraic function on $(0, \infty)$. Since $f(\rho)$ is monotone decreasing, it is locally bounded on $(0, \infty)$. Lemma 5.2 applied to $S = [0, \infty], E = (0, 1]$ yields

$$|f(\rho)| \leq C\rho^{-m} \text{ for } 0 < \rho \leq 1. \tag{3}$$

From (1) and (3) we get

$$\text{Norm}_*(p, D', C', \rho) \leq C\rho^{-m} \text{ for all } p \in W, \rho \in (0, 1].$$

Note that D', C', C, m depend only on c_1, D, n.

By definition of the Norm, this means the following. Let $p \in W, \rho \in (0, 1]$ be given. Then there exists $q \in H^{D'}$ with $q(0) = 1, \|q\| \leq C'$, such that for all $f \in H^D$ we can find an $\tilde{f} \in H^{D'}$ that satisfies

$$\tilde{f} = qf \text{ on } V(p), \text{ and} \tag{a}$$

$$\|\tilde{f}\| \leq C\rho^{-m} \max_{Q_\rho \cap V(p)} |f|. \tag{b}$$

Now pick a small constant $\rho_1 > 0$ depending on c_1, D, n. Lemma 6.3(a) shows that $V(p) \cap Q_{2\rho_1} = \{x \in Q_{2\rho_1} : p(x) = 0\}$. Also, $q \in H^{D'}$, $q(0) = 1, \|q\| \leq C'$ imply that $\frac{1}{2} \leq q \leq 2$ on $Q_{2\rho_1}$ if ρ_1 is small enough. Hence, (a) and (b) above yield :

$$f = \tilde{f}/q \text{ on } \{x \in Q_{2\rho_1} : p(x) = 0\} \text{ where } \tilde{f}, q \in H^{D'} \tag{α}$$

$$\frac{1}{2} \leq q \leq 2 \text{ on } Q_{2\rho_1} \tag{β}$$

$$\max_{Q_{2\rho_1}} |\tilde{f}| \leq \overline{C} \max\{|f(x)| : x \in Q_{\rho_1} \text{ and } p(x) = 0\}. \tag{γ}$$

Here, $\overline{C}$ depends only on c_1, n, D since ρ_1 only depends on c_1, n, D. We have proved $(\alpha), (\beta), (\gamma)$ for $p \in W, f \in H^D$.

Next, let p, c, C, D be as in the statement of the extension theorem. We have $p/\|p\| \in W$ with c_1 depending only on c, C, D, n. Applying $(\alpha), (\beta), (\gamma)$ to $p/\|p\|$, we obtain the conclusions of the extension theorem for the value $\rho = \rho_1$. Thus, it remains only to pass from $\rho = \rho_1$ to $\rho \le \rho_1$. Now, if p satisfies the hypotheses of the extension theorem, so also does $\tilde{p}(x) = \dfrac{\rho_1}{\rho} p\left(\dfrac{\rho}{\rho_1} x\right)$ for $0 < \rho \le \rho_1$.

Applying the conclusions of the extension theorem to the polynomial $\tilde{p}$ and the value $\rho = \rho_1$ gives us the extension theorem for the polynomial p and any given value of ρ $(0 < \rho \le \rho_1)$.

Proof of Theorem 1. — Let $f, P, Q, \Gamma, \psi, D, d, c, C$ be as in the statement of Theorem 1. Thus, conditions (I) $\cdots$ (IV) hold. Denote by c_*, C_*, C'_* etc. a positive constant depending only on c, C, d, D.

We have to prove (A), (B), and (C). The first step is to reduce matters to the case in which ψ satisfies

$$(1) \qquad |\psi'(x)| \le \frac{1}{2} \quad \text{for } |x| \le 1.$$

In fact, $Q(x, \psi(x)) = 0$, $\left|\dfrac{\partial Q}{\partial x}\right| \le C_*$, and $\left|\dfrac{\partial Q}{\partial y}\right| \ge c_*$ at $(x, \psi(x))$ for $|x| \le 1$. Hence, $\left(\dfrac{\partial Q}{\partial x}\right) + \psi'(x)\left(\dfrac{\partial Q}{\partial y}\right) = 0$ and therefore $|\psi'(x)| \le C^1_*$ for $|x| \le 1$. If $2C^1_* \le 1$, then we have (1). Otherwise, set

$$\underline{\psi}(x) = \psi(x)/(2C^1_*), \underline{\Gamma} = \{(x, \underline{\psi}(x)) : |x| \le 1\},$$
$$\underline{Q}(x, y) = Q(x, 2C^1_* y), \underline{P}(x, y) = P(x, 2C^1_* y).$$

One checks easily that $\underline{\psi}, \underline{\Gamma}, \underline{Q}$ again satisfy conditions (I) $\cdots$ (IV) with constants depending only on c, C, D. Moreover, (1) holds for $\underline{\psi}$. Also, $f(x) = P(x, \psi(x)) = \underline{P}(x, \underline{\psi}(x))$. Thus, to prove (A), (B), (C) we may replace P, Q, Γ, ψ, by $\underline{P}, \underline{Q}, \underline{\Gamma}, \underline{\psi}$. So we may assume that (1) holds.

For $x_0 \in [-1, 1]$ and $\delta > 0$, define

$$I_{x_0,\delta} = \{x \in \mathbb{R} : |x - x_0| < \delta\} \text{ and } U_{x_0,\delta} = \{(x, y) \in \mathbb{R}^2 : |x - x_0| < \delta \text{ and } |y - \psi(x_0)| < \delta\}.$$

Assume that $I_{x_0,2\delta} \subset [-1, 1]$. Then by (1), $x \in I_{x_0,2\delta}$ implies $(x, \psi(x)) \in U_{x_0,2\delta}$. Also, $\left|\dfrac{\partial Q}{\partial y}\right| \ge c$ at $y = \psi(x)$ and $\left|\dfrac{\partial^2 Q}{\partial y^2}\right| \le C_*$ in $U_{x_0,2\delta}$. Hence

$(x,y) \in U_{x_0,2\delta}$ implies $\left|\frac{\partial Q}{\partial y}\right| \geq c - C_*|y-\psi(x)| \geq c - C_* \cdot (4\delta) > 0$ provided we take

(2) $$\delta < c_*^1 \text{ for a small enough } c_*^1.$$

Therefore for fixed $x \in I_{x_0,2\delta}$, we can have $Q(x,y) = 0, (x,y) \in U_{x_0,2\delta}$ for at most one y. This condition holds when $y = \psi(x)$, so we have proven that

(3) $$\{(x,y) \in U_{x_0,2\delta} : Q(x,y) = 0\} = \{(x,\psi(x)) : x \in I_{x_0,2\delta}\}, \text{ and}$$

(4) $$\{(x,y) \in U_{x_0,\delta} : Q(x,y) = 0\} = \{(x,\psi(x)) : x \in I_{x_0,\delta}\}.$$

Let ρ_1 be as in the statement of the Extension Theorem, with $n = 2$, and with $d+D$ in place of D. Thus ρ_1 is a small constant of the form c_*. Assume

(5) $$\delta < \rho_1.$$

Then the extension theorem shows that there are polynomials $F(x,y)$, $G(x,y)$ of degree at most D', satisfying the conditions:

(6) $$P = F/G \quad \text{on} \quad \{(x,y) \in U_{x_0,2\delta} : Q(x,y) = 0\}$$

(7) $$\frac{1}{2} \leq G \leq 2 \text{ on } U_{x_0,2\delta}.$$

(8) $$\max_{U_{x_0,2\delta}} |F| \leq C_* \max\{|P(x,y)| : (x,y) \in U_{x_0,\delta} \text{ and } Q(x,y) = 0\}.$$

Now we fix δ to be a small constant $c_* < \frac{1}{20}$, small enough to satisfy (2) and (5). Thus, (3), (4), (6), (7), (8) hold provided $I_{x_0,2\delta} \subset [-1,1]$. From (3) and (6) and the definition of f, we get

(9) $$f(x) = \frac{F(x,\psi(x))}{G(x,\psi(x))} \text{ for } x \in I_{x_0,2\delta}.$$

From (3) and (7) we get

(10) $$\frac{1}{2} \leq G(x,\psi(x)) \leq 2 \text{ for } x \in I_{x_0,2\delta}.$$

Also, (7) and the fact that $\delta = c_*$ and G is a polynomial of degree at most D' yield the bound

(11) $$|G_x(x,y)|, |G_y(x,y)| \leq C_*, (x,y) \in U_{x_0,2\delta},$$

for the partial derivations of G.

Since $\frac{d}{dx}\{G(x,\psi(x))\} = G_x(x,\psi(x)) + \psi'(x)G_y(x,\psi(x))$, equations (1), (3), (11) yield

(12) $$\left|\frac{d}{dx}\{G(x,\psi(x))\}\right| \leq C_* \text{ for } x \in I_{x_0,2\delta}.$$

Next, (4), (8) and the definition of f yield

$$\max_{U_{x_0,2\delta}} |F| \le C_* \max_{I_{x_0,\delta}} |f|. \tag{13}$$

Since F is a polynomial of degree at most D' and δ has the form c_*, (13) implies the bounds

$$\max_{U_{x_0,2\delta}} (|F_x| + |F_y|) \le C_* \max_{I_{x_0,\delta}} |f| \tag{14}$$

for the partial derivatives of F. From (3) and (13) we get

$$|F(x,\psi(x))| \le C_* \max_{I_{x_0,\delta}} |f| \text{ for all } x \in I_{x_0,2\delta}. \tag{15}$$

Also, since $\frac{d}{dx}\{F(x,\psi(x))\} = F_x(x,\psi(x)) + \psi'(x)F_y(x,\psi(x))$, equations (1), (3), (14) imply

$$\left|\frac{d}{dx}\{F(x,\psi(x))\}\right| \le C_* \max_{I_{x_0,\delta}} |f| \text{ for all } x \in I_{x_0,2\delta}. \tag{16}$$

Immediately from (9), (10), (12), (15), (16) we obtain the basic estimates

$$\max_{I_{x_0,2\delta}} |f| \le C_* \max_{I_{x_0,\delta}} |f| \text{ and} \tag{17}$$

$$\max_{I_{x_0,2\delta}} |f'| \le C_* \max_{I_{x_0,\delta}} |f|. \tag{18}$$

We have proven (17) and (18) for $\delta = c_* < \frac{1}{20}$, assuming $I_{x_0,2\delta} \subset [-1,1]$.

Now it is trivial to complete the proof of Theorem 1. Using (17) and induction on k, we see that

$$\max_{[-1,1]} |f| \le C_*^k \max_{[-1+k\delta,1-k\delta]} |f| \text{ for } 0 \le k \le \frac{2}{3}\delta^{-1}.$$

Taking $k > \frac{1}{2}\delta^{-1}$, we obtain the first conclusion of Theorem 1. From (18) we obtain at once $\max_{[-1,1]} |f'| \le C_* \max_{[-1,1]} |f|$ which is the second conclusion of Theorem 1.

Next we divide $[-1,1]$ into $2N$ equal subintervals I_ν, where N is a large integer to be picked in a moment. For each I_ν we have

$$\max_{I_\nu} |f| \le \frac{1}{N} \max_{I_\nu} |f'| + N \int_{I_\nu} |f(x)|dx.$$

Applying this to the I_ν containing a point where $|f|$ is maximized on $[-1,1]$, we find that

$$\max_{[-1,1]} |f| \le \frac{1}{N} \max_{[-1,1]} |f'| + N \int_{-1}^{1} |f(x)|dx. \tag{19}$$

Putting the second conclusion of Theorem 1 into (19), we obtain

$$\max_{[-1,1]} |f| \leq \frac{C_*}{N} \max_{[-1,1]} |f| + N \int_{-1}^{1} |f(x)| dx. \tag{20}$$

We pick N to be the least integer larger than $2C_*$. The first term on the right in (20) can then be absorbed into the left-hand side, leaving us with the estimate

$$\frac{1}{2} \max_{[-1,1]} |f| \leq (2C_* + 1) \int_{-1}^{1} |f(x)| dx.$$

This proves the final conclusion of Theorem 1.

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