ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 43, nº 5 (1993), p. 1253-1266

<http://www.numdam.org/item?id=AIF_1993__43_5_1253_0>

Ann. Inst. Fourier, Grenoble
43, 5 (1993), 1253-1266

EXOTIC PARAMETRIZATION PROBLEMS

by Leon EHRENPREIS

Dedicated to my good friend, Bernard Malgrange

Given a system of differential equations

$$\overrightarrow{P}(x,D)F = \overrightarrow{g} \tag{1}$$

on R^n meaning

$$P_j(x,D)F = g_j, \quad j = 1,2,\cdots,r$$

there is an associated dimension. If the P_j have constant coefficients then this is the dimension of the algebraic variety

$$V : P_j(i\widehat{x}) = 0 \quad j = 1,2,\cdots,r. \tag{2}$$

If the P_j have analytic coefficients then the dimension is the maximum dimension of any analytic manifold $L \subset R^n$ (local) such that any analytic function h on L is the restriction to L of a solution F of $\overrightarrow{P}F = 0$ near L. We can determine these L by the Cartan-Kahler theory. Such maximal manifolds L are called $\overrightarrow{P}$ *Lagrangians*. We refer to dim L as dim $\overrightarrow{P}$. For $n = 2m$ and $\overrightarrow{P} = \overline{\partial}$ a $\overrightarrow{P}$ Lagrangian is a totally real subspace of R^n which is given the complex structure associated to $\overline{\partial}$, that is, a $\overline{\partial}$ Lagrangian is [essentially, that is, up to an element of $GL(n,R)$] a usual Lagrangian.

If $\overrightarrow{P}$ has C^∞ coefficients then I do not know a general definition of $\overrightarrow{P}$ Lagrangian or of the dimension associated to $\overrightarrow{P}$.

Work supported by National Science Foundation Grant No. DMS-9106262.
Key-words : Boundary value problems – Overdetermined systems.
A.M.S. Classification : 35N10.

For the usual boundary problems associated to $\overrightarrow{P}$ such as the Cauchy and Dirichlet problems, (CP) and (DP), the data is given on a $\overrightarrow{P}$ Lagrangian L. (Sometimes that does not suffice and we have to give data on a finite number of lower dimensional varieties $L_0 \subset L_1 \subset \cdots \subset L$.)

The CP and DP are ways of parametrizing solutions of $\overrightarrow{P} F = 0$. We use the term exotic parametrization problem to indicate a way of parametrizing solutions by data on manifolds M whose dimension is different from the dimension of $\overrightarrow{P}$. (We shall deal with situations in which all data are given on a single manifold; the more general case when we need additional submanifolds presents only some technical difficulties.)

There are two ways in which $\dim M \neq \dim \overrightarrow{P}$, namely

(a) $\dim M > \dim \overrightarrow{P}$.

(b) $\dim M < \dim \overrightarrow{P}$.

In case (a) we shall consider two situations:

(a1) M is smooth.

(a2) M is "polygonal".

For the $\bar{\partial}$ system (a1) was pioneered by Hartogs and Lewy (see [E4], [L]). Early work on (a2) bears the names of Hartogs and Bernstein (see [AR]).

When $\dim M > \dim L$ the solutions of $\overrightarrow{P} F = 0$ satisfy differential equations on M. Thus we want to parametrize solutions of $\overrightarrow{P} F = 0$ by solutions of the "induced system" $\overrightarrow{P}_M f_M = 0$ on M. If $M = R^n$ then $\overrightarrow{P}_M = \overrightarrow{P}$ and everything is trivial.

This article contains the germs of our ideas. Full details appear in the author's book: *The Radon Transform* [E2].

We now clarify what we mean by Parametrization Problem. We start with differential operators $\partial_1, \cdots, \partial_s$ (of arbitrary order) which are "normal derivatives" meaning that they cannot be expressed in terms of tangential derivatives to M. The parametrization problem is related to the map

$$\alpha : F \to \left[\overrightarrow{P} F, \{\partial_j F|_M\}\right].$$

α is a map of a suitable function or distribution space W on R^n into $W^{\overrightarrow{P}} \oplus [W(M)]^s$ where $W^{\overrightarrow{P}}$ is the subset of W^r satisfying the compatibility relations of the P_j and $W(M)$ is a space of functions or distributions on M related to W. For example if W is the space of C^∞ functions on R^n then $W(M)$ is the space of C^∞ functions on M.

In case (a) the elements of $W(M)$ satisfy differential equations.

In case (b) there are infinitely many ∂_j so $[W(M)]^s$ must be interpreted in a somewhat different manner which we shall explain presently.

Denote by α' the adjoint of α. Thus α' is a map

$$\alpha' : (\overrightarrow{T}, \{S_j\}) \to \overrightarrow{P}'\overrightarrow{T} + \sum \partial'_j S_j.$$

In the right side of this formula S_j are considered as elements of W' using the adjoint of restriction to M.

In case $s = \infty$ we regard $\sum \partial'_j S_j$ as an element of a suitably defined topological tensor-like product $\widetilde{W'}(M)\widetilde{\otimes} U'$ of a suitable subspace $\widetilde{W'}(M)$ of $[W'(M)]^\infty$ with some space of functions or distributions U' in the directions "orthogonal to M"; U' is a space spanned by $\{\partial'_j\}$ in a suitable topological sense. We shall present some examples of this tensor product below. As we shall see, we do not get an actual topological tensor product but some modification of it.

Our work on (a1) is inspired by Lewy's theorem (see [L]) that if M is a hypersurface then, under suitable "convexity" conditions, a solution of $\overline{\partial}_M f_M$ on M can be extended on an a priori "inside" neighborhood to a solution of $\overline{\partial} F = 0$. Actually Lewy's work applies to some M with codimension $M > 1$.

Our first task is to define the induced system $\overrightarrow{P}_M$. This is done most simply formal using power series. Suppose first that M is a smooth (local) hypersurface defined by $\chi(x) = 0$. We choose some coordinates θ on M and we use χ to define the "orthogonal" coordinate. We study formal power series solutions $h = \sum h_j(\theta)\chi^j$ of $\overrightarrow{P} h = 0$. The $h_j(\theta)$ are assumed to be smooth. (For $\overline{\partial}$ the calculations appear in [E6].)

Proposition 1. — *Suppose $\overrightarrow{P}$ has holomorphic coefficients. For generic M there is an integer N and an $N \times N$ matrix $\boxed{P}^N_M$ of differential operators in θ such that*

(3) $$\overrightarrow{P} h = 0 \text{ implies } \boxed{P}^N_M \ \overrightarrow{h}^N_M = 0.$$

Here $\overrightarrow{h}^N_M = (h_0(\theta), \cdots, h_{N-1}(\theta))$. Moreover for any $\overrightarrow{h}^N_M$ satisfying (3) there is a unique formal power series solution h of $\overrightarrow{h} = 0$ starting with $\overrightarrow{h}^N_M$.

It is reasonable to regard $\boxed{P}_M^N \ \vec{h}_M^N = 0$ as the equation induced by $\vec{P}$ on M. For the $\vec{P} = \bar{\partial}$ the induced system is Lewy's. For the pluriharmonic operator $\partial\bar{\partial}$ an analog of Lewy's condition was formed by several mathematicians. For the general $\partial^k\bar{\partial}^\ell$ (symmetric derivatives) a beautiful formalism for writing $\boxed{P}_M^N$ in an explicit fashion was developed by my student J. Wang in his thesis (see [W]).

Remark. — Consider the Cauchy-Riemann equation in 2 complex variables $z_j = x_j + y_j$. The restriction of a holomorphic function to the surface $S : y \cdot y = 1$, $x \cdot y = 0$ satisfies independent differential identities of all odd orders. These do not arise from holomorphic tangent vectors but rather from pairs, triples, ... of nontangential holomorphic vectors which meet near S.

This example shows that, in general, there is no finite N for M of codimension ≥ 1. (It arose in joint work with P. Kuchment.)

So much for the formal theory. When do formal solutions extend to actual solutions?

One way of thinking of this extension question is in the spirit of hyperbolicity. Thus we want to use fundamental solutions to construct the extension.

DEFINITION. — *A* mouth *for $\vec{P}$ is a smooth submanifold (local or global) Λ such that there exists an operator ${}_\Lambda P$ in the left ideal generated by P_j with the properties:*

(i) *${}_\Lambda P$ is defined on Λ, meaning it is in the algebra generated by tangent vectors to Λ.*

(ii) *For each point $\lambda \in \Lambda$ there exists a fundamental solution ${}_\Lambda e_\lambda$ in $\mathcal{D}'(\Lambda)$ for ${}_\Lambda P$ with singularity at λ*

$$ {}_\Lambda P {}_\Lambda e_\lambda = \delta_\lambda. \tag{4} $$

For the d system every Λ is a mouth. For the $\bar{\partial}$ system holomorphic curves are the primary example of mouths.

Suppose M is an hypersurface and $q \in M$. We say that M is $\vec{P}$ mouth convex at q if there is a family of mouths Λ which intersect M compactly and non tangentially, which depend smoothly on parameters, and such that

$$ \lim \Lambda \cap M = q. \tag{5} $$

For $\overrightarrow{P} = \overline{\partial}$ mouth convexity is strict 1 pseudoconvexity.

For any point λ in the interior of any $\Lambda \cap M$, if there were an extension H of a formal solution h on M to λ, then its value would be determined at λ by ${}_{\Lambda}e_{\lambda}$. We might expect that, conversely, if M were $\overrightarrow{P}$ mouth convex at each $q \in M$ then these values would fit together to form a solution.

We conjecture that such is the case but, thus far, we have been able to prove a result of this type only in the elliptic constant coefficient case. One of the difficulties is that λ might belong to many mouths and we have to know that the proposed extension does not depend on which mouth we use.

In order to simplify the statement of our result let us assume that M and the h_j are real analytic.

THEOREM 2. — *Suppose h is a formal solution to $\overrightarrow{P}$ (elliptic, constant coefficients) with real analytic h_j. Suppose M is $\overrightarrow{P}$ mouth convex at each $q \in M$. Then h extends to a real analytic solution of $\overrightarrow{P}H = 0$ in some one sided a priori neighborhood of M.*

Remark 1. — We can replace ellipticity by a suitable unique continuation property.

Remark 2. — Theorem 2 does not follow from the Cartan-Kahler theory because the extension is to an *a priori* (not depending on h) one sided neighborhood of M.

(a2) The results we have in mind have their origin in the work of Bernstein and of Hartogs (see [AR]) on separate analyticity and edge-of-the-wedge theorems. For the $\overline{\partial}$ system the separate analyticity result we have in mind states that if f is defined on some neighborhood of zero in R^m and if for each j and for each fixed (small) $x_1^0, \cdots, x_{j-1}^0, x_{j+1}^0, \cdots, x_m^0$ the function $f(x_1^0, \cdots, x_j, \cdots, x_m^0)$ of x_j has a holomorphic extension to some fixed complex neighborhood of zero then f extends to a holomorphic function on a complex neighborhood of zero in $R^n = C^m$.

To put this in our present framework of parametrization, we think of f as being defined on some set M of real dimension $m+1$. M is formed by keeping $m-1$ coordinates real and complexifying the other variable. Thought of as an edge-of-the-wedge theorem, the edge is R^m (locally) as usual but the wedges are degenerate, being $R^m \oplus$ the various axes in the imaginary space. (Here and in what follows we could replace the whole axes by the positive axes. The result would be holomorphicity for Im x in the positive orthant. More generally we obtain the convexity of the complement of the analytic wave front set.)

The extension to general $\vec{P}(D)$ (constant coefficients) goes as follows: The real space R^m is a Lagrangian for $\bar{\partial}$ which is, in fact, noncharacteristic (see Chapter IX of [E1]). Let Ω be a linear noncharacteristic Lagrangian for $\vec{P}$. Let $\vec{h}$ be a candidate for Cauchy Data on Ω for the Cauchy Problem for $\vec{P}$.

Now let M be the union of the spans $\Omega + \alpha_j$ where α_j are suitable positive half-lines (finite in number) through the origin, with α_j not contained in Ω.

The separate analyticity hypothesis is that $\vec{h}$ extends to a solution $\vec{\tilde{h}}$ of $\vec{P}$ on

$$M = \cup(\Omega + \alpha_j). \tag{6}$$

This means that

$$\vec{\tilde{h}}$$

is a formal solution of $\vec{P}$ on M whose Cauchy Data on Ω is $\vec{h}$. This is meaningful because it can be shown that $\vec{\tilde{h}}$ is determined by $\vec{h}$.

Our general separate analyticity theorem is

THEOREM 3. — *We can find a finite number of positive half-lines α_j so that if $\vec{h} \in C^\infty$ extends to a formal solution $\vec{\tilde{h}}$ on M then $\vec{h}$ is the Cauchy Data for a C^∞ solution F on a full neighborhood of zero in the positive orthant.*

In case $\vec{h}$ is globally defined and small at infinity and Ω is linear, Theorem 3 is easily proven using Fourier transform. The whole difficulty in Theorem 3 lies in the local theorem. To prove Theorem 3 we have to appeal to a form of the *nonlinear Fourier transform*: If $g(x)$ is a function defined on R^n which is small at infinity then we define the nonlinear Fourier transform $\mathcal{G}$ of order ℓ by

$$\mathcal{G}(\widehat{p}, \widehat{x}) = \int g(x) e^{i\widehat{p}\cdot p + i\widehat{x}\cdot x}\, dx. \tag{7}$$

Here $p = (p_1, \cdots, p_N)$ is an enumeration of the

$$N = \binom{n+\ell-1}{\ell} \tag{8}$$

monomials of degree ℓ and $\widehat{p}$ is a vector with N components which are complex numbers. Thus for any fixed $\widehat{p}$ the function $\widehat{p}{\cdot}p(x)$ is a homogeneous polynomial of degree ℓ and we get all such homogeneous polynomials in this fashion.

The nonlinear Fourier transform is useful in proving Theorem 3 because the polynomials $\widehat{p}\cdot p$ serve the purpose of a cut off. [Actually we need a more general form of (7) in which we use all polynomials of degree $\leq \ell$, not only those of degrees ℓ and 1.]

The nonlinear Fourier transform is a refinement of the FBI transform which is essentially the nonlinear Fourier transform for $\ell = 2$. It was introduced at about the same time as the FBI transform by the author and Paul Malliavin (see [E7], [E2]).

Observe that the nonlinear Fourier transform satisfies the relations

$$\left[\frac{1}{i}\frac{\partial}{\partial \widehat{p}_j} - p_j\left(\frac{1}{i}\frac{\partial}{\partial \widehat{x}}\right)\right]\mathcal{G} = 0. \tag{9}$$

For any $j = 1, 2, \cdots, N$. We call these the heat (or Schroedinger) equations. There are also Plücker equations involving only $\widehat{p}$ derivatives, for example

$$\left[\frac{\partial^2}{\partial \widehat{p}_{j_1}\partial \widehat{p}_{j_2}} - \frac{\partial^2}{\partial \widehat{p}_{k_1}\partial \widehat{p}_{k_2}}\right]\mathcal{G} = 0 \tag{10}$$

whenever

$$p_{j_1}p_{j_2} = p_{k_1}p_{k_2}. \tag{11}$$

Finally

$$\mathcal{G}(0, \widehat{x}) = \widehat{g}(\widehat{x}) \tag{12}$$

is the ordinary (linear) Fourier transform of g.

To prove Theorem 3 we start with property (12). We express a suitable cut-off of $\overrightarrow{h}$ in terms of its linear Fourier transform. Using (12) and the heat equations we can shift the contour and express $\overrightarrow{h}$ in terms of an integral of its nonlinear Fourier transform over a favorable contour. The fact that $\overrightarrow{h}$ extends to the formal solution $\overrightarrow{\tilde{h}}$ enables us to derive suitable inequalities on this new contour. These inequalities imply that this integral representation of $\overrightarrow{h}$ converges in an open set in R^n thus establishing Theorem 3.

We can go further and drop the assumption that Ω is linear. We start with a real analytic $\overrightarrow{P}$ Lagrangian Ω and proposed CD $\overrightarrow{h}$ on Ω. Assume that $\overrightarrow{h}$ extends to a formal solution of $\overrightarrow{P}$ on some real analytic $M \supset \Omega$. In some cases we can characterize such $\overrightarrow{h}$ using the nonlinear Fourier transform. To define the nonlinear Fourier transform we pick a point $q \in \Omega$ and use the nonlinear Fourier transform on the tangent plane T_p to Ω at p combined with a suitable local map of Ω onto T_p. In this way we can determine if there is an extension theorem.

Our results are not yet complete. They can be applied to give another proof of Theorem 2. We start with a $\overrightarrow{P}$ Lagrangian L in M and a suitable Cauchy Problem with data on L. Using the nonlinear Fourier transform we can study the Cauchy Data of h. Then the technique of proof of Theorem 3 yields an extension.

We call this process *Fourier transform on Lagrangians.*

We note in passing the relation between the nonlinear Fourier transform and the nonlinear Radon transform.

The nonlinear Radon transform studies integrals of the form

$$\int_{\widehat{p}^0 \cdot p = s} g(x)\, dx = (\mathbf{R}g)(\widehat{p}^0, s). \tag{13}$$

It is easy to see that the knowledge of $(\mathbf{R}g)(\widehat{p}^0, s)$ for all s is the same as the knowledge of $\mathcal{G}(t\widehat{p}^0, 0)$ for all t. Thus the determination of g from part of its nonlinear Radon transform amounts to determining $\mathcal{G}$ from suitable data on the $\widehat{p}$-axis (i.e. $\widehat{x} = 0$). This determination is possible because $\mathcal{G}$ satisfies the Plücker and heat equations.

We now pass to case (b) for which dim $M <$ dim $\overrightarrow{P}$. The simplest example of this occurs for $\overrightarrow{P} = \overline{\partial}$ and M a point (say the origin). Then we need infinitely many ∂_j and, in fact, it is clear that we can take

$$\{\partial_j\} = \left\{ \frac{\partial^q}{\partial x_1^{q_1} \cdots \partial x_m^{q_m}} \right\}. \tag{14}$$

By Fourier transform the operators ∂_j become

$$\{\widehat{\partial}_j\} = \{\widehat{x}_1^{q_1} \cdots \widehat{x}_m^{q_m}\}. \tag{15}$$

It is clear in what sense the $\widehat{\partial}_j$ span the space $\widehat{\mathcal{H}}_0'$ which is the Fourier transform of the dual of the space of locally holomorphic functions.

We pass to a highly nontrivial example. Let $r = 1$ and let $\overrightarrow{P}$ be the wave operator

$$\square = \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}. \tag{16}$$

We have written $\Box$ in this form to indicate that M is the Y axis and $t = (t_1, t_2)$ is the "time axis". We search for differential operators ∂_j in t so that the map

$$\alpha' : \{\partial_j' S_j\} \to \sum \partial_j' S_j \tag{17}$$

is an isomorphism of some topological tensor-like product $\mathcal{U}'(t) \otimes \mathcal{E}'(Y)$ with $\mathcal{E}'/\Box\mathcal{E}'$. This is the adjoint of the map

$$\alpha : F \to \{\partial_j F|_Y\} \tag{18}$$

defined on solutions of $\Box F = 0$. (The inhomogeneous equation $\Box F = g$ is much more difficult than $\Box F = 0$.) Recall the result of Malgrange and Ehrenpreis (see [M1], [E3]) that $\mathcal{E}'/\Box\mathcal{E}'$ is the dual of the kernel of $\Box$.

The simplest type of ∂_j are constant coefficient operators in t. Then the Fourier transform of (17) becomes

$$\widehat{\alpha}' : \{\widehat{\partial}_j' \widehat{S}_j\} \to \sum \widehat{\partial}_j' \widehat{S}_j. \tag{19}$$

The $\widehat{\partial}'$ are polynomials in $\widehat{t}$ and $\widehat{S}_j \in \widehat{\mathcal{E}}'(Y)$. We want this to be a topological isomorphism with $\widehat{\mathcal{E}}'/\widehat{\Box}\widehat{\mathcal{E}}'$. The Fundamental Principle (see [E1]) describes $\widehat{\mathcal{E}}'/\widehat{\Box}\widehat{\mathcal{E}}'$ intrinsically: It is the space $\widehat{\mathcal{E}}'(V)$ of entire functions on $V = \{\widehat{y}^2 = \widehat{t}_1^2 + \widehat{t}_2^2\}$ which satisfy the growth conditions induced on V from the growth conditions of $\widehat{\mathcal{E}}'$.

Thus the desired isomorphism involves writing

$$H(\widehat{y}, \widehat{t}) = \sum \widehat{\partial}_j'(\widehat{t}) \widehat{S}_j(\widehat{y}) \quad \text{on } V. \tag{20}$$

Here $H \in \widehat{\mathcal{E}}'(V)$ and $\widehat{S}_j(\widehat{y}) \in \widehat{\mathcal{E}}'(\widehat{y})$.

Now a large part of V is defined by

$$\begin{aligned} \widehat{t}_1 &= \widehat{y} \cos\theta \\ \widehat{t}_2 &= \widehat{y} \sin\theta. \end{aligned} \tag{21}$$

Thus one might guess that a reasonable choice for $\{\partial_j\}$ is given by

$$\partial_j^{\pm} = \left(\frac{\partial}{\partial t_1} \pm i \frac{\partial}{\partial t_2} \right)^j \qquad j = 0, 1, 2, \cdots \tag{22}$$

for then

$$\begin{aligned} \widehat{\partial}_j^{\pm\prime}(\widehat{t}) &= (\widehat{t}_1 \pm i\widehat{t}_2)^j \\ &= \widehat{y}^j e^{\pm ij\theta} \quad \text{on } V. \end{aligned} \tag{23}$$

This means that we can solve equation (20) by fixing $\widehat{y}$ and then solving

$$H(\widehat{y}, \widehat{y}\cos\theta, \widehat{y}\sin\theta) = \sum \widehat{y}^j e^{\pm ij\theta} \widehat{S}_j^{\pm}(\widehat{y}). \tag{24}$$

The solution is given by Fourier series

$$\hat{y}^j \hat{S}_j^{\pm}(\hat{y}) = \int H(\hat{y}, \hat{y}\cos\theta, \hat{y}\sin\theta) e^{\mp ij\theta}\, d\theta. \tag{25}$$

Of course one has to check that $\hat{S}_j$ defined by (25) belong to $\hat{\mathcal{E}}'(Y)$ and one has to characterize those $\{\hat{S}_j\}$ which arise in this fashion.

THEOREM 4. — *The $\hat{S}_j^{\pm}$ corresponding to $H \in \hat{\mathcal{E}}'$ are characterized by being functions in $\hat{\mathcal{E}}'(Y)$ satisfying*

$$|\hat{S}_j^{\pm}(\hat{y})| \leq c^{j+1}(1+|\hat{y}|)^c e^{c|\hat{y}|}/j! \tag{26}$$

for all $\hat{y}$ and, for any q,

$$|\hat{S}_j^{\pm}(\hat{y})| \leq c^{q+1}(1+|\hat{y}|)^c e^{c|\mathrm{Im}\,\hat{y}|} j^{-q} |\hat{y}|^{q-j}. \tag{27}$$

Because of the term $|\hat{y}|^{q-j}$ the space $\hat{\mathcal{E}}'(V)$ is not quite in the form of a topological tensor product.

The proof of Theorem 4 can be reduced to the problem of characterizing the Fourier series coefficients on the circles $|\hat{t}|$ =const. of functions in $\hat{\mathcal{E}}'(t)$.

From Theorem 4 we can, by duality, characterize the data

$$\left\{ \left(\frac{\partial}{\partial t_1} \pm i \frac{\partial}{\partial t_2} \right)^j F(0,y) \right\} \tag{28}$$

for $F \in \mathcal{E}(t,y)$, $\Box F = 0$.

We refer to the parametrization problem defined by the data (28) as the Watergate Problem (WP) and (28) as the Watergate Data (WD). The reason is that the WP tells what information one has to gather if he stays fixed and wants to know everything– just as at Watergate people stayed at their phones and, eventually, (thought that) they knew everything.

The WP has several interesting applications. In the first place it enables us to find the solutions of the wave equation which vanish inside the light cone. Such a solution F must have all its WD supported at the origin, e.g.

$$\left(\frac{\partial}{\partial t_1} + i \frac{\partial}{\partial t_2} \right)^{j_0} F(0,y) = \delta_0^{(\ell)} \tag{29}$$

all other WD= 0. (Actually this does not fit into Theorem 4 which deals with C^∞ solutions of $\Box$. But the same ideas apply to solutions which are distributions of finite order.) We can construct these solutions explicitly. If $j_0 - \ell$ is large enough they are C^3 in R^3. Moreover they are homogeneous under scalar multiplication. As such, by separation of variables, they define eigenfunctions of the Laplacian on the hyperboloid of one sheet

$$y^2 - t_1^2 - t_2^2 = -1.$$

Because of the C^3 regularity at the light cone such functions are L^2 on this hyperboloid. Since this hyperboloid is an orbit of $SO(1,2)$, they belong to the discrete series of representations of $SL(2,R)$ (see [E5], [E2]).

Certainly a great number of interesting representations of semi simple Lie groups arise in this fashion. We are now studying the scope of this method.

The WP we just studied is called the compact WP because the real parts of $V \cap \{\widehat{y} = c\}$ (c real) are compact. If we write the wave operator in the form

$$\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2} \tag{30}$$

then this is no longer the case. If we want to find a WP with data on this Y axis then we have to use Fourier integral expansions in ζ on the quadrants defined by $\widehat{t_1} = \pm\widehat{t_2}$:

$$\begin{aligned} \widehat{t_1} &= \pm\,\widehat{y}\cosh\zeta \\ \widehat{t_2} &= \widehat{y}\sinh\zeta. \end{aligned} \tag{31}$$

and

$$\begin{aligned} \widehat{t_1} &= \widehat{y}\sinh\zeta \\ \widehat{t_2} &= \pm\,\widehat{y}\cosh\zeta. \end{aligned}$$

There are many difficulties in extending the compact analysis to this case:

(i) The coordinate system (31) degenerates on the cone $\widehat{t_1} = \pm\widehat{t_2}$. This degeneration is much more complicated than in the compact case where degeneration is only at a point.

(ii) The "normal differential operators" $\{\partial_z\}$ that arise in this theory are the Fourier transforms in t of $(t_1 \pm t_2)^{iz}$. These operators must be defined by analytic continuation in z. Moreover the ∂_z are <u>not</u> local operators.

We have not succeeded in characterizing the Fourier integral coefficients (in ζ) of $H \in \widehat{\mathcal{D}}(\widehat{t})$. [They are not defined, directly, for $H \in \widehat{\mathcal{E}}'(\widehat{t})$.] But we have solved the problem for the Schwartz space $\mathcal{S}(\widehat{t})$. The result is quite complicated. Using this theorem we can construct a WP for the wave operator $\square$ for the space S' with data on the Y axis. We refer the reader to Chapters III and IV of [E2] for details.

There is an interesting bi-product of our theory when it is applied to a type of periodic solutions of $\square F = 0$. Instead of choosing the Y axis as above, we choose a different space-like axis Y_d which is fixed by an infinite subgroup Γ_d of the modular group $\Gamma = \left\{\begin{matrix} a & b \\ c & d \end{matrix}\right\}$ considered as a subgroup of $SL(2,R)$ which is essentially $SO^+(1,2)$. [Any $g \in SL(2,R)$ defines a transformation $X \longrightarrow g^t X g$ of real symmetric 2×2 matrices. Such matrices X can be thought of as R^3 and since g preserves $\det X$ this essentially identifies $SL(2,R)$ with $SO^+(1,2)$.] Y_d is thus fixed by a hyperbolic subgroup Γ_d of Γ which we can think of as the unit group of the quadratic form $t_1^2 - dt_2^2$.

Since Γ_d fixes Y it fixes the Minkowski normal. Suppose H is Γ invariant (or even Γ_d invariant). Then H is periodic in ζ for each $\widehat{y}$. Instead of using Fourier integrals in the t direction we can use Fourier series. This Fourier series WP is not much more difficult than the compact WP.

An interesting Γ invariant distribution is

$$\widehat{\Theta} = \sum \delta_{k\ell m}, \quad k^2 = \ell^2 - m^2. \tag{32}$$

Since $\widehat{\Theta}$ is Γ invariant, so is its Fourier transform Θ. (We use the Minkowski inner product to define the Fourier transform.) We can compute the Fourier series WD of Θ.

THEOREM 5. — *For space-like Y_d the Mellin transform in scalar multiplication of the Fourier series WD of Θ is the set of ζ functions with Grossencharaktere for the field $Q(\sqrt{d})$. The Mellin transform of the WD on a time-like Y_d (which is fixed by an elliptic rather than a hyperbolic group) are the zeta functions with Grossencharaktere).*

We plan to investigate the scope of these ideas in higher dimension.

Exotic Parametrization Problems arise in other contexts. One interesting example is a variant of the edge-of-the-wedge theorem which is different from the separate analyticity of Theorem 3 because the data is given on a lower dimensional set.

For the $\overline{\partial}$ system the result takes the following form (see [KN][(*)]): Let Γ be a proper open convex cone in imaginary space and let $f^{\pm}$ be

(*) This result was pointed out to the author by Yeren Xu.

holomorphic in the tube over $\pm\Gamma$. Suppose $f^{\pm}$ and all their derivatives have continuous limits on a line Y in the real space and that these limits are equal. Suppose Y is "noncharacteristic" in the sense that, when we identify real and imaginary space, $Y \subset \Gamma \cup -\Gamma \cup \{0\}$. Then there is an entire function f which agrees with $f^{\pm}$ in the respective tubes over $\pm\Gamma$.

Actually in [KN] the result is proven when Y is a real analytic curve. Yeren Xu extended this result to C^{∞} curves. All these results hold locally.

How do we extend this result to general $\overrightarrow{P}(D)$? The real space becomes a Lagrangian L which is noncharacteristic (see Chapter IX of [E1]). Call T the orthogonal to L. We suppose that $\overrightarrow{P}$ is elliptic. Let Γ be a proper open convex cone $\subset T$.

As before we let $f^{\pm}$ be solutions of $\overrightarrow{P}(D)f^{\pm} = 0$ in the respective tubes over $\pm\Gamma$, meaning $\pm\Gamma + L$.

A linear variety $Y \subset L$ is called $\overrightarrow{P}$ noncharacteristic for Γ if, for every point α,

$$\operatorname{Proj}_L \left[(i\Gamma + L)V\right](Y^{\perp} + \alpha) \tag{33}$$

is compact. Here Γ is the dual cone to Γ in T and $Y^{\perp}$ is the orthogonal to Y in L. Γ, L, and Y are real, and V and α are complex.

It is readily verified that, for $\overrightarrow{P} = \overline{\partial}$ and L =real space, this notion of noncharacteristic agrees with our previous notion.

THEOREM 6. — *Let $Y \subset L$ be $\overrightarrow{P}$ noncharacteristic for Γ. If $f^{\pm}$ have smooth limits at the "edge", meaning $t = 0$, all of whose derivatives are equal on Y then there is a solution f of $\overrightarrow{P}(D)f = 0$ in the whole space which equals $f^{\pm}$ in the respective tubes over $\pm\Gamma$. In particular if all derivatives of f^{+} vanish on Y then $f^{+} \equiv 0$. The same result is true locally.*

The crucial point in the proof is that the compactness of (33) means that polynomials are dense on these sets. The polynomials come from the Fourier transforms of the derivatives of $f^{\pm}$ on Y. The occurrence of $Y^{\perp}$ in (33) is a consequence of the fact that restriction to Y is the Fourier transform (in $y \in Y$) of integration over the sets $Y^{\perp} + y$.

The uniqueness statement in Theorem 6 is closely related to Theorem 9.30 of [E1]. It is used to reduce Theorem 6 to an edge-of-the-wedge theorem for $\overrightarrow{P}$ where the "edge" is L. Such a result can be proven by nonlinear Fourier analysis.

Remark 1. — If we assume only that for each differential operator δ the limit of δf^+ on Y is holomorphic (even entire) then f^+ need not have a continuation.

Remark 2. — In case $f^\pm$ are globally defined and are suitably small at infinity it is possible to prove Theorem 6 using standard Fourier techniques. It is only for large $f^\pm$ (or locally defined $f^\pm$) that one needs suitable nonlinear Fourier analysis.

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