FRANCESCO AMOROSO

f- transfinite diameter and number theoretic applications

Annales de l'institut Fourier, tome 43, nº 4 (1993), p. 1179-1198

<http://www.numdam.org/item?id=AIF_1993__43_4_1179_0>

Ann. Inst. Fourier, Grenoble
43, 4 (1993), 1179-1198

f-TRANSFINITE DIAMETER AND NUMBER THEORETIC APPLICATIONS

by Francesco AMOROSO

1. Introduction.

Given a compact set $X \subset \mathbf{C}$ its transfinite diameter $t(X)$ is defined as the limit of

$$t_n(X) = \max_{x \in X^n} |V_n(x)|^{2/n(n-1)},$$

where $V_n(x) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$ is the Vandermonde determinant. We generalize this quantity introducing a weight as follows. Let $f: X \to \mathbf{R}^+$ be an upper semi-continuous function and put

$$t_{f,n}(X) = \max_{x \in X^n} \left(|V_n(x)|^{2/n(n-1)} \prod_{1 \leq i \leq n} f(x_i)^{2/n} \right).$$

As in the classical case, this sequence converges to a real number $t_f(X)$ which we shall call the "f-transfinite diameter". In the classical case it is well known that $-\log t(X)$ is the minimum over the set $\mathcal{M}_X^+$ of all unitary measures concentrated on X of the following quadratic functional

$$I[\lambda] = \int\int \log \frac{1}{|x-y|} \, d\lambda(x) d\lambda(y)$$

(see [HI] Theorem 16.4.44, p.284). The same result is still true for our generalization : $-\log t_f(X)$ is the minimum over $\mathcal{M}_X^+$ of

$$I_f[\lambda] = \int\int \log \frac{1}{f(x)} + \log \frac{1}{|x-y|} + \log \frac{1}{f(y)} \, d\lambda(x) d\lambda(y).$$

Key words : Transfinite diameter – Integer transfinite diameter – Capacities.
AMS classification : 41A10 – 31C15 – 11J82.

These new quantities arise in analytic number theory in the elementary approach to the Prime Number Theorem and in diophantine approximation in the study of the least deviation from zero of integral polynomials.

For the first field of applications, let us denote by $\psi(x)$ the Chebyshev ψ-function and let $\psi_1(x) = \sum_{h \le x} \psi(h)$. It is known that the Prime Number Theorem is equivalent to $\psi_1(x) \sim x^2/2$. The determinant

$$D_n = \mathrm{Det}\Big(\int_0^1 x^{i+j}\, dx \Big)_{0 \le i,j \le n-1}$$

is a rational number whose denominator is bounded by

$$\exp\big(\psi_1(2n) - \psi_1(n)\big).$$

On the other hand

$$D_n = \frac{1}{n!} \int_0^1 \cdots \int_0^1 V_n(x)^2\, dx_1 \cdots dx_n \le t_n([0,1])^{2n(n-1)}$$

(see [SZ], (2.1.9) p. 23) and so

$$\overline{\lim} \frac{1}{n^2} \log D_n \le 2 \log t([0,1])) = -2 \log 4.$$

Therefore, we obtain

$$\psi_1(x)/x^2 \ge \frac{\log 4}{3} = 0.46209, \qquad x \gg 1.$$

Considering for $r > 0$ the more general determinants

$$\begin{aligned} D_{r,n} &= \mathrm{Det}\Big(\int_0^1 x^{i+j} \{x(1-x)\}^{2r(n-1)}\, dx \Big)_{0 \le i,j \le n-1} \\ &= \frac{1}{n!} \int_0^1 \cdots \int_0^1 V_n(x)^2 \prod_{1 \le i \le n} \{x_i(1-x_i)\}^{2r(n-1)}\, dx_1 \cdots dx_n \\ &\le t_{x^r(1-x)^r, n}([0,1])^{2n(n-1)} \end{aligned}$$

M. Nair ([N] Theorem 2; see also [C]) obtains

$$\psi_1(x)/x^2 \ge 0.49517, \qquad x \gg 1.$$

The second number theoretic domain where our quantities are applied is the following. For a compact set $X \subset \mathbf{C}$, let us define its integer transfinite diameter as

$$t_{\mathbf{Z}}(X) = \inf_{\substack{P \in \mathbf{Z}[x],\\ \deg P \ge 1}} |P|_{\infty, X}^{1/\deg P},$$

where

$$|P|_{\infty,X} = \sup_{x\in X} |P(x)|$$

is the norm of uniform convergence. In [A], we obtain good estimations for $t_{\mathbf{Z}}(I)$, $I = [a,b] \subset \mathbf{R}$ being a real interval, via Minkowski's theorem by using asymptotic bounds for

$$D_{r_0,r_1,n}(I) = \operatorname{Det}\Big(\int_I x^{i+j}(x-a)^{\frac{2r_0(n-1)}{1-r_0-r_1}}(b-x)^{\frac{2r_1(n-1)}{1-r_0-r_1}}\,dx\Big)_{0\le i,j\le n-1},$$

$(r_0, r_1 > 0,\ r_0 + r_1 < 1)$. For example, in the case $I = [0,1]$ our method gives the upper bound

$$t_{\mathbf{Z}}([0,1]) < 0.42477.$$

If the function f is more complicated than $(x-a)^{\frac{2r_0(n-1)}{1-r_0-r_1}}(b-x)^{\frac{2r_1(n-1)}{1-r_0-r_1}}$, there are no explicit formulas for the determinant above. In spite of that, we can deal with t_f, which can be evaluated giving the solution of the variational problem

$$\min_{\lambda\in\mathcal{M}^+_{[0,1]}} I_f[\lambda],$$

instead of directly estimating $D_{I,f,n}$. This can be done using the link between t_f and $t_{\mathbf{Z}}$, which we shall give in Corollary 3.1.

More generally, given a compact set $X \subset \mathbf{C}$ and an upper semi-continuous function $f\colon X \to \mathbf{R}^+$, we define the integer f-transfinite diameter $t_{\mathbf{Z},f}(X)$ of X as

$$\lim_{n\in\mathbf{N}} \inf_{\substack{P\in\mathbf{Z}[x],\\ P\not\equiv 0,\, \deg P\le n}} \max_{x\in X} |P(x)|^{1/n}\cdot f(x)$$

(see §3 for the proof of the existence of this limit). If $f\equiv 1$, $t_{\mathbf{Z},f}(X) = t_{\mathbf{Z}}(X)$ provided that $t_{\mathbf{Z}}(X)\le 1$ (see [A] §3 for instance). The integer f-transfinite diameter plays an important rôle in the study of rational approximations of logarithms. Let $l \ne m$ be positive rational numbers; we say that μ is an irrationality measure for $\log(l/m)$ if

$$\left|\left(\log\frac{l}{m}\right) - \frac{p}{q}\right| > C(\mu)q^{-\mu}$$

for $p, q\in\mathbf{Z}$ with $q > 0$. We also define $\mu(\log l/m)$ as the infimum of the set of irrationality measures of $\log(l/m)$. Let

$$\alpha = (\sqrt{l}+\sqrt{m})^2, \qquad \beta = (\sqrt{l}-\sqrt{m})^2;$$

following a method first developed by G. Rhin ([RH1]) and R. Dvornicich – C. Viola ([DV]) independently, we can give good irrationality measures for $\log l/m$ if we are able to exhibit polynomials with integer coefficients which are *very close* to zero in $[0,\beta]$ and *not too big* in $[\beta,\alpha]$ (see Theorem 4.1 below). In §4 we obtain good estimates for $t_{\mathbf{Z},f_c}([0,\alpha])$, where $f_c\colon [0,\alpha]\to\mathbf{R}$ is the real function defined by

$$f_c(x)=\begin{cases}1 & \text{if } x\in[0,\beta),\\ \exp(-c) & \text{if } x\in[\beta,\alpha].\end{cases}$$

This allows us to find some explicit functions $F_{l,m}(t)$ defined on a compact set $K\subset\mathbf{R}^k$ ($k=2,4$ or 5) having the following property : if $t\in K$ and $F_{l,m}(t)>0$, then this number is an irrationality measure for $\log(l/m)$. Numerical computations will give :

ξ	$\mu(\xi)\le$	ξ	$\mu(\xi)\le$
$\log 2$	3.991	$\log 3$	16.960
$\log 5/3$	6.851	$\log 2/3$	3.402
$\log 3/4$	3.154	$\log 4/5$	3.017
$\log 7/5$	5.456		

Our measures for $\log 2$ and $\log 5/3$ improve Rhin's results $\mu(\log 2)\le 4.0765$ ([RH1]) and $\mu(\log 5/3)\le 7.224$ ([RH2]). For $\log 2$, Rukhadze [RU] obtains $\mu(\log 2)\le 3.893$ with another method (see also [HA]). For $\log 3$, the best result is $\mu(\log 3)\le 7.616$ ([RH1]) which arises as a particular case of a linear independence measure between 1, $\log 2$, $\log 3$. Finally all our results improve those of Alladi and Robinson ([AR]).

2. Associated kernels.

We start with some classical notations from potential theory. A *kernel* k on $\mathbf{C}$ will be a lower semi-continuous function $k\colon \mathbf{C}\times\mathbf{C}\to\mathbf{R}\cup\{+\infty\}$. For a signed measure λ on $\mathbf{C}$ with compact support $S(\lambda)$, we define its *potential* (with respect to k) by

$$U_k^\lambda(x)=\int k(x,y)\,d\lambda(y),\qquad x\in\mathbf{C}$$

and its *energy* by

$$I_k[\lambda]=\int\!\!\int k(x,y)\,d\lambda(x)d\lambda(y)=\int U_k^\lambda(x)\,d\lambda(x).$$

Given a non-empty compact set $X \subset \mathbf{C}$, its *Wiener k-capacity* will be the real number (possibly $+\infty$)

$$W_k(X) = \inf I_k[\lambda],$$

where the infimum is taken over the set $\mathcal{M}_X^+$ of all positive measures concentrated on X of total mass $\lambda(\mathbf{C}) = \lambda(X) = 1$. Using the lower semicontinuity of k, we see that the infimum is actually a minimum (see [FU] Theorem 2.3, p. 154); each minimizing measure $\lambda \in \mathcal{M}_X^+$ will be called a *capacitary measure* (with respect to k and X). For an arbitrary set $A \subset \mathbf{C}$, we take $W_k(A)$ equal to $\sup W_k(X)$, where X ranges over the compact subsets contained in A. It is easy to see that $W_k(A) = +\infty$ if and only if the interior measure $\mu_*(A)$ of A is zero for any measure μ of finite energy. We shall say that a property $P(x)$ involving a variable point $x \in X$ (X being a compact set) is true *k-nearly everywhere* (= k-n.e.) if the set of points A where P fails to hold, satisfies $W_k(A) = +\infty$.

From now on we make the following assumptions on k :

- k is *symmetric* : $k(x, y) = k(y, x)$;
- k is *positive definite* in the following "weak" sense : for any signed measure λ with compact support and total mass zero, we have $I_k[\lambda] \geq 0$ and the equality holds if and only if $\lambda = 0$.

THEOREM 2.1. — *Let X be a compact set with $W_k(X) < +\infty$. With the previous assumptions on k, there is only one capacitary measure λ. It is the unique measure $\lambda \in \mathcal{M}_X^+$ for which there exists a real constant W such that*

$$\begin{array}{lll} \text{(a)} & U_k^\lambda(x) \geq W & n.e. \text{ in } X \\ \text{(b)} & U_k^\lambda(x) \leq W & \forall x \in S(\lambda). \end{array}$$

Moreover, the constant W is equal to $W_k(X)$.

Proof. — According to [FU], Theorem 2.4, p.159, any capacitary measure satisfies (a) and (b) for $A = W_k(X)$. Let us assume that $\lambda \in \mathcal{M}_X^+$ satisfies (a) and (b) for a constant A and let μ be any capacitary measure. Then, using (b),

$$W_k(X) \leq I_k[\lambda] = \int U_k^\lambda(x)\, d\lambda(x) \leq A,$$

and, from (a),

$$\begin{aligned} I_k[\lambda] &= I_k[\lambda] + I_k[\mu] - 2\int U_k^\lambda(x)\,d\mu(x) \\ &\le A + W_k(X) - 2A = W_k(X) - A \le 0. \end{aligned}$$

So $A = W_k(X)$ and $\lambda = \mu$.

Q.E.D.

In view of our estimations, we consider kernels k_f of type

$$k_f(x,y) = \log\frac{1}{f(x)} + \log\frac{1}{|x-y|} + \log\frac{1}{f(y)}$$

where f is an upper semi-continuous function. For simplicity we shall write U_f^λ, $I_f[\lambda]$, $W_f(X)$, etc. instead of $U_{k_f}^\lambda$, $I_{k_f}[\lambda]$, $W_{k_f}(X)$, etc. Such kernels are obviously lower semi-continuous and symmetric; they are also strictly positive definite in the previous sense. In fact, if λ has zero mass, $I_f[\lambda] = I_{k_2}[\lambda]$, where $k_2 = \log\dfrac{1}{|x-y|}$ is the classical logarithmic kernel, which is strictly positive definite (see [L], Theorem 1.16, p.80). Finally, we remark that for these kernels every finite set has Wiener capacity $+\infty$.

We consider the following special case. Let $X = [a,b]$ be a real interval, fix some points $a = a_0 < a_1 < \cdots < a_k = b$ in X and fix $2k$ real numbers $r_0, \ldots, r_k, c_1, \ldots, c_{k-1}$ with $r_h > 0$. Put

$$f(x) = \prod_{h=0}^{k} |x-a_h|^{\frac{r_h}{1-r}} \exp\Big(-\frac{1}{1-r}\sum_{h=1}^{k-1} c_h\chi_{[a_h,a_{h+1}]}(x)\Big)$$

where χ_* is the characteristic function of $*$ and $r = r_0 + \cdots + r_k$. In this case we are able to find an explicit formula for $W_f([a,b])$. We start by expressing the $2k$ parameters r_h, c_h in terms of new parameters $x_1, \ldots, x_{2k}$ with

$$a = a_0 < x_1 < x_2 < a_1 < \cdots < a_{k-1} < x_{2k-1} < x_{2k} < a_k = b.$$

We put

$$r_h = \prod_{i=1}^{2k} |a_h - x_i|^{1/2} \prod_{\substack{i=0\\ i\neq h}}^{k} |a_h - a_i|^{-1}, \quad h = 1, \ldots, k$$

and

$$c_h = \sum_{i=1}^{h} (-1)^{k-i} \int_{x_{2i}}^{x_{2i+1}} \prod_{j=1}^{2k} |x - x_j|^{1/2} \prod_{j=0}^{k} (x - a_j)^{-1}\,dx$$

(the integrals are Cauchy's integrals). Now we can state our result on the f-capacity of $[a,b]$, which generalizes (and specifies) an idea of G.V. Chudnovsky (see [C], pp. 97-100).

THEOREM 2.2. — *The capacitary measure λ of $[a,b]$ with respect to the kernel*

$$\log\frac{1}{f(x)}+\log\frac{1}{|x-y|}+\log\frac{1}{f(y)}$$

is concentrated on the union L of the intervals $L_h = [x_{2h-1}, x_{2h}]$ with density

$$\frac{1}{\pi(1-r)}\prod_{h=1}^{2k}|x-x_h|^{1/2}\prod_{h=0}^{k}|x-a_h|^{-1}.$$

Moreover

$$\begin{aligned}(1)\quad (1-r)^2W_f([a,b]) &\\ = -(1-r_k)\big(r_k\log(b-a)(b-x_{2k}) &+ (1-r_k)\log(x_{2k}-a)\big)\\ &+\sum_{h=0}^{k-1} r_h^2\log(x_{2h+1}-a_h)+2\sum_{0\le i<j\le k} r_ir_j\log(a_j-a_i)\\ &+\sum_{h=0}^{k} r_h\int_{a_h}^{x_{2h+1}}\left\{(-1)^{k-h}g(x)-\frac{r_h}{x-a_h}\right\}dx\\ &+(1-r_k)\int_{x_{2k}}^{+\infty}\left\{g(x)-\frac{r_k}{x-a_k}-\frac{1-r_k}{x-a}\right\}dx\\ &-\sum_{h=1}^{k-1} c_h\left(r_h-\frac{(-1)^{k-h}}{\pi}\int_{x_{2h+1}}^{x_{2h+2}} g\,dx\right)+(1-r_k)c_{k-1}\end{aligned}$$

where

$$g(x)=\prod_{h=1}^{2k}|x-x_h|^{1/2}\prod_{h=0}^{k}(x-a_h)^{-1}.$$

Proof. — Let

$$\Delta(x)=\prod_{h=1}^{2k}|x-x_h|,\qquad s(x)=\begin{cases}(-1)^h, & \text{if } x\in L_h\,;\\ 0, & \text{otherwise}\end{cases}$$

and put $x_0 = -\infty$, $x_{2k+1} = +\infty$. We start with the following lemma, proved in [M], chapter 11.

LEMMA 2.2. — *For any real number $x \neq x_h$ and for any polynomial $P \in \mathbf{R}[x]$ of degree $\leq k-1$ we have*

$$\frac{1}{\pi}\int \frac{P(y)}{\Delta(y)^{1/2}}\frac{s(y)dy}{y-x} = \begin{cases} 0, & \text{if } x \in L; \\ (-1)^{h+1}\frac{P(x)}{\Delta(x)^{1/2}}, & \text{if } x \in (x_{2h}, x_{2h+1}) \\ & \text{for some } h = 0,\dots,k, \end{cases}$$

where the integral is a Cauchy's integral.

Let now

$$\lambda = \left\{\frac{1}{\pi(1-r)}\prod_{h=1}^{2k}|x-x_h|^{1/2}\prod_{h=0}^{k}|x-a_h|^{-1}\right\}\chi_L(x)dx = \frac{|g(x)|}{\pi(1-r)}\chi_L(x)dx.$$

For any $y \in L_h$ we have

$$\begin{aligned}\frac{|g(y)|}{\pi(1-r)} &= \frac{(-1)^{k-h}}{\pi(1-r)\Delta(y)^{1/2}}\left\{\prod_{i=1}^{2k}(y-x_i)\prod_{i=0}^{k}(y-a_h)^{-1}\right\} \\ &= \frac{(-1)^{k-h}}{\pi(1-r)\Delta(y)^{1/2}}\left\{P(y)+\sum_{i=0}^{k}\frac{(-1)^{k-i}r_i\Delta(a_i)^{1/2}}{(y-a_i)}\right\}\end{aligned}$$

for some monic polynomials P of degree $k-1$. Hence, for any $x \neq a_h$,

$$\begin{aligned}\phi(x) &= \int\frac{d\lambda(y)}{y-x} - \sum_{i=0}^{k}\frac{r_i}{1-r}\frac{1}{x-a_i} = \frac{(-1)^k}{\pi(1-r)}\int\frac{P(y)}{\Delta(y)^{1/2}}\frac{s(y)dy}{y-x} \\ &+\sum_{i=0}^{k}\left\{\frac{(-1)^i r_i\Delta(a_i)^{1/2}}{\pi(1-r)(x-a_i)}\int\frac{1}{\Delta(y)^{1/2}}\left[\frac{1}{y-x}-\frac{1}{y-a_i}\right]s(y)dy\right. \\ &\qquad\left. - \frac{r_i}{(1-r)(x-a_i)}\right\}.\end{aligned}$$

Lemma 2.1 gives

$$\frac{(-1)^i r_i\Delta(a_i)^{1/2}}{\pi(1-r)(x-a_i)}\int\frac{1}{\Delta(y)^{1/2}}\frac{s(y)dy}{y-a_i} = -\frac{r_i}{(1-r)(x-a_i)}, \qquad i=0,\dots,k,$$

hence

$$\phi(x) = \frac{(-1)^k}{\pi(1-r)}\int\frac{P(y)}{\Delta(y)^{1/2}}\frac{s(y)dy}{y-x}+\sum_{i=0}^{k}\frac{(-1)^i r_i\Delta(a_i)^{1/2}}{\pi(1-r)(x-a_i)}\int\frac{1}{\Delta(y)^{1/2}}\frac{s(y)dy}{y-x}.$$

If x is in the interior of L, the same lemma gives $\phi(x) = 0$. Otherwise, if $x \in (x_{2h}, x_{2h+1})$, $x \neq a_h$,

$$\phi(x) = \frac{(-1)^{k+h+1}}{(1-r)\Delta(x)^{1/2}}\left\{P(x) + \sum_{i=0}^{k} \frac{(-1)^{k-i} r_i \Delta(a_i)^{1/2}}{(x-a_i)}\right\} = \frac{(-1)^{k+h+1} g(x)}{1-r}. \tag{2}$$

In particular,

$$\begin{aligned}\int d\lambda(y) &= \left[\lim_{x\to+\infty}(-x)\phi(x)\right] - \frac{r}{1-r} \\ &= \lim_{x\to+\infty}\frac{xg(x)}{1-r} - \frac{r}{1-r} = 1,\end{aligned}$$

hence $\lambda \in \mathcal{M}^+_{[a,b]}$ and its potential U_f^λ is constantly equal to some W_h on L_h, since $\dfrac{dU_f^\lambda}{dx} = \phi(x) = 0$ on the interior of L. For $h = 0, \dots k-1$, let us consider the function

$$\psi_h(x) = U_f^\lambda(x) + \frac{r_h}{1-r}\log|x-a_h| - \frac{1}{1-r}\sum_{i=1}^{k-1} c_i \chi_{[a_i,a_{i+1}]}(x)$$

which is continuous on $[x_{2h}, x_{2h+1}]$ (on $(-\infty, x_1]$, if $h = 0$) and differentiable on $(x_{2h}, x_{2h+1})\backslash\{a_h\}$. From (2) we have

$$\frac{d\psi_h}{dx} = \phi(x) + \frac{r_h}{1-r}\cdot\frac{1}{x-a_h} = -\frac{1}{1-r}\left\{(-1)^{k-h}g(x) - \frac{r_h}{x-a_h}\right\}, \tag{3}$$

for any $x \in (x_{2h}, x_{2h+1})\backslash\{a_h\}$. Hence, for $h = 1, \dots, k-1$,

$$\begin{aligned}W_{h+1} - W_h &= U_f^\lambda(x_{2h+1}) - U_f^\lambda(x_{2h}) \\ &= \psi_h(x_{2h+1}) - \psi_h(x_{2h}) + \frac{1}{1-r}(c_h - c_{h-1}) \\ &\quad - \frac{r_h}{1-r}\log\frac{x_{2h+1}-a_h}{a_h - x_{2h}} \\ &= -\frac{1}{1-r}\int_{x_{2h}}^{x_{2h+1}}\left\{(-1)^{k-h}g(x) - \frac{r_h}{x-a_h}\right\}dx \\ &\quad + \frac{(-1)^{k-h}}{1-r}\int_{x_{2h}}^{x_{2h+1}} g(x)\,dx - \frac{r_h}{1-r}\log\frac{x_{2h+1}-a_h}{a_h-x_{2h}} = 0.\end{aligned}$$

This yelds $U_f^\lambda(x) \equiv W \in \mathbf{R}$ for any $x \in L = S(\lambda)$. Moreover, from (2) we get

$\dfrac{dU_f^\lambda}{dx} = \phi(x) > 0$ for $x_{2h} < x < a_h$ and $\dfrac{dU_f^\lambda}{dx} = \phi(x) < 0$ for $a_h < x < x_{2h+1}$ $(h = 0,\dots,k)$; hence $U_f^\lambda(x) \geq W$ for any $x \in [a,b]\backslash\{a_0,\dots,a_k\}$. By Theorem 2.1, λ is the f-capacitary measure of $[a,b]$ and $W_f([a,b]) = W$; the first assertion of Theorem 2.2 is proved. For the second, let

$$(4)\ F = \int \log\frac{1}{f(y)}\,d\lambda(y) = \sum_{h=0}^{k}\frac{r_h}{1-r}\int\log\frac{1}{|y-a_h|}\,d\lambda(y) + \frac{1}{\pi(1-r)^2}\sum_{h=1}^{k-1} c_h\int_{x_{2h+1}}^{x_{2h+2}}(-1)^{k-h}g(x)\,dx.$$

For $h = 0,\dots,k-1$, we have, using (3),

$$\begin{aligned}(5_h)\ -\frac{1}{1-r}\int_{a_h}^{x_{2h+1}}&\left\{(-1)^{k-h}g(x) - \frac{r_h}{x-a_h}\right\}dx\\ &= W + \frac{r_h}{1-r}\log|x_{2h+1}-a_h| - \frac{c_h}{1-r}\\ &\quad-\int\log\frac{1}{|y-a_h|}\,d\lambda(y) + \sum_{\substack{0\leq i\leq k\\ i\neq h}}\frac{r_i}{1-r}\log|a_h-a_i| - F.\end{aligned}$$

Let now

$$\psi_k(x) = U_f^\lambda(x) + \frac{r_k}{1-r}\log|x-a_k| + \frac{1-r_k}{1-r}\log|x-a| - \frac{1}{1-r}\sum_{i=1}^{k-1}c_i\chi_{[a_i,a_{i+1}]}(x)$$

which is continuous on $[x_{2k},+\infty)$ and differentiable on $(x_{2k},+\infty)\backslash\{a_0\}$. We have

$$\begin{aligned}\lim_{x\to+\infty}\psi_k(x) &= F\\ \frac{d\psi_k}{dx} &= \phi(x) + \frac{r_k}{1-r}\cdot\frac{1}{x-a_k} + \frac{1-r_k}{1-r}\cdot\frac{1}{x-a}\\ &= -\frac{1}{1-r}\left\{g(x) - \frac{r_k}{x-a_k} - \frac{1-r_k}{x-a}\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}(5_k)\quad -\frac{1}{1-r}&\int_{a_k}^{+\infty}\left\{g(x) - \frac{r_k}{x-a_k} - \frac{1-r_k}{x-a}\right\}dx\\ &= -\int\log\frac{1}{|y-a_k|}\,d\lambda(y) + \sum_{i=0}^{k-1}\frac{r_i}{1-r}\log|a_k-a_i| - \frac{1-r_k}{1-r}\log(b-a).\end{aligned}$$

Similarly,

$$
\begin{aligned}
(6)\qquad & -\frac{1}{1-r}\int_{x_{2k}}^{+\infty}\left\{g(x)-\frac{r_k}{x-a_k}-\frac{1-r_k}{x-a}\right\}dx \\
& = F-W-\frac{r_k}{1-r}\log(b-x_{2k})-\frac{1-r_k}{1-r}\log(x_{2k}-a)+\frac{c_{k-1}}{1-r}.
\end{aligned}
$$

Now, from equations $(5_0),\ldots,(5_k)$ and (4), we easily obtain

$$
\begin{aligned}
&(1-r_k)F-(r-r_k)W \\
&=\frac{1}{1-r}\Bigg\{-(1-r_k)r_k\log(b-a)+\sum_{h=0}^{k-1} r_h^2\log(x_{2h+1}-a_h) \\
&\qquad\qquad +2\sum_{0\le i<j\le k} r_i r_j\log(a_j-a_i) \\
&\qquad +\sum_{h=0}^{k} r_h\int_{a_h}^{x_{2h+1}}\left\{(-1)^{k-h}g(x)-\frac{r_h}{x-a_h}\right\}dx \\
&\qquad -\sum_{h=1}^{k-1} c_h\left(r_k-\frac{(-1)^{k-h}}{\pi}\int_{x_{2h+1}}^{x_{2h+2}} g\,dx\right)\Bigg\}.
\end{aligned}
$$

Our claim (1) is established taking into account (6).

Q.E.D.

Remark. — The previous result remains true if $r_h=0$ for some index h (in this case, necessarily $a_h=x_{2h}$ or $a_h=x_{2h+1}$ and all the quantities which appear in the formulation of Theorem 2.2 still have a meaning, except perhaps for $r_h\log(x_{2h+1}-a_h)$ and for $r_k\log(b-x_{2k})$, which we take $=0$). This can be verified directly or by using a continuity argument.

3. Connections between t_f, W_f and $t_{\mathbf{Z},f}$.

As announced in §1, the Wiener f-capacity and the f-transfinite diameter are closely related.

THEOREM 3.1. — *For any compact set $X\subset\mathbf{C}$ and for any upper semi-continuous function $f\colon X\mapsto\mathbf{R}^+$ the sequence of real numbers $t_{f,n}(X)$ converges to a real number $t_f(X)$. Moreover, $t_f(X)=\exp\{-W_f(X)\}$.*

For the proof, see [HI], Theorem 16.4.4, p. 284, although this author considers only the classical case $f = 1$.

Let f be as before and let us define

$$t_{\mathbf{Z},f,n}(X) = \inf_{\substack{P\in\mathbf{Z}[x],\\ P\not\equiv 0,\, \deg P\leq n}} \max_{x\in X} |P(x)|^{1/n}\cdot f(x).$$

We have $t_{\mathbf{Z},f,n+m}(X)^{n+m} \leq t_{\mathbf{Z},f,n}(X)^n \cdot t_{\mathbf{Z},f,m}(X)^m$, hence it is easy to see that the sequence $t_{\mathbf{Z},f,n}$ converges to its infimum. We define the integer f-transfinite diameter of X as

$$t_{\mathbf{Z},f}(X) = \lim_{n\in\mathbf{N}} t_{\mathbf{Z},f,n}(X) = \inf_{n\in\mathbf{N}} t_{\mathbf{Z},f,n}(X).$$

The next theorem, which generalizes a classical result of Fekete explains the links between t_f and $t_{\mathbf{Z},f}$.

THEOREM 3.2. — *Let $X \subset \mathbf{C}$ be a compact set, symmetric with respect to the real axis (i.e. $\overline{X} = X$) and let $f\colon X \to \mathbf{R}^+$ be an upper semi-continuous function such that $f(\overline{x}) = f(x)$ for any $x \in X$. Then, $t_{\mathbf{Z},f}(X) \leq \sqrt{t_f(X)}$.*

Proof. — The proof is a consequence of the following two lemmas, which, on the whole, are classical.

LEMMA 3.1. — *Let $X \subset \mathbf{C}$ be a compact set and let $f\colon X \to \mathbf{R}^+$ be an upper semi-continuous function. For $h,n \in \mathbf{N}$, $0 \leq h \leq n$, put $M_{h,n} = \min|P\cdot f^n|_{\infty,X}$, where the minimum is taken over the set of monic polynomials $P \in \mathbf{C}[x]$ with degree $\leq h$. Then the sequence $\left\{\Big(\prod_{0\leq j\leq n-1} M_{j,n-1}\Big)^{2/n(n-1)}\right\}_{n\in\mathbf{N}}$ converges to $t_f(X)$.*

Proof. — We use the same arguments as in [HI] p.269-270. Let $x_1,\dots,x_n \in X$ such that

$$t_{f,n}(X)^{n(n-1)/2} = |V_n(x)| \prod_{1\leq i\leq n} f(x_i)^{n-1}.$$

Then

$$t_{f,n}(X)^{n(n-1)/2} = \prod_{j=1}^{n} f(x_j)^{n-1} \prod_{i=1}^{j-1} |x_j - x_i| \geq \prod_{j=0}^{n-1} M_{j,n-1}.$$

On the other hand, let P_h $(h = 1, \dots, n-1)$ be a monic polynomial of degree h such that $M_{h,n-1} = |P_h \cdot f^{n-1}|_{\infty,X}$. Using Hadamard's inequality for determinants we find

$$\begin{aligned} t_{f,n}(X)^{n(n-1)/2} &= \left|\mathrm{Det}\left(x_i^j f(x_i)^{n-1}\right)_{0\le i,j\le n-1}\right| \\ &= \left|\mathrm{Det}\left(P_j(x_i) f(x_i)^{n-1}\right)_{0\le i,j\le n-1}\right| \\ &\le \prod_{h=0}^{n-1} \sqrt{\sum_{j=1}^{n} \left|P_h(x_j) f(x_j)^{n-1}\right|^2} \le n^{n/2} \cdot \prod_{j=0}^{n-1} M_{j,n-1}. \end{aligned}$$

Q.E.D.

LEMMA 3.2. — *Let $X \subset \mathbf{C}$ be a compact set, symmetric with respect to the real axis and let $f_1, \dots, f_n \colon X \to \mathbf{C}$ be linearly independent functions such that $f_i(\overline{x}) = \overline{f_i(x)}$. Put*

$$M_h = \inf_{\lambda_1,\dots,\lambda_{h-1}\in\mathbf{R}} \sup_X |\lambda_1 f_1 + \cdots + \lambda_{h-1} f_{h-1} + f_h|.$$

Then there exist integers $\lambda_1, \dots, \lambda_n$ such that

$$\sup_X |\lambda_1 f_1 + \cdots + \lambda_n f_n| \le n(M_1 \cdots M_n)^{\frac{1}{n}}.$$

Proof (see also [FE] and [S]). — Let us consider the symmetric convex set

$$C = \{(x_1, \dots, x_n) \in \mathbf{R}^n \text{ such that } |x_1 f_1 + \cdots + x_n f_n|_{\infty,X} \le 1\}.$$

For $\delta > 1$ and $1 \le i < j \le n$ let $\lambda_{i,j} \in \mathbf{C}$ be such that

$$|\lambda_{1,j} f_1 + \cdots + \lambda_{j-1,j} f_{j-1} + f_j|_{\infty,X} \le \delta M_h, \qquad j = 1, \dots, n;$$

from our hypotheses on X and $f_1, \dots, f_n$ we can assume $\lambda_{i,j} \in \mathbf{R}$. Let us consider the linear map $L \colon \mathbf{R}^n \to \mathbf{R}^n$ defined by $L(x) = Ax$, where A is the $n \times n$ matrix $A = (\lambda_{i,j})$ $(\lambda_{i,j} = 0$ for $i > j$ and $\lambda_{i,i} = 1)$. The image via L of the parallelepiped

$$P = \{(x_1, \dots, x_n) \in \mathbf{R}^n \text{ such that } |x_i| \le (\delta n M_i)^{-1},\ i = 1, \dots, n\}$$

is contained in C. So, taking $\delta \to 1$,

$$\mathrm{Vol}(C) \ge \left(\frac{2}{n}\right)^n \frac{1}{M_1 \cdots M_n}.$$

Hence, Minkowski's Convex Body Theorem gives

$$n(M_1\cdots M_n)^{\frac{1}{n}}C\cap\mathbf{Z}^n\neq\{(0,\dots,0)\}$$

and our assert follows. Q.E.D.

In general, it is not convenient to apply Theorem 3.2 directly. For example, it only gives the trivial bound $t_{\mathbf{Z}}([0,1])\leq 1/2$. It is better to introduce first some arithmetical information and this may be done using the following simple result :

THEOREM 3.3. — *Let X and f be as in the previous theorem and let $Q_1,\dots,Q_k\in\mathbf{Z}[x]$. Given $r_1,\dots,r_k\geq 0$, put*

$$r=\sum_{h=1}^{k}r_h\deg Q_h,\qquad g(x)^{1-r}=f(x)\prod_{h=1}^{k}|Q_h(x)|^{r_h}.$$

Then

$$t_{\mathbf{Z},f}(X)\leq t_{\mathbf{Z},g}(X)^{1-r}.$$

Proof. — For any $\delta>1$ and for any sufficiently large $n\in\mathbf{N}$, we can find a polynomial $P_n\in\mathbf{Z}[x]$ of degree $\leq m_n=\big[(1-r)n\big]-\sum_{h=1}^{k}\deg Q_h$ such that

$$\max_{x\in X}|P(x)|^{1/m_n}g(x)\leq t_{\mathbf{Z},g}(X)^{\delta}.$$

The polynomial $R_n(x)=P_n(x)\prod_{h=1}^{k}Q_h(x)^{[r_hn]+1}\in\mathbf{Z}[x]$ has degree $\leq n$. For any $x\in X$ we have

$$|R_n(x)|^{1/n}f(x)=\big(|P_n(x)|^{1/m_n}g(x)\big)^{a_n}f(x)^{b_n}\prod_{h=1}^{k}|Q_h(x)|^{c_{h,n}},$$

where a_n, b_n and c_n are positive numbers defined by

$$a_n=\frac{m_n}{n},\quad b_n=1-\frac{m_n}{(1-r)n},\quad c_{h,n}=\frac{[r_hn]+1}{n}-\frac{r_hm_n}{(1-r)n}.$$

Therefore,

$$t_{\mathbf{Z},f}(X)\leq t_{\mathbf{Z},g}(X)^{\delta a_n}\cdot\max\{1,|f|_{\infty,X}\}^{b_n}\cdot\prod_{h=1}^{k}\max\{1,|Q_h|_{\infty,X}\}^{c_{h,n}}.$$

Using

$$\lim_{n\in\mathbf{N}} a_n = (1-r), \qquad \lim_{n\in\mathbf{N}} b_n = \lim_{n\in\mathbf{N}} c_{h,n} = 0$$

we get $t_{\mathbf{Z},f}(X) \leq t_{\mathbf{Z},g}(X)^{(1-r)\delta}$. Our claim follows taking $\delta \to 1$.

Q.E.D.

In particular, combining Theorems 3.2 and 3.3, we obtain

COROLLARY 3.1. — *Let Q_h, r_h, r and g be as in Theorem 3.3 and assume $t_g(X) \leq 1$. Then*

$$t_{\mathbf{Z}}(X) \leq t_g(X)^{(1-r)/2}.$$

For example, with

$$g(x) = \{x(1-x)\}^{\frac{r_0}{1-r}} |1-2x|^{\frac{r_1}{1-r}}$$

($r_0, r_1 > 0$, $r = 2r_0 + r_1$), Corollary 3.1 gives

$$\log t_{\mathbf{Z}}([0,1]) \leq (1-r)\log t_{\mathbf{Z},g}([0,1]) \leq \frac{1-r}{2}\log t_g([0,1])$$

and the last quantity can be evaluated by Theorem 2.2. A proper choice of the parameters again gives the inequality $t_{\mathbf{Z}}([0,1]) < 0.42477$ proved in [A]. The choice of further polynomial factors leads to negligible improvements.

4. Irrationality measures for logarithms.

The link between integer transfinite diameter and irrationality measures is given by the following

THEOREM 4.1 (G.Rhin, 1989). — *Let a, b be two positive integers and put*

$$\alpha = (\sqrt{a}+\sqrt{b})^2, \qquad \beta = (\sqrt{a}-\sqrt{b})^2.$$

For $c > 0$, let $f_c = \exp(-c\chi_{[\beta,\alpha]})$ and let $\varepsilon(c) = \log t_{\mathbf{Z},f_c}([0,\alpha])$. Then, if $\varepsilon(c) < -1$, the number $\log a/b$ is irrational and its irrationality measure is bounded by $-c/(1+\varepsilon(c))$.

Sketch of the Proof (see also [RH2]). — Let $\varepsilon = \varepsilon(c)$ and assume that $\varepsilon < -1$. For $\delta > 1$ we can find a polynomial $P \in \mathbf{Z}[x]$ of degree $\leq m$ such that

$$\varepsilon' = \log |P|_{\infty,[0,\beta]} \leq \delta\varepsilon m, \qquad \log |P|_{\infty,[\beta,\alpha]} \leq cm + \delta\varepsilon m.$$

Let us consider the linear map $\sigma\colon \mathbf{R}[x] \to \mathbf{R}[x]$ defined by $\sigma x^h = (b-a)^h L_h(x)$, where

$$L_h(x) = \frac{1}{n!}\frac{d^h}{dx^h} x^h(1-x)^h \in \mathbf{Z}[x]$$

is the h-th Legendre polynomial. For $n \in \mathbf{N}$, let $P_n(x) = \sigma P^n$, which is a polynomial with integer coefficients and degree $\leq nm$. Repeated integration by parts gives

$$I_n = \int_0^1 \frac{(a-b)P_n(x)}{b+(a-b)x}\,dx = \int_0^1 P\left(\frac{(a-b)^2 x(1-x)}{b+(a-b)x}\right)^n \frac{(a-b)dx}{b+(a-b)x}.$$

The function

$$x \mapsto \frac{(a-b)^2 x(1-x)}{b+(a-b)x}$$

maps $[0,1]$ onto $[0,\beta]$, so we deduce

$$\log |I_n| = \varepsilon'(n + o(n)), \qquad n \to +\infty.$$

Moreover $I_n = q_n \log(a/b) - p_n$, where p_n, q_n are rational numbers whose denominators are bounded by

$$d_{nm+1} = \operatorname{lcm}\{1, \ldots, nm+1\} = \exp(nm + o(n))$$

(see [AR], for instance). The Laplace formula gives

$$|q_n| = \frac{1}{\pi}\int_0^\pi P_n\big(a + b + 2\sqrt{ab}\cos\theta\big)\,d\theta \leq \exp(cnm + \delta\varepsilon nm).$$

By standard facts about the estimate of irrationality measures, we obtain our claim.

Q.E.D.

Let α, β, c, f_c and $\varepsilon(c)$ as above and consider the function

$$f_{r,c}(x) = \big(x^r f_c\big)^{\frac{1}{1-r}} = x^{\frac{r}{1-r}} \exp\Big(-\frac{c}{1-r}\chi_{[\beta,\alpha]}(x)\Big).$$

Theorems 3.3, 3.2 and 3.1 give

$$\begin{aligned}\varepsilon(c) = \log t_{\mathbf{Z},f_c}([0,\alpha]) &\le (1-r)\log t_{\mathbf{Z},f_{r,c}}([0,\alpha])\\ &\le \frac{1-r}{2}\log t_{f_{r,c}}([0,\alpha]) = -\frac{1-r}{2}W_{f_{r,c}}([0,\alpha]).\end{aligned}$$

The last quantity is easily evaluated using Theorem 2.2 with $k=2$, $c_1 = c$ and

$$\begin{aligned}&a_0 = 0, \qquad a_1 = \beta, \qquad a_2 = \alpha,\\ &x_1 \in (0,\beta), \qquad x_2 = \beta, \qquad x_3 \in (\beta,\alpha), \qquad x_4 = \alpha.\end{aligned}$$

This leads to the following generalization of a theorem of K. Alladi and L. Robinson (see [AR] theorem 1).

THEOREM 4.2. — *Let a, b, α, β be as in Theorem 4.1 and let t_1,t_2 be two positive real numbers with $t_1 < \beta < t_2 < \alpha$. Let also $r = \sqrt{\dfrac{t_1 t_2}{\alpha\beta}}$, $g(x) = \dfrac{1}{x}\sqrt{\left|\dfrac{(x-t_1)(x-t_2)}{(x-\alpha)(x-\beta)}\right|}$ and $c = \displaystyle\int_\beta^{t_2} g(x)\,dx$. Then, if*

$$\varepsilon = \frac{1}{2(1-r)}\left\{\log\alpha - r^2\log t_1 - r\int_0^{t_1} g(x) - \frac{r}{x}\,dx - \int_\alpha^{+\infty} g(x) - \frac{1}{x}\,dx \right.$$
$$\left. -c\Big(1 + \frac{1}{\pi}\int_{t_2}^{\alpha} g(x)\,dx\Big)\right\} < -1,$$

the irrationality measure of $\log(a/b)$ is bounded by $-c/(1+\varepsilon)$.

Given a and b, the best values for the parameters t_1 and t_2 can be found using the optimization routine DBCONF in the IMSL library of FORTRAN subprograms. The following table shows some explicit results :

ξ	$\mu(\xi)$	(t_1, t_2)	(r, c)
$\log 2$	4.047	$(0.1010\,7211, 5.0374\,9422)$	$(0.7135, 3.6353)$
$\log 3$	26.817	$(0.0687\,0930, 7.1179\,7508)$	$(0.3497, 3.6326)$
$\log 5/3$	7.158	$(0.0984\,3191, 13.9891\,8578)$	$(0.5867, 4.5371)$
$\log 2/3$	3.402	$(0.0772\,0888, 8.9403\,8240)$	$(0.8308, 4.5708)$
$\log 3/4$	3.154	$(0.0603\,1302, 12.9096\,0087)$	$(0.8824, 5.2203)$
$\log 4/5$	3.017	$(0.0490\,8116, 16.8985\,0597)$	$(0.9107, 5.7166)$
$\log 7/5$	5.456	$(0.0937\,5426, 21.3483\,7215)$	$(0.7074, 5.1610)$

All the above results can be checked using a personal computer. Some improvement is obtained considering more complicated functions f. Let $\gamma = 1/([1/\beta]+1) \in (0,\beta)$ and let

$$f_{r_0,r_1,c_1,c_2}(x) = x^{\frac{r_0}{1-r}}(x/\gamma - 1)^{\frac{r_1}{1-r}} \exp\Big(-\frac{1}{1-r}(c_1\chi_{[\beta,\gamma]}(x) + c_2\chi_{[\gamma,\alpha]}(x))\Big), \qquad r = r_0 + r_1.$$

Using Theorems 3.3, 3.2, 3.1 and 2.2 with $k = 3$ and

$$a_0 = 0, \quad a_1 = \gamma, \quad a_2 = \beta, \quad a_3 = \alpha,$$
$$x_1 = t_1, \quad x_2 = t_2, \quad x_3 = t_3, \quad x_4 = \beta, \quad x_5 = t_4, \quad x_6 = \alpha,$$

we obtain

THEOREM 4.3. — *Let a, b, α, β, γ be as above and let t_1, t_2, t_3, t_4 be four positive real numbers with $t_1 < t_2 < \gamma < t_3 < \beta < t_4 < \alpha$. Put*

$$r_0 = \frac{1}{\gamma}\sqrt{\frac{t_1t_2t_3t_4}{\alpha\beta}}, \qquad r_1 = \frac{1}{\gamma}\sqrt{\frac{(\gamma - t_1)(\gamma - t_2)(t_3-\gamma)(t_4-\gamma)}{(\alpha-\gamma)(\beta-\gamma)}};$$

$$c_1 = r_1 \log\frac{t_3-\gamma}{\gamma - t_2} + \int_{t_2}^{t_3} g(x) - \frac{r_1}{x-\gamma}\,dx, \qquad c_2 = c_1 + \int_{\beta}^{t_4} g(x)\,dx$$

and

$$g(x) = \frac{1}{x(x-\gamma)}\sqrt{\left|\frac{(x-t_1)(x-t_2)(x-t_3)(x-t_4)}{(x-\alpha)(x-\beta)}\right|}.$$

Then, if

$$\begin{aligned}\varepsilon = \frac{1}{2(1-r)}\Bigg\{ &\log\alpha - r_0^2\log t_1 - r_1^2\log(t_3-\gamma) - 2r_0r_1\log\gamma \\ &-r_0\int_0^{t_1} -g(x) - \frac{r_0}{x}\,dx - r_1\int_\gamma^{t_3} g(x) - \frac{r_1}{x-\gamma}\,dx - \int_\alpha^{+\infty} g(x) - \frac{1}{x}\,dx \\ &+c_1\Big(r_1 - \frac{1}{\pi}\int_{t_3}^{\beta} g(x)\,dx\Big) - c_2\Big(1 + \frac{1}{\pi}\int_{t_4}^{\alpha} g(x)\,dx\Big)\Bigg\} \\ &-r_1\log\gamma + \max\{0, c_1\} < -1,\end{aligned}$$

the irrationality measure of $\log(a/b)$ *is bounded by*

$$-(c_2 - \max\{0,c_1\})/(1+\varepsilon).$$

Numerically :

ξ	$\mu(\xi)$	(t_1, t_2, t_3, t_4)
$\log 2$	4.001	(0.1082 8477, 0.1629 2110, 0.1693 4883, 5.3168 2544)
$\log 3$	16.960	(0.1136 4578, 0.4578 5023, 0.5267 3509, 7.1932 1262)
$\log 5/3$	6.851	(0.1136 1007, 0.2446 9414, 0.2531 5690, 14.5506 5964)

Finally, considering

$$f_{r_0,r_1,r_2,c_1,c_2}(x)$$
$$= x^{\frac{r_0}{1-r}}(x/\gamma - 1)^{\frac{r_1}{1-r}}Q(x)^{\frac{r_2}{1-r}} \exp\Big(- \frac{1}{1-r}\big(c_1\chi_{[\gamma,\beta]}(x) + c_2\chi_{[\gamma,\alpha]}(x)\big)\Big)$$

where $r = r_0 + r_1 + 2r_2$ and $Q(x) = (x - \beta)(x - \alpha) \in \mathbf{Z}[x]$ is the minimal polynomial of α and β, some other small improvements can be obtained. For example, we find that

$$\mu(\log 2) < 3.991$$

with $\underline{t} =$ (0.1104 9544, 0.1631 0685, 0.1692 5754, 0.1715 2667, 5.2726 2890, 5.8241 7429).

BIBLIOGRAPHIE

[A] F. AMOROSO, Sur le diamètre transfini entier d'un intervalle réel, Ann. Inst. Fourier, 40-4 (1990), 885-911.

[AR] K. ALLADI and L. ROBINSON, Legendre polynomials and irrationality, J. reine angew. Math., 318 (1980), 137-155.

[C] G.V. CHUDNOVSKY, Number Theoretic Applications of Polynomials with Rational Coefficients Defined by Extremality Conditions, Arithmetic and Geometry, Vol I, ed. M.Artin and J.Tate, Birkhäuser. Progress in Math., 35 (1983), 61-105.

[DV] R. DVORNICICH and C. VIOLA, Some remarks on Beukers'integrals, Colloquia Math. Soc. János Bolyai, 51 (1987), 637-657.

[FE] M. FEKETE, Über die Verteilung der Wurzeln gewisen algebraischen Gleichungen mit ganzzahligen Koeffizienten, Mathematische Zeitschrift, 17 (1923), 228-249.

[FU] B. FUGLEDE, On the theory of potentials in locally compact spaces, Acta Math., 103 (1960), 139-215.

[HA] M. HATA, Legendre type polynomials and irrationality measures, J. reine angew. Math., 407 (1990), 99-125.

[HI] E. HILLE, Analytic function theory, Gim and Company, Boston, 1962.

[L] N.S. LANDKOF, Foundations of modern potential theory, Springer-Verlag, Berlin, 1972.

[M] N.I. MUSKHELISHVLI, Singular integral equations, Noordhoff Internationals Publishing, Leyden, 1977.

[N] M. NAIR, A new method in elementary prime number theory, J. London Math. Soc., (2), 25 (1982), 385-391.

[RH1] G. RHIN, Approximants de Padé et mesures effectives d'irrationalité, Séminaire de Théorie des Nombres, Paris 1985-86, ed. C. Goldstein, Birkhäuser, Progress in Math., 71 (1987), 155-164.

[RH2] G. RHIN, Diamètre transfini et mesures d'irrationalité des logarithmes, notes of lectures given at the University of Pise in March 1989.

[RU] E.A. RUKHADZE, A lower bound for the approximation of ln 2 by rational numbers (Russian), Vestnik Moskov. Univ., Ser. I Math. Mekh., no. 6 (1987), 25-29.

[S] I.N. SANOV, Function with integral parameters, deviating the last from zero (en russe), Leningrad. Gos. Univ., Ucen. Zap. Ser. Mat. Nauk, 111 (1949), 32-46.

[SZ] G. SZEGÖ, Orthogonal Polynomials, 4^{th} ed., American Math. Soc., New York, 1975.

Manuscrit reçu le 15 juillet 1992.

Francesco AMOROSO,
Dipartimento di Matematica
Via Buonarroti 2
56127 Pisa (Italy).