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QUANTIZATIONS AND SYMBOLIC CALCULUS OVER THE p-ADIC NUMBERS

by Shai HARAN

INTRODUCTION

We shall be concerned with functions $f : V_0 \to \mathbb{C}$, defined in some vector space V_0 over the p-adic numbers $\mathbb{Q}_p$ and taking values in the complex numbers $\mathbb{C}$. One of the most basic problems encountered when trying to imitate the classical theory — where the domain V_0 is a vector space over $\mathbb{R}$ or $\mathbb{C}$ — is the lack of derivatives. Indeed, the derivation $\partial/\partial x$ is nothing but $\mathcal{F}^{-1}x\mathcal{F}$, where $\mathcal{F}$ is the Fourier transform, and «x» is multiplication by the function x which is an additive homomorphism from V_0 to $\mathbb{C}$; and there are no such homomorphisms from V_0 to $\mathbb{C}$ when V_0 is a vector space over $\mathbb{Q}_p$.

This problem repeatedly makes its appearance in various disguises; for example, given a unitary representation of a p-adic analytic group on a Hilbert space, one cannot associate with it the derived representation of a p-adic Lie algebra. From a different perspective, the derivatives $(\partial/\partial x)^n$ correspond to the extra poles of the ∞-component of the zeta function, while the p-components have a unique pole.

There are thus no differential operators over $\mathbb{Q}_p$. But as we will show in this paper, there is a meaningful theory of pseudodifferential operators over the p-adics, which parallels the classical theory over the real numbers $\mathbb{R}$. In fact, the theory over $\mathbb{Q}_p$ is better behaved than the one over $\mathbb{R}$, in as much as in all estimates the numbers $2 = [\mathbb{C} : \mathbb{R}]$ for the reals can be replaced by $\infty = [\overline{\mathbb{Q}}_p : \mathbb{Q}_p]$ for the p-adics, and one encounters here the phenomenon expounded in [15], [16].

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The pseudodifferential operator $\hat{\rho}(f)$ associated with the «symbol» f, a function defined on the «phase space» $V = V_0 \oplus V_0$, is given via the action of the Heisenberg group (cf. 0.1.10). The composition of operators correspond to «twisted multiplication» of their symbols : $\hat{\rho}(f_1)\hat{\rho}(f_2) = \hat{\rho}(f_1 \# f_2)$. The main goal of the symbolic calculus is to control the twisted multiplication over appropriately defined symbol classes, thus enabling the comparison of the resulting function algebra and operator algebra. Classically, the symbol classes are defined using the Mihlin condition which incorporates the global growth condition with the local smoothness condition on the symbols. To define symbol classes in the p-adic setting we encounter once again the lack of derivatives. Fortunately, this time the problem can be overcome if one uses in place of derivatives a Besov condition.

Chapter 0 is devoted to the Heisenberg group and its fundamental representation. For the analogous theory over $\mathbb{R}$ see [6], [9], [21], [23], [32]; some of the material over $\mathbb{Q}_p$ can be found in [11], [36]. In paragraph 0, we fix our notations and recall the Schrödinger model, the Weyl quantization procedure and twisted multiplication. In paragraph 0.2, we review the lattice model and the Bargmann isomorphism [1], and we describe the Wick symbols [4]. In paragraph 0.3, with an eye towards connection with [14], [15], we offer a third realization of the fundamental representation in the «positive» subspace of L_2 of the p-adic circle, or the space of «p-adic particles», giving explicit formulas for the Hermite functions, and explaining why the Tœplitz operators over $\mathbb{Q}_p$ have index zero. In paragraph 0.4, we give a «mannerized» version of the lattice model, and we present the Wigner transform, the uncertainty principle and the wave-packet-expansions [7] in the p-adic setting.

Chapter 1 is devoted to the exposition of the various functions spaces over $\mathbb{Q}_p$. For the analogous theory over $\mathbb{R}$, we refer to [3], [25], [33], [34]; a good reference for some of the material over $\mathbb{Q}_p$ can be found in [31]. In paragraph 1.1, we recall the Bessel potentials and Sobolev spaces. In paragraph 1.2, we explain the problematics of these — the lack of the Leibniz rule for differentiation, and offer a way out via Besov spaces. After reviewing the elementary properties of the ordinary Besov spaces, we present the concept of a metric-covering, and the associated «mannerized» Besov spaces. In paragraph 1.3, we define the «smooth» functions over $\mathbb{Q}_p$, and the (analytic) Schwartz spaces. We quote the Schwartz kernel theorems and define our symbol classes (in analogy with [18]). In paragraph 1.4, we collect the basic properties of temperate metric-coverings needed in the sequel, and offer as examples the Tœplitz symbols (discussed over the reals in [12], [17], [21], [30], [35]) and the pseudodifferential symbols.

The Tœplitz symbols $\Sigma^\alpha(V)$ consist of functions f on V which satisfy

$$|f(x)| \le C_0 \cdot |1, px|^\alpha,$$

and for all $\beta > 0$,

$$|f(x) - f(x+y)| \le C_\beta \cdot |1, px|^{\alpha-\beta} \cdot |y|^\beta$$

for $|y| \le |1, px|$. We note that our (analytic) Schwartz space, S, which can be defines as the intersection over all α of Σ^α, is different from the usual (algebraic) Schwartz space of locally constant compactly supported functions that one encounters in the literature, though it shares much of its nice properties such as nuclearity, closure under multiplication, convolution and Fourier transform.

Chapter 2 is devoted to the proof of the main theorem of the calculus (2.2.8) and the basic boundedness theorems, along the lines of [18], [19]. In paragraph 2.1, we prepare the grounds by giving the short range estimate (2.1.6), the error estimate (2.1.11) and the Long range estimate (2.1.24) and (2.1.27). Then in paragraph 2.2, we put these together and prove the main theorem of the calculus (2.2.8). In paragraph 2.3, we establish the continuity of our operators on S and S', (2.3.1), and their boundedness on L_2 (2.3.9), and give a proof of the Calderón-Vaillancourt estimate [5] (along the lines of [21]).

In chapter 3 we establish the basic properties of elliptic operators. In paragraph 3.1, we prove the sharp Gårding inequality (3.1.11) and the improved Fefferman-Phong inequality (3.1.9) (cf. [8]), as consequences of the wave-packets theory (cf. [7]). In paragraph 3.2,, we show that every elliptic symbol defines a Fredholm operator of index zero (3.2.9). In paragraph 3.3, we consider positive elliptic symbols, show that they give rise to self-adjoint operators with discrete spectra (3.3.1); study their complex powers (3.3.21) (in analogy with [28]); and determine the precise asymptotic behavior of their eigenvalues (3.3.14) (see [10], [13], [17], [20], [27], [30] for some of the literature on the analogous theory over $\mathbb{R}$).

Thus the operators on $L_2(V_0)$ corresponding to symbols in $\Sigma^\alpha(V)$ are bounded for $\alpha \le 0$, Hilbert-Schmidt for $\alpha < -\dim V_0$, and trace-class for $\alpha < -2\dim V_0$. For f in $\Sigma^\alpha(V)$, we can write $f = \tilde{f} + f_0$, where $\tilde{f}(x)$ is the average of f over the unit ball centered at x, and f_0 is in the

Schwartz space. The operator $\hat{\rho}(f)$ is thus the sum of $\hat{\rho}(\tilde{f})$, an explicitly diagonalizable operator, and of $\hat{\rho}(f_0)$, a trace-class operator (in fact, a «smoothening» operator in the sense that it maps S' into S). As a corollary we see that if $f \geq 0$ is positive, then the operator $\hat{\rho}(f)$ is positive modulo trace-class operators and is bounded from below. If f in $\Sigma^\alpha(V)$ is elliptic, in the sense that $|f(x)| \geq C \cdot |1, px|^\alpha$ for x outside a compact set, then $\hat{\rho}(f)$ is a Fredholm operator of index zero; if f is real valued, then $\hat{\rho}(f)$ is an essentially self-adjoint operator, if, moreover, $\alpha > 0$, then $\hat{\rho}(f)$ has a discrete point spectrum tending to infinity and there is an orthonormal basis for $L_2(V_0)$ consisting of $\hat{\rho}(f)$-eigenvectors all of which are in the Schwartz space. Assuming further that both f and $\hat{\rho}(f)$ are positive, we can defined their zeta functions

$$\zeta_{\hat{\rho}(f)}(s) = \mathrm{tr}\big(\hat{\rho}(f)^{-s}\big) \quad \text{and} \quad \zeta_f(s) = \int_V f(x)^{-s}\,\mathrm{d}x$$

both of which are holomorphic in the right half plane $\mathrm{Re}(s) > \dfrac{1}{\alpha}\dim V$, have meromorphic continuation and moreover, their difference is entire. Denoting by $N(\lambda)$ the number of eigenvalues $\leq \lambda$, we have the following precise estimate :

$$N(\lambda) = \mathrm{vol}\big\{x \mid f(x) \leq \lambda\big\} + \mathrm{O}(\lambda^\varepsilon), \quad \text{for any } \varepsilon > 0.$$

Our motivation in developing the theory of pseudodifferential operators in an arithmetical situation was derived from [14], [15], where it was shown that such a theory could have bearings on the problem of the Riemann hypothesis. Much of what we do here carries over to the global situation of the adeles, where the zeta function not only controls the symbolic calculus, but to some extent is controlled by it.

We note that the content of this paper generalizes easily to the case of an arbitrary local field. Other future applications of our theory could include the local Howe's duality conjectures. The theory can also be made to work over varieties where it might be useful as a tool in defining the zeta functions of such varieties by analytic means (a basic ingredient for trying to do Tate's thesis for such varieties). Since we wrote this paper, we have found some attempts to construct quantum mechanics over the p-adics in the literature of physics, e.g. [42], but these are always wrong and lead to strange phenomenon (e.g. the existence of two vacuum states), because the physicists were insensitive to the basic difference between the p-adics and the reals – the two dimensional absolute value, which for the reals is the L_2

one : $|x, y| = (|x|^2 + |y|^2)^{1/2}$, should be replaced in the p-adic setting by its L_∞ analog : $|x, y| = \max(|x|, |y|)$.

In the one dimensional theory, $V_0 = \mathbb{Q}_p$, $V = \mathbb{Q}_p \oplus \mathbb{Q}_p$, we can take as a basic group of symmetries the group $G = \mathrm{Sp}(1, \mathbb{Q}_p) = \mathrm{SL}(2, \mathbb{Q}_p)$, and identify $V \setminus \{(0,0)\}$ with the symmetric space G/N, where N is the subgroup of matrices of the form $\left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$, the action of G on V, and the resulting action on symbols, correspond to the metaplectic representation on $L_2(\mathbb{Q}_p)$. In a similar fashion we can follow the Unterbergers, cf. [39], [40], [41], and take as our phase space G/K, where $K = \mathrm{SL}(2, \mathbb{Z}_p)$ is a maximal compact subgroup, obtaining quantizations which are relevant to the other discrete series representations of G; or we can take as phase space $G/T = \mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_p) \setminus \text{diagonal}$, where $T = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right\}$ is a maximal torus, obtaining quantizations relevant to the principal and the complementary series. Again, these constructions could be carried in an adelic situation, and will ultimately lead to the quantization of $G(\mathbb{A})/G(\mathbb{Q})$, i.e. quantization of automorphic forms.

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0. THE HEISENBERG GROUP

0.1. The Schrödinger model and twisted multiplication.

We denote by ψ the additive character of $\mathbb{Q}_p$, given by

$$\psi : \mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p \simeq \mathbb{Z}[p^{-1}]/\mathbb{Z} \xhookrightarrow{\exp(2\pi i x)} \mathbb{C}^*. \tag{0.1.1}$$

We let $\mathrm{d}x$ denote Haar measure on $\mathbb{Q}_p^{\oplus d}$, self-dual with respect to $\psi(x \cdot y) = \psi\big(\sum_{i=1}^{d} x_i \cdot y_i\big)$, so the Fourier transform

$$\mathcal{F}_d\varphi(y) = \int \varphi(x)\bar{\psi}(xy)\,\mathrm{d}x$$

satisfy $\mathcal{F}_d\mathcal{F}_d\varphi(x) = \varphi(-x)$. The symbol ϕ will denote the characteristic function of $\mathbb{Z}_p^{\oplus d}$, so that $\mathcal{F}_d\phi = \phi$. We shall usually assume $p \neq 2$, and leave it to the reader to make the easy modifications needed for $p = 2$.

We let $V, \langle\ \rangle$ denote a symplectic vector space over $\mathbb{Q}_p$, and $\mathbb{H} = V \times \mathbb{Q}_p$ the associated *Heisenberg group*, with multiplication law given by

$$(0.1.2)\quad (v_1, t_1) \cdot (v_2, t_2) = \big(v_1 + v_2, t_1 + t_2 + \tfrac{1}{2}\langle v_1, v_2\rangle\big), \quad v_i \in V,\ t_i \in \mathbb{Q}_p.$$

$\mathbb{H}$ is two-step nilpotent group, with center $\mathbb{Q}_p$, and the Stone-von Neumann theorem states that it has a unique irreducible representation with central character ψ (up-to-equivalence). *The Schrödinger model* for this representation, $\rho : \mathbb{H} \to U(L_2(V_0))$, is associated with any complete polarization $V = V_0 \oplus V_1$, V_i Lagrangian subspaces. Fixing coordinates we can focus our attention on the case

$$V_0 = V_1 = \mathbb{Q}_p^{\oplus d}, \quad \big\langle (x_1, y_1), (x_2, y_2)\big\rangle = x_1 y_2 - x_2 y_1,$$

so that ρ is given by

$$(0.1.3)\qquad \rho(x, y, t)\,\varphi(z) = \psi\big(t + x(z - \tfrac{1}{2}y)\big) \cdot \varphi(z - y).$$

The integrated representation, $\rho : L_1(\mathbb{H}) \to \mathcal{B}(L_2(V_0))$, factors through the algebra

$$(0.1.4)\qquad L_1(\mathbb{H}, \psi) = \Big\{ f \in L_1(\mathbb{H}) \mid f(h \cdot t) = \bar{\psi}(t) \cdot f(h) \\ \text{for } t \in \mathbb{Q}_p = \text{center}(\mathbb{H}) \Big\}.$$

Identifying V with the subspace of $\mathbb{H}$, via the *isotopic cross-section* $v \mapsto (v, 0)$, we can identify $L_1(\mathbb{H}, \psi)$ with $L_1(V)$ via restriction. Thus for $f \in L_1(V)$ and $\varphi \in L_2(V_0)$,

$$(0.1.5)\qquad \rho(f)\,\varphi(z) = \int_V \mathrm{d}v\, f(v)\,\rho(v, 0)\,\varphi(z) = \int_{V_0} \mathrm{d}y\, K_f(z, y)\,\varphi(y)$$

with associated kernel given by

$$(0.1.6)\qquad K_f(z, y) = \int \mathrm{d}x\, f(x, z - y)\,\psi\big(\tfrac{1}{2}x(z + y)\big).$$

We have $\rho(f_1)\rho(f_2) = \rho(f_1 \natural f_2)$ with *twisted convolution* $\natural$ given by

$$(0.1.7)\qquad f_1 \natural f_2(v) = \int_V \mathrm{d}v'\, f_1(v')\, f_2(v - v')\,\psi\big(\tfrac{1}{2}\langle v', v - v'\rangle\big).$$

Since (0.1.6) is essentially a partial Fourier transform, we see that ρ induces a unitary isomorphism from the algebra $L_2(V), \natural$ onto the algebra of Hilbert-Schmidt operators $\mathcal{K}_2(L_2(V_0))$; hence it takes $L_1(V)$, $\natural$ into the algebra of compact operators $\mathcal{K}(L_2(V_0))$. We have

$$(0.1.8)\qquad \begin{cases} \|f_1 \natural f_2\|_{L_1} \leq \|f_1\|_{L_1} \cdot \|f_2\|_{L_1}\,, \\ \|f_1 \natural f_2\|_{L_2} \leq \|f_1\|_{L_2} \cdot \|f_2\|_{L_2}\,, \\ \|f_1 \natural f_2\|_{L_\infty} \leq \|f_1\|_{L_2} \cdot \|f_2\|_{L_2}. \end{cases}$$

We denote the *symplectic Fourier transform*, or *isotropic symbol*, by

$$(0.1.9)\qquad \mathcal{F}f(v) = \hat{f}(v) = \int \mathrm{d}v'\, f(v')\, \psi(\langle v', v\rangle).$$

$\mathcal{F}$ is a unitary operator on $L_2(V)$, satisfying $\mathcal{F}^2 = \mathrm{id}$. The *Weyl quantization procedure* associates with a function f on V the operator $\hat{\rho}(f) = \rho(\hat{f})$ on $L_2(V_0)$, which is given by

$$(0.1.10)\qquad \hat{\rho}(f)\,\varphi(x) = \iint \mathrm{d}y\,\mathrm{d}\xi\, f\big(\xi, \tfrac{1}{2}(x+y)\big)\, \psi\big(\xi(x-y)\big)\, \varphi(y).$$

We have $\hat{\rho}(f_1)\hat{\rho}(f_2) = \hat{\rho}(f_1 \# f_2)$, where *twisted multiplication* $\#$ is given by

$$(0.1.11)\qquad \begin{aligned} f_1 \# f_2(v) &= \mathcal{F}(\mathcal{F}f_1 \natural \mathcal{F}f_2)(v) \\ &= \iint \mathrm{d}v_1\,\mathrm{d}v_2\, f_1(v+v_1)\, f_2(v+v_2)\, \bar{\psi}\big(2\langle v_1, v_2\rangle\big) \\ &= \psi\big(Q(D)\big)\, f_1 \otimes f_2(v,v). \end{aligned}$$

Here $f_1 \otimes f_2(v_1, v_2) = f_1(v_1) \cdot f_2(v_2)$, and $\psi(Q(D))$ is the unitary operator on $L_2(V \oplus V) = L_2(V) \otimes V_2(V)$ given by

$$(0.1.12)\qquad \psi\big(Q(D)\big) = \mathcal{F} \otimes \mathcal{F}\, \psi\big(Q(v_1, v_2)\big) \mathcal{F} \otimes \mathcal{F}$$

with $Q(v_1, v_2) = \frac{1}{2}\langle v_1, v_2\rangle$ the quadratic form on $V \oplus V$ associated with the symplectic form $\langle\ ,\ \rangle$ on V.

The adjoint operator $\hat{\rho}(f)^*$ is given by :

(0.1.13) $\hat{\rho}(f)^* = \hat{\rho}(\bar{f})$, $\bar{f}$ the complex conjugate of f.

(0.1.14) If $f(\xi, x) = f(x)$ is independent of ξ, then $\hat{\rho}(f)\varphi(x) = f(x) \cdot \varphi(x)$.

(0.1.15) If $f(\xi, x) = f(\xi)$ is independent of x, then $\hat{\rho}(f)\varphi(x) = \mathcal{F}_d^{-1} f(\xi) \mathcal{F}_d \varphi(x)$.

Setting, for example,

$$\Phi^{a,b}_{x_0,y_0}(x,y) = p^{-(a+b)\cdot d}\,\phi\big(p^a(x-x_0)\big)\,\phi\big(p^b(y-y_0)\big),$$
$$\widetilde{\Phi}^{a,b}_{x_0,y_0}(x,y) = \phi\big(p^{-b}(x-x_0)\big)\,\phi\big(p^{-a}(y-y_0)\big)\,\bar{\psi}\big((x-x_0)(y-y_0)\big),$$

a straightforward calculation gives :

$$\Phi^{a_1,b_1}_{x_1,y_1} \# \Phi^{a_2,b_2}_{x_2,y_2} = \begin{cases} \Phi^{a_1,b_1}_{x_1,y_1}\cdot\Phi^{a_2,b_2}_{x_2,y_2} \\ \quad = \Phi^{a_1,b_2}_{x_1,y_2}\cdot\Phi^{a_2,b_1}_{x_2,y_1} & \text{if } a_1+b_2,\ b_1+a_2\geq 0, \\ \Phi^{a_1,b_2}_{x_1,y_2}\cdot\overline{\widetilde{\Phi}^{a_2,b_1}_{x_2,y_1}} & \text{if } a_1+b_2\geq 0\geq b_1+a_2, \\ \widetilde{\Phi}^{a_1,b_2}_{x_1,y_2}\cdot\Phi^{a_2,b_1}_{x_2,y_1} & \text{if } a_1+b_2\leq 0\leq b_1+a_2, \\ \widetilde{\Phi}^{a_1,b_2}_{x_1,y_2}\cdot\overline{\widetilde{\Phi}^{a_2,b_1}_{x_2,y_1}} & \text{if } a_1+b_2, b_1+b_2\leq 0. \end{cases} \tag{0.1.16}$$

For a lattice $\mathcal{O}\subseteq V$, we denote the *dual lattice* by

$$\mathcal{O}^\vee = \{v\in V \mid \langle v,\mathcal{O}\rangle \subseteq \mathbb{Z}_p\} \tag{0.1.17}$$

and say that $\mathcal{O}$ is *certain* if $\mathcal{O}^\vee\subseteq\mathcal{O}$. If $\mathcal{O}$ is a certain lattice, and f_1, f_2 are $\mathcal{O}$-locally-constant, i.e., they factor through $V/\mathcal{O}$, then an easy calculation shows that $f_1 \# f_2 = f_1\cdot f_2$, i.e., twisted multiplication reduces to ordinary multiplication. More generally, if $V = \coprod_j \mathfrak{A}_j$ is written as a disjoint union of translates of certain lattices, $\mathfrak{A}_j = x_j + \mathcal{O}_j$, $\mathcal{O}_j \supseteq \mathcal{O}_j^\vee$, and f_1, f_2 are constant on each $\mathfrak{A}_j$, then again

$$f_1 \# f_2 = f_1\cdot f_2. \tag{0.1.18}$$

0.2. The lattice model.

If $\mathcal{O}\subseteq V$ is a certain lattice, then we can always find a *self-dual* lattice $\mathcal{O}^0 = (\mathcal{O}^0)^\vee$, such that $\mathcal{O}^\vee\subseteq\mathcal{O}^0\subseteq\mathcal{O}$. The representation ρ can be realized in the *$\mathcal{O}$–lattice model*

$$\rho \cong \rho_\psi \overset{\text{def}}{=\!=} \operatorname{ind}^{\mathbb{H}}_{\mathcal{O}^0\times\mathbb{Q}_p}(\psi) = \operatorname{ind}^{\mathbb{H}}_{\mathcal{O}\times\mathbb{Q}_p}(\rho_{\mathcal{O}}) \tag{0.2.1}$$

where ψ is viewed as a character of the abelian subgroup $\mathcal{O}^0\times\mathbb{Q}_p/\mathbb{Z}_p\subseteq \mathbb{H}/\mathbb{Z}_p$, via $\psi(v,t) = \psi(t)$, and $\rho_{\mathcal{O}}$ is the representation $\operatorname{ind}^{\mathcal{O}\times\mathbb{Q}_p}_{\mathcal{O}^0\times\mathbb{Q}_p}(\psi)$ of $\mathcal{O}\times\mathbb{Q}_p$, which is finite dimensional of dimension

$$[\mathcal{O}:\mathcal{O}^0] = [\mathcal{O}:\mathcal{O}^\vee]^{1/2}. \tag{0.2.2}$$

In particular for $V = \mathbb{Q}_p^{\oplus 2d}$, $\mathcal{O}_0 = \mathbb{Z}^{\oplus 2d}$, we obtain a realization of ρ in $L_2(V)_\psi$ where the subscript ψ indicates that we are looking on the subspace of $L_2(V)$ consisting of *ψ-holomorphic* functions, i.e., functions $f \in L_2(V)$ satifying the following « Cauchy-Riemann equation »

$$f(v+v_0) = f(v)\cdot \psi\big(\tfrac{1}{2}\langle v, v_0\rangle\big) \quad \text{for all } v \in V,\ v_0 \in \mathcal{O}_0. \tag{0.2.3}$$

The representation $\rho_\psi : \mathbb{H} \to U(L_2(V)_\psi)$ is given by

$$\rho_\psi(v,t)\, f(v') = \psi\big(t + \tfrac{1}{2}\langle v, v'\rangle\big)\, f(v'-v). \tag{0.2.4}$$

To obtain *the Bargmann isomorphism* $B : L_2(V)_\psi \xrightarrow{\sim} L_2(V_0)$, intertwining ρ_ψ and ρ, we notice that $L_2(V)_\psi$ is also gotten by taking the coefficients of ρ with respect to $\phi \in L_2(V_0)$. Thus

$$Bf(x) = \rho(f)\,\phi(x) = \big(f, \overline{\rho(v)\,\phi(x)}\big)_{L_2(V)}, \tag{0.2.5}$$

$$B^{-1}\varphi(v) = \big(\varphi, \rho(v)\phi\big)_{L_2(V_0)}. \tag{0.2.6}$$

For any $v \in V$ there is a unique function $\Phi_v \in L_2(V)_\psi$ which is supported in $v + \mathcal{O}_0$, and normalized by $\Phi_v(v) = 1$; the functions $B\Phi_v = \rho(v)\phi$, $v \in V/\mathcal{O}_0$, form an orthonormal basis for $L_2(V_0)$; in particular we have

$$\Phi_0 = B^{-1}\phi = \phi \otimes \phi = \text{characteristic function of } \mathcal{O}_0. \tag{0.2.7}$$

For $f \in L_2(V)_\psi$ we have $f(v) = (f, \Phi_v)$, i.e., $L_2(V)_\psi$ is a *reproducing-kernel-Hilbert space.* The orthogonal projection $P_\psi : L_2(V) \to L_2(V)_\psi$ is given by

$$P_\psi f(v) = f \natural \Phi_0(v) = \int_{\mathcal{O}_0} dv_0\, f(v-v_0)\, \psi\big(\tfrac{1}{2}\langle v, v_0\rangle\big). \tag{0.2.8}$$

For a function $f \in L_\infty(V)$, letting M_f denote multiplication by f, we obtain the operator $P_\psi M_f$ on $L_2(V)_\psi$ having *anti-Wick symbol* f; but an easy calculation gives

$$P_\psi M_f = M_{\tilde f}, \quad \tilde f(v) = \int_{\mathcal{O}_0} dv_0\, f(v-v_0), \tag{0.2.9}$$

i.e. the algebra of such operators reduces to the *commutative* algebra $L_\infty(V/\mathcal{O}_0)$ acting on $L_2(V)_\psi$ by multiplication.

For any bounded operator $T \in \mathcal{B}(L_2(V)_\psi)$, we denote its *Wick symbol* by

$$ET(v) = (T\Phi_v, \Phi_v). \tag{0.2.10}$$

E is a norm-decreasing positive projection $\mathcal{B}(L_2(V)_\psi) \twoheadrightarrow L_\infty(V/\mathcal{O}_0)$. The integrated representation associated with ρ_ψ is given by

$$\rho_\psi(f)\, f_0 = f \natural f_0, \qquad f_0 \in L_2(V)_\psi, \tag{0.2.11}$$

and the Weyl quantization $\hat{\rho}_\psi(f) = \rho_\psi(\hat{\rho}) = B^{-1}\hat{\rho}(f)B$ is given by

$$\hat{\rho}_\psi(f)\, f_0 = f \# f_0, \qquad f_0 \in L_2(V)_\psi. \tag{0.2.12}$$

The relations between the Weyl, Wick, and anti-Wick symbols are easy to obtain

$$\begin{cases} E\hat{\rho}_\psi(f)(v) = \tilde{f}(v), \\ M_f = \hat{\rho}_\psi(f) \quad \text{for } f \in L_\infty(V/\mathcal{O}_0). \end{cases} \tag{0.2.13}$$

0.3. The Tœplitz model.

In this section we take $d = 1$, so that we are dealing with operators on $L_2(\mathbb{Q}_p)$ and $L_2(\mathbb{Q}_p \oplus \mathbb{Q}_p)_\psi$. Choose $\varepsilon \in \mathbb{Z}_p^* \setminus (\mathbb{Z}_p^*)^2$, and identify $V \cong \mathbb{Q}_p[\sqrt{\varepsilon}\,]$, the unramified quadratic extension of $\mathbb{Q}_p$, via $(x, y) \leftrightarrow x + \sqrt{\varepsilon}\, y$; thus $\mathcal{O}_0 = \mathbb{Z}_p[\sqrt{\varepsilon}]$ is the ring of integers of V, and the associated norm is given by

$$|x, y| = \left|x + \sqrt{\varepsilon}\, y\right| = |x^2 - \varepsilon y^2|^{1/2} = \max\{|x|, |y|\}. \tag{0.3.1}$$

(0.3.2) Let
$$\begin{aligned} U_1 &= \ker\{\mathbb{N} : V^* \to \mathbb{Q}_p^*\} \\ &= \{x + \sqrt{\varepsilon}\, y \in \mathcal{O}_0^* \mid x^2 - \varepsilon y^2 = 1\} = \mu_{p+1} \times \exp(\sqrt{\varepsilon}\, p\, \mathbb{Z}_p). \end{aligned}$$

By Hilbert lemma 90, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} * & \longrightarrow & U_1 & \longrightarrow & \mathcal{O}_0^* & \xrightarrow{\mathbb{N}u = u\cdot\bar{u}} & \mathbb{Z}_p^* & \longrightarrow & * \\ & & {\scriptstyle u^2}\big\downarrow & & \big\| & & \big\uparrow{\scriptstyle x^2} & & \\ * & \longleftarrow & U_1 & \longleftarrow & \mathcal{O}_0^* & \longleftarrow & \mathbb{Z}_p^* & \longleftarrow & * \\ & & u/\bar{u} & \longleftarrow\!\shortmid & u & & & & \end{array}$$

Hence we may also identify

$$U_1 \cong \mathcal{O}_0^*/\mathbb{Z}_p^* = V^*/\mathbb{Q}_p^* = \mathbb{P}^1(\mathbb{Q}_p). \tag{0.3.4}$$

U_1 may also be identified with the isometry group of the Hertmitian form on V, given by

$$\begin{cases} \bar{v}_1 \cdot v_2 = q_\varepsilon(v_1, v_2) + \sqrt{\varepsilon}\,\langle v_1, v_2\rangle \\ \text{with symmetric part } q_\varepsilon\big(x_1 + \sqrt{\varepsilon}\,y_1, x_2 + \sqrt{\varepsilon}\,y_2\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad = x_1x_2 - \varepsilon y_1 y_2, \\ \text{and symplectic part } \langle v_1, v_2\rangle = \operatorname{tr}\big((2\sqrt{\varepsilon})^{-1}\,\bar{v}_1 v_2\big). \end{cases} \tag{0.3.5}$$

Note that the metric induced on U_1 via $U_1 \hookrightarrow V$ is given, under the above identification, by any of the following metrices

$$\begin{cases} \delta\big(\xi_1\, \mathrm{e}^{\sqrt{\varepsilon} p w_1}, \xi_2\, \mathrm{e}^{\sqrt{\varepsilon} p w_2}\big) = \begin{cases} 1 & \text{if } \xi_1 \neq \xi_2, \\ |pw_1 - pw_2| & \text{if } \xi_1 = \xi_2, \end{cases} \\ \qquad\qquad\qquad\qquad\qquad \text{where } U_1 \simeq \mu_{p+1} \times \exp(\sqrt{\varepsilon}\, p\mathbb{Z}_p)\,; \\ \delta(z_1, z_2) = |z_1/\bar{z}_1 - z_2/\bar{z}_2|, \quad \text{where } U_1 \simeq V^*/\mathbb{Q}_p^* \\ \delta(x_1 : y_1, x_2 : y_2) = \dfrac{|x_1y_2 - x_2y_1|}{|x_1, y_1| \cdot |x_2, y_2|}, \quad \text{where } U_1 \simeq \mathbb{P}^1(\mathbb{Q}_p). \end{cases} \tag{0.3.6}$$

The group $\mathrm{Sp}(V) = \mathrm{SL}_2(\mathbb{Q}_p)$ acts on $\mathbb{H}$ as automorphisms inducing the identity on the center, hence by the uniqueness of the representation ρ, we obtain a projective representation R of $\mathrm{Sp}(V)$ such that

$$R_{\mathcal{A}}\rho(v)R_{\mathcal{A}}^{-1} = \rho(\mathcal{A}v), \qquad \mathcal{A} \in S_p(V),\ v \in V. \tag{0.3.7}$$

R in fact lifts to a true representation of the double cover of $\mathrm{Sp}(V)$, the *metaplectic group* $\widehat{\mathrm{Sp}}(V)$. We have an embedding

$$U_1 \hookrightarrow \mathrm{SL}_2(\mathbb{Z}_p), \quad x + \sqrt{\varepsilon}\, y \longmapsto \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}, \tag{0.3.8}$$

by which we view U_1 as a maximal *abelian* subgroup of $\mathrm{Sp}(V)$; in the terminology of [22] : (U_1, U_1) *is a dual reductive pair.* The metaplectic group splits over $\mathrm{SL}_2(\mathbb{Z}_p)$, hence we obtain a true representation R of $\mathrm{SL}_2(\mathbb{Z}_p)$, and a posteriori of U_1. This can be summarized in the following diagram :

$$\begin{array}{ccccccc} & & & & \widehat{\mathrm{Sp}}(V) & \xrightarrow{R} & U\big(L_2(\mathbb{Q}_p)\big) \\ & & & \nearrow & \downarrow & & \\ U_1 & \hookrightarrow & \mathrm{SL}_2(\mathbb{Z}_p) & \hookrightarrow & \mathrm{Sp}(V) & & \end{array} \tag{0.3.9}$$

Explicitly, $R : U_1 \to U(L_2(\mathbb{Q}_p))$ is given by the *Mehler kernel*

$$(0.3.10) \qquad R_{x+\sqrt{\varepsilon}\,y}(z_1, z_2) = \frac{(x,y)}{\gamma(y)\cdot|y|^{1/2}}\,\psi\Big(\frac{x}{2y}(z_1^2+z_2^2) - \frac{z_1z_2}{y}\Big)$$

for $x+\sqrt{\varepsilon}\,y \in U_1 \setminus \{\pm 1\}$ and $R_{\pm 1}\varphi(z) = \varphi(\pm z)$.

Here (x,y) denotes the quadratic symbol, $\gamma(y)$ denotes Weil's root of unity, and they are given by $(p \neq 2)$

$$(0.3.11) \qquad (p^{j_1}\varepsilon^{k_1}, p^{j_2}\varepsilon^{k_2}) = (-1)^{\frac{1}{2}(p-1)\,j_1j_2+k_1j_2+j_1k_2},$$

$$(0.3.12) \qquad \gamma(p^j\varepsilon^k) = \begin{cases} 1 & \text{if } j \text{ is even,}\\ (-i)^{\frac{1}{2}(p-1)+2k} & \text{if } j \text{ is odd.}\end{cases}$$

Denote the dual group of U_1 by $\mathcal{Z} = (U_1)^\vee \cong (\mu_{p+1})^\vee \times \mathbb{Q}_p/\mathbb{Z}_p$, and write a typical character $\chi \in \mathcal{Z}$ as

$$(0.3.13) \qquad \chi(\xi\, e^{\sqrt{\varepsilon}pw}) = \chi_t(\xi)\cdot\bar{\psi}(\tfrac{1}{2}q_\chi pw),$$

where $\quad \chi_t \in (\mu_{p+1})^\vee$ and $q_\chi \in \mathbb{Q}_p/p^{-1}\mathbb{Z}_p$.

We put $\mathcal{Z} = \coprod_{N\geq 0} \mathcal{Z}^{(N)}$, $\mathcal{Z}^{(0)} = \{\chi^0\}$ the identity, $\mathcal{Z}^{(1)} = \{\chi_t \neq \chi^0\}$ the non-trivial tame characters, and for $N \geq 2 : \mathcal{Z}^{(N)} = \{\chi;\ |q_\chi| = p^N\}$. A straightforward calculation gives that $L_2(\mathbb{Q}_p)$ decomposes under the U_1 action given by R as

$$(0.3.14) \qquad L_2(\mathbb{Q}_p) = \bigoplus_{\chi\in\mathcal{Z}^+} \mathbb{C}\cdot\phi^\chi, \quad R_u\phi^\chi = \chi(u)\cdot\phi^\chi,$$

where $\mathcal{Z}^+ = \coprod_{N\geq 0} \mathcal{Z}^{(2N)}$. In particular, $\phi^{\chi_0} = \phi$ is the unique U_1-invariant vector. For $\chi \in \mathcal{Z}^{(2N)}, N \geq 1$, the *Hermite function* ϕ^χ is the unique χ-covariant vector (up-to a constant of absolute value one), and is given explicitly by

$$(0.3.15) \qquad \phi^\chi = (1+p^{-1})^{1/2}p^{N/2}\int_{U_1} d^*u\,\bar{\chi}(u)\,\rho(u\cdot z_\chi)\,\phi$$

where $q_\chi = \mathbb{N}z_\chi$, and d^*u is Haar measure on U_1 of total mass 1.

Comparing the orthonormal basis $\{\phi^\chi \mid \chi \in \mathcal{Z}^+\}$ with the orthonormal basis $\{\rho(x,y)\phi \mid x,y \in \mathbb{Q}_p/\mathbb{Z}_p\}$ of the lattice model, we note that

$$(0.3.16)\qquad \#\mathcal{Z}^{(2N)} = p^{2N}(1-p^{-2}) = \#\Big\{(x,y) \in V/\mathcal{O}_0\,;\ |x,y| = p^N\Big\},\quad N \geq 1,$$

and each ϕ^χ is a linear combination of $p^N(1+p^{-1})$ $\rho(x,y)\phi$'s, and vice versa. More precisely,

$$(0.3.17)\qquad \Phi^\chi(v) = B^{-1}\phi^\chi(v) = \big(\phi^\chi, \rho(v)\phi\big)$$

is supported in $V^\chi = \big\{v \in V \mid \mathbb{N}v \equiv q_\chi \ (\operatorname{mod} p^{-N}\mathbb{Z}_p)\big\} = U_1 \cdot z_\chi + \mathcal{O}_0$, where it is given by

$$(0.3.18)\qquad \Phi^\chi(u \cdot z_\chi + v_0) = \big((1+p^{-1})p^N\big)^{-1/2} \cdot \bar{\chi}(u) \cdot \psi\big(\tfrac{1}{2}\langle u \cdot z_\chi, v_0\rangle\big),$$

with $u \in U_1$ and $v_0 \in \mathcal{O}_0$.

Using (0.3.14) we obtain the *Tœplitz model* $\rho_\mathcal{T} : \mathbb{H} \longrightarrow U(L_2(U_1)^+)$, where $L_2(U_1)^+ \subseteq L_2(U_1)$ is the subspace corresponding to $\ell_2(\mathcal{Z}^+) \subseteq \ell_2(\mathcal{Z})$ under Fourier transform $L_2(U_1) \simeq \ell_2(\mathcal{Z})$. The orthogonal projection $P^+ : L_2(U_1) \twoheadrightarrow L_2(U_1)^+$ is easily seen to be given by the singular integral

$$(0.3.19)\qquad P^+f(u_0) = \text{p.v.}\int_{U_1} \mathrm{d}^*u\, f(u)\,\tilde{\delta}(u,u_0)^{-1},$$

where $\tilde{\delta}(u,u_0) = (-p)^{-N}$ if $\delta(u,u_0) = p^{-N}$.

If $f \in C(U_1)^*$ is a continuous non-vanishing function, the associated *Tœplitz operator* P^+M_f is Fredholm of index zero. Indeed, any such function is homotopic to $\mathbb{1}$, the constant function 1, since U_1 is totally disconnected and compact. This can be partially explained by considering the problem from a dual point of view : if $\chi \in \mathcal{Z}^{(N)}$, translation by χ followed by the projection $\widehat{P^+} : \ell_2(\mathcal{Z}) \twoheadrightarrow \ell_2(\mathcal{Z}^+)$ induces a Fredholm operator $\widehat{P^+\chi}$ on $\ell_2(\mathcal{Z}^+)$, which has index zero since,

$$(0.3.20)\qquad \dim\ker \widehat{P^+\chi} = \dim\operatorname{coker} \widehat{P^+\chi} = p^{N-1},\qquad N \geq 1.$$

Remark. — The material in paragraph 0.3. works equally well for the case of a totally ramified quadratic extension of $\mathbb{Q}_p$.

0.4. Wave packets.

We can consider more general coefficients of the representation ρ than those of ϕ. Thus for $\varphi_1, \varphi_2 \in L_2(V_0)$ let $c_{\varphi_1,\varphi_2}(v) = (\varphi_2, \rho(v)\varphi_2)$. After Fourier inversion we obtain

$$c_{\varphi_1,\varphi_2}(x,y) = \int \mathrm{d}z\, \bar{\psi}(xz)\, \varphi_1\big(z + \tfrac{1}{2}y\big)\, \overline{\varphi_2\big(z - \tfrac{1}{2}y\big)}. \tag{0.4.1}$$

A direct calculation gives

$$(c_{\varphi_1,\varphi_2}, c_{\varphi_3,\varphi_4})_{L_2(V)} = (\varphi_1, \varphi_3)_{L_2(V_0)} \cdot \overline{(\varphi_2, \varphi_4)_{L_2(V_0)}} \tag{0.4.2}$$

showing that ρ is square integrable modulo the center. The Schwartz inequality gives

$$\|c_{\varphi_1,\varphi_2}\|_{L_\infty} \leq \|\varphi_1\|_{L_2} \cdot \|\varphi_2\|_{L_2}. \tag{0.4.3}$$

Hence the map $\varphi_1 \otimes \varphi_2 \mapsto c_{\varphi_1,\varphi_2}$ induces

$$c : L_2(V_0) \otimes \overline{L_2(V_0)} \longrightarrow L_2(V) \cap C_0(V).$$

Using (0.4.2) we get $(\varphi', \rho(c_{\varphi_1,\varphi_2})\varphi) = (c_{\varphi',\varphi}, c_{\varphi_1,\varphi_2}) = (\varphi', (\varphi,\varphi_2)\cdot\varphi_1)$, i.e.,

$$\rho(c_{\varphi_1,\varphi_2}) \text{ is the rank 1 operator } \rho(c_{\varphi_1,\varphi_2})\varphi = (\varphi, \varphi_2) \cdot \varphi_1. \tag{0.4.4}$$

Hence $c_{\varphi_1,\varphi_2} \natural\, c_{\varphi_3,\varphi_4} = \overline{(\varphi_2, \varphi_3)} \cdot c_{\varphi_1,\varphi_4}$.

Taking the symplectic Fourier transform of c_{φ_1,φ_2}, we obtain after Fourier inversion

$$\hat{c}_{\varphi_1,\varphi_2}(\xi, x) = \int \mathrm{d}z\, \bar{\psi}(\xi z)\, \varphi_1\big(x + \tfrac{1}{2}z\big)\, \overline{\varphi_2\big(x - \tfrac{1}{2}z\big)}. \tag{0.4.5}$$

$\hat{c}_{\varphi_1,\varphi_2}$ satisfies (0.4.2), (0.4.3), and again induces

$$\hat{c} : L_2(V_0) \otimes \overline{L_2(V_0)} \longrightarrow L_2(V) \cap C_0(V).$$

Moreover, by direct verification we have

$$\hat{c}_{\mathcal{F}_d\varphi_1,\mathcal{F}_d\varphi_2}(\xi, x) = \hat{c}_{\varphi_1,\varphi_2}(x, -\xi), \tag{0.4.6}$$

$$\overline{\hat{c}_{\varphi_1,\varphi_2}} = \hat{c}_{\varphi_2,\varphi_1}. \tag{0.4.7}$$

We define the *Wigner transform* by $W\varphi = \hat{c}_{\varphi,\varphi} \in L_2(V) \cap C_0(V)$, and by (0.4.7) it is real valued. It also satisfies the following properties :

$$\int \mathrm{d}\xi \, W\varphi(\xi, x) = \left|\varphi(x)\right|^2, \tag{0.4.8}$$

$$\int \mathrm{d}x \, W\varphi(\xi, x) = \left|\mathcal{F}_d\varphi(\xi)\right|^2, \tag{0.4.9}$$

$$W\varphi_1 = W\varphi_2 \iff \varphi_1 = \lambda \cdot \varphi_2 \text{ for some } \lambda,\ |\lambda| = 1. \tag{0.4.10}$$

Note that $W\varphi(\xi, x) \neq 0$ implies that there exists z such that $x \pm z \in \operatorname{supp}\varphi$, and hence $x \in \Delta(\operatorname{supp}\varphi) \stackrel{\text{def}}{=} \{\frac{1}{2}(y_1 + y_2) \mid y_i \in \operatorname{supp}\varphi\}$, thus we have

$$\operatorname{proj}_x(\operatorname{supp} W\varphi) \subseteq \Delta(\operatorname{supp}\varphi), \tag{0.4.11}$$

and using (0.4.6),

$$\operatorname{proj}_\xi(\operatorname{supp} W\varphi) \subseteq \Delta(\operatorname{supp}\mathcal{F}_d\varphi). \tag{0.4.12}$$

We have the following inequalities :

$$\begin{aligned} \|W\varphi\|_{L_\infty} \leq \|\varphi\|_{L_2}^2 &= \iint \mathrm{d}x \, \mathrm{d}\xi \, W\varphi(\xi, x), \\ &\leq \|W\varphi\|_{L_\infty} \cdot \operatorname{vol}(\operatorname{supp} W\varphi) \end{aligned} \tag{0.4.13}$$

from which we obtain the *uncertainly principle*

$$\operatorname{vol}(\operatorname{supp} W\varphi) \geq \frac{\|\varphi\|_{L_2}^2}{\|W\varphi\|_{L_\infty}} \geq 1. \tag{0.4.14}$$

We have further

$$W\varphi = c_{\phi,\phi} = \Phi_0 = \text{characteristic function of } \mathcal{O}_0, \tag{0.4.15}$$

$$W(\rho(v_0)\varphi)(v) = W\varphi(v - v_0), \tag{0.4.16}$$

$$W(\mathcal{R}_{\mathcal{A}}\varphi)(v) = W\varphi(\mathcal{A}^{-1}v), \quad \mathcal{A} \in \operatorname{Sp}(V), \text{ by (0.3.7)}. \tag{0.4.17}$$

Since $\operatorname{Sp}(V)$ acts transitively on the collection of self-dual lattices, we deduce using (0.4.15), (0.4.16), (0.4.17) that for any translate of a self-dual lattice $\mathfrak{A} = v_0 + \mathcal{O}, \mathcal{O}^\vee = \mathcal{O}$, there exists a unit vector $\varphi_{\mathfrak{A}} \in L_2(V_0)$ such that $W\varphi_{\mathfrak{A}} =$ characteristic function of $\mathfrak{A}$; by (0.4.10) $\varphi_{\mathfrak{A}}$ is unique up to a constant of absolute value one.

For any unit vector $\varphi_0 \in L_2(V_0)$, the map $C_{\varphi_0}(\varphi) = c_{\varphi,\varphi_0}$ is by (0.4.2) an isometry $c_{\varphi_0} : L_2(V_0) \hookrightarrow L_2(V)$. The adjoint map

$$C^*_{\varphi_0} : L_2(V) \twoheadrightarrow L_2(V_0)$$

is given by $C^*_{\varphi_0} f(x) = \rho(f)\varphi_0(x)$, $f \in L_2(V) \cap L_1(V)$. Since $C^*_{\varphi_0} C_{\varphi_0} = \mathrm{id}_{L_2(V_0)}$, we obtain the *wave packet expansion of* φ as superposition of the $\rho(v)\varphi_0$'s

$$\varphi(x) = \int \mathrm{d}v \, \big(\varphi, \rho(v)\varphi_0\big) \cdot \rho(v)\,\varphi_0(x). \tag{0.4.18}$$

For $f \in L_\infty(V) \cap L_1(V)$, denoting by M_f multiplication by f, we get the operator $C^*_{\varphi_0} M_f C_{\varphi_0}$ on $L_2(V_0)$. We have

$$\begin{aligned}
&(C^*_{\varphi_0} M_f C_{\varphi_0}\varphi_1, \varphi_2)_{L_2(V_0)} \\
&\quad = (M_f C_{\varphi_0}\varphi_1, C_{\varphi_0}\varphi_2)_{L_2(V)} \\
&\quad = \int \mathrm{d}v\, f(v)\, c_{\varphi_1,\varphi_0}(v)\, \overline{c_{\varphi_2,\varphi_0}}(v) \\
&\quad = \int \mathrm{d}v\, f(v)\, (\varphi_1, \rho(v)\varphi_0) \cdot \overline{(\varphi_2, \rho(v)\varphi_0)} \\
&\quad = \int \mathrm{d}v\, f(v)\, \big(\hat{c}_{\varphi_1,\varphi_2}, W(\rho(v)\varphi_0)\big) \quad \text{by (0.4.2)} \\
&\quad = \int \mathrm{d}v \int \mathrm{d}v_1\, f(v)\, \hat{c}_{\varphi_1,\varphi_2}(v_1)\, W\varphi_0(v_1 - v) \quad \text{by (0.4.16)} \\
&\quad = \int \mathrm{d}v_1\, f * W\varphi_0(v_1)\, \hat{c}_{\varphi_1,\varphi_2}(v_1) \\
&\quad = \int \mathrm{d}v\, \mathcal{F}(f * W\varphi_0)(-v)(\varphi_1, \rho(v)\varphi_2) \\
&\quad = \int \mathrm{d}v\, \mathcal{F}(f * W\varphi_0)(v)(\rho(v)\varphi_1, \varphi_2) = \big(\hat{\rho}(f * W\varphi_0)\varphi_1, \varphi_2\big).
\end{aligned}$$

(0.4.19) Hence : $C^*_{\varphi_0} M_f C_{\varphi_0} = \hat{\rho}(f * W\varphi_0)$.

(0.4.20) If $V = \coprod_j \mathfrak{A}_j$ is written as a disjoint union of translates of self-dual lattices, $\mathfrak{A}_j = x_j + \mathcal{O}_j$, $\mathcal{O}_j^\vee = \mathcal{O}_j$, and if for each j, $\varphi_j \in L_2(V_0)$ is a unit vector such that $W\varphi_j = \Phi_j$ is the characteristic function of $\mathfrak{A}_j$, then the φ_j form an orthonormal basis for $L_2(V_0)$. Indeed, we have by (0.4.2), for $j_1 \neq j_2$

$$\left|(\varphi_{j_1}, \varphi_{j_2})_{L_2(V_0)}\right|^2 = (\Phi_{j_1}, \Phi_{j_2})_{L_2(V)} = 0\,;$$

moreover, letting $\varphi_j^0 = \rho(-x_j)\varphi_j$, so that by (0.4.16), $\Phi_j^0 = W\varphi_j^0$ is the characteristic function of $\mathcal{O}_j$, $\Phi_j^0 * \Phi_j = \Phi_j$, and for any locally constant compactly supported function φ in $L_2(V_0)$, we have by (0.4.19)

$$\begin{aligned}
\text{(0.4.21)} \qquad \varphi &= \hat{\rho}(\mathbb{1})\,\varphi = \sum_j \hat{\rho}(\Phi_j)\,\varphi = \sum_j \hat{\rho}(\Phi_j^0 * \Phi_j)\,\varphi \\
&= \sum_j C_{\varphi_j}^*\, M_{\Phi_j^0}\, C_{\varphi_j}\varphi \\
&= \sum_j C_{\varphi_j}^*\left[\Phi_j^0(v)\cdot(\varphi,\rho(v)\varphi_j)_{L_2(V_0)}\right] \\
&= \sum_j \int_V \mathrm{d}v\, \Phi_j^0(v)\cdot(\varphi,\rho(v)\varphi_j)_{L_2(V_0)}\cdot\rho(v)\varphi_j \\
&= \sum_j (\varphi,\varphi_j)_{L_2(V_0)}\cdot\varphi_j.
\end{aligned}$$

If f is a function on V, assuming a constant value, say λ_j, on each of the $\mathfrak{A}_j$'s, so that $f = \sum_j \lambda_j \cdot \Phi_j$, we have as in (0.4.21) : $\hat{\rho}(f)\varphi_j = \lambda_j \cdot \varphi_j$, i.e., $\hat{\rho}(f)$ is diagonalizable with respect to the orthonormal basis φ_j with associated eigenvalues the λ_j's.

1. FUNCTION SPACES OVER $\mathbb{Q}_p$

1.1. Soboleff spaces.

The local zeta function of $\mathbb{Q}_p$ is denoted by

$$\text{(1.1.1)} \qquad \zeta(s) = (1-p^{-s})^{-1} = \int_{\mathbb{Q}_p^*} \phi(x)\,|x|^s\,\mathrm{d}^*x, \quad \operatorname{Re} s > 0.$$

It is a meromorphic function of s, $(2\pi i/\log p)\mathbb{Z}$ periodic, with a unique simple pole at $s = 0$. We denote by I^α the operator of multiplication by the function

$$\text{(1.1.2)} \qquad |1,x|^{-\alpha} = \max\{1,|x|\}^{-\alpha}$$

and we denote by J^α the operator $\mathcal{F}_1^{-1} I^\alpha \mathcal{F}_1$, which for $\mathrm{Re}\,\alpha > 0$ is given by convolution with the L_1 function

$$J^\alpha(x) = \mathcal{F}_1\big(|1,\xi|^{-\alpha}\big)(x) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\big(|x|^{\alpha-1} - p^{\alpha-1}\big)\,\phi(x), \tag{1.1.3}$$

where $\mathrm{Re}\,\alpha > 0$ and $\alpha \not\equiv 1 \mod (2\pi i/\log p)\mathbb{Z}$.

The J^α's form an entire family of distributions, and we have $J^\alpha * J^\beta = J^{\alpha+\beta}$. For real $\alpha > 0$, the J^α's form a semi-group of probability measures associated with the negative definite function

$$\log|1,\xi| = \int \big(1 - \psi(x\xi)\big)\, d\sigma(x), \tag{1.1.4}$$

where the Lévi measure $\sigma(x)$ is given by the divisor function

$$\sigma(x) = \log p \cdot \sum_{n\geq 0} p^n \cdot \phi(p^{-n}x)\cdot \tag{1.1.5}$$

Note that, since J^α is supported in $\mathbb{Z}_p$, and I^β is $\mathbb{Z}_p$-locally-constant, we have

$$I^\beta J^\alpha = J^\alpha I^\beta. \tag{1.1.6}$$

All this extends to the higher dimensional case $V_0 = \mathbb{Q}_p^{\oplus d}$; using the metric $|x_1,\dots,x_d| = \max\{|x_1|,\dots,|x_d|\}$ we have commuting operators I^β, $J^\alpha = \mathcal{F}_d^{-1} I^\alpha \mathcal{F}_d$; J^α forming an entire family of distributions, $J^\alpha * J^\beta = J^{\alpha+\beta}$; for $\alpha > 0, J^\alpha$ being a probability measure supported in $\mathbb{Z}_p^{\oplus d}$; and

$$J^\alpha(x) = \frac{\zeta(d-\alpha)}{\zeta(\alpha)}\big(|x|^{\alpha-d} - p^{\alpha-d}\big)\cdot\phi(x), \tag{1.1.7}$$

for $\mathrm{Re}\,\alpha > 0,\ \alpha \not\equiv d \mod (2\pi i/\log p)\mathbb{Z}$,

$$J^d(x) = \frac{1}{\zeta(d)}\big(1 - \log_p|x|\big)\cdot\phi(x). \tag{1.1.8}$$

Note that

$$J^\alpha \in L_q(V_0) \quad \text{for} \quad \mathrm{Re}\,\alpha > d\cdot(1-q^{-1}) \tag{1.1.9}$$

and in particular : $J^\alpha \in L_1$, $\mathrm{Re}\,\alpha > 0$; $J^\alpha \in L_2$, $\mathrm{Re}\,\alpha > \frac{1}{2}d$; $J^\alpha \in L_\infty$, $\mathrm{Re}\,\alpha > d$.

The *Soboleff spaces* (or *Lebesgue spaces of Bessel potentials* [31]) are defined by

$$L_q^\alpha(V_0) = J^\alpha(L_q(V_0)), \quad \|f\|_{L_q^\alpha} = \|J^{-\alpha}f\|_{L_q} \tag{1.1.10}$$

and in a similar way, we can define $L_q^\alpha(\mathfrak{A})$, where $\mathfrak{A} \subseteq V_0$ is any domain such that $\mathfrak{A} + \mathbb{Z}_p^{\oplus d} = \mathfrak{A}$. We also define the *weighted Soboleff spaces* by

$$L_q^{\alpha|\beta}(V_0) = J^\alpha I^\beta(L_q(V_0)), \quad \|f\|_{L_q{}^{\alpha|\beta}} = \|I^{-\beta}J^{-\alpha}f\|_{L_q}. \tag{1.1.11}$$

The spaces L_q^α, $L_q^{\alpha|\beta}$ are Banach spaces isomorphic to L_q, and they satisfy the following list of basic properties [31] :

(1.1.12) Embedding $L_{q_1}^{\alpha_1}(V_0) \hookrightarrow L_{q_2}^{\alpha_2}(V_0)$ continuously for

$$0 < q_2^{-1} \le q_1^{-1} < 1, \quad d \cdot (q_1^{-1} - q_2^{-1}) \le \alpha_1 - \alpha_2\,;$$
$$0 \le q_2^{-1} \le q_1^{-1} \le 1, \quad d \cdot (q_1^{-1} - q_2^{-1}) < \alpha_1 - \alpha_2.$$

(1.1.13) Duality $(L_q^\alpha)' = L_{q'}^{-\alpha}$ for $1 \le q < \infty$, $q^{-1} + q'^{-1} = 1$.

(1.1.14) Relation with the Riesz potential

$$R^\alpha = \mathcal{F}|\xi|^{-\alpha} = \frac{\zeta(d-\alpha)}{\zeta(\alpha)}|x|^{\alpha-d}, \quad \text{for } \operatorname{Re}\alpha > 0,$$

$L_q^\alpha(V_0) \cong$ domain of R^α acting on $L_q(V_0)$, $\operatorname{Re}\alpha > 0$,

i.e. $\|f\|_{L_q^\alpha} \approx \|f\|_{L_q} + \|R^{-\alpha}f\|_{L_q}$.

In particular, let us note that we have

$$L_2^\alpha(V_0) \hookrightarrow C_0(V_0), \quad \text{for } \alpha > \tfrac{1}{2}d, \tag{1.1.15}$$

and that we have *quasinuclear* (i.e. Hilbert-Schmidt) embeddings

(1.1.16) $L_2^{\alpha_1}(\mathfrak{A}) \hookrightarrow L_2^{\alpha_2}(\mathfrak{A})$, $\alpha_1 - \alpha_2 > \tfrac{1}{2}d$, $\mathfrak{A}$ bounded domain,

(1.1.17) $L_2^{\alpha_1|\beta_1}(V_0) \hookrightarrow L_2^{\alpha_2|\beta_2}(V_0)$, $\alpha_1 - \alpha_2 > \tfrac{1}{2}d$, $\beta_1 - \beta_2 > \tfrac{1}{2}d$.

Recalling the Bargmann isomorphism $B : L_2(V)_\psi \xrightarrow{\sim} L_2(V_0)$, where $V = V_0 \oplus V_0$, (0.2.5), (0.2.6), we note that using (0.1.14), (0.1.15), and (0.2.13), we have

$$B^{-1} I^\alpha B = M_{|1,x|^{-\alpha}}, \tag{1.1.18}$$

$$B^{-1} J^\alpha B = M_{|1,\xi|^{-\alpha}}. \tag{1.1.19}$$

It becomes natural to define the *Schrödinger operators* by

$$B^{-1} K^\alpha B = M_{|1,\xi,x|^{-\alpha}} \tag{1.1.20}$$

and the associated spaces

$$L_q^{\{\alpha\}}(V_0) = K^\alpha(L_q(V_0)), \quad \|f\|_{L_q{}^{\{\alpha\}}} = \|K^{-\alpha} f\|_{L_q} \tag{1.1.21}$$

since $|1,\xi|^{-\alpha} \cdot |1,x|^{-\alpha} \le |1,\xi,x|^{-\alpha} \le |1,\xi|^{-\alpha}$ and $|1,x|^{-\alpha}$, for $\alpha > 0$, we have

$$L_q^{\alpha|\alpha} \hookrightarrow L_q^{\{\alpha\}} \hookrightarrow L_q^{\alpha|0} \cap L_q^{0|\alpha}. \tag{1.1.22}$$

The operators I^α, J^α, K^α belong to the algebra $\mathrm{BL}_\infty(V/\mathcal{O}_0)B^{-1}$, for $\mathrm{Re}\,\alpha \ge 0$, and hence they admit the lattice basis $\{\rho(v)\phi \mid v \in V/\mathcal{O}_0\}$ as eigenfunctions. In particular, the operators $I^\beta J^\alpha$, and K^α, are trace class in the following domains, with trace

$$\mathrm{tr}(I^\beta J^\alpha) = \frac{\zeta(\alpha - d)}{\zeta(\alpha)} \cdot \frac{\zeta(\beta - d)}{\zeta(\beta)}, \quad \mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > d, \tag{1.1.23}$$

$$\mathrm{tr}(K^{2\alpha}) = \frac{\zeta(2(\alpha - d))}{\zeta(2\alpha)}, \quad \mathrm{Re}\,\alpha > d. \tag{1.1.24}$$

The operators $K^{2\alpha}$, for $d = 1$, also admit the Hermite functions $\{\phi^\chi\}$ as eigenfunctions :

$$K^{2\alpha} \phi^\chi = (\mathrm{cond}\,\chi)^{-\alpha} \cdot \phi^\chi \tag{1.1.25}$$

where $\mathrm{cond}\,\chi = p^N$ for $\chi \in \mathcal{Z}^{(N)}$.

One can further generalize the above spaces by considering more general «metrics» i.e. functions $g : V_0 \to \{0\} \cup p^{\mathbb{Z}}$ satisfying

$$\begin{cases} g(v_1 + v_2) \le \max\{g(v_1), g(v_2)\}, & v_i \in V_0, \\ g(\lambda \cdot v) = |\lambda| \cdot g(v), \lambda \in \mathbb{Q}_p, & v \in V_0, \\ g(v) = 0 \iff v = 0. \end{cases} \tag{1.1.26}$$

Recall that such metrics correspond bijectively to lattices $\mathcal{O} \subseteq V_0$, the correspondence given by

$$\mathcal{O} = \{v \in V \mid g(v) \leq 1\}, \quad g(v) = \inf\{|\lambda|\,;\ \lambda^{-1} \cdot v \in \mathcal{O}\}. \tag{1.1.27}$$

If $\mathcal{O}_i \subseteq V_i$ corresponds to $g_i, i = 1, 2$, then $\mathcal{O}_1 \oplus \mathcal{O}_2 \subseteq V_1 \oplus V_2$ corresponds to

$$g_1 \otimes g_2(v_1, v_2) = \max\{g_1(v_1), g_2(v_2)\}. \tag{1.1.28}$$

If $\mathcal{O} \subseteq V$ corresponds to g, the dual lattice $\mathcal{O}^\vee \subseteq V^\vee$, defined in the dual vector space by

$$\mathcal{O}^\vee = \{v^\vee \in V^\vee \mid \langle v^\vee, \mathcal{O}\rangle \subseteq \mathbb{Z}_p\} \tag{1.1.29}$$

corresponds to the dual-metric $g^\vee$, given by

$$g^\vee(v^\vee) = \inf\Big\{\frac{|\langle v^\vee, v\rangle|}{g(v)}\,;\ v \in V\Big\}. \tag{1.1.30}$$

This holds in particular when V is a symplectic vector space identified with its own dual by means of the symplectic form.

Given V, g we can define the operators

$$\begin{cases} I_g^\alpha = \text{multiplication by } \max\{1, g(x)\}^{-\alpha}, \\ J_g^\alpha = \mathcal{F}^{-1} I_{g^\vee}^\alpha \mathcal{F}, \\ R_g^\alpha = \mathcal{F}^{-1} g^\vee(\xi)^{-\alpha} \mathcal{F}, \\ K_g^\alpha = B I_{g^\vee \otimes g}^\alpha B^{-1}, \end{cases} \tag{1.1.31}$$

and the associated spaces

$$L_q^\alpha(V, g),\ L_q^{\alpha|\beta}(V, g),\ L_q^{\{\alpha\}}(V, g),\ \text{and } L_q^\alpha(\mathfrak{A}, g) \tag{1.1.32}$$

for a domain $\mathfrak{A}$ such that $\mathfrak{A} + \mathcal{O} \subseteq \mathfrak{A}$ (note that since $I_{g^\vee}^\alpha$ is $\mathcal{O}^\vee$-locally-constant, $\operatorname{supp} J_g^\alpha \subseteq \mathcal{O}$). Since any metrics are equivalent by an affine transformation (i.e. given g_1, g_2 on V, there exists $\mathcal{A} \in \mathrm{GL}(V)$ such that $g_1(v) = g_2(\mathcal{A}v)$), it is clear that the results (1.1.12) through (1.1.17) remain valid for the spaces (1.1.32).

1.2. Besov spaces.

The problem with operators J^α, and the spaces L^α_q, is that unlike the real case, where we have $J^{-2N} = (1 + \partial^2/\partial x^2)^N$, we lack in the p-adic setting an analogue of the *Leibnitz rule*

$$\left(\frac{\partial}{\partial x}\right)^N (f_1 \cdot f_2) = \sum_{0 \le j \le N} \binom{N}{j} \left(\frac{\partial}{\partial x}\right)^j f_1 \cdot \left(\frac{\partial}{\partial x}\right)^{N-j} f_2$$

which follows upon taking the Fourier transform from the *Newton rule* :

$$(1.2.1) \qquad (y_1 + y_2)^N = \sum_{0 \le j \le N} \binom{N}{j} y_1^j \cdot y_2^{N-j}.$$

Thus, for example, in the p-adic setting $f_1, f_2 \in L^\alpha_\infty$ does not imply $f_1 \cdot f_2 \in L^\alpha_\infty$. To see the problem more clearly, we notice that we can write for $\operatorname{Re}\alpha > 0$ and $y_i \in \mathbb{Q}_p$,

$$(1.2.2) \quad \begin{cases} |y_1 + y_2|^\alpha = \text{p.v.} \displaystyle\int_0^{2\pi/\log p} dt \cdot \frac{\log p}{2\pi} \\ \qquad \times \left\{ \zeta(it) - \dfrac{1}{2} \dfrac{\zeta(1)}{\zeta(1+\alpha)} + \dfrac{\zeta(-1)\zeta(-(1+\alpha))}{\zeta(\alpha)} \right\} \\ \qquad \times \left\{ |y|^{\alpha+it} \cdot |y_2|^{-it} + |y_1|^{-it} \cdot |y_2|^{\alpha+it} \right\} \\ + \dfrac{\zeta(1)\zeta(1+\alpha)}{\zeta(-\alpha)} \displaystyle\sum_{\chi \neq 1} \chi(-1) \cdot (\operatorname{cond}\chi)^{-(1+\alpha)} \\ \quad \times \displaystyle\int_0^{2\pi/\log p} dt \cdot \frac{\log p}{2\pi} \cdot \chi(y_1)\, |y_1|^{\alpha/2+it} \cdot \bar{\chi}(y_2)\, |y_2|^{\alpha/2-it} \end{cases}$$

where χ varies over all non-trivial characters of $\mathbb{Z}_p^*$, but the sum $\sum_\chi$ is actually finite for $y_1 + y_2 \neq 0$. Thus, (1.2.2) like (1.2.1), expresses $|y_1 + y_2|^\alpha$ as «sum» of $\chi_1(y_1) \cdot \chi_2(y_2)$, with $\chi_1 \cdot \chi_2 = |\ \ |^\alpha$.

To overcome this problem we introduce the *Besov spaces* $B^\alpha_{q,r}$ (or *Lipschitz spaces*) of functions that are well approximated by locally-constant-functions. We shall use the following notations : V is a $\mathbb{Q}_p$-vector space with a metric g corresponding to the lattice $\mathcal{O} \subseteq V$, and $\mathcal{O}^\vee \subseteq V^\vee$ is the dual lattice. We let

(1.2.3)
$$\begin{cases} \phi_{g,k} = \text{Fourier transform of the characteristic function of } p^{-k}\mathcal{O}^{\vee}, \quad k \geq 0; \\ \varphi_{g,k} = \phi_{g,k} - \phi_{g,k-1} \text{ for } k \geq 1, \ \varphi_{g,0} = \phi_{g,0}; \\ \mathrm{d}_g x = \text{Haar measure in } V \text{ normalized by } \mathrm{d}_g x(\mathcal{O}) = 1; \\ L_q = L_q(V, \mathrm{d}_g x); \\ \mathrm{d}_g^* y = \zeta(d) \cdot g(y)^{-d} \cdot \mathrm{d}_g y, \quad d = \dim V, \text{ so that } \mathrm{d}_g^* y \text{ is the «multiplicative Haar measure» with } \mathrm{d}_g^* y(p^k\mathcal{O} \setminus p^{k+1}\mathcal{O}) = 1; \\ L_r^* = L_r(\mathcal{O}, \mathrm{d}_g^* y); \\ \ell_r \text{ is the usual } \ell_r\text{-space on } k \in \{0, 1, 2, \ldots\}. \end{cases}$$

The Besov space $B^{\alpha}_{q,r}(V, g)$ consists of functions f (or distributions, if $\alpha < 0$) such that

(1.2.4)
$$\|f\|_{B^{\alpha}_{q,r}} = \left\| p^{k\alpha} \|\varphi_{g,k} * f(x)\|_{L_q} \right\|_{\ell_r} < \infty.$$

We shall only consider these spaces for $1 \leq q \leq \infty$, $1 \leq r \leq \infty$. The $B^{\alpha}_{q,r} = B^{\alpha}_{q,r}(V, g)$ are Banach spaces with the following basic properties :

Embeddings

(1.2.5) $B^{\alpha}_{q,r_1} \hookrightarrow B^{\alpha}_{q,r_2}, \quad r_1 \leq r_2;$

(1.2.6) $B^{\alpha_1}_{q,\infty} \hookrightarrow B^{\alpha_2}_{q,1}, \quad \alpha_1 > \alpha_2;$

(1.2.7)
$$\begin{cases} B^{\alpha}_{q,q} \hookrightarrow L^{\alpha}_q \hookrightarrow B^{\alpha}_{q,2}, & 1 < q \leq 2, \\ B^{\alpha}_{q,2} \hookrightarrow L^{\alpha}_q \hookrightarrow B^{\alpha}_{q,q}, & 2 \leq q < \infty; \end{cases}$$

(1.2.8) $B^{\alpha + d(1/q - 1/q_0)}_{q,r} \hookrightarrow B^{\alpha}_{q_0,r}, \quad q \leq q_0;$

(1.2.9) Interpolation $[B^{\alpha_0}_{q_0,r_0}, B^{\alpha_1}_{q_1,r_1}]_{\theta} = B^{\alpha}_{q,r}$, for the complex method; here $0 < \theta < 1$, $q^{-1} = (1-\theta) \cdot q_0^{-1} + \theta \cdot q_1^{-1}$, $r^{-1} = (1-\theta) \cdot r_0^{-1} + \theta \cdot r_1^{-1}$, $\alpha = (1-\theta) \cdot \alpha_0 + \theta \cdot \alpha_1 \alpha_0 \neq \alpha_1$;

(1.2.10) $(L^{\alpha_0}_q, L^{\alpha_1}_q)_{\theta,r} = B^{\alpha}_{q,r}$, for the real method;

(1.2.11) Duality $(B^{\alpha}_{q,r})' = B^{-\alpha}_{q',r'}$ with $q^{-1} + {q'}^{-1} = 1 = r^{-1} + {r'}^{-1}$;

(1.2.12) Lifting $J_g^\alpha : B_{q,r}^{\alpha_0} \xrightarrow{\sim} B_{q,r}^{\alpha_0+\alpha}$;

(1.2.13) Characterization via approximation : for $\alpha > 0$,

$$\|f\|_{B_{q,r}^\alpha} \cong \|f\|_{L_q} + \Big\| p^{k\alpha} \cdot \|f(x) - \phi_{g,k} * f(x)\|_{L_q} \Big\|_{\ell_r}.$$

(1.2.14) Characterization via differences : for $\alpha > 0$,

$$\|f\|_{B_{q,r}^\alpha} \cong \|f\|_{L_q} + \Big\| g(y)^{-\alpha} \cdot \|f(x) - f(x+y)\|_{L_q} \Big\|_{L_r^*};$$

(1.2.15) Pointwise multiplier : for $\alpha > 0$,

$$\|f_1 \cdot f_2\|_{B_{q,r}^\alpha} \leq \mathrm{C}\, \|f_1\|_{B_{\infty,r}^\alpha} \cdot \|f_2\|_{B_{q,r}^\alpha}.$$

(1.2.16) $B_{q,r}^{d/q+\alpha}$ and $B_{q,1}^{d/q}$ are quasi Banach algebras of continuous functions.

(1.2.17) *Remark.* — Note that if $\operatorname{supp} f \subseteq \mathcal{O}$ then $\operatorname{supp} \varphi_{g,k} * f \subseteq \mathcal{O}$. Hence we can define $B_{q,r}^\alpha(\mathcal{O}, g) = \{f \in B_{q,r}^\alpha(V, g) \mid \operatorname{supp} f \subseteq \mathcal{O}\}$, and similary we can define $B_{q,r}^\alpha(\mathfrak{A}, g)$ for any domain such that $\mathfrak{A} + \mathcal{O} = \mathfrak{A}$. All the above properties remain valid for the spaces $B_{q,r}^\alpha(\mathfrak{A}, g)$. When $\mathfrak{A} = x + \mathcal{O}$, we shall write $B_{q,r}^\alpha(\mathfrak{A})$ for $B_{q,r}^\alpha(\mathfrak{A}, g)$.

(1.2.18) *Remark.* — The only discrepancy from the analogous spaces over the real numbers is (1.2.14); for the reals, one has to use instead of the difference $\Delta_y f(x) = f(x) - f(x+y)$, the N-fold-difference

$$\Delta_y^N f(x) = \sum_{0 \leq j \leq n} \binom{N}{j} (-1)^j f(x + jy),$$

N an integer greater than α. The proofs of (1.2.5) through (1.2.12) are immediate translations of the corresponding proofs in the real case (e.g. [3]). The proof of the equivalence of the norms (1.2.4), (1.2.13), (1.2.14) follows as in [31], theorem (2.2), p. 180. The proof of (1.2.15), (1.2.16) follows as in the real case (cf. [25]), or as a consequence of (1.2.14).

We shall also need the *Besov spaces of a product space* given by

(1.2.19)
$$\begin{cases} B_{q,r}^{\alpha_1,\alpha_2}(V_1 \times V_2, g_1 \otimes g_2) \stackrel{\text{def}}{=} B_{q,r}^{\alpha_1}(V_1, g_1) \otimes B_{q,r}^{\alpha_2}(V_2, g_2), \\ B_{q,r}^{\alpha_1,\alpha_2}(\mathfrak{A}_1 \times \mathfrak{A}_2, g_1 \otimes g_2) \stackrel{\text{def}}{=} B_{q,r}^{\alpha_1}(\mathfrak{A}_1, g_1) \otimes B_{q,r}^{\alpha_2}(\mathfrak{A}_2, g_2), \end{cases}$$

with $\mathfrak{A}_i + \mathcal{O}_i = \mathfrak{A}_i$, where $\otimes$ denotes the completed (in the crossed-product topology) tensor product. These are again Banach spaces, satisfying analogous properties, their norm being given by

$$\|f\|_{B^{\alpha_1,\alpha_2}_{q,r}} = \Big\| p^{k_1\alpha_1+k_2\alpha_2} \big\| (\varphi_{g_1,k_1} \otimes \varphi_{g_2,k_2}) * f \big\|_{L_q} \Big\|_{\ell_r}. \tag{1.2.20}$$

In particular, we can restate the pointwise multiplier assertions of (1.2.16) as trace theorems :

$$\text{Res} : B^{d/q+\alpha,d/q+\alpha}_{q,r}(V \times V, g \otimes g) \longrightarrow B^{d/q+\alpha}_{q,r}(V, g), \quad \alpha > 0, \tag{1.2.21}$$

$$\text{Res} : B^{d/q,d/q}_{q,1}(V \times V, g \otimes g) \longrightarrow B^{d/q}_{q,1}(V, g) \tag{1.2.22}$$

are continuous, where Res denotes the restriction to the diagonal. The spaces in (1.2.21), (1.2.22) are spaces of continuous functions, and we can view the δ-functions as elements of their dual. We have

$$\|\delta_{x_1,x_2}\|_{(B^{d/2,d/2}_{2,1})'} = \|\delta_{x_1,x_2}\|_{B^{-d/2,-d/2}_{2,\infty}} = 1. \tag{1.2.23}$$

We can also define the *weighted Besov space* $B^{\alpha|\beta}_{q,r}(V, g)$ given by the norm

$$\|f\|_{B^{\alpha|\beta}_{q,r}} = \Big\| p^{k\alpha} \big\| I_g^{-\beta}(\varphi_{g,k} * f)(x) \big\|_{L_q} \Big\|_{\ell_r} = \|I_g^{-\beta} f\|_{B^{\alpha}_{q,r}}, \tag{1.2.24}$$

noting that $I_g^{-\beta}(\varphi_{g,k} * f) = \varphi_{g,k} * (I_g^{-\beta} f)$, since $\operatorname{supp} \varphi_{g,k} \subseteq \mathcal{O}$.

$B^{\alpha|\beta}_{q,r}$ will satisfy the analogous properties, e.g.

- interpolation $\big[B^{\alpha_0|\beta_0}_{q_0 r_0}, B^{\alpha_1|\beta_1}_{q_1 r_1}\big]_\theta = B^{\alpha|\beta}_{q,r}$, as in (1.2.9) with $\beta = (1-\theta)\beta_0 + \theta \cdot \beta_1$;
- duality $\big(B^{\alpha|\beta}_{q,r}\big)' = B^{-\alpha|-\beta}_{q',r'}$;
- lifting $I_g^\beta : B^{\alpha|\beta_0}_{q,r} \xrightarrow{\sim} B^{\alpha|\beta_0+\beta}_{q,r}$;
- pointwise multiplier $\|f_1 \cdot f_2\|_{B^{\alpha|\beta_1+\beta_2}_{q,r}} \le \mathrm{C} \cdot \|f_1\|_{B^{\alpha|\beta_1}_{\infty,r}} \cdot \|f_2\|_{B^{\alpha|\beta_2}_{q,r}}$, for $\alpha > 0$;

etc.

A *metric-covering* of V is an assignment for each $x \in V$ of a metric g_x such that

$$g_x(y) \le 1 \implies g_x = g_{x+y}. \tag{1.2.25}$$

Letting $\mathcal{O}_x$ denote the lattice correponding to g_x, and setting $\mathfrak{A}_x = x + \mathcal{O}_x$, i.e.

$$\mathfrak{A}_x = \{y \in V \mid g_x(x-y) \leq 1\} \tag{1.2.26}$$

we can choose a sequence $x_j \in V$ such that $V = \coprod_j \mathfrak{A}_{x_j}$, a disjoint union. Conversely, given a covering $V = \coprod_j \mathfrak{A}_j$, by disjoint sets $\mathfrak{A}_j$, of the form $\mathfrak{A}_j = x_j + \mathcal{O}_j$, $\mathcal{O}_j$ a lattice, we obtain a metric covering by setting

$$g_x = \text{the metric corresponding to } \mathcal{O}_j, \text{ for } x \in \mathfrak{A}_j. \tag{1.2.27}$$

A *g-continuous weight* is a function $m : V \to \mathbb{R}^+$, such that for some constant C,

$$g_x(y) \leq 1 \implies C^{-1} \cdot m(x) \leq m(x+y) \leq C \cdot m(x). \tag{1.2.28}$$

Given such g and m, we define the Banach space $B^\alpha_{q,r}(m,g)$, with norm

$$\|f\|_{B^\alpha_{q,r}(m,g)} = \sup_{x \in V} \frac{1}{m(x)} \cdot \|f_x\|_{B^\alpha_{q,r}(\mathfrak{A}_x, g_x)} \tag{1.2.29}$$

where f_x denote the restriction of f to $\mathfrak{A}_x$.

Choosing a sequence $x_j \in V$ such that $V = \coprod_j \mathfrak{A}_j$, $\mathfrak{A}_j = x_j + \mathcal{O}_{x_j}$, and putting for $x \in \mathfrak{A}_j : \widetilde{m}(x) = m(x_j) = m_j$, it follows from (1.2.28) that $B^\alpha_{q,r}(m,g) = B^\alpha_{g,r}(\widetilde{m},g)$. Hence we may as well assume from the start that m is *g-locally-constant* in the sense that

$$g_x(y) \leq 1 \implies m(x) = m(x+y) \tag{1.2.30}$$

and then the norm of $B^\alpha_{q,r}(m,g)$ is given by

$$\|f\|_{B^\alpha_{q,r}(m,g)} = \sup_j \frac{1}{m_j} \|f_j\|_{B^\alpha_{q,r}(\mathfrak{A}_j, g_j)} \tag{1.2.31}$$

where f_j denotes the restriction of f to $\mathfrak{A}_j$.

The spaces $B^\alpha_{q,r}(m,g)$ will inherit many of the properties of $B^\alpha_{q,r}(V)$, in particular :

Embeddings

$$B^{\alpha}_{q_1,r_1}(m,g) \hookrightarrow B^{\alpha}_{q_2,r_2}(m,g), \quad r_1 \le r_2\,; \tag{1.2.32}$$

$$B^{\alpha_1}_{q,\infty}(m,g) \hookrightarrow B^{\alpha_2}_{q,1}(m,g), \quad \alpha_1 > \alpha_2\,; \tag{1.2.33}$$

$$B^{\alpha+d(1/q-1/q_0)}_{q,r}(m,g) \hookrightarrow B^{\alpha}_{q_0,r}(m,g), \quad q \le q_0\,; \tag{1.2.34}$$

$$B^{\alpha}_{q,r}(m_1,g) \hookrightarrow B^{\alpha}_{q,r}(m_2,g), \quad m_1 < \mathrm{C}\cdot m_2. \tag{1.2.35}$$

Liftings

$$J^{\alpha}_g : B^{\alpha_0}_{q,r}(m,g) \xrightarrow{\sim} B^{\alpha_0+\alpha}_{q,r}(m,g), \quad \text{where } J^{\alpha}_g f(x) = J^{\alpha}_{g_x} f(x)\,; \tag{1.2.36}$$

$$I_m : B^{\alpha}_{q,r}(m_0,g) \xrightarrow{\sim} B^{\alpha}_{q,r}(m\cdot m_0,g), \quad \text{where } I_m f(x) = m(x)\cdot f(x). \tag{1.2.37}$$

Pointwise multiplier : for $\alpha > 0$,

$$\|f_1\cdot f_2\|_{B^{\alpha}_{q,r}(m_1\cdot m_2,g)} \le \mathrm{C}\cdot\|f_1\|_{B^{\alpha}_{\infty,r}(m_1,g)}\cdot\|f_2\|_{B^{\alpha}_{q,r}(m_2,g)} \tag{1.2.38}$$

and characterizations via approximation and difference similar to (1.2.13) and (1.2.14) of which we note : for $\alpha > 0$,

$$\|f\|_{B^{\alpha}_{\infty,\infty}(m,g)} \cong \sup_{x\in V}\frac{1}{m(x)}|f(x)| + \sup_{\substack{x_0,\,x_1\in V\\ 0<g_{x_0}(x_0-x_1)\le 1}} \frac{1}{m(x_0)}\cdot\frac{|f(x_0)-f(x_1)|}{g_{x_0}(x_0-x_1)^{\alpha}}. \tag{1.2.39}$$

1.3. Symbol classes and the analytic Schwartz space.

For a lattice $\mathcal{O} \subseteq V$ we denote by $C_\infty(\mathcal{O})$ the Fréchet space defined by the norms $\|f\|_{L^{\alpha}_{\infty}}$, $\alpha > 0$. We can also use the norms $\|f\|_{L^{\alpha}_{q}}$, $\alpha \ge 0$, for any $1 \le q \le \infty$, and similary we can use the norms $\|f\|_{B^{\alpha}_{q,r}}$, $\alpha \ge 0$, for any $1 \le q, r \le \infty$. In particular, we see that $C_\infty(\mathcal{O})$ is an algebra under pointwise multiplication, consisting of continuous functions, and that it is a nuclear space.

(1.3.1). — For any open set $\mathfrak{A} \subseteq V$, we denote by $C_\infty(\mathfrak{A})$ the Fréchet space defined by the norms $\|f\|_{L^\alpha_\infty(x+\mathcal{O})}$, $\alpha \geq 0$, $x + \mathcal{O} \subseteq \mathfrak{A}$ a translate of a lattice contained in $\mathfrak{A}$. Again we can use the norms associated with $L^\alpha_q(x+\mathcal{O})$, or with $B^\alpha_{q,r}(x+\mathcal{O})$, and $C_\infty(\mathfrak{A})$ is a nuclear algebra of continuous functions. In particular, we have $C_\infty(\mathfrak{A}_1 \times \mathfrak{A}_2) = C_\infty(\mathfrak{A}_1) \otimes C_\infty(\mathfrak{A}_2)$. Just as $C_\infty(V) = \varprojlim C_\infty(\mathcal{O})$, the inverse limit taken over all lattices $\mathcal{O} \subseteq V$ with respect to the restriction mapping $C_\infty(\mathcal{O}_1) \to C_\infty(\mathcal{O}_2)$, $\mathcal{O}_1 \supseteq \mathcal{O}_2$, we can define the space of compactly supported smooth functions $C_{\infty,c}(V) = \varinjlim C_\infty(\mathcal{O})$, where now we take the direct limit with respect to the mappings $C_\infty(\mathcal{O}_2) \hookrightarrow C_\infty(\mathcal{O}_1)$, $\mathcal{O}_1 \supseteq \mathcal{O}_2$, given by extending by zero. $C_{\infty,c}(V)$ is similary a nuclear algebra of compactly supported continuous functions.

(1.3.2). — We define the (*analytic*) *Schwartz space* $\mathcal{S}(V)$ to be the Fréchet space defined by the norms $\|f\|^{\alpha|\beta}_{L_\infty}$, $\alpha, \beta \geq 0$. Equivalently, we can use the norms associated with $L^{\alpha|\beta}_q$, $\alpha, \beta \geq 0$, for any q; or the norms of $L^{\{\alpha\}}_q$, $\alpha \geq 0$, associated with the Schrödinger operator (1.1.21); or the norms of $B^{\alpha|\beta}_{q,r}(V, g)$, $\alpha, \beta \geq 0$, for any $1 \leq q, r \leq \infty$ and any metric g on V. $\mathcal{S}(V)$ is a nuclear algebra of continuous functions. Fixing some non-degenerate quadratic form, giving an identification $V \cong V^\vee$ of V with its dual, we can view the Fourier transform $\mathcal{F}$ as a mapping $\mathcal{F} : L^{\alpha|\beta}_2(V) \xrightarrow{\sim} L^{\beta|\alpha}_2(V)$, hence $\mathcal{F}$ defines an automorphism of $\mathcal{S}(V)$, which consequently forms an algebra also under convolution. We have again $\mathcal{S}(V_1 \oplus V_2) = \mathcal{S}(V_1) \otimes \mathcal{S}(V_2)$. Denoting by $\mathcal{S}_{\rm alg}(V)$ the *algebraic Schwartz space* of locally-constant compactly-supported functions, we have *proper* dense inclusions

$$\mathcal{S}_{\rm alg} \subseteq C_{\infty,c} \subseteq \mathcal{S} \subseteq \mathcal{S}' \subseteq C'_{\infty,c} \subseteq \mathcal{S}'_{\rm alg}.$$

We note that $\mathcal{S}_{\rm alg}$, $C_{\infty,c}$, $\mathcal{S}$, C_∞, as well as their duals, are all stable under the action of the Heisenberg group given by ρ.

We have the *Schwartz kernel theorems* (cf. [33] for the similar proof over $\mathbb{R}$) :

$$(1.3.3) \qquad \begin{cases} \mathrm{Hom}\big(C_\infty(\mathfrak{A}_1), C'_\infty(\mathfrak{A}_2)\big) \cong C'_\infty(\mathfrak{A}_2 \times \mathfrak{A}_1), \\ \mathrm{Hom}\big(\mathcal{S}(V_1), \mathcal{S}'(V_2)\big) \cong \mathcal{S}'(V_2 \oplus V_1). \end{cases}$$

An operator A on functions on V_0 will be called *smoothening* if

$$A : \mathcal{S}'(V_0) \longrightarrow \mathcal{S}(V_0)$$

continuously, or equivalently if A is given by a kernel $A(x,y) \in \mathcal{S}(V_0 \oplus V_0)$. A still equivalent condition for A to be smoothering is that $A = \hat{\rho}(f)$ with $f \in \mathcal{S}(V_0^\vee \oplus V_0) = \mathcal{S}(V)$, since $A(x,y)$ and $f(\xi,x)$ are obtained from each other by a partial Fourier transform which takes $\mathcal{S}(V_0 \oplus V_0)$ isomorphically onto $\mathcal{S}(V_0^\vee \oplus V_0)$.

Given a metric-covering g, and a g-locally-constant weight function m, we define the *symbol class* $S(m,g)$ to be the Fréchet space defined by the norms $\|f\|_{B^\alpha_{\infty,\infty}(m,g)}$, $\alpha > 0$. These norms are given in an equivalent form in (1.2.39), so that $f \in S(m,g)$ if and only if for some constants C, C_α, we have

$$|f(x)| \le C \cdot m(x) \quad \text{for all } x \in V, \tag{1.3.4}$$

$$|f(x_0) - f(x_1)| \le C_\alpha \cdot m(x_0) \cdot g_{x_0}(x_0 - x_1)^\alpha, \tag{1.3.5}$$

for all $x_0, x_1 \in V$, $g_{x_0}(x_0 - x_1) \le 1$, and for any given $\alpha > 0$.

Using the pointwise-multiplier, or by using (1.3.4), (1.3.5), we have :

$$f_i \in S(m_i, g) \Longrightarrow f_1 \cdot f_2 \in S(m_1 \cdot m_2, g). \tag{1.3.6}$$

Note that $C_{\infty,c} \subseteq S(m,g) \subseteq C_\infty$, but the first inclusion is not dense, hence a continuous linear form on $S(m,g)$ is not determined by its restriction to $C_{\infty,c}$; such a form will be called *weakly continuous* if its restriction to a bounded subset of $S(m,g)$ is continuous in the C_∞ topology.

1.4. Temperate metric-coverings.

We return to a symplectic vector space $V, \langle\ \rangle$, and we shall deal with a metric-covering g_x, corresponding to $V = \coprod_j \mathfrak{A}_j$, $\mathfrak{A}_j = x_j + \mathcal{O}_j$, where g_x is always assumed to be *certain* : $g_x \le g_x^\vee$, or equivalently $\mathcal{O}_j \supseteq \mathcal{O}_j^\vee \cdot g_x$ will be said to be *temperate* if there exist n_0, $N \ge 0$, such that

$$g_{x_0}^\vee(x) \le p^{n_0} \cdot g_{x_1}^\vee(x) \cdot g_{x_1}^\vee(1, x_1 - x_0)^N, \quad x_0, x_1, x \in V, \tag{1.4.1}$$

where $g^\vee(1,x) = \max\{1, g^\vee(x)\}$. Using the definition (1.1.30) of the dual metric, one can write (1.4.1) in the equivalent form,

$$g_{x_1}(x) \le p^{n_0} \cdot g_{x_0}(x) \cdot g_{x_1}^\vee(1, x_1 - x_0)^N. \tag{1.4.2}$$

Taking $x = x_0 - x_1$ in (1.4.1) we have in particular,

$$g_{x_0}^{\vee}(1, x_0 - x_1) \leq p^{n_0} \cdot g_{x_1}^{\vee}(1, x_1 - x_0)^{N+1}. \tag{1.4.3}$$

From (1.4.1) we may also deduce that

$$g_{x_1}^{\vee}(x_1 - x_0) \leq p^{n_0} \cdot g_{x_1+x_2-x}^{\vee}(x_1 - x_0) \cdot g_{x_1+x_2-x_0}^{\vee}(1, x_2 - x_0)^N,$$

$$\begin{aligned} g_{x_1+x_2-x_0}^{\vee}(x_1 - x_0) &\leq p^{n_0} \cdot g_{x_2}^{\vee}(x_1 - x_0) \cdot g_{x_2}^{\vee}(1, x_1 - x_0)^N \\ &\leq p^{n_0} \cdot g_{x_2}^{\vee}(1, x_1 - x_0)^{N+1} \end{aligned}$$

and similary with x_1, x_2 interchanged, from which we obtain

$$g_{x_1}^{\vee}(x_1 - x_0) \leq p^{(2+N)n_0} \cdot g_{x_2}^{\vee}(1, x_1 - x_0)^{N+1} \cdot g_{x_1}^{\vee}(1, x_2 - x_0)^{N \cdot (N+1)}. \tag{1.4.4}$$

Multiplying (1.4.4) with the similar expression obtained by interchanging x_1, x_2 we finally get,

$$\begin{aligned} & g_{x_1}^{\vee}(x_1 - x_0) \cdot g_{x_2}^{\vee}(x_2 - x_0) \\ & \quad \leq p^{2(2+N)n_0} \cdot g_{x_2}^{\vee}(1, x_1 - x_0)^{(N+1)^2} \cdot g_{x_1}^{\vee}(1, x_2 - x_0)^{(N+1)^2}. \end{aligned} \tag{1.4.5}$$

LEMMA (1.4.6). — *For* $\alpha > d(2N+1)$,

$$\sum_j g_{x_j}^{\vee}(1, x_j - x)^{-\alpha} \leq p^{2dn_0} \cdot \zeta\big(\alpha - d(2N+1)\big).$$

Proof. — Set $\mathcal{J}(k) = \{j \mid g_{x_j}^{\vee}(1, x_j - x) \leq p^k\}$. For $j \in \mathcal{J}(k)$ we have

$$g_{x_j}(y) \leq p^{n_0 + kN} g_x(y)$$

by (1.4.2), and hence $x_j + p^{n_0+kN} \cdot \mathcal{O}_x \subseteq x_j + \mathcal{O}_j = \mathfrak{A}_j$. Similarly, using $g_x \leq g_x^{\vee}$ and (1.4.1), for $j \in \mathcal{J}(k)$,

$$g_x(x - x_j) \leq g_x^{\vee}(x - x_j) \leq p^{n_0} \cdot g_{x_j}^{\vee}(1, x_j - x)^{N+1} \leq p^{n_0 + k\cdot(N+1)},$$

and hence $\mathfrak{A}_j \subseteq x + p^{-[n_0+k(N+1)]} \cdot \mathcal{O}_x$. Denoting by vol the Haar measure in V normalized by $\mathrm{vol}(\mathcal{O}_x) = 1$, we get from the above two inclusions :

$$\begin{aligned} \#\mathcal{J}(k) &= \sum_{i \in \mathcal{J}(k)} 1 = \sum_{i \in \mathcal{J}(k)} \mathrm{vol}(x_j + p^{n_0+kN}\mathcal{O}_x) \cdot p^{d(n_0+kN)} \\ &\leq \mathrm{vol}\Big(\coprod_{j \in \mathcal{J}(k)} \mathfrak{A}_j \Big) \cdot p^{d(n_0+kN)} \\ &\leq \mathrm{vol}\big(x + p^{-[n_0+k(N+1)]}\mathcal{O}_x\big) \cdot p^{d(n_0+kN)} \\ &= p^{d(n_0+k(N+1))} \cdot p^{d(n_0+kN)} = p^{2dn_0} \cdot p^{d(2N+1)k}. \end{aligned}$$

Hence

$$\begin{aligned}\sum_j g_{x_j}^{\vee}(1, x_j - x)^{-\alpha} &= \sum_{j\geq 0} \#\big(\mathcal{J}(k) \setminus \mathcal{J}(k-1)\big) \cdot p^{-k\cdot\alpha} \\ &\leq p^{2dn_0} \cdot \sum_{k\geq 0} p^{(d(2N+1)-\alpha)\cdot k} \\ &= p^{2dn_0} \cdot \zeta\big(\alpha - d(2N+1)\big). \qquad \square\end{aligned}$$

Using (1.4.5) we deduce the following

COROLLARY (1.4.7). — *For $\beta > d\cdot(2N+1)\cdot(N+1)^2$,*

$$\sum_{j_1,j_2} g_{x_{j_1}}^{\vee}(1, x_{j_2} - x)^{-\beta} \cdot g_{x_{j_2}}^{\vee}(1, x_{j_1} - x)^{-\beta} < \infty.$$

The weight function m will be called *temperate* if there exist C and β, such that

$$m(x_1) \leq C \cdot m(x_0) \cdot g_{x_1}^{\vee}(1, x_1 - x_0)^{\beta}. \tag{1.4.8}$$

If m is temperate, so is m^α for all $\alpha \geq 0$, and using (1.4.3) also for $\alpha \leq 0$. We also have

$$h(x) \overset{\text{def}}{=} \sup_y \frac{g_x(y)}{g_x^{\vee}(y)} \quad \text{is temperate.} \tag{1.4.9}$$

Indeed, using (1.4.1) and (1.4.2), we have

$$h(x_1) \leq p^{2n_0} \cdot h(x_0) \cdot g_{x_1}^{\vee}(1, x_1 - x_0)^{2N}.$$

Given two temperate weights m_1, m_2, then $m_1 \cdot m_2$ is also temperate; moreover we have

$$m_1(x_1)\cdot m_2(x_2) \leq C_1 \cdot C_2 \cdot m_1(x_0) m_2(x_0) g_{x_1}^{\vee}(1, x_1 - x_0)^{\beta_1} \cdot g_{x_2}^{\vee}(1, x_2 - x_0)^{\beta_2}$$

which by (1.4.5) is smaller than

$$C_0 \cdot m_1(x_0) \cdot m_2(x_0) \cdot g_{x_2}^{\vee}(1, x_1 - x_0)^{\beta_0} \cdot g_{x_1}^{\vee}(1, x_2 - x_0)^{\beta_0}$$

for some new constants C_0, β_0. Hence,

$$g_{x_1}^{\vee}(1, x_2 - x_0)^{-\beta_0} \cdot g_{x_2}^{\vee}(1, x_1 - x_0)^{-\beta_0} \leq C_0 \cdot \frac{m_1(x_0)\cdot m_2(x_0)}{m_1(x_1) m_2(x_2)}. \tag{1.4.10}$$

When g and m are temperate, we have $1/m(x) \leq C/m(0) \cdot g_0^\vee(1,x)^\beta$, and using (1.4.2) we easily obtain $B^{\alpha|\beta+N\alpha}_{\infty,\infty}(V,g_0) \hookrightarrow B^\alpha_{\infty,\infty}(m,g)$, and hence $\mathcal{S}(V) \hookrightarrow S(m,g)$. Conversely, given g, the functions $m_0(x) = g_0(1,x)^{-1}$, $m_1(x) = g_x^\vee(1,x)^{-1}$, are temperate, and an easy estimate gives $B^\alpha_{\infty,\infty}(m_0^\beta \cdot m_1^{N\cdot\alpha}, g) \hookrightarrow B^{\alpha|\beta}_{\infty,\infty}(V,g_0)$. Thus we conclude that

$$\mathcal{S}(V) = \bigcap_m S(m,g), \tag{1.4.11}$$

where the intersection is taken over all temperate weights m, i.e. the topology of $\mathcal{S}(V)$ is defined by all the norms $B^\alpha_{\infty,\infty}(m,g)$, $\alpha > 0$, m temperate.

Beside the uniform metric-covering $g_x = g_0 \leq g_0^\vee$, which we studied in paragraph 0.2, basic metrics are the *Tœplitz* metric and the *pseudodifferential* metric which we next describe. The *Tœplitz symbols* are defined by

$$\Sigma^\alpha = S(m^\alpha, g), \quad g_x(y) = |1,px|^{-1} \cdot |y|, \quad m^\alpha(x) = |1,px|^\alpha. \tag{1.4.12}$$

It is easily verified that g_x is a metric-covering (i.e. it satifies (1.2.25)), that it is certain (i.e. $g_x(y) \leq g_x^\vee(y) = |1,px| \cdot |y|$) with $h(x) = |1,px|^{-2} = m^{-2}(x)$, and that it is temperate; indeed, we have a stronger inequality than (1.4.1)

$$\begin{cases} g_{x_0}^\vee(x) \leq g_{x_1}^\vee(x) \cdot \big|1, p(x_0 - x_1)\big|, \\ \big|1, p(x_0 - x_1)\big| \leq g_{x_1}^\vee(1, x_1 - x_0). \end{cases} \tag{1.4.13}$$

This follows from the basic inequality

$$|1,y_0| \leq |1,y_1| \cdot |1, y_0 - y_1|. \tag{1.4.14}$$

Explicitly, $f \in \Sigma^\alpha$ if and only if

$$|f(x)| \leq C_0 \cdot |1,px|^\alpha \tag{1.14.15}$$

for all $x \in V$, and for all $\beta > 0$:

$$\big|f(x) - f(x+y)\big| \leq C_\beta \cdot |1,px|^{\alpha-\beta} \cdot |y|^\beta \quad \text{for } |y| \leq |1,px|.$$

Letting $C^{\alpha}_{\infty}(V) = \bigcap_{\beta>0} B^{\beta|\alpha}_{\infty,\infty}(V)$ denote the Fréchet space defined by the norms of $B^{\beta|\alpha}_{\infty,\infty}$, $\beta > 0$, we note that since

$$\begin{aligned}
&\|f\|_{B^{\beta}_{\infty,\infty}(m^\alpha,g)} \\
&= \sup_x |1,px|^{-\alpha} \cdot \big|f(x)\big| + \sup_{\substack{x,y \\ |y|\le|1,px|}} |1,px|^{\beta-\alpha}\, |y|^{-\beta} \big|f(x) - f(x+y)\big| \\
&\ge \sup_x |1,px|^{-\alpha} \cdot \big|f(x)\big| + \sup_{\substack{x,y \\ |y|\le 1}} |1,px|^{-\alpha}\, |y|^{-\beta} \big|f(x) - f(x+y)\big| \\
&= \|f\|_{B^{\beta|-\alpha}_{\infty,\infty}}
\end{aligned}$$

we have

$$\Sigma^{\alpha} \hookrightarrow C^{-\alpha}_{\infty}(V). \tag{1.4.16}$$

In particular, Σ^0 consists of bounded smooth functions, and Σ^α for $\alpha < 0$ consists of smooth functions vanishing at infinity. Letting $\Sigma^{-\infty} = \bigcap_\alpha \Sigma^\alpha$ denote the Fréchet space defined by the norms of all the Σ^α's, we have by (1.4.11) and (1.4.16) :

$$\Sigma^{-\infty} = \mathcal{S}(V). \tag{1.4.17}$$

The *pseudodifferential symbols* are defined by

$$\begin{cases} S^\alpha = S(m^\alpha, g), \\ g_{(\xi,x)}(\eta, y) = \max\{|1,p\xi|^{-1} \cdot |\eta|, |y|\}, \\ m^\alpha(\xi,x) = |1,p\xi|^\alpha, \end{cases} \tag{1.4.18}$$

where (ξ, x) and (η, y) refer to the symplectic coordinates associated with the complete polarization $V = V_0 \oplus V_1$. Again it is easy to verify that $g_{\xi,x}$ is a metric-covering, certain with $h(\xi,x) = |1,p\xi|^{-1} = m^{-1}(\xi,x)$, and that it is temperate with in fact a stronger inequality

$$\begin{cases} g^{\vee}_{(\xi_0,x_0)}(\eta,y) \le g^{\vee}_{(\xi_1,x_1)}(\eta,y) \cdot \big|1,p(\xi_0-\xi_1)\big|, \\ \big|1,p(\xi_0-\xi_1)\big| \le g^{\vee}_{(\xi_1,x_1)}(1,\xi_0-\xi_1,x_0-x_1). \end{cases} \tag{1.4.19}$$

Explicitly, $f \in S^\alpha$ if and only if

$$\begin{cases} \big|f(\xi,x)\big| \le C_0 \cdot |1,p\xi|^\alpha \quad \text{for all } (\xi,x) \in V, \\ \text{and for all } \beta > 0 : \big|f(\xi,x) - f(\xi+\eta, x+y)\big| \\ \qquad\qquad \le C_\beta \cdot |1,p\xi|^\alpha \cdot \max\{|1,p\xi|^{-1} \cdot |\eta|, |y|\}^\beta \\ \text{for } |\eta| \le |1,p\xi| \text{ and } |y| \le 1. \end{cases} \tag{1.4.20}$$

For the norm associated with (1.4.20) for a fixed β, we have

$$\begin{aligned}
&\sup_{x,\xi} |1,p\xi|^{-\alpha}\cdot\big|f(\xi,x)\big| \\
&\qquad + \sup_{\substack{x,y,\xi,\eta \\ |y|\le 1,\, |\eta|\le|1,p\xi|}} |1,p\xi|^{-\alpha}\max\big\{|1,p\xi|^{-1}|\eta|,|y|\big\}^{-\beta}\cdot\big|f(\xi,x)-f(\xi+\eta,x+y)\big| \\
&\ge \sup_{x,\xi}|1,p\xi|^{-\alpha}\big|f(\xi,x)\big| \\
&\qquad + \sup_{\substack{x,y,\xi,\eta \\ |y,\eta|\le 1}} |1,p\xi|^{-\alpha}\cdot|\eta,y|^{-\beta}\cdot\big|f(\xi,x)-f(\xi+\eta,x+y)\big| \\
&= \|I_\xi^\alpha f\|_{B^\beta_{\infty,\infty}},
\end{aligned}$$

with $I_\xi^\alpha f(\eta,y)=|1,p\eta|^{-\alpha}\cdot f(\eta,y)$. Hence, letting

$$C_\infty^{(-\alpha,0)}(V)=\bigcap_{\beta>0} I_\xi^{-\alpha}B^\beta_{\infty,\infty},$$

the Fréchet space defined by the norms $\|I_\xi^\alpha f\|_{B^\beta_{\infty,\infty}}$, $\beta>0$, we have

$$S^\alpha \hookrightarrow C_\infty^{(-\alpha,0)}(V). \tag{1.4.21}$$

In particular, $S^0\subseteq C^0_\infty$ consists of bounded smooth functions, and S^α for $\alpha<0$ consists of smooth functions vanishing at ∞ in the ξ-direction.

2. THE SYMBOLIC CALCULUS

2.1. Estimates for twisted multiplication.

We fix a temperate, certain, metric-covering g_x in our $2d$-dimensional symplectic vector space V, so $V=\coprod_j \mathfrak{A}_j$, $\mathfrak{A}_j=x_j+\mathcal{O}_j$, $\mathcal{O}_j\supseteq\mathcal{O}_j^\vee$. In $V\oplus V$ we have the metric covering $g\otimes g$ corresponding to

$$V\oplus V=\coprod_{j_1,j_2}\mathfrak{A}_{j_1,j_2}=\mathfrak{A}_{j_1}\times\mathfrak{A}_{j_2}=(x_{j_1},x_{j_2})+\mathcal{O}_{j_1}\oplus\mathcal{O}_{j_2}.$$

We begin by giving *a short range estimate* for the restriction to $\mathfrak{A}_j$ of

$$\begin{aligned}
f_1\# f_2(x) &= \psi\big(Q(D)\big)f_1\otimes f_2(x,x) \\
&= \mathcal{F}\otimes\mathcal{F}\psi\big(Q(v_1,v_2)\big)\mathcal{F}\otimes\mathcal{F}f_1\otimes f_2(x,x),
\end{aligned} \tag{2.1.1}$$

where $f_1, f_2 \in B^{\alpha}_{\infty,\infty}(\mathfrak{A}_j)$, $x \in \mathfrak{A}_j$, and Q is as in (0.1.12). We have for $x_1, x_2 \in \mathfrak{A}_j$

$$(2.1.2)\quad \begin{cases} \big|\psi(Q(D))f_1 \otimes f_2(x_1,x_2)\big| \\ \quad \leq \big\|\psi(Q(D))f_1 \otimes f_2\big\|_{B^{d,d}_{2,1}(V\oplus V, g_j\otimes g_j)} \quad \text{by (1.2.23),} \\ \quad \leq \|f_1 \otimes f_2\|_{B^{d,d}_{2,1}(V\oplus V, g_j\otimes g_j)} \quad \text{(since } \psi(Q(D)) \text{ is unitary} \\ \qquad\qquad \text{on } L_2 \text{ and commutes with convolution)} \\ \quad = \|f_1 \otimes f_2\|_{B^{d,d}_{2,1}(\mathfrak{A}_{j,j})} \text{ since } \operatorname{supp} f_i \subseteq \mathfrak{A}_j \\ \quad = \|f_1\|_{B^{d}_{2,1}(\mathfrak{A}_j)} \cdot \|f_2\|_{B^{d}_{2,1}(\mathfrak{A}_j)}. \end{cases}$$

Using (2.1.2) we also obtain

$$(2.1.3)\quad \begin{cases} \big\|\varphi_{g_j,k_1} \otimes \varphi_{g_j,k_2} * \psi(Q(D))f_1 \otimes f_2\big\|_{L_\infty(\mathfrak{A}_{j,j})} \\ \quad = \big\|\psi(Q(D))\varphi_{g_j,k_1} * f_1 \otimes \varphi_{g_j,k_2} * f_2\big\|_{L_\infty(\mathfrak{A}_{j,j})} \\ \quad \leq \|\varphi_{g_j,k_1} * f_1\|_{B^{d}_{2,1}(\mathfrak{A}_j)} \cdot \|\varphi_{g_j,k_2} * f_2\|_{B^{d}_{2,1}(\mathfrak{A}_j)} \\ \quad = p^{k_1\cdot d}\|\varphi_{g_j,k_1} * f_1\|_{L_2(\mathfrak{A}_j)} \cdot p^{k_2\cdot d}\|\varphi_{g_j,k_2} * f_2\|_{L_2(\mathfrak{A}_j)}. \end{cases}$$

Multiplying (2.1.3) by $p^{(k_1+k_2)\alpha}$, and taking ℓ_r-norm, we conclude that

$$(2.1.4)\quad \big\|\psi(Q(D))f_1 \otimes f_2\big\|_{B^{\alpha,\alpha}_{\infty,r}(\mathfrak{A}_{j,j})} \leq \|f_1\|_{B^{d+\alpha}_{2,r}(\mathfrak{A}_j)} \cdot \|f_2\|_{B^{d+\alpha}_{2,r}(\mathfrak{A}_j)}.$$

Using (1.2.21), we obtain from (2.1.4)

$$(2.1.5)\quad \|f_1 \# f_2\|_{B^{\alpha}_{\infty,r}(\mathfrak{A}_j)} \leq c_\alpha \cdot \|f_1\|_{B^{d+\alpha}_{2,r}(\mathfrak{A}_j)} \cdot \|f_2\|_{B^{d+\alpha}_{2,r}(\mathfrak{A}_j)}.$$

Note that Haar measure is normalized by (1.2.3) as $dx(\mathfrak{A}_j) = 1$, so that $L_\infty(\mathfrak{A}_j) \hookrightarrow L_2(\mathfrak{A}_j)$ with norm 1, and hence $\|f\|_{B^{\alpha}_{2,r}(\mathfrak{A}_j)} \leq \|f\|_{B^{\alpha}_{\infty,\alpha}(\mathfrak{A}_j)}$, so that (2.1.5) gives

$$(2.1.6)\quad \begin{cases} \|f_1 \# f_2\|_{B^{\alpha}_{2,r}(\mathfrak{A}_j)} \\ \quad \leq \|f_1 \# f_2\|_{B^{\alpha}_{\infty,r}(\mathfrak{A}_j)} \\ \quad \leq c_\alpha \cdot \|f_1\|_{B^{d+\alpha}_{2,r}(\mathfrak{A}_j)} \cdot \|f_2\|_{B^{d+\alpha}_{2,r}(\mathfrak{A}_j)} \\ \quad \leq c_\alpha \cdot \|f_1\|_{B^{d+\alpha}_{\infty,r}(\mathfrak{A}_j)} \cdot \|f_2\|_{B^{d+\alpha}_{\infty,r}(\mathfrak{A}_j)}. \end{cases}$$

To estimate the «error», that is the difference between $f_1 \# f_2$ and $f_1 \cdot f_2$ we notice that for $y \in \mathbb{Q}_p$ we have

$$(2.1.7)\quad \big|\psi(y) - 1\big| \leq \begin{cases} 2 & \text{if } y \notin \mathbb{Z}_p, \\ 0 & \text{if } y \in \mathbb{Z}_p, \end{cases}$$

and hence in particular for any $\gamma \geq 1$

(2.1.8) $$\left|\psi(y) - 1\right| \leq |y|^{\gamma}.$$

Also, using $g_j^{\vee\vee} = g_j$, we have

(2.1.9)
$$\begin{aligned}\sup_{\substack{g_j^{\vee}(x_1)=p^{k_1}\\ g_j^{\vee}(x_2)=p^{k_2}}} \left|\langle x_1, x_2\rangle\right| &= p^{(k_1+k_2)} \sup_{x_1,x_2} \frac{|\langle x_1, x_2\rangle|}{g_j^{\vee}(x_1)\cdot g_j^{\vee}(x_2)}\\ &= p^{(k_1+k_2)} \sup_{x} \frac{g_j(x)}{g_j^{\vee}(x)} = p^{(k_1+k_2)}\cdot h_j\end{aligned}$$

where h_j is defined as in (1.4.9) with the metric g_j.

(2.1.10) Applying these remarks we can now estimate

$$\left|\psi(Q(D))f_1 \otimes f_2(x_1, x_2) - f_1 \otimes f_2(x_1, x_2)\right|$$

$$\leq \left\|(\psi(Q(D)) - \delta)f_1 \otimes f_2\right\|_{B_{2,1}^{d,d}(V\oplus V, g_j\otimes g_j)} \qquad \text{by (1.2.23)}$$

$$= \sum_{k_1,k_2\geq 0} p^{(k_1+k_2)d}\left\|\widehat{\varphi}_{g_j,k_1} \otimes \widehat{\varphi}_{g_j,k_2}\cdot\left(\psi\left(\tfrac{1}{2}\langle x_1, x_2\rangle\right) - 1\right)\mathcal{F}f_1 \otimes \mathcal{F}f_2\right\|_{L_2(V\oplus V)}$$

(by the definitions (1.2.4), (0.1.12) and Plancherel)

$$\leq h_j^{\gamma} \sum_{k_1,k_2\geq 0} p^{(k_1+k_2)\cdot(d+\gamma)}\left\|\widehat{\varphi}_{g_j,k_1} \otimes \widehat{\varphi}_{g_j,k_2}\cdot\mathcal{F}f_1 \otimes \mathcal{F}f_2\right\|_{L_2(V\oplus V)}$$

(by using (2.1.8) and (2.1.9), with any $\gamma \geq 1$),

$$= h_j^{\gamma}\cdot\|f_1\|_{B_{2,1}^{d+\gamma}(\mathfrak{A}_j)}\cdot\|f_2\|_{B_{2,1}^{d+\gamma}(\mathfrak{A}_j)}$$

(by Plancherel again, and the definition (1.2.4)).

Using (2.1.10), we can now deduce as in (2.1.3), (2.1.4), (2.1.5) and (2.1.6), that we have

(2.1.11)
$$\begin{cases} \|f_1 \# f_2 - f_1\cdot f_2\|_{B_{2,r}^{\alpha}(\mathfrak{A}_j)}\\ \leq \|f_1 \# f_2 - f_1\cdot f_2\|_{B_{\infty,r}^{\alpha}(\mathfrak{A}_j)}\\ \leq c_\alpha\cdot h_j^{\gamma}\cdot\|f_1\|_{B_{2,r}^{d+\gamma+\alpha}(\mathfrak{A}_j)}\cdot\|f_2\|_{B_{2,r}^{d+\gamma+\alpha}(\mathfrak{A}_j)}\\ \leq c_\alpha\cdot h_j^{\gamma}\cdot\|f_1\|_{B_{\infty,r}^{d+\gamma+\alpha}(\mathfrak{A}_j)}\cdot\|f_2\|_{B_{\infty,r}^{d+\gamma+\alpha}(\mathfrak{A}_j)}.\end{cases}$$

We next give the *long range estimate* for the restriction to $\mathfrak{A}_j$ of $f_1 \# f_2$, where $\operatorname{supp} f_i \subseteq \mathfrak{A}_{j_i}$, and at least one of the j_i's is different from j. We recall from (0.1.11) that we can write $\psi(Q(D))f_1 \otimes f_2$ also as a convolution

$$\begin{aligned}(2.1.12)\qquad & \psi(Q(D))f_1 \otimes f_2(x_1, x_2) \\ &= \iint_{\mathfrak{A}_{j_1} \times \mathfrak{A}_{j_2}} dx'\, dx''\, f_1(x')\, f_2(x'')\, \bar{\psi}(2\langle x' - x_1, x'' - x_2\rangle).\end{aligned}$$

We set

$$(2.1.13)\qquad \widetilde{g}^{\vee}_{j_1,j_2}(1,x) = \min_{y_2 \in \mathcal{O}_{j_2}} g^{\vee}_{j_1}(1, x + y_2)$$

so that we have

$$(2.1.14)\qquad \widetilde{g}^{\vee}_{j_1,j_2} \quad \text{is} \quad \mathcal{O}^{\vee}_{j_1} + \mathcal{O}_{j_2}\text{-locally constant, and}$$

$$(2.1.15)\qquad 1 \le \widetilde{g}^{\vee}_{j_1,j_2}(1,x) \le g^{\vee}_{j_1}(1,x).$$

Hence if we put

$$(2.1.16)\qquad \widetilde{J}^{-\beta}_{j_1,j_2} = \mathcal{F}\widetilde{g}^{\vee}_{j_1,j_2}(1,x)^{\beta}\mathcal{F}$$

then $\widetilde{J}^{-\beta}_{j_1,j_2}$ acting on $\mathcal{S}'(V)$, preserves by (2.1.14) $C'_{\infty}(\mathfrak{A}_{j_1})$, the distributions supported in $\mathfrak{A}_{j_1}$; moreover, it commutes with convolution, and by (2.1.15), (1.2.12), and Plancherel, we have for $\beta > 0$

$$(2.1.17)\qquad \|\widetilde{J}^{-\beta}_{j_1,j_2} f_1\|_{B^{\alpha}_{2,r}(\mathfrak{A}_{j_1})} \le \|J^{-\beta}_{g_{j_1}} f_1\|_{B^{\alpha}_{2,r}(\mathfrak{A}_{j_1})} = \|f_1\|_{B^{\alpha+\beta}_{2,r}(\mathfrak{A}_{j_1})}.$$

Similary we can define $\widetilde{g}^{\vee}_{j_1,j_2}$, and $\widetilde{J}^{-\beta}_{j_1,j_2}$, and they satisfy the same properties.

Notice that we have in the distributional sense,

$$(2.1.18)\qquad \mathcal{F}_{x'}\bar{\psi}(2\langle x' - x_1, x'' - x_2\rangle) = \psi(2\langle x_1, x'' - x_2\rangle) \cdot \delta_{2(x''-x_2)}$$

and hence, letting $\widetilde{J}^{-\beta}_{j_1,j_2}$ act through the variable x', we have

$$\begin{aligned}(2.1.19)\qquad & \widetilde{J}^{-\beta}_{j_1,j_2}\bar{\psi}(2\langle x' - x_1, x'' - x_2\rangle) \\ &= \widetilde{g}^{\vee}_{j_1,j_2}(1, x'' - x_2)^{\beta}\bar{\psi}(2\langle x' - x_1, x'' - x_2\rangle).\end{aligned}$$

Putting,

$$(2.1.20)\qquad \begin{aligned} g^{\vee}_{j_1}(x_2,\mathfrak{A}_{j_2}) &= \min_{x''\in\mathfrak{A}_{j_2}} g^{\vee}_{j_1}(1,x''-x_2) \\ &= \widetilde{g}^{\vee}_{j_1,j_2}(1,x''-x_2), \quad \text{for any } x''\in\mathfrak{A}_{j_2}, \end{aligned}$$

we obtain from (2.1.12) and (2.1.19),

$$(2.1.21)\qquad \begin{aligned} &\psi\big(Q(D)\big)f_1\otimes f_2(x_1,x_2) \\ &\quad = g^{\vee}_{j_1}(x_2,\mathfrak{A}_{j_2})^{-\beta}\cdot\psi\big(Q(D)\big)\big[\widetilde{J}^{-\beta}_{j_1,j_2}f_1\otimes f_2\big](x_1,x_2). \end{aligned}$$

Similarly, using $\widetilde{g}^{\vee}_{j_2,j_1}$ and $\widetilde{J}^{-\beta}_{j_2,j_1}$, and the analogous formula to (2.1.19), we have

$$(2.1.22)\qquad \begin{aligned} &\psi\big(Q(D)\big)f_1\otimes f_2(x_1,x_2) \\ &\quad = g^{\vee}_{j_1}(x_2,\mathfrak{A}_{j_2})^{-\beta}g^{\vee}_{j_2}(x_1,\mathfrak{A}_{j_1})^{-\beta} \\ &\qquad\quad \cdot\psi\big(Q(D)\big)\big[\widetilde{J}^{-\beta}_{j_1,j_2}f_1\otimes\widetilde{J}^{-\beta}_{j_2,j_1}f_2\big](x_1,x_2). \end{aligned}$$

Now we can estimate, as in the short range-estimate (2.1.2),

$$(2.1.23)\qquad \begin{cases} \big|\psi(Q(D))f_1\otimes f_2(x_1,x_2)\big| \\ = g^{\vee}_{j_1}(x_2,\mathfrak{A}_{j_2})^{-\beta}\cdot g^{\vee}_{j_2}(x_1,\mathfrak{A}_{j_1})^{-\beta} \\ \qquad \cdot\big|\psi(Q(D))\widetilde{J}^{-\beta}_{j_1,j_2}f_1\otimes\widetilde{J}^{-\beta}_{j_2,j_1}f_2(x_1,x_2)\big| \\ \le g^{\vee}_{j_1}(x_2,\mathfrak{A}_{j_2})^{-\beta}\cdot g^{\vee}_{j_2}(x_1,\mathfrak{A}_{j_1})^{-\beta} \\ \qquad \cdot\big\|\psi(Q(D))\widetilde{J}^{-\beta}_{j_1,j_2}f_1\otimes\widetilde{J}^{-\beta}_{j_2,j_1}f_2\big\|_{B^{d,d}_{2,1}(V\oplus V,g_{j_1}\otimes g_{j_2})} \\ \le g^{\vee}_{j_1}(x_2,\mathfrak{A}_{j_2})^{-\beta}\cdot g^{\vee}_{j_2}(x_1,\mathfrak{A}_{j_1})^{-\beta} \\ \qquad \cdot\big\|\widetilde{J}^{-\beta}_{j_1,j_2}f_1\otimes\widetilde{J}^{-\beta}_{j_2,j_1}f_2\big\|_{B^{d,d}_{2,1}(V\oplus V,g_{j_1}\otimes g_{j_2})} \\ \le g^{\vee}_{j_1}(x_2,\mathfrak{A}_{j_2})^{-\beta}\cdot g^{\vee}_{j_2}(x_1,\mathfrak{A}_{j_1})^{-\beta} \\ \qquad \cdot\|f_1\|_{B^{d+\beta}_{2,1}(\mathfrak{A}_{j_1})}\cdot\|f_2\|_{B^{d+\beta}_{2,1}(\mathfrak{A}_{j_2})} \quad \text{by (2.1.17).} \end{cases}$$

Using the fact that by our normalization we have $L_\infty(\mathfrak{A}_j,g_j)\hookrightarrow L_2(\mathfrak{A}_j,g_j)$ with norm 1, and the embedding (1.2.6), we obtain from (2.1.23) that for any $\beta'>\beta+d$

$$(2.1.24)\qquad \begin{aligned} \big|\psi(Q(D))f_1\otimes f_2(x_1,x_2)\big| &\le C\cdot g^{\vee}_{j_1}(x_2,\mathfrak{A}_{j_2})^{-\beta}g^{\vee}_{j_2}(x_1,\mathfrak{A}_{j_1})^{-\beta} \\ &\qquad \|f_1\|_{B^{\beta'}_{\infty,\infty}(\mathfrak{A}_{j_1})}\|f_2\|_{B^{\beta'}_{\infty,\infty}(\mathfrak{A}_{j_2})}. \end{aligned}$$

This, of course gives an estimate for $\|f_1 \# f_2\|_{L_\infty(\mathfrak{A}_j)}$, but to estimate the Besov norms of $f_1 \# f_2$ we need a further argument since we have different metrics g_j and g_{j_i}. We remark that since we have, by (1.4.2), for any $x_i \in \mathfrak{A}_j$

$$g_{j_i}(y) \le g_j(y) \cdot p^{n_0} \cdot g_{j_i}^{\vee}(x_i, \mathfrak{A}_{j_i})^N \tag{2.1.25}$$

an easy estimate gives

$$\begin{aligned} \sup_{g_j(y)\le 1} g_j(y)^{-\alpha} \|(\delta_0 - \delta_y) * f_i\|_{B^{\beta}_{\infty,\infty}(V, g_{j_i})} \\ \le 6 \cdot p^{n_0\alpha} g_{j_i}^{\vee}(x_i, \mathfrak{A}_{j_i})^{N\alpha} \cdot \|f_i\|_{B^{\alpha+\beta}_{\infty,\infty}(V, g_{j_i})}. \end{aligned} \tag{2.1.26}$$

Hence from (2.1.24) and (2.1.26) we deduce, since $\psi(Q(D))$ commutes with translations, that for $x \in \mathfrak{A}_j$, $\beta' > \beta + d$, we have

$$\left\{ \begin{aligned} &\sup_{g_j(y)\le 1} g_j(y)^{-\alpha} \big| f_1 \# f_2(x) - f_1 \# f_2(x+y) \big| \\ &\le \sup_{g_j(y)\le 1} g_j(y)^{-\alpha} \left|\psi(Q(D)) f_1 \otimes (\delta_0 - \delta_{-y}) * f_2(x,x)\right| \\ &\quad + \sup_{g_j(y)\le 1} g_j(y)^{-\alpha} \left|\psi(Q(D))(\delta_0 - \delta_y) * f_1 \otimes f_2(x+y, x+y)\right| \\ &\le C \cdot g_{j_1}^{\vee}(x, \mathfrak{A}_{j_2})^{-\beta} g_{j_2}^{\vee}(x, \mathfrak{A}_{j_1})^{-\beta} \cdot \|f_1\|_{B^{\beta'}_{\infty,\infty}(\mathfrak{A}_{j_1})} \\ &\qquad \cdot \sup_{g_j(y)\le 1} g_j(y)^{-\alpha} \big\|(\delta_0 - \delta_y) * f_2\big\|_{B^{\beta'}_{\infty,\infty}(V, g_{j_2})} \\ &\quad + C \cdot g_{j_1}^{\vee}(x, \mathfrak{A}_{j_2})^{-\beta} g_{j_2}^{\vee}(x, \mathfrak{A}_{j_1})^{-\beta} \\ &\qquad \cdot \sup_{g_j(y)\le 1} g_j(y)^{-\alpha} \big\|(\delta_0 - \delta_y) * f_1\big\|_{B^{\beta'}_{\infty,\infty}(V, g_{j_1})} \cdot \|f_2\|_{B^{\beta'}_{\infty,\infty}(\mathfrak{A}_{j_2})} \\ &\le C' \cdot g_{j_1}^{\vee}(x, \mathfrak{A}_{j_2})^{-\beta} g_{j_2}^{\vee}(x, \mathfrak{A}_{j_1})^{-\beta} g_{j_2}^{\vee}(x, \mathfrak{A}_{j_2})^{N\alpha} \\ &\qquad \cdot \|f_1\|_{B^{\beta'}_{\infty,\infty}(\mathfrak{A}_{j_1})} \cdot \|f_2\|_{B^{\beta'+\alpha}_{\infty,\infty}(\mathfrak{A}_{j_2})} \\ &\quad + C' g_{j_1}^{\vee}(x, \mathfrak{A}_{j_2})^{-\beta} g_{j_2}^{\vee}(x, \mathfrak{A}_{j_1})^{-\beta} g_{j_1}^{\vee}(x, \mathfrak{A}_{j_1})^{N\alpha} \\ &\qquad \cdot \|f_1\|_{B^{\beta'+\alpha}_{\infty,\infty}(\mathfrak{A}_{j_1})} \cdot \|f_2\|_{B^{\beta'}_{\infty,\infty}(\mathfrak{A}_{j_2})} \\ &\le C'' \cdot g_{j_1}^{\vee}(x, \mathfrak{A}_{j_2})^{-\beta} g_{j_2}^{\vee}(x, \mathfrak{A}_{j_1})^{-\beta} g_{j_1}^{\vee}(x, \mathfrak{A}_{j_1})^{N\alpha} g_{j_2}^{\vee}(x, \mathfrak{A}_{j_2})^{N\alpha} \\ &\qquad \cdot \|f_1\|_{B^{\beta'+\alpha}_{\infty,\infty}(\mathfrak{A}_{j_1})} \cdot \|f_2\|_{B^{\beta'+\alpha}_{\infty,\infty}(\mathfrak{A}_{j_2})} \end{aligned} \right. \tag{2.1.27}$$

where the constant C'' depends only on α, $(\beta - \beta')$, n_0 and d.

2.2. The main theorem of symbolic calculus.

We now give a global estimate for $f_1 \# f_2$, where $f_i \in S(m_i, g)$ and the weights m_1, m_2 are temperate, i.e. they satisfy (1.4.8). We let $f_{i,(j)}$ denote the restriction of f_i to $\mathfrak{A}_j$, and we note that the partial sums $\sum_{j \le M} f_{i,(j)}$ are bounded in $S(m_i, g)$, and they converge to f_i in $C_\infty(V)$, since they are ultimately equal to f_i on any compact set. Letting β_0 denote the constant in (1.4.10), we deduce from (2.1.24) that for any $\beta' > \beta + d + \beta_0$,

$$
\begin{aligned}
(2.2.1)\quad & \left| f_{1,(j_1)} \# f_{2,(j_2)}(x) \right| \\
& \le C \cdot g_{j_1}^{\vee}(x, \mathfrak{A}_{j_2})^{-\beta} g_{j_2}^{\vee}(x, \mathfrak{A}_{j_1})^{-\beta} \cdot m_1(x) m_2(x) \\
& \quad \cdot \left\| m_1^{-1} f_{1,(j_1)} \right\|_{B^{\beta'}_{\infty,\infty}(\mathfrak{A}_{j_1})} \cdot \left\| m_2^{-1} f_{2,(j_2)} \right\|_{B^{\beta'}_{\infty,\infty}(\mathfrak{A}_{j_2})}.
\end{aligned}
$$

Taking β sufficiently large so that (1.4.7) applies, we obtain

$$
\begin{aligned}
(2.2.2)\quad & \sum_{j_1, j_2} \left| f_{1,(j_1)} \# f_{2,(j_2)}(x) \right| \\
& \le C' \cdot m_1 \cdot m_2(x) \cdot \|f_1\|_{B^{\beta'}_{\infty,\infty}(m_1, g)} \cdot \|f_2\|_{B^{\beta'}_{\infty,\infty}(m_2, g)}.
\end{aligned}
$$

From the estimate (2.2.2), being valid for all f_i in a bounded subset of $S(m_i, g)$ we deduce that the form $f_1 \otimes f_2 \mapsto f_1 \# f_2$ defined on $C_{\infty,c}(V) \otimes C_{\infty,c}(V) = C_{\infty,c}(V \oplus V)$ extends uniquely to a weakly continuous linear form on $S(m_1, g) \otimes S(m_2, g)$, given by

$$
f_1 \# f_2(x) = \sum_{j_1, j_2} f_{1,(j_1)} \# f_{2,(j_2)}(x).
$$

Similarly, note that the functions

$$
(2.2.3)\qquad \widetilde{m}_i(y) = m_i(y) \cdot g_y^{\vee}(x, \mathfrak{A}_y)^{-N\alpha}
$$

are again temperate, with the corresponding constants in (1.4.8) depending only on m_i, g and α (and independent of x) ; letting β_α denote the associated constant in (1.4.10) (with m_i replaced by $\widetilde{m}_i$), and using (1.4.5), we deduce from (2.1.27) that for any $\beta' > \beta + d + \beta_\alpha + (N+1)^2 \cdot N\alpha$, we have

$$
\begin{aligned}
(2.2.4)\quad & \left\| f_{1,(j_1)} \# f_{2,(j_2)}(x) \right\|_{B^{\alpha}_{\infty,\infty}(\mathfrak{A}_x)} \\
& \le C''' \cdot g_{j_1}^{\vee}(x, \mathfrak{A}_{j_2})^{-\beta} g_{j_2}^{\vee}(x, \mathfrak{A}_{j_1})^{-\beta} m_1 m_2(x) \\
& \quad \cdot \left\| m_1^{-1} f_{1,(j_1)} \right\|_{B^{\beta'+\alpha}_{\infty,\infty}(\mathfrak{A}_{j_1})} \cdot \left\| m_2^{-1} f_{2,(j_2)} \right\|_{B^{\beta'+\alpha}_{\infty,\infty}(\mathfrak{A}_{j_2})}.
\end{aligned}
$$

Taking again a β satisfying (1.4.7), we conclude that with some new constant C

$$(2.2.5)\quad \sum_{j_1,j_2} \|f_{1,(j_1)} \# f_{2,(j_2)}\|_{B^{\alpha}_{\infty,\infty}(\mathfrak{A}_x)} \leq C \cdot m_1 m_2(x) \cdot \|f_1\|_{B^{\beta'+\alpha}_{\infty,\infty}(m_1,g)} \cdot \|f_2\|_{B^{\beta'+\alpha}_{\infty,\infty}(m_2,g)}.$$

Thus the map $f_1 \otimes f_2 \mapsto f_1 \# f_2$ is weakly continuous when viewed as a map

$$S(m_1, g) \otimes S(m_2, g) \longrightarrow S(m_1 \cdot m_2, g).$$

Moreover, since the function $x \mapsto h(x)$ of (1.4.9) is temperate, and since for $x \notin \mathfrak{A}_{j_i}$

$$(2.2.6)\qquad 1 \leq \min_{y \in \mathfrak{A}_{j_i}} g_{j_i}(x-y) \leq h_{j_i} \cdot g^{\vee}_{j_i}(x, \mathfrak{A}_{j_i}),$$

we see that, using (1.4.5), we can bound for $x \notin \mathfrak{A}_{j_1} \cap \mathfrak{A}_{j_2}$,

$$(2.2.7)\qquad g^{\vee}_{j_1}(x, \mathfrak{A}_{j_2})^{-\beta_1 \cdot \gamma} g^{\vee}_{j_2}(x, \mathfrak{A}_{j_1})^{-\beta_1 \cdot \gamma} \leq h(x)^{\gamma}$$

for some $\beta_1 > 0$. Hence, by taking β large, we can replace $m_1 \cdot m_2(x)$ in the right hand side of (2.2.4) by $h^{\gamma} \cdot m_1 \cdot m_2(x)$, for $x \notin \mathfrak{A}_{j_1} \cap \mathfrak{A}_{j_2}$, and a posteriori we can similarly replace $m_1 \cdot m_2$ by $h^{\gamma} \cdot m_1 \cdot m_2$ in (2.2.5) if we sum only over $j_1 \neq j_2$. When $j_1 = j_2 = j$, we can use the estimate (2.1.11). Thus we conclude that the map $f_1 \otimes f_2 \mapsto f_1 \# f_2 - f_1 \cdot f_2$ is weakly continuous when viewed as a map $S(m_1, g) \otimes S(m_2, g) \to S(h^{\gamma} \cdot m_1 \cdot m_2, g)$, for any γ. We summarize the above discussion in the following

MAIN THEOREM (2.2.8). — *Given a temperate, certain, metric-covering g in the symplectic space V, and two temperate weights m_1, m_2, then*

$$\# : S(m_1, g) \otimes S(m_2, g) \longrightarrow S(m_1 m_2, g)$$

is weakly continuous. If $h(x) = \sup_y g_x(y)/g^{\vee}_x(y)$, then for any γ the map $f_1 \otimes f_2 \mapsto f_1 \# f_2 - f_1 \cdot f_2$ is weakly continuous

$$S(m_1, g) \otimes S(m_2, g) \longrightarrow S(h^{\gamma} \cdot m_1 \cdot m_2, g).$$

2.3. Bounds for operators.

In this section we study the continuity of the operators $\hat{\rho}(f)$ on the spaces $\mathcal{S}(V_0) \hookrightarrow L_2(V_0) \hookrightarrow \mathcal{S}'(V_0)$, for $f \in S(m,g)$, g,m temperate. We first remark that since by (0.1.13) $\hat{\rho}$ takes complex conjugates to adjoints, the continuity in $\mathcal{S}'(V_0)$ follows from the continuity in $\mathcal{S}(V_0)$. Secondly, we note that the Bargman isomorphism $B^{-1} : L_2(V_0) \xrightarrow{\sim} L_2(V_0)_\psi$ induces an isomorphism $B^{-1} : \mathcal{S}(V_0) \xrightarrow{\sim} \mathcal{S}(V_0)_\psi = \mathcal{S}(V) \cap L_2(V)_\psi$ onto the subspace of $\mathcal{S}(V)$ consisting of ψ-holomorphic functions. By the description of $\mathcal{S}(V)$ given in (1.4.11), and the description of $B^{-1}\hat{\rho}(f)B$ given in (0.2.12), we deduce from our main theorem (2.2.8) the continuity in $\mathcal{S}(V_0)$. Thus we have :

THEOREM (2.3.1). — *For a temperate certain-metric-covering g, and a temperate weight m, the operators $\hat{\rho}(f)$ are continuous in $\mathcal{S}(V_0)$, and in $\mathcal{S}'(V_0)$, for every $f \in S(m,g)$. Moreover, the map $f \mapsto \hat{\rho}(f)$ induces weakly continuous embeddings*

$$\left\{ \begin{array}{l} S(m,g) \hookrightarrow \mathrm{Hom}\big(\mathcal{S}(V_0), \mathcal{S}(V_0)\big) = \mathcal{S}(V_0) \otimes \mathcal{S}'(V_0), \\ S(m,g) \hookrightarrow \mathrm{Hom}\big(\mathcal{S}'(V_0), \mathcal{S}'(V_0)\big) = \mathcal{S}'(V_0) \otimes \mathcal{S}(V_0). \end{array} \right. \tag{2.3.2}$$

For the continuity on $L_2(V_0)$ we take $f \in S(\mathbb{1}, g)$ and write $f = \sum_i f_i$ for the corresponding decomposition of f with respect to the metric covering g. We shall use the lemma of Cotlar-Knapp-Stein-Calderón-Vaillancourt, so that we need to estimate the operator norms

$$\begin{aligned} (2.3.3)\quad \big\|\hat{\rho}(f_i)^*\hat{\rho}(f_j)\big\|_{\mathrm{op}} &= \|\hat{\rho}(\bar{f}_i \# f_j)\|_{\mathrm{op}} = \big\|\rho(\mathcal{F}(\bar{f}_i \# f_j))\big\|_{\mathrm{op}} \\ &\le \big\|\mathcal{F}(\bar{f}_i \# f_j)\big\|_{L_1(V)} \le C_\alpha \cdot \big\|\bar{f}_i \# f_j\big\|_{B^\alpha_{\infty,\infty}(\mathbf{1},g)} \end{aligned}$$

for α sufficiently large. From (2.2.4) we obtain for $\beta' \gg \beta, \alpha$

$$\big\|\bar{f}_i \# f_j\big\|_{B^\alpha_{\infty,\infty}(\mathfrak{A}_x, g_x)} \le C \cdot g_i^\vee(x, \mathfrak{A}_j)^{-\beta} g_j^\vee(x, \mathfrak{A}_i)^{-\beta} \cdot \|f\|^2_{B^{\beta'}_{\infty,\infty}(\mathbf{1},g)}. \tag{2.3.4}$$

By using repeatedly (1.4.1), and the inequality of (1.1.26), one easily obtains that for some constants C_0, M, and all $x \in V$

$$g_i^\vee(\mathfrak{A}_i, \mathfrak{A}_j) \stackrel{\mathrm{def}}{=} \min_{\substack{y' \in \mathfrak{A}_i \\ y'' \in \mathfrak{A}_j}} g_i^\vee(1, y' - y'') \le C_0 \cdot g_i^\vee(x, \mathfrak{A}_j)^M \cdot g_j^\vee(x, \mathfrak{A}_i)^M. \tag{2.3.5}$$

Putting (2.3.3), (2.3.4), (2.3.5) together we obtain, with some new constant C, for $\gamma \gg \beta \gg 0$,

$$\left\|\hat{\rho}(f_i)^*\hat{\rho}(f_j)\right\|_{\mathrm{op}}^{1/2} \leq C \cdot g_i^{\vee}(\mathfrak{A}_i, \mathfrak{A}_j)^{-\beta} \cdot \|f\|_{B^{\gamma}_{\infty,\infty}(\mathbb{1},g)}. \tag{2.3.6}$$

By an analogous proof to the proof of Lemma (1.4.6) we have, for $\beta \gg 0$

$$\sum_i g_i^{\vee}(\mathfrak{A}_i, \mathfrak{A}_j)^{-\beta} < \infty, \quad \sum_j g_j^{\vee}(\mathfrak{A}_i, \mathfrak{A}_j)^{-\beta} < \infty. \tag{2.3.7}$$

From (2.3.6) and (2.3.7), we obtain by applying the lemma of Cotlar et al.,

$$\left\|\hat{\rho}(f)\right\|_{\mathrm{op}} \leq C \cdot \|f\|_{B^{\gamma}_{\infty,\infty}(\mathbb{1},g)}, \quad \text{for } \gamma \text{ sufficiently large.} \tag{2.3.8}$$

THEOREM (2.3.9). — *For a temperate certain metric-covering g, and a temperate bounded weight m, the operators $\hat{\rho}(f)$ are bounded on $L_2(V_0)$ for every $f \in S(m,g)$. Moreover, for some C, γ we have*

$$\left\|\hat{\rho}(f)\right\|_{\mathrm{op}} \leq C \cdot \|f\|_{B^{\gamma}_{\infty,\infty}(m,g)}. \tag{2.3.10}$$

If m vanishes at infinity, the operators $\hat{\rho}(f)$ with $f \in S(m,g)$ are all compact.

Proof. — The inequality in (2.3.10) follows immediately from (2.3.8), since if $m \leq C$, then $\|f\|_{B^{\gamma}_{\infty,\infty}(\mathbb{1},g)} \leq C \cdot \|f\|_{B^{\gamma}_{\infty,\infty}}(m,g)$. The compactness assertion follows since the operators $\hat{\rho}(f_i)$ are all compact, and by (2.3.8)

$$\Big\|\hat{\rho}(f) - \sum_{i \leq j} \hat{\rho}(f_i)\Big\|_{\mathrm{op}} \leq C \cdot \Big\|\sum_{i>j} f_i\Big\|_{B^{\gamma}_{\infty,\infty}(\mathbb{1},g)} \leq C \cdot \|f\|_{B^{\gamma}_{\infty,\infty}(m,g)} \cdot \sup_{i>j} m_i$$

which converges to 0 as $j \to \infty$ by assumption. □

(2.3.11) *Remark.* — We note that if f_j is a *bounded* sequence in $S(\mathbb{1},g)$ converging to f in $C_\infty(V)$ then for any $\varphi \in L_2(V_0)$, $\hat{\rho}(f_j)\varphi \to \hat{\rho}(f)\varphi$ in $L_2(V_0)$. Indeed, we may assume $f = 0$, and in view of the uniform bound (2.3.8) we can take $\varphi \in \mathcal{S}(V_0)$, then by the weak-continuity assertion of (2.3.1), $\hat{\rho}(f_j)\varphi$ is bounded in $\mathcal{S}(V_0)$, and converges to zero in $\mathcal{S}'(V_0)$, hence a posteriori in $\mathcal{S}(V_0)$ and in $L_2(V_0)$.

The inequality (2.3.8) can be sharpened in various ways, one of which we describe next. For a self-dual lattice $\mathcal{O} = \mathcal{O}^{\vee}$, denoting by Φ

its characteristic function, we know from (0.4.15), (0.4.17) that there is a unit vector $\varphi_0 \in L_2(V_0)$ such that $W\varphi_0 = \Phi = \widehat{\Phi}$. If f is such that $\operatorname{supp}\hat{f} \subseteq \mathcal{O}$, then by (0.4.19), we have $\hat{\rho}(f) = \hat{\rho}(f * \Phi) = C^*_{\varphi_0} M_f C_{\varphi_0}$, and hence $\|\hat{\rho}(f)\|_{\mathrm{op}} \leq \|f\|_{L_\infty}$. This also holds for f such that $\operatorname{supp}\hat{f} \subseteq x + \mathcal{O}$, since the function $t_x\hat{f}(y) = \hat{f}(y+x) = \mathcal{F}(\bar{\psi}_x \cdot f)(y)$ is supported in $\mathcal{O}$, and so

$$\|\hat{\rho}(f)\|_{\mathrm{op}} = \|\rho(-\tfrac{1}{2}x)\,\hat{\rho}(f)\,\rho(-\tfrac{1}{2}x)\|_{\mathrm{op}}$$
$$= \|\hat{\rho}(\bar{\psi}_x \cdot f)\|_{\mathrm{op}} \leq \|\bar{\psi}_x \cdot f\|_{L_\infty} = \|f\|_{L_\infty}.$$

Thus for a self-dual metric covering $g_x = g_x^\vee$, corresponding to $V = \coprod_j \mathfrak{A}_j$, $\mathfrak{A}_j = x_j + \mathcal{O}_j$, $\mathcal{O}_j = \mathcal{O}_j^\vee$, letting Φ_j denote the characteristic function of $\mathfrak{A}_j$, we have for any function f

$$\|\hat{\rho}(f)\|_{\mathrm{op}} = \Big\|\hat{\rho}\Big(\sum_j \widehat{\Phi}_j * f\Big)\Big\|_{\mathrm{op}} \leq \sum_j \|\hat{\rho}(\widehat{\Phi}_j * f)\|_{\mathrm{op}}$$
$$\leq \sum_j \|\widehat{\Phi}_j * f\|_{L_\infty} \stackrel{\mathrm{def}}{=} \|f\|_{B^\vee_{\infty,1}(g)}.$$

Here the right hand defines a norm for a Banach space $B^\vee_{\infty,1}(g)$ analogous to a Besov space. If for example $g_x = g$ is independent of x, i.e., $g = g^\vee$ is a self-dual metric corresponding to a lattice $\mathcal{O} \subseteq V$ and $V = \coprod_{x \in V/\mathcal{O}} (x + \mathcal{O})$ is the associated covering, then an easy calculation gives

$$\|\widehat{\Phi}_x * f\|_{L_\infty} \leq g(1,x)^{-\alpha} \cdot \|J_g^{-\alpha} f\|_{L_\infty}$$

and since $\sum_{x \in V/\mathcal{O}} g(1,x)^{-\alpha} = \zeta(\alpha - 2d)/\zeta(\alpha)$ is finite for $\alpha > 2d = \dim V$, we obtain

$$\|\hat{\rho}(f)\|_{\mathrm{op}} \leq \|f\|_{B^\vee_{\infty,1}(g)} \leq \frac{\zeta(\alpha - 2d)}{\zeta(\alpha)} \cdot \|f\|_{L_\infty{}^\alpha(V,g)}. \tag{2.3.12}$$

From this we deduce the following

THEOREM (2.3.13). — *For a self-dual metric g, the operators $\hat{\rho}(f)$ are bounded on $L_2(V_0)$ for every $f \in L^\alpha_\infty(V,g)$, $\alpha > \dim V$. If such an f vanishes at infinity, then $\hat{\rho}(f)$ is compact.*

For a g-locally constant function m, we define the operator $J_g^m = \hat{\rho}(m^{-1})$; if $m \geq 1$, it is bounded on $L_2(V_0)$ by (2.3.8). We define the m-th *Soboleff space* by

$$L_2^{(m)}(V_0) = J_g^m\big(L_2(V_0)\big), \quad \|\varphi\|_{L_2^{(m)}(V_0)} = \|\hat{\rho}(m)\varphi\|_{L_2(V_0)}. \tag{2.3.14}$$

Since for g-locally constant functions m_1, m_2 we have $m_1 \# m_2 = m_1 \cdot m_2$, we deduce that $J_g^{m_1} J_g^{m_2} = J_g^{m_1 \cdot m_2}$, and in particular, we see that $J_g^m = J_g^{m^{1/2}} \cdot (J_g^{m^{1/2}})^*$ is positive, and $J_g^m J_g^{m^{-1}} = \mathrm{id}$, so that J_g^m is invertible. For example, taking g and m as in (1.4.12) (or as in (1.4.18)) we get essentially the operator K^α of (1.1.20) (respectively, the usual J^α), and the space $L_2^{\{\alpha\}}(V_0)$ of (1.1.21) (respectively, $L_2^\alpha(V_0)$). For a temperate weight m_0 if $f \in S(m_0, g)$, we have $f \# m_0^{-1} \in S(\mathbb{1}, g)$, hence by (2.3.8) $\hat{\rho}(f \# m_0^{-1}) = \hat{\rho}(f) J_g^{m_0}$ is bounded.

Similarly replacing f by $m_1^{-1} \# f \# m_1 \in S(m_0, g)$, we obtain the

COROLLARY (2.3.15). — *For temperate weights m_0, m_1, every f in $S(m_0, g)$ defines a bounded operator $\hat{\rho}(f) : L_2^{(m_0 m_1)}(V_0) \to L_2^{(m_1)}(V_0)$.*

3. ELLIPTIC OPERATORS

3.1. The Gårding inequalities.

Fix a temperate certain metric covering g in V corresponding to $V = \coprod_j \mathfrak{A}_j$, $\mathfrak{A}_j = x_j + \mathcal{O}_j$, $\mathcal{O}_j \supseteq \mathcal{O}_j^\vee$, and write $h_j = p^{-a_j}$ for the quantity defined in (1.4.9) for the metric g_j. For each j choose a self-dual lattice $\mathcal{O}_j^0$ such that $\mathcal{O}_j \supseteq \mathcal{O}_j^0 \supseteq \mathcal{O}_j^\vee$, and moreover such that we have

$$p^{b_j} \mathcal{O}_j \supseteq \mathcal{O}_j^0, \tag{3.1.1}$$

with $b_j = \left[\frac{1}{2} a_j\right] =$ (the greatest integer $< \frac{1}{2} a_j$). To see that such a choice for $\mathcal{O}_j^0$ is possible, write in some symplectic basis

$$\mathcal{O}_j = p^{-e_1}\mathbb{Z}_p \oplus \cdots \oplus p^{-e_d}\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p, \quad 0 \le e_1 \le \cdots \le e_d,$$

$$\mathcal{O}_j^\vee = \mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p \oplus p^{e_1}\mathbb{Z}_p \oplus \ldots \oplus p^{e_d}\mathbb{Z}_p, \quad \text{so } e_1 = a_j, \text{ and take}$$

$$\mathcal{O}_j^0 = p^{-c_1}\mathbb{Z}_p \oplus \cdots \oplus p^{-c_d}\mathbb{Z}_p \oplus p^{c_1} \oplus \cdots \oplus p^{c_d}\mathbb{Z}_p, \quad \text{with } c_i = \left[\tfrac{1}{2} e_i\right].$$

We denote by Φ_j the characteristic function of $\mathcal{O}_j^0$, which is by (0.4.15), (0.4.17) the Wigner transform of a unit vector $\phi_j \in L_2(V_0)$, $\Phi_j = W\varphi_j$. For a function f in V, we denote, as usual, by f_j the restriction of f to $\mathfrak{A}_j$, and we let $\tilde{f}(x) = \Phi_j * f_j(x)$ for $x \in \mathfrak{A}_j$, $\tilde{f}$ is the average of f over the $x + \mathcal{O}_j^0$'s. We have the following :

LEMMA (3.1.2). — *For $f \in S(m,g)$, we have $f - \tilde{f} \in S(mh^{\gamma},g)$, for any γ.*

Proof. — It is enough to prove that for any $\alpha, \gamma > 0$, we have for all j

$$h_j^{-\gamma} \|f_j - \tilde{f}_j\|_{B^{\alpha}_{\infty,\infty}(\mathfrak{A}_j)} \leq 2 \cdot \|f_j\|_{B^{\alpha+2\gamma}_{\infty,\infty}(\mathfrak{A}_j)}. \tag{3.1.3}$$

With the notations of (1.2.3), using (3.1.1), we have

$$\phi_{g_j,k} * f = \phi_{g_j,k} * \tilde{f}, \quad \text{and hence} \quad \varphi_{g_j,k} * f = \varphi_{g_j,k} * \tilde{f}, \tag{3.1.4}$$

for $k = 0, 1, \ldots, b_j$. From (3.1.4) we obtain

$$\begin{aligned}
h_j^{\gamma} &\|f_j - \tilde{f}_j\|_{B^{\alpha}_{\infty,\infty}(\mathfrak{A}_j)} \\
&= p^{\gamma \cdot a_j} \sup_{k \geq 0} p^{\alpha \cdot k} \sup_{x \in \mathfrak{A}_j} \left|\varphi_{g_j,k} * (f_i - \tilde{f}_j)(x)\right| \\
&= p^{\gamma \cdot a_j} \sup_{k > b_j} p^{\alpha \cdot k} \sup_{x \in \mathfrak{A}_j} \left|\varphi_{g_j,k} * (f_j - \tilde{f}_j)(x)\right| \\
&\leq p^{\gamma \cdot a_j} \sup_{k > b_j} p^{\alpha \cdot k} \big[\sup_{x \in \mathfrak{A}_j} |\varphi_{g_j,k} * f_j(x)| + \sup_{x \in \mathfrak{A}_j} |\varphi_{g_j,k} * \Phi_j * f_j(x)|\big] \\
&\leq 2 \cdot p^{\gamma \cdot a_j} \sup_{k > b_j} p^{\alpha \cdot k} \sup_{x \in \mathfrak{A}_j} |\varphi_{g_j,k} * f_j(x)| \\
&\leq 2 \cdot \sup_{k > b_j} p^{(\alpha+2\gamma)k} \sup_{x \in \mathfrak{A}_j} \left|\varphi_{g_j,k} * f_j(x)\right| \\
&\leq 2 \cdot \sup_{k > 0} p^{(\alpha+2\gamma)k} \sup_{x \in \mathfrak{A}_j} \left|\varphi_{g_j,k} * f_j(x)\right| = 2 \cdot \|f_j\|_{B^{\alpha+2\gamma}_{\infty,\infty}(\mathfrak{A}_j)}.
\end{aligned}$$

□

The operator $\hat{\rho}(\tilde{f})$ corresponding to $\tilde{f}$ is expressed by (0.4.19) as

$$\hat{\rho}(\tilde{f}) = \sum_j \hat{\rho}(f_j * \Phi_j) = \sum_j \hat{\rho}(f_j * W\phi_j) = \sum_j C^*_{\phi_j} M_{f_j} C_{\phi_j}. \tag{3.1.5}$$

In particular, if $f \geq 0$ is positive, $\hat{\rho}(\tilde{f})$ is positive in the sense that

$$(\hat{\rho}(\tilde{f})\varphi, \varphi)_{L_2(V_0)} \geq 0 \quad \text{for all } \varphi \in \mathcal{S}(V_0). \tag{3.1.6}$$

Thus we have obtained the following

COROLLARY (3.1.7). — *If $f \in S(m,g)$ is positive, $f \geq 0$, then $f = \tilde{f} + f_0$, where the operator $\hat{\rho}(\tilde{f})$, is positive, and $f_0 \in S(mh^{\gamma},g)$ for all γ.*

COROLLARY (3.1.8). — *If $f \in S(m,g)$ is positive, then for any γ*

$$\big(\hat\rho(f)\varphi,\varphi\big)_{L_2(V_0)} \geq -C\cdot \|\varphi\|^2_{L_2^{(m^{1/2}\cdot h^\gamma)}(V_0)}$$

for all $\varphi \in \mathcal{S}(V_0)$.

Proof. — Writing $f = \tilde f + f_0$ as in (3.1.7), we have

$$\begin{aligned}\big(\hat\rho(f),\varphi,\varphi\big)_{L_2(V_0)} &\\ &\geq \big(\hat\rho(f_0)\varphi,\varphi\big)_{L_2(V_0)} := \Big(\hat\rho(m^{-1/2}h^{-\gamma})\hat\rho(f_0)\varphi,\hat\rho(m^{1/2}h^\gamma)\varphi\Big)_{L_2(V_0)}\\ &\geq -\big\|\hat\rho(m^{-1/2}h^{-\gamma})\hat\rho(f_0)\varphi\big\|_{L_2(V_0)}\cdot\big\|\hat\rho(m^{1/2}h^\gamma)\varphi\big\|_{L_2(V_0)}\\ &\geq -C\cdot\|\varphi\|^2_{L_2^{(m^{1/2}h^\gamma)}(V_0)}\end{aligned}$$

where C denotes the norm of $\hat\rho(f_0) : L_2^{(m^{1/2}h^\gamma)}(V_0) \to L_2^{(m^{-1/2}h^{-\gamma})}$, which is bounded by (2.3.15). □

As a special case of (3.1.8) we note that we have

COROLLARY (3.1.9). — *If $f \in S(h^{-\gamma},g)$ is positive, then, for all $\varphi \in \mathcal{S}(V_0)$,*

$$\big(\hat\rho(f)\varphi,\varphi\big)_{L_2(V_0)} \geq -C\cdot\|\varphi\|^2_{L_2(V_0)}.$$

We say that $f \in S(m,g)$ is *strongly-elliptic* if for some $A>0$, we have

(3.1.10) $\qquad \operatorname{Re} f(x) \geq A\cdot m(x)$ for x outside a compact subset of V.

SHARP GÅRDING INEQUALITY (3.1.11). — *For a strongly-elliptic f in $S(m,g)$, we have for any γ :*

$$\operatorname{Re}\big(\hat\rho(f)\varphi,\varphi\big)_{L_2(V_0)} \geq A\cdot\|\varphi\|_{L_2^{(m^{1/2})}(V_0)} - C\cdot\|\varphi\|_{L_2^{(m^{1/2}h^\gamma)}(V_0)}.$$

Proof. — Set $f_1 = \operatorname{Re} f - A\cdot m + \Phi$, where $\Phi \in C_{\infty,c}(V)$ is large enough to make $f_1 \geq 0$. Applying (3.1.8) to f_1, and noting that by (0.1.13) $\operatorname{Re}(\hat\rho(f)\varphi,\varphi)_{L_2(V_0)} = (\hat\rho(\operatorname{Re} f)\varphi,\varphi)_{L_2(V_0)}$, we get the desired result. □

We can write $\mathfrak{A}_j = \coprod_i \mathfrak{A}_{ji}$, with $\mathfrak{A}_{ji} = x_{ji} + \mathcal{O}_j^0, i = 1,\ldots,[\mathcal{O}_j,\mathcal{O}_j^0]$, so that the decomposition $V = \coprod_{ji}\mathfrak{A}_{ji}$ corresponds to a self-dual metric covering $\tilde g_x$ with $g_x \leq \tilde g_x \leq g_x^\vee$. For any $f \in S(m,g)$, denoting by $f_{ji} = \tilde f(x_{ji})$ the average of f over $\mathfrak{A}_{ji}$, the operator $\hat\rho(\tilde f)$ is diagonalizable with respect to the orthonormal basis $\varphi_{ji} \in L_2(V_0)$, $W\varphi_{ji}$ = characteristic function of $\mathfrak{A}_{ji}$, cf. (0.4.20) : $\hat\rho(\tilde f)\varphi_{ji} = f_{ji}\cdot\varphi_{ji}$. Thus we have « almost diagonalized » $\hat\rho(f)$.

3.2. Fredholm operators.

By the Stone-von Neumann theorem, $\hat{\rho} : L_2(V) \xrightarrow{\sim} \mathcal{K}_2(L_2(V_0))$ is an isometry onto the ideal of Hilbert-Schmidt operators. Since an operator is trace-class if and only if it is a product of two Hilbert-Schmidt operators, we conclude that

$$(3.2.1) \qquad \hat{\rho}(f) \text{ is trace-class, if and only if } f \in L_2(V) \# L_2(V),$$

and note that $L_2(V) \# L_2(V) \subseteq L_2(V) \cap C_0(V)$.

Given an invertible trace-class operator $\mathcal{K}$, if $\mathcal{K}^{-1}\hat{\rho}(f)$ is bounded then $\hat{\rho}(f)$ is trace-class, so we can use the criteria for boundedness of paragraph 2.3 to get criteria for $\hat{\rho}(f)$ to be trace-class. In particular, using (1.1.23), (1.1.24) we get :

LEMMA (3.2.2). — *The operator $\hat{\rho}(f)$ will be of trace-class provided either $\hat{\rho}(f \# g(1,\xi,x)^{\alpha})$ is a bounded operator for some metric g in V, and some $\alpha > 2d = \dim V$, or if $\hat{\rho}(f \# g_0(1,\xi)^{\alpha_1} \cdot g_0(1,x)^{\alpha_2})$ is a bounded operator for some metric g_0 in V_0, and some $\alpha_1, \alpha_2 > d$.*

For example, using the first criteria of the lemma, we see that the Tœplitz symbols Σ^{α} correspond to trace-class operators if $\alpha < -2d$.

A symbol $f \in S(m,g)$ will be called *elliptic* if

$$(3.2.3) \qquad \left|f(x)\right| \geq A \cdot m(x) \text{ for } x \text{ outside a compact set } K \text{ of } V.$$

For such an f we define

$$(3.2.4) \qquad f_K^{-1}(x) = \begin{cases} f(x)^{-1} & \text{if } x \in V \setminus K, \\ 0 & \text{if } x \in K, \end{cases}$$

where we can take K to be a finite union of the sets $\mathfrak{A}_j$ corresponding to the metric-covering g. Since we have for $x \in V \setminus K$, and for y such that

$$\begin{aligned} g_x(y) &\leq 1, \\ \left|f_K^{-1}(x)\right| &\leq A^{-1} \cdot m(x)^{-1}, \\ g_x(y)^{-\alpha}\left|f_K^{-1}(x) - f_K^{-1}(x+y)\right| &= \frac{g_x(y)^{-\alpha}|f(x+y) - f(x)|}{|f(x)f(x+y)|} \\ &\leq A^{-2} \cdot C_{\alpha} \cdot m(x)^{-1}, \end{aligned}$$

we see that $f_K^{-1} \in S(m^{-1}, g)$. By the main theorem (2.2.8) we get,

$$\mathbb{1} - f \# f_K^{-1}, \quad \mathbb{1} - f_K^{-1} \# f \in S(h^\gamma, g) \quad \text{for all } \gamma. \tag{3.2.5}$$

If h vanishes at infinity we conclude from (2.3.9), (2.3.15), that $\hat\rho(f)$ defines a Fredholm operator : $L_2^{(m)}(V_0) \to L_2(V_0)$. Assume now that g is *global*, in the sense that we have

$$h(x) \leq C \cdot g_0(1, x)^{-\delta} \quad \text{for some } \delta > 0. \tag{3.2.6}$$

We first remark that from (3.2.6) we obtain for any temperate weight m,

$$\begin{cases} m(x) \leq C_1 \cdot m(0) \cdot g_x^\vee(1,x)^{\beta_1} & \text{by (1.4.8)} \\ \quad \leq C_2 \cdot g_0^\vee(1,x)^{\beta_2} & \text{by (1.4.3)} \\ \quad \leq C_3 \cdot h(x)^{-\beta_3} & \text{by (3.2.6)} \end{cases} \tag{3.2.7}$$

so that $\bigcap_{\gamma>0} S(mh^\gamma, g) = \bigcap_{\gamma>0} S(h^\gamma, g) = \mathcal{S}(V)$ by (1.4.11).

To calculate the index of $\hat\rho(f)$, $f \in S(m,g)$ elliptic, we may assume that $m = \mathbb{1}$ by the logarithmic law for the index. So let $f \in S(\mathbb{1}, g)$, $|f(x)| \geq A$ for $x \notin K$, and assume that (3.2.6) holds, so that

$$\mathbb{1} - f \# f_K^{-1}, \quad \mathbb{1} - f_K^{-1} \# f \in \mathcal{S}(V),$$

and they correspond by (3.2.2) to trace-class operators. Hence we have,

$$\begin{cases} \operatorname{Index}(\hat\rho(f)) \\ \quad = \operatorname{tr}\big(\mathbb{1} - \hat\rho(f_K^{-1} \# f)\big) - \operatorname{tr}\big(\mathbb{1} - \hat\rho(f \# f_K^{-1})\big) \\ \quad = \operatorname{tr}\big(\hat\rho(f \# f_K^{-1}) - \hat\rho(f_K^{-1} \# f)\big) \\ \quad = \displaystyle\int_V \mathrm{d}x \big(f \# f_K^{-1}(x) - f_K^{-1} \# f(x)\big). \end{cases} \tag{3.2.8}$$

Writing $f = \tilde f + f_0$, $f_K^{-1} = \tilde f_K^{-1} + f_1$ as in Lemma (3.1.2), with $f_0, f_1 \in \mathcal{S}(V)$ by (3.2.7), so that $\hat\rho(f_i)$ are trace-class, we obtain from (3.2.8)

$$\operatorname{Index}\big(\hat\rho(f)\big) = \int_V \mathrm{d}x \big(\tilde f \# \tilde f_K^{-1}(x) - \tilde f_K^{-1} \# \tilde f(x)\big).$$

But since $\tilde f$ and $\tilde f_K^{-1}$ are $\tilde g$-locally-constant, we have $\tilde f \# \tilde f_K^{-1} = \tilde f_K^{-1} \# \tilde f = f \cdot f_K^{-1}$, and we have proved the following

THEOREM (3.2.9). — *For global g, every elliptic $f \in S(m,g)$ defines a Fredholm operator of index zero*

$$\hat\rho(f) : L_2^{(m)}(V_0) \to L_2(V_0).$$

3.3. Zeta functions and spectral asymptotics.

In this section we fix a global metric-covering g, temperate and certain, so that we have from (3.2.7) : $\bigcap_{\gamma>0} S(mh^{\gamma}, g) = \mathcal{S}(V)$, for any temperate weight m. Let $f \in S(m,g)$ be a real valued elliptic symbol. By (0.1.13), $\hat{\rho}(f)$ is formally self-adjoint. From (2.3.15) we know that $\hat{\rho}(f)$ is bounded as an operator from $L_2^{(m)}(V_0)$ to $L_2(V_0)$. Let us denote by $\mathcal{A}$ the unbounded operator on $L_2(V_0)$ defined by $\hat{\rho}(f)$, with domain $L_2^{(m)}(V_0)$.

Note that we can write $f_K^{-1} \# f = \mathbb{1} - R_0$, with $R_0 \in \mathcal{S}(V)$ as in (3.2.5), hence if $\varphi \in L_2(V_0)$ is any element such that $\hat{\rho}(f)\varphi \in L_2(V_0)$, then

$$\hat{\rho}(m)\varphi = \hat{\rho}(m \# f_K^{-1})\,\hat{\rho}(f)\,\varphi + \hat{\rho}(m \# R_0)\,\varphi,$$

showing that $\hat{\rho}(m)\varphi \in L_2(V_0)$, and so $\varphi \in L_2^{(m)}(V_0)$. From the continuity of $\hat{\rho}(f)$ in $\mathcal{S}'(V_0)$, we deduce that the operator $\mathcal{A}$ is closed. We claim that $\mathcal{A}$ is self-adjoint, and to prove it we shall show that $\mathcal{A}$ equals the closure of its restriction to $\mathcal{S}(V_0)$. Let Φ_k denote the partial sums of the characteristic functions of the sets $\mathfrak{A}_j$ corresponding to the metric-covering g. The Φ_k are a bounded sequence in $S(\mathbb{1}, g)$ covering to $\mathbb{1}$ in $C_\infty(V)$, and by (2.3.11) we have for any $\varphi \in L_2(V_0)$, $\hat{\rho}(\Phi_k)\varphi \to \varphi$ in $L_2(V_0)$; moreover, since $\Phi_k \in \mathcal{S}(V)$ the operators $\hat{\rho}(\Phi_k)$ are smoothening, and so $\hat{\rho}(\Phi_k)\varphi \in \mathcal{S}(V_0)$. For $\varphi \in L_2^{(m)}(V_0)$ we have

$$\mathcal{A}\hat{\rho}(\Phi_k)\varphi = \hat{\rho}(f \# \Phi_k \# f_K^{-1})\hat{\rho}(f)\,\varphi + \hat{\rho}(f \# \Phi_k \# R_0)\,\varphi.$$

The $f \# \Phi_k \# f_K^{-1}$, and $f \# \Phi_k \# R_0$, are bounded in $S(\mathbb{1}, g)$, and converge to $f \# f_K^{-1}$, and to $f \# R_0$, in $C_\infty(V)$, respectively, hence by (2.3.11) we have

$$\mathcal{A}\hat{\rho}(\Phi_k)\varphi \longrightarrow \hat{\rho}(f \# f_K^{-1})\hat{\rho}(f)\varphi + \hat{\rho}(f \# R_0)\varphi = \mathcal{A}\varphi$$

in $L_2(V_0)$. Thus $\mathcal{A}$ is indeed self-adjoint. We note that if $f \geq 0$ is positive, so that by Gårding inequality (3.1.11) $\mathcal{A}$ is bounded from below, and hence for some real number λ_0, $(\mathcal{A} - \lambda_0)$ is invertible. Its inverse $(\mathcal{A} - \lambda_0)^{-1}$ is a self-adjoint injective operator with image $L_2^{(m)}(V_0)$. Assume now that $m^{-1} \in C_0(V)$ vanishes at infinity. From Theorem (2.3.9) we see that $(\mathcal{A} - \lambda_0)^{-1}$ is a compact operator. From the spectral theorem for compact, self-adjoint, injective operators, we deduce that there is an orthonormal basis $\{\phi_j\}$ for $L_2(V_0)$ consisting of eigenfunctions for $(\mathcal{A} - \lambda_0)^{-1}$, with real non-zero eigenvalues μ_j converging to 0. Hence the ϕ_j's are eigenfunctions of $\mathcal{A}$ with the real eigenvalues $\lambda_j = \lambda_0 + \mu_j^{-1}$, converging to $+\infty$. Moreover, writing

$$(f - \lambda_j)_{K_j}^{-1} \# (f - \lambda_j) = \mathbb{1} - R_{\lambda_j},$$

with $R_{\lambda_j} \in \mathcal{S}(V)$, we have $\phi_j = \rho(R_{\lambda_j})\phi_j$ is in $\mathcal{S}(V_0)$. We summarize the above discussion by the following

THEOREM (3.3.1). — *For a global certain temperate metric-covering g, and a temperate weight m, every real-valued elliptic $f \in S(m,g)$ defines an essentially self-adjoint operator $\hat{\rho}(f)$. Denoting by $\mathcal{A}$ the unique self-adjoint extension of $\hat{\rho}(f)$,* Domain$(\mathcal{A}) = L_2^{(m)}(V_0)$, *and if $f \geq 0$ is positive, $\mathcal{A}$ is bounded from below. If moreover, $m^{-1} \in C_0(V)$ vanishes at infinity, then $\mathcal{A}$ has a discrete point spectrum, and there is an orthonormal basis for $L_2(V_0)$ consisting of eigenfunctions $\phi_j \in \mathcal{S}(V_0), \mathcal{A}\phi_j = \lambda_j \cdot \phi_j$, with eigenvalues λ_j converging to $+\infty$.*

To study the asymptotic behaviour of the eigenvalues λ_j, we may assume that $f(x) > c \cdot m(x) > \varepsilon > 0$, and moreover that $\min \lambda_j > \varepsilon$, since we can always achieve this by adding to f a large constant. For $\lambda \in \mathbb{C}\backslash[\varepsilon, \infty]$ we can write

$$\begin{cases} (f - \lambda)^{-1} \# (f - \lambda) = \mathbb{1} - R_\lambda \\ (f - \lambda) \# (f - \lambda)^{-1} = \mathbb{1} - R'_\lambda \end{cases} \tag{3.3.2}$$

with $R_\lambda, R'_\lambda \in \mathcal{S}(V)$. From (3.3.2) we obtain

$$\begin{aligned} (\mathcal{A} - \lambda)^{-1} &- \hat{\rho}((f - \lambda)^{-1}) \\ &= \hat{\rho}\big(R_\lambda \# (f - \lambda)^{-1}\big) + \hat{\rho}(R_\lambda)(\mathcal{A} - \lambda)^{-1}\hat{\rho}(R'_\lambda). \end{aligned} \tag{3.3.3}$$

For $\lambda \notin \{\lambda_j\}$ the operator $(\mathcal{A} - \lambda)^{-1}$ is bounded, so that the right hand side of (3.3.3) is a smoothening operator and can be written as $\hat{\rho}(r_{\{\lambda\}})$ for some $r_{\{\lambda\}} \in \mathcal{S}(V)$; thus we conclude that for $\lambda \notin \{\lambda_j\}$,

$$(\mathcal{A} - \lambda)^{-1} = \hat{\rho}(f_{\{\lambda\}}) \tag{3.3.4}$$

for some $f_{\{\lambda\}} \in S(m^{-1}, g)$ with $f_{\{\lambda\}} - (f - \lambda)^{-1} = r_{\{\lambda\}} \in \mathcal{S}(V)$.

Note that for $\mathrm{Re}(\lambda) \leq \varepsilon_1 < \varepsilon$ we have the inequalities

$$\begin{cases} \sup\limits_{x \in \mathfrak{A}_j} \big|(f - \lambda)^{-1}(x)\big| \leq |c \cdot m_j - \lambda|^{-1} \leq |\varepsilon - \lambda|^{-1}, \\ \sup\limits_{x_1, x_2 \in \mathfrak{A}_j} \quad g_j(x_1 - x_2)^{-\alpha}\big|(f - \lambda)^{-1}(x_1) - (f - \lambda)^{-1}(x_2)\big| \\ \quad = \sup\limits_{x_1, x_2 \in \mathfrak{A}_j} \dfrac{g_j(x_1 - x_2)^{-\alpha}|f(x_1) - f(x_2)|}{|f(x_1) - \lambda| \cdot |f(x_2) - \lambda|} \\ \quad \leq c_\alpha \cdot |c \cdot m_j - \lambda|^{-2} \leq c_\alpha |\varepsilon - \lambda|^{-2}. \end{cases} \tag{3.3.5}$$

Hence in particular,

$$\|(f-\lambda)^{-1}\|_{B^{\alpha}_{\infty,\infty}(1,g)} = \mathrm{O}(|\lambda|^{-1}) \quad \text{for } |\lambda| \to \infty,\ \mathrm{Re}(\lambda) \le \varepsilon_1. \tag{3.3.6}$$

From (3.3.5) and the error estimates of the main theorem (2.1.11), (2.2.4), (2.2.7), a straightforward calculation gives for

$$R_\lambda = (f-\lambda)^{-1}\cdot f - (f-\lambda)^{-1}\,\#\, f, \quad \mathrm{Re}(\lambda) \le \varepsilon_1 < \varepsilon,$$

the inequalities

$$\begin{cases} \|R_\lambda\|_{B^{\alpha}_{\infty,\infty}(h^\gamma,g)} \le C_{\alpha,\gamma}, & \text{and similarly} \\ \|R'_\lambda\|_{B^{\alpha}_{\infty,\infty}(h^\gamma,g)} \le C_{\alpha,\gamma}. \end{cases} \tag{3.3.7}$$

From (3.3.6) and (3.3.7), combined with (2.3.8), and with (2.3.15), we have as $|\lambda| \to \infty$, $\mathrm{Re}(\lambda) \le \varepsilon_1$,

$$\begin{cases} \|\hat{\rho}((f-\lambda)^{-1})\|_{\mathrm{op}} = \mathrm{O}(|\lambda|^{-1}), \\ \|\hat{\rho}(R_\lambda)\|_{\mathrm{op}(h^\gamma)} = \mathrm{O}(1), \\ \|\hat{\rho}(R'_\lambda)\|_{\mathrm{op}(h^\gamma)} = \mathrm{O}(1), \end{cases} \tag{3.3.8}$$

where $\|R\|_{\mathrm{op}(h^\gamma)}$ denotes the norm of $R : L_2^{(h^\gamma)}(V_0) \to L_2^{(h^{-\gamma})}(V_0)$. From (3.3.8) we have for the right hand side of (3.3.3) the estimate

$$\|\hat{\rho}(r_{\{\lambda\}})\|_{\mathrm{op}(h^\gamma)} = \|(\mathcal{A}-\lambda)^{-1} - \hat{\rho}((f-\lambda)^{-1})\|_{\mathrm{op}(h^\gamma)} = \mathrm{O}(|\lambda|^{-1}) \tag{3.3.9}$$

as $|\lambda| \to \infty$, $\mathrm{Re}(\lambda) \le \varepsilon_1$.

For $\mathrm{Re}(s) < 0$ we have

$$\mathcal{A}^s = \frac{1}{2\pi i}\int_{\varepsilon_1 - i\infty}^{\varepsilon_1 + i\infty} \lambda^s (\mathcal{A}-\lambda)^{-1}\,\mathrm{d}\lambda \tag{3.3.10}$$

which converges in the operator norm since $\|\mathcal{A} - (\varepsilon_1 + it)\|_{\mathrm{op}} = \mathrm{O}(|t|^{-1})$. From (3.3.5) we see that for $\mathrm{Re}(s) < 0$ we have

$$f^s = \frac{1}{2\pi i}\int_{\varepsilon_1 - i\infty}^{\varepsilon_1 + i\infty} \lambda^s (f-\lambda)^{-1}\,\mathrm{d}\lambda \in S(m^{\mathrm{Re}(s)}, g). \tag{3.3.11}$$

From (3.3.9) we see that for $\mathrm{Re}(s) < 0$

$$r^{\{s\}} = \frac{1}{2\pi i}\int_{\varepsilon_1 - i\infty}^{\varepsilon_1 + i\infty} \lambda^s r_{\{\lambda\}}\, \mathrm{d}\lambda \in \mathcal{S}(V) \tag{3.3.12}$$

since the norms $\| \ \|_{\mathrm{op}(h^\lambda)}$, $\lambda > 0$, define the space $\mathcal{S}(V)$ by the remarks after (1.3.3).

Combining (3.3.10), (3.3.11), (3.3.12) with (3.3.4) we conclude that for $\mathrm{Re}(s) < 0$

$$\begin{cases} \mathcal{A}^s = \hat{\rho}(f^{\# s}) \quad \text{with } f^{\# s} = f^s + r^{\{s\}} \in S\big(m^{\mathrm{Re}(s)}, g\big), \\ r^{\{s\}} \in \mathcal{S}(V). \end{cases} \tag{3.3.13}$$

Since for s an integer (3.3.13) holds with $f^{\# s} = f \# \cdots \# f$, s times, we see that (3.3.13) actually holds for all $s \in \mathbb{C}$. Thus we have proved the following

THEOREM (3.3.14). — *For a global certain temperate metric covering g, a temperate weight m with $m^{-1} \in C_0(V)$, and a positive elliptic symbol $f \in S(m,g)$, denoting by $\mathcal{A}$ the unique self-adjoint extension of $\hat{\rho}(f)$, and assuming that $\mathcal{A}$ is positive, $\mathcal{A}$ generates a holomorphic group of operators $\mathcal{A}^s$ (bounded only for* $\mathrm{Re}(s) \leq 0$) *and given by $\mathcal{A}^s = \hat{\rho}(f^{\# s})$, $f^{\# s} = f^s + r^{\{s\}} \in S(m^{\mathrm{Re}(s)}, g)$, $r^{\{s\}} \in \mathcal{S}(V)$.*

Note in particular that $s \mapsto r^{\{s\}}$ is an entire holomorphic function of s with values in $\mathcal{S}(V)$; for $\mathrm{Re}(s) \leq 0$ this follows from (3.3.9) (3.3.12), and we have a «step by step» holomorphic continuation by means of the recursion formula

$$r^{\{s+1\}} = f \# r^{\{s\}} + (f \# f^s - f \cdot f^s).$$

Assume now that we have

$$m^{-1}(x) \leq C \cdot g_0(1,x)^{-\alpha_0} \quad \text{for some } \alpha_0 > 0. \tag{3.3.15}$$

It follows from Lemma (3.2.2), and from Theorem (2.3.9), that the symbols in $S(m^{-s}, g)$ correspond to trace-class operators for $s > 2d/\alpha_0$. Thus we can define the *zeta function*,

$$\zeta_{\mathcal{A}}(s) = \mathrm{tr}(\mathcal{A}^{-s}) = \sum_j \lambda_j^{-s} = \int_V \mathrm{d}x\, f^{\#(-s)}(x), \tag{3.3.16}$$

which is a holomorphic function of s in the right half plane $\mathrm{Re}(s) > 2d/\alpha_0$. Similarly we can define

$$\zeta_f(s) = \int_V \mathrm{d}x\, f(x)^{-s} \tag{3.3.17}$$

which is holomorphic in the same right half plane $\mathrm{Re}(s) > 2d/\alpha_0$. Setting

$$N_{\mathcal{A}}(\lambda) = \#\{\lambda_j \le \lambda\},$$

$$N_f(\lambda) = \mathrm{vol}\{x \mid f(x) \le \lambda\},$$

we have the Mellin transform expressions for our zetas

$$\zeta_{\mathcal{A}}(s) = \int \lambda^{-s}\, \mathrm{d}N_{\mathcal{A}}(\lambda), \tag{3.3.18}$$

$$\zeta_f(s) = \int \lambda^{-s}\, \mathrm{d}N_f(\lambda). \tag{3.3.19}$$

Their difference is given via (3.3.14) by

$$\zeta_{\mathcal{A}}(s) - \zeta_f(s) = \int \lambda^{-s}\, \mathrm{d}\big(N_{\mathcal{A}}(\lambda) - N_f(\lambda)\big) = \int_V \mathrm{d}x\, r^{\{-s\}}(x), \tag{3.3.20}$$

and is an entire function of s. Applying the inverse Mellin transform we obtain the following

THEOREM (3.3.21). — *For a global certain temperate metric-covering g, a temperate weight m such that*

$$m(x) \ge C \cdot g_0(1,x)^{\alpha_0}, \qquad \alpha_0 > 0,$$

and a positive elliptic $f \in S(m,g)$, denoting by λ_j the eigenvalues of the self-adjoint extension $\mathcal{A}$ of $\hat{\rho}(f)$, we have for every $\varepsilon > 0$

$$N_{\mathcal{A}}(\lambda) = \#\{\lambda_j \le \lambda\} = \mathrm{O}\big(\lambda^{2d/\alpha_0+\varepsilon}\big), \quad \text{as } \lambda \to +\infty.$$

Moreover, we have for every $\varepsilon > 0$

$$N_{\mathcal{A}}(\lambda) = \mathrm{vol}\{x \mid f(x) \le \lambda\} + \mathrm{O}(\lambda^{\varepsilon}), \quad \text{as } \lambda \to +\infty.$$

This theorem should be compared with the remark at the end of paragraph 3.1. Using the notation of the same paragraph, $\tilde{f}$ is $\tilde{g}$-locally constant and the operator $\widetilde{\mathcal{A}} = \tilde{\rho}(\tilde{f})$ is explicitly diagonalizable; denoting by $\tilde{\lambda}_j$ its eigenvalues, we have $\{\tilde{\lambda}_j\} = \tilde{f}(V)$, so that for any positive real λ we have an equality $\#\{\tilde{\lambda}_j\,;\ |\tilde{\lambda}_j| \leq \lambda\} = \mathrm{vol}\{x\,;\ |\tilde{f}(x)| \leq \lambda\}$ if finite, and more precisely we have an equality of measures

$$\sum_j \delta_{\tilde{\lambda}_j} = \tilde{f}_*(\mathrm{d}x). \tag{3.3.22}$$

Here there are no ellipticity or positivity assumptions on f. If however the assumptions of Theorem (3.3.21) are satisfied, then $\tilde{f}$ is also positive elliptic, and so for any $\varepsilon > 0$, combining (3.3.21) with (3.1.2) we have as $\lambda \to +\infty$,

$$\begin{cases} N_{\mathcal{A}}(\lambda) = \mathrm{vol}\big\{x \mid f(x) \leq \lambda\big\} + \mathrm{O}(\lambda^{\varepsilon}) \\ \quad = \mathrm{vol}\big\{x \mid \tilde{f}(x) \leq \lambda\big\} + \mathrm{O}(\lambda^{\varepsilon}) \\ \quad = N_{\widetilde{\mathcal{A}}}(\lambda) + \mathrm{O}(\lambda^{\varepsilon}). \end{cases} \tag{3.3.23}$$

For particular examples of Theorem (3.3.21) one can take the Tœplitz symbols defined in (1.4.12).

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