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FURTHER REMARKS ON THE WINDING NUMBER

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In our previous paper [2], we defined the winding number for regular closed curves on two manifolds, and gave an algebraic method for finding the winding number of a regular simple closed curve when its homotopy class is known. In this paper, we extend the definition to piecewise regular curves, and thereby give a geometric method of computation which also casts some light on the meaning of the winding number. We also elucidate the algebraic technique somewhat, and thereby find an alternate algebraic method of computation. The definitions, notations, and numbering of propositions of the previous paper will be retained.

1. A geometric interpretation of the winding number.

The winding number is defined axiomatically, and then the existence is proved by means of the following integral formula :

$$w(\mathrm{C}) \equiv \frac{1}{2\pi} \int_{\mathrm{C}} d(\dot{\mathrm{C}} - \mathrm{F}) \quad \mathrm{mod}\ \chi$$

where C is a regular closed curve. Let us call a curve C piecewise regular if C is a continuous mapping of [0, 1] into M such that there is a finite sequence of points $t_0 = 0, t_1, \ldots, t_k = 1$ such that C is differentiable and has nonzero tangent vector on each interval $[t_i, t_{i+1}]$, and satisfies one further condition to be given later. Then the tangent vector is uniquely defined

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except possibly at the points t_i, where there are a left hand and a right hand tangent. Let θ_i be the angle measured from the left hand tangent at t_i to the right hand tangent at this point, with sign defined by the orientation of M. The additional condition required is that $|\theta_i| < \pi$. Then we define

$$w(C) \equiv \frac{1}{2\pi} \sum_{i=0}^{k-1} \left\{ \int_{t_i}^{t_{i+1}} d(\dot{C} - F) + \theta_{i+1} \right\} \quad \text{mod } \chi.$$

Recall that for M a surface of genus g, we have supposed $\pi_1(M)$ written in terms of generators A_i, $i = 1, \ldots, 2g$, subject to the relation

$$\prod_{i=1}^{g} A_{2i-1}A_{2i}A_{2i-1}^{-1}A_{2i}^{-1} = 1.$$

Suppose further that Γ_i are pairwise disjoint regular Jordan curves such that Γ_i is homotopic to $A_{2i-1}A_{2i}A_{2i-1}^{-1}A_{2i}^{-1}$, that is, the Γ_i are boundaries of the handles.

Proposition 3. — *If* C *is a piecewise regular Jordan curve, not homologous to zero, which does not cut any* Γ_i, *then* $w(C) = 0$.

This generalizes a theorem proved earlier for regular curves on the torus [1].

Proof. — Since the singularity of the vector field F may be moved about at will without changing the winding number, we may suppose that some Γ_i separates C from the singularity. Cut the surface along each Γ_i, and consider the handle H which contains C. Embed H in the torus, and extend F to a nonsingular field F_i on the torus. Since

$$w(A_{2i-1}) = w(A_{2i}) = 0,$$

this field may be used to define the winding number on the torus. Hence, $w(C) = 0$, by a slight modification of the proof given in [1].

Now let C be any regular Jordan curve on the surface. By suitably deformong the Γ_i, we may assume that at each point where C meets Γ_i they actually cross. Since both C and Γ_i are Jordan curves, each crossing point corresponds to exactly one parameter value on each curve. The minimum number of points of intersection of C with Γ_i under these 'conditions is well defined, and may be found by the methods of [4]. More precisely, if we know the homotopy class of C in terms of the generators A_i, we can find a cyclically ordered set of points $P_1, \ldots, P_{2r}$ on C such that:

1. The cyclic ordering is consistent with the parametrization of C.

2. $\{P_j\}$ is a set of all intersections of C with the union of the Γ_i, minimal under the assumptions.

3. For each Γ_i, it is known which P_j lie on Γ_i and what their cyclic order on Γ_i is. Notice that each time C enters and leaves a given handle, the corresponding points of intersection are adjacent in the cyclic ordering on C. Hence, we may construct a family of piecewise regular curves as follows:

A. Begin at a point where C enters the i-th handle, follow C until it leaves, then turn left along Γ_i and return to the initial point. Thus we construct curves $C_1, \ldots, C_r$.

B. All other points of C lie on no handle. Begin at any one, follow C until it meets Γ_i, turn left and follow Γ_i until you reach the corresponding exit point, turn left again, etc. Thus we construct a single curve C_* which contains all the P_j plus all those points of C not on any C_i. (In fact, we get C_* from C by taking a short cut each time we come to a handle boundary.) Notice that C_* lies on the sphere with g holes, hence may be considered as a plane curve.

Proposition 4. — *Let $w_p(C_*)$ be the winding number of C_* considered as a plane curve, and let $\sum_{i=1}^{g-1} c_i^* \Gamma_i$ be the homology class of C_* on the sphere with g holes. Then*

$$w(C) = -r + w_p(C_*) - 2\sum_{i=1}^{g-1} c_i^* \mod \chi$$

This proposition is a modification of one previously conjectured by us [3].

Proof. — The sum $w(C_*) + \Sigma w(C_i)$ is equal to $w(C)$ plus half the number of points of intersection, that is, to $w(C) + r$. In fact, the sum of the jumps in angle at each P_i is π, and there are $2r$ points P_i. By proposition 3, $w(C_i) = 0$. Hence $w(C) = -r + w(C_*)$. Suppose now that the singular point of F is on the g-th handle. Span $\Gamma_1, \ldots, \Gamma_{g-1}$ with discs D.

and extend F to a field F_* on the resulting disc such that F_* has on each D_i exactly one singular point, necessarily of degree $+2$. Then $w(C_*)$ is equal to $w_p(C_*)$ plus the variation along C_* of the angle between F_* and a field of parallel vectors in the plane, that is,

$$w(C_*) = w_p(C_*) - 2\Sigma c_i^*.$$

2. Algebraic computation of the winding number.

We have previously given an algorithm for finding the winding number of a regular simple closed curve when a representation of its homotopy class in terms of the generators A_i is given. This algorithm is based on the study of a diagram representing a neighborhood of the base point Q of the fundamental group, which diagram occurs originally in connection with the definition of regular generating system [2, p. 273]. The diagram consists of a pair of concentric circles, D_1 inside D_2, having Q for center. A segment of radius going out from D_1 to D_2 serves as beginning and end for a linear ordering of the points of each circle, and none of the curves to be constructed is allowed to cross this segment. Each A_i crosses each circle once in leaving Q and once in returning to Q. The point where A_ν enters is denoted by ν/i, and the point where it leaves by ν/o. These points are arranged on D_1 and D_2 according to the following principles:

I. For each ν, the linear order of the four points $A_{2\nu-1}/i$, $A_{2\nu}/i$, $A_{2\nu-1}/o$, $A_{2\nu}/o$ should be the same on the two circles.

II. All entering arrows should be grouped on one semicircle of D_1, and all leaving arrows on the other.

III. The order on D_2 is that required by the structure of the fundamental polygon bounded by the A_ν and A_ν^{-1}. In or previous paper, one arrangement of points was given which satisfies these conditions. We now give another, which seems to be more convenient, since the resulting algorithm is more symmetric:

On D_2

$$1/o,\ 2/i,\ 1/i,\ 2/o,\ \ldots,\ 2g-1/o,\ 2g/i,\ 2g-1/i,\ 2g/o.$$

On D_1

$$1/o,\ 3/o,\ \ldots,\ 2g-1/o,\ 2/i,\ 4/i,\ \ldots,\ 2g/i,\ 1/i,\ 3/i,\ \ldots,\ 2g-1/i,\ 2/o,\ 4/o,\ \ldots,\ 2g/o.$$

As before, we introduce the figure 8 curve inside D_1 to handle the curves A_ν^{-1}. There are two essentially different ways to do this. Let us choose the one opposite to what we used before. Then, by following methods analogous to those used before, we prove the following proposition:

PROPOSITION 5. — *Replace the schema of the theorem by the one indicated immediately above. Also redefine the integer t by the rule*

$t = 1$ for the sequences

$$\begin{array}{ll} A_{2j-1}A_{2i-1}^{-1} & i > j \\ A_{2j}A_{2i-1}^{-1} & \\ A_{2j}A_{2i}^{-1} & i > j \end{array}$$

$t = -1$ for the sequences which consist of the same symbols in the inverse order, and $t = 0$ otherwise. Define the integer s by means of the revised schema. Then the winding number is equal to $s + t$ as before.

We remark that one may construct other algorithms according to his taste, provided the rules I, II, and III are followed.

In conclusion, we would like to pose the following problems about the winding number:

1. The winding number is not invariant under self-homeomorphisms of the surface, but it is invariant under isotopies. Let G be the group of self-homeomorphisms modulo isotopies. What can we say about the subgroup of G consisting of elements which leave the winding number unchanged?

2. Given a set of singular points and their indices, a homotopy class of vector fields under homotopies leaving the singular points fixed, and a set of free homotopy classes of curves supposed to contain periodic solutions of the vector field, what information can we draw from the winding number about the existence or non-existence of a vector field consistent with these data. Certain obvious cases of non-existence are known [3], but the general problem seems difficult.

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