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ON THE CONTRACTION OF THE DISCRETE SERIES OF SU (1,1)

by C. CISHAHAYO [(1)] and S. DE BIÈVRE

1. Introduction.

It is well known that the Lie algebra of the 1+1-dimensional Poincaré group $\mathcal{P}^{1,1} = SO(1,1) \otimes_s \mathbb{R}^2$ is a contraction of the $su(1,1)$ Lie-algebra. We show in this paper in which sense the positive energy massive representations of $\mathcal{P}^{1,1}$ are contractions of the discrete series of representations of $SU(1,1)$ (Theorem 7.4).

Contractions of Lie algebras were first studied systematically in [IW]. More details and examples can be found in [Gi]. The basic idea comes from physics. When a physical theory, invariant under some Lie group, contains a parameter, then very often the theory obtained for the limiting value of the parameter has a different invariance group. Galilean physics, for example, is obtained in the limit when the speed of light is taken to infinity in a Poincaré invariant theory. Similarly, physical theories on flat Minkowski spacetime ought to be obtainable from their curved spacetime versions in the limit when the curvature is taken to zero [Fr]. In [AAG], [DBE] and [GH] various aspects of this phenomenon were studied for the case when the curved spacetime is the two-dimensional Anti-de Sitter(AdS) spacetime. The identity component of the isometry group of the AdS spacetime is $SO_0(2,1) \cong SU(1,1)/\mathbb{Z}_2$. To massive particles on this spacetime are associated discrete series representations of $SO_0(2,1)$ (see [DBE] and refererences therein). On the other hand, massive particles on Minkowski

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spacetime are described by positive energy massive representations of the Poincaré group $\mathcal{P}^{1,1}$. It is then natural to ask in which precise mathematical sense the latter are obtainable from the former in the limit of zero curvature. This question is answered in this paper.

We remark that the contraction of Lie algebras and of their representations has attracted considerable attention in the literature. On the other hand, the behaviour of the unitary irreducible representations of their associated Lie groups under contractions has not been studied as thoroughly. Some notable exceptions are the works of Mickelsson and Niederle [MN] and of Dooley and Rice [D] [DR1] [DR2]. In [MN], the first proper definition of the contraction of unitary representations of Lie groups appears. The contraction of the principal series of $SO_0(n,1)$ to representations of the Euclidean group $E(n)$ and of the Poincaré group $\mathcal{P}^{n-1,1}$ is then established. Here $E(n)$ is a contraction of $SO_0(n,1)$ along $SO(n)$ and $\mathcal{P}^{n-1,1}$ is a contraction of $SO_0(n,1)$ along $SO(n-1,1)$. In [DR1] and [DR2] the irreducible representations of a Cartan motion group $K \otimes_s V$ associated with a Riemannian symmetric pair (G, K) are obtained as the limit of the irreducible unitary principal series representations of G. Here $K \otimes_S V$ is a contraction of G along K, and this work partially generalizes the results of [MN]. In [D] the same results are reviewed in the light of the Kostant-Kirillov orbit method. It is suggested there that this viewpoint might provide a useful framework for studying contractions of group representations by exploiting the geometry of the coadjoint orbits. This program is realized in the present paper to study the contraction of the discrete series of $SU(1,1)$ to positive energy massive representations of $\mathcal{P}^{1,1} = SO(1,1) \otimes_s \mathbb{R}^2$. It will be seen that the models of the representations that are provided by the orbit method, together with the *Kählerian* character of the corresponding $SU(1,1)$ orbits are the key ingredients needed to establish our results.

The rest of this paper is organized as follows. In section 2, we recall the essentials on the contraction of Lie algebras and Lie groups, and describe the contraction of $SU(1,1)$ to $\mathcal{P}^{1,1}$. In section 3 we give a precise definition of the contraction of representations which generalizes the one used in [MN] and [DR2]. In section 4 we briefly describe the discrete series of $SU(1,1)$ using the method of orbits. In section 5 we show how the contraction deforms the orbits of $SU(1,1)$ in $su(1,1)^*$ associated with the discrete series to orbits of $\mathcal{P}^{1,1}$ associated with its positive energy massive representations. In section 6 we describe the latter, using again the method of orbits. Finally, in section 7 we show how the geometric picture provided in sections 4-6 permits a very natural formulation and proof of our central result, Theorem 7.4.

To conclude, we remark that $\mathcal{P}^{1,1}$ is the motion group associated with the semi simple pseudo-Riemannian symmetric pair $(SO_0(2,1), SO(1,1))$. In addition, the discrete series representations we contract here are obtained via holomorphic induction from the corresponding maximally compact subgroup $SO(2)$. This suggests a generalization of our results which is partially exploited for the contraction of $SO_0(3,2)$ to $\mathcal{P}^{3,1}$ in [E] and [EDB].

2. The contraction of $SU(1,1)$ to $\mathcal{P}^{1,1}$.

The definition of the contraction of Lie algebras goes back to [IW]. A coordinate independent definition was given in [Sa] and [D]. We follow essentially the treatment of [D].

DEFINITION 2.1. — *Let $\mathcal{G}_1 = (V, [\cdot,\cdot]_1)$ and $\mathcal{G}_2 = (V, [\cdot,\cdot]_2)$ be two Lie algebras constructed on the same vector space V. We say $\mathcal{G}_2$ is a contraction of $\mathcal{G}_1$ if there exists a family Φ_κ, $\kappa \in (0,1]$, of invertible linear transformations of V so that*

$$\lim_{\kappa\to 0} \Phi_\kappa^{-1}[\Phi_\kappa \cdot x, \Phi_y]_1 = [x,y]_2,\ \forall x,\ y \in V. \tag{2.1}$$

One can also say that $\mathcal{G}_1$ is a deformation of $\mathcal{G}_2$. A special case, of particular interest for us, is the Inönü-Wigner contraction, or contraction along a subalgebra. Suppose there exists a subalgebra $\mathcal{K}$ of $\mathcal{G}_1$ and a vector space complement V_c to $\mathcal{K}$ in V, i.e.

$$V = \mathcal{K} \oplus V_c \tag{2.2a}$$

so that

$$[\mathcal{K}, V_c]_1 \subset V_c\,. \tag{2.2b}$$

Note that in applications V_c is almost never a subalgebra of $\mathcal{G}_1$. Then we can construct, $\forall x, y \in V$,

$$[x,y]_2 = [x_k, y_k]_1 + [x_k, y_c]_1 + [x_c, y_k]_1, \tag{2.3a}$$

where

$$x = x_k + x_c;\ x_k \in \mathcal{K}; x_c \in V_c, \tag{2.3b}$$

and similarly for $y \in V$. The bracket $[\cdot,\cdot]_2$ is a Lie bracket and we write $\mathcal{G}_2 = (V, [\cdot,\cdot]_2)$ for the corresponding Lie algebra, which is a semi-direct sum of $(\mathcal{K}, [\cdot,\cdot]_1)$ and the abelian Lie algebra V_c.

Now introduce

$$\Phi_\kappa(x) = x_k + \kappa x_c, \kappa \in (0,1]. \tag{2.4}$$

It is then easily verified that $\mathcal{G}_2$ is a contraction of $\mathcal{G}_1$. One says that $\mathcal{G}_2$ is a contraction of $\mathcal{G}_1$ along $\mathcal{K}$. Note that, restricted to $\mathcal{K}$, both brackets coincide.

Suppose now that G_1 is a Lie group with $\mathcal{G}_1$ as its Lie algebra and K a subgroup of G_1 having $\mathcal{K}$ as its Lie algebra. We can then construct the semi-direct product group

$$G_2 = K \otimes_s V_c\,, \tag{2.5}$$

where K acts on V_c with the adjoint representation. Then it is clear that $\mathcal{G}_2$ is the Lie algebra of G_2 and one can construct[D]

$$\Pi_\kappa : K \otimes_s V_c \to G_1$$

$$(k, v) \to (\exp_{G_1} \kappa v)k\,. \tag{2.6a}$$

It is not hard to verify that

$$T_e \Pi_\kappa = \Phi_\kappa\,. \tag{2.6b}$$

In this sense we can say G_2 is a contraction of G_1 along K.

We now turn to the case of interest in this paper and show that the Poincaré group $\mathcal{P}^{1,1} = SO(1,1) \otimes_s \mathbf{R}^2$ is a contraction of $SU(1,1)$ along one of its hyperbolic subgroups. The elements g of $SU(1,1)$ can be written as

$$g = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}, \ |\alpha|^2 - |\beta|^2 = 1, \alpha, \beta \in \mathbf{C} \tag{2.7}$$

and a basis for its Lie algebra $su(1,1) \cong \mathbf{R}^3$ is

$$e_{50} = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \ e_{15} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \ e_{01} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2.8}$$

with

$$[e_{50}, e_{01}] = -e_{15}\,; \ [e_{50}, e_{15}] = e_{01}\,; \ [e_{15}, e_{01}] = -e_{50}\,. \tag{2.9}$$

The compact subgroup is generated by e_{50}, while e_{01} and e_{15} generate a hyperbolic subgroup. The unusual indexation comes from the physical interpretation based on the homomorphism $SU(1,1) \to SO_0(2,1)$[DBE]. We take $\mathcal{K} = \text{span}\{e_{01}\}$ and $V_c = \text{span}\{e_{50}, e_{15}\}$ so that

$$[e_{50}, e_{15}]_2 = 0\,;\ [e_{50}, e_{01}]_2 = -e_{15}\,;\ [e_{15}, e_{01}]_2 = -e_{50} \tag{2.10}$$

which is easily verified to be the Poincaré Lie algebra.

3. Contracting representations.

Having defined the contraction of Lie algebras and Lie groups, we can now study the behaviour of their representations under the contraction procedure. Let $\mathcal{H}$ be a Hilbert space, carrier space of an unitary representation U of G_2. Let J be a subset of (0,1] accumulating at 0 and let $(\mathcal{H}_\kappa, U_\kappa), \kappa \in J$, be a family of unitary representations of G_1. Let $\{\mathcal{D}_\kappa, \kappa \in J\}$ be a family of dense subsets of $\mathcal{H}$ and $\{I_\kappa, \kappa \in J\}$ a family of linear injective maps

$$I_\kappa : \mathcal{D}_\kappa \subset \mathcal{H} \to \mathcal{H}_\kappa, \kappa \in J. \tag{3.1}$$

We then have the following definition :

DEFINITION 3.2. — *The representation $(\mathcal{H}, U, G_2)$ is a contraction of the family $(\mathcal{H}_\kappa, U_\kappa, G_1)$ if there exists a dense subset $\mathcal{D}$ of $\mathcal{H}$ and I_κ as above so that $\forall\, \phi \in \mathcal{D}$, $\forall\, g \in G_2$*

(3.2*a*) (i) $\forall \kappa \in J$ *sufficiently small*, $\phi \in \mathcal{D}_\kappa$ *and* $U_\kappa(\Pi_\kappa(g))I_\kappa\phi \in I_\kappa(\mathcal{D}_\kappa)$;

(3.2*b*) (ii) $\lim_{\kappa \to 0} ||I_\kappa^{-1} U_\kappa(\Pi_\kappa(g)) I_\kappa \phi - U(g)\phi||_{\mathcal{H}} = 0.$

This definition is close to the one of [MN], but less restrictive. In [MN], the authors require the existence of a family of isometries

$$A_\kappa : \mathcal{H}_\kappa \to A_\kappa(H_\kappa) \subset \mathcal{H} \tag{3.3}$$

which play the role of our I_κ^{-1}. It is then required that $U_\kappa A_\kappa(\mathcal{H}_\kappa) = \mathcal{H}$. These requirements are too strong for our purposes, as we explain in the remark after Lemma 7.1.

We also wish to stress the fact that the same family $(\mathcal{H}_\kappa, U_\kappa, G_1)$ can contract to many different $(\mathcal{H}, U, G_2)$, even if all representations involved are irreducible. In fact, this is the rule rather than the exception as a study of the known examples shows. It will also be the case in the situation studied in this paper.

4. The discrete series of $SU(1,1)$.

We give a short description of the discrete series of the representations of $SU(1,1)$, in a formulation convenient for our purposes.

First, let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc and let for each E_0

$$\omega_{E_0} = -2iE_0 \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2} \,. \tag{4.1}$$

The pair (D, ω_{E_0}) is a symplectic manifold and the action

$$SU(1,1) \times D \to D$$

$$(g, z) \to g \cdot z \equiv \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \tag{4.2}$$

leaves ω_{E_0} invariant. In fact, this action is transitive and is globally and strongly Hamiltonian [LM] and its generators are the hamiltonian vector fields associated to the functions

$$L_{50}(z, \bar{z}) = \frac{E_0(1+|z|^2)}{1-|z|^2} \,, \tag{4.3a}$$

$$L_{01}(z, \bar{z}) = -E_0 \frac{z+\bar{z}}{1-|z|^2} \,, \tag{4.3b}$$

$$L_{15}(z, \bar{z}) = -iE_0 \frac{z-\bar{z}}{1-|z|^2} \,. \tag{4.3c}$$

The corresponding moment map $L : D \to su(1,1)^*$, defined by

$$L(z) \cdot e_{\mu\nu} = L_{\mu\nu}(z) \,, \tag{4.4}$$

maps D into a coadjoint orbit in $su(1,1)^*$. The map L is a diffeomorphism of D onto one sheet of the two-sheeted hyperboloid in $su(1,1)^* \cong \mathbb{R}^3$, determined by

$$(L_{50})^2 - (L_{15})^2 - (L_{01})^2 = E_0^2 \,, \; L_{50} \geq E_0, \tag{4.5}$$

where we used the notation $L_{50}\, e_{50}^* + L_{15}\, e_{15}^* + L_{01}\, e_{01}^* \in su(1,1)^*$, with $e_{\mu\nu}^*$ the dual basis of $e_{\mu\nu}$ and $L_{\mu\nu} \in \mathbb{R}$. We shall use the notation $\mathcal{O}_{E_0}^+$ for the surface determined in (4.5). The orbit method of Kostant-Kirillov [Ki] [Ko] [Wo] associates to each of these coadjoint orbits a representation of the discrete series of $SU(1,1)$, provided E_0 is a half integer greater or equal than 1. Remembering that $SO_0(2,1) \cong SU(1,1)/\mathbb{Z}_2$, these representations restrict to representations of $SO_0(2,1)$, provided E_0 is an integer. When explicitly executing the Kostant-Kirillov construction, the representation Hilbert spaces $\mathcal{H}_{E_0}$ are realized as closed reproducing kernel subspaces of $L^2(D, \omega_{E_o})$ defined as follows [DBE] :

(4.6*a*) $\psi \in \mathcal{H}_{E_0}$ iff (i) $\psi \in L^2(D, \omega_{E_0})$

(4.6*b*) (ii) $\exists\, f$ analytic on D, so that $\psi(z,\overline{z}) = (1-|z|^2)^{E_0} f(z)$.

Condition (4.6*b*) corresponds to a polarisation which in this case is *Kählerian* and positive, whence the emergence of the analytic functions. Explicitly, the unitary irreducible representation of $SU(1,1)$ on $\mathcal{H}_{E_0}$ is

(4.7)
$$(U_{E_0}(g)\psi)(z,\bar{z}) = \left(\frac{1-|z|^2}{1-|g^{-1}\cdot z|^2}\right)^{E_0} (-\bar{\beta}z+\alpha)^{-2E_0}\psi(g^{-1}\cdot z, \overline{g^{-1}\cdot z}).$$

The link of this representation with the more usual formulation in terms of Bargman spaces of analytic functions [Pe] is readily made via the unitary transformation

$$\mathcal{V}:\ L^2(D,\omega_{E_0}) \to L^2(D,(1-|z|^2)^{2E_0}\omega_E)$$

(4.8)
$$\psi \to (1-|z|^2)^{-E_0}\,\psi.$$

The advantage of the formulation in (4.6)-(4.7) in the context of the contraction of $SU(1,1)$ to $\mathcal{P}^{1,1}$ will become clear in sections 6-7. We now first describe the behaviour of the orbits $\mathcal{O}_{E_0}^+$ under contraction.

5. The contraction of coadjoint orbits.

As announced in the introduction, we wish to establish that each massive, positive energy representation of $\mathcal{P}^{1,1}$ can be obtained via contraction from the discrete series of $SU(1,1)$. We expect $E_0 \to \infty$ as $\kappa \to 0$. We have seen that each $(\mathcal{H}_{E_0}, U_{E_0})$ is associated to a coadjoint orbit of $SU(1,1)$. It is then natural to ask how the latter behave under contraction. We shall see that they approximate in some sense coadjoint orbits of $\mathcal{P}^{1,1}$. This leads one to conjecture that the discrete series contracts to the unitary irreducible representation of $\mathcal{P}^{1,1}$ corresponding to those orbits. This is indeed what we prove in section 7.

Recall that we identified both $\mathcal{L}(\mathcal{P}^{1,1})$ and $\mathcal{L}(SU(1,1)) = su(1,1)$ with $\mathbb{R}^3$ by using the basis e_{50}, e_{15}, e_{01}. Similarly, we can identify both duals $\mathcal{L}(\mathcal{P}^{1,1})^*$ and $su(1,1)^*$ with $\mathbb{R}^{3*}$ by using the dual basis $e^*_{\mu\nu}$. We shall write $L_{50}\, e^*_{50} + L_{15}\, e^*_{15} + L_{01}\, e^*_{01}$ for elements of $su(1,1)^*$ and $H\, e^*_{50} + P\, e^*_{15} + K\, e^*_{01}$ for elements of $\mathcal{L}(\mathcal{P}^{1,1})^* \cong \mathbb{R}^{3*}$. The coadjoint action of both groups on $\mathbb{R}^{3*}$ is then readily computed and one finds the following orbits.

For $SU(1,1)$, they are subsets of the surfaces

$$(L_{50})^2 - (L_{15})^2 - (L_{01})^2 = r\,,\ r \in \mathbb{R}\,. \tag{5.1}$$

If $r > 0$, this corresponds to a two-sheeted hyperboloid, each sheet an $\mathrm{Ad}^*_{SU(1,1)}$ orbit. If $r < 0$, one obtains a one-sheeted hyperboloid. Finally, for $r = 0$, one obtains three orbits : two cones and the origin. Similarly, the coadjoint action of $\mathcal{P}^{1,1}$ foliates $\mathbb{R}^{3*}$ into orbits, determined by

$$H^2 - P^2 = r\,,\ r \in \mathbb{R}\,. \tag{5.2}$$

For $r \neq 0$, this surface splits into two hyperbolic cylinders; for $r = 0$ into five disjoint orbits : four half planes $(H = \pm P, H > 0, H < 0)$ and the origin.

Consider the orbit $\mathcal{O}^m$ given by

$$H^2 - P^2 = m^2\,,\ H \geq m \tag{5.3}$$

for some $m > 0$. Let $\beta_m = m e_{50}$ and consider the following family of submanifolds of $\mathbb{R}^{3*}$:

$$\begin{aligned}(\Phi_\kappa^{-1})^*(\mathrm{Ad}^*_{SU(1,1)}(\Phi^*_\kappa(\beta_m)) &= \mathcal{Q}_{m,\kappa} \\ &= (\Phi_\kappa^{-1})^*\mathcal{O}^+_{\frac{m}{\kappa}}\,.\end{aligned} \tag{5.4}$$

They are given by

$$\kappa^{-2}H^2 - \kappa^{-2}P^2 - K^2 = \left(\frac{m}{\kappa}\right)^2 \tag{5.5a}$$

or

$$H^2 - P^2 - \kappa^2 K^2 = m^2, H > 0\,. \tag{5.5b}$$

Comparing (5.5b) to (5.3), one sees that the $\mathcal{Q}_{m,\kappa}$ "approximate" $\mathcal{O}^m$ as κ tends to zero. A more precise statement of this observation can be found in [DBE]. The method of orbits associates to each of the $SU(1,1)$ orbits $\mathcal{O}^+_{\frac{m}{\kappa}}$ (with $\frac{m}{\kappa}$ half integer) a representation $(\mathcal{H}_{\frac{m}{\kappa}}, U_{\frac{m}{\kappa}}, SU(1,1))$ of the discrete series described in the previous section, and to each $\mathcal{O}^m$ a unitary irreducible representation $(\mathcal{H}^m, U^m, \mathcal{P}^{1,1})$ of $\mathcal{P}^{1,1}$ [R]. We shall describe the latter in the next section and prove in section 7 that the $(\mathcal{H}_{\frac{m}{\kappa}}, U_{\frac{m}{\kappa}}, SU(1,1))$ contract to $(\mathcal{H}^m, U^m, \mathcal{P}^{1,1})$.

6. The massive positive energy representations of $\mathcal{P}^{1,1}$.

The representation $(\mathcal{H}^m, U^m, \mathcal{P}^{1,1})$ associated via the Kirillov-Kostant method of orbits to $\mathcal{O}^m$ [R] [Ra] [DBE] can be written down in several ways. However, since $\mathcal{O}^m$ does not admit a positive invariant *Kähler* polarization, the Hilbert space $\mathcal{H}^m$ cannot be realized as a subspace of $L^2(\mathcal{O}^m, \omega^m)$, where ω^m is the symplectic two-form on $\mathcal{O}^m$ (for details, see for example [DBE]). We choose here a realization of $(\mathcal{H}^m, U^m)$ that is particularly well-suited for our purposes, and which is essentially contained in [DBE]. We shall describe $\mathcal{H}^m$ as a space of functions on a subset of $\mathcal{O}^m$ which arises naturally.

Note indeed that the sets $\mathcal{Q}_{m,\kappa}$ defined in (5.5b) have a common intersection $\mathcal{E}_m$ for all $\kappa > 0$, given by

$$K = 0\,,\, H > 0\,,\, H^2 - P^2 = m^2. \tag{6.1}$$

Note that $\mathcal{E}_m$ is the orbit of β_m under the Ad* action of $K = SO(1,1)$, which explains the existence of a diffeomorphism between $i(-1,1) \subset D$ and $\mathcal{E}_m$:

$$T = (\Phi_\kappa^{-1})^* \circ L : i(-1,1) \subset D \to \mathcal{E}_m \subset \mathcal{Q}_{m,\kappa} \tag{6.2}$$

as is readily verified upon noting that $L_{01}(z, \bar{z}) = 0$ if $z = i\lambda$ (see (4.3)-(4.4)). Moreover, $\forall k \in K = SO(1,1), \forall z \in i(-1,1)$, we have

$$T(k \cdot z) = \mathrm{Ad}_k^* T(z), \tag{6.3}$$

so that T intertwines the action of K on $i(-1,1) \subset D$ and on $\mathcal{E}_m$. It is convenient to introduce a coordinate τ on $\mathcal{E}_m$ by

$$H = m \cosh \tau, \tag{6.4a}$$

(6.4b)
$$P = m \sinh \tau,$$

and one verifies that

(6.5)
$$He_{50}^* + Pe_{15}^* = \mathrm{Ad}^*_{\exp \tau e_{01}} \beta_m.$$

The $SO(1,1)$-invariant measure on $\mathcal{E}_m$ is then $d\tau$. On the other hand, from (6.2) and (4.3) one sees that

(6.6a)
$$T(i\lambda) = \frac{m(1+\lambda^2)}{1-\lambda^2} e_{50}^* + m \frac{2\lambda}{1-\lambda^2} e_{15}^*$$

so that

(6.6b)
$$\lambda = \tanh \frac{\tau}{2} \, .$$

It was shown in [DBE] that applying the orbit method to $\mathcal{O}^m$ leads to a realization of $\mathcal{H}^m$ as

(6.7)
$$\mathcal{H}^m \cong L^2(\mathcal{E}_m, d\tau) \cong L^2 \left((-1,1), \frac{2d\lambda}{1-\lambda^2} \right)$$

where the last equivalence follows from (6.6). The representation of $\mathcal{P}^{1,1} = SO(1,1) \otimes_s \mathbb{R}^2$ on $\mathcal{H}^m$ is then

(6.8)
$$(U^m(a; \exp\varphi e_{01})\phi)(\lambda) = e^{-i\frac{m(1+\lambda^2)}{1-\lambda^2} a^0 - \mathrm{im} \frac{2\lambda}{1-\lambda^2} a^1} \phi \left(\frac{\lambda \cosh \frac{\varphi}{2} + \sinh \frac{\varphi}{2}}{\cosh \frac{\varphi}{2} + \lambda \sinh \frac{\varphi}{2}} \right) .$$

7. Contraction of the discrete series.

We show in this section that the family $(\mathcal{H}_{\frac{m}{\kappa}}, U_{\frac{m}{\kappa}}, SU(1,1))$, with $\frac{m}{\kappa}$ half integer, contracts to $(\mathcal{H}^m, U^m, \mathcal{P}^{1,1})$. For that purpose, we need to construct appropriate maps I_κ from $\mathcal{H}^m$ to $\mathcal{H}_{\frac{m}{\kappa}}$ (see (3.1)). We shall actually first construct their inverses.

Recall from section 4 that the $\mathcal{H}_{\frac{m}{\kappa}}$ are subspaces $L^2(D, \omega_{\frac{m}{\kappa}})$. Moreover, in view of (4.6b), each $\psi \in \mathcal{H}_{\frac{m}{\kappa}}$ is uniquely determined by its restriction to $i(-1,1)$. Hence we can define the injective map

(7.1a)
$$I_\kappa^{-1} : \psi \in \mathcal{H}_{\frac{m}{\kappa}} \to (I_\kappa^{-1}\psi) \in C^\infty((-1,1))$$

by

(7.1b)
$$(I_\kappa^{-1}\psi)(\lambda) = \psi(i\lambda).$$

Remark that $I_\kappa^{-1}\psi$ has an analytic extension to D. We are interested in $\mathcal{D}_\kappa = (\mathcal{I}\mathrm{m}\, I_\kappa^{-1}) \cap L^2\left((-1,1), \dfrac{2d\lambda}{1-\lambda^2}\right)$.

LEMMA 7.1.

(i) If $\kappa' \leq \kappa$, *then*, $\mathcal{D}_\kappa \subset \mathcal{D}_{\kappa'}$

(ii) $\forall n \in \mathbb{N}, (1-\lambda^2)^{\frac{m}{\kappa}}\lambda^n \in \mathcal{D}_\kappa$

(iii) $\bigcup\limits_{\kappa>0} \mathcal{D}_\kappa$ *is dense in* $L^2\left((-1,1), \dfrac{2d\lambda}{1-\lambda^2}\right)$.

Remark. — (i) says that the $\mathcal{D}_\kappa$ grow as κ shrinks, which will be important in the sequel. It is clear that $\bigcup\limits_{\kappa>0} \mathcal{D}_\kappa$ can not be equal to $L^2\left((-1,1); \dfrac{2d\lambda}{1-\lambda^2}\right)$ since all elements of $\mathcal{D}_\kappa$ are analytic functions.

Proof. — Let $\phi \in \mathcal{D}_\kappa$. It follows from (7.1*b*) and (4.6*b*) that there exists an analytic function f on D so that

(7.2*a*) $$(a)\ \psi(z) \equiv (1-|z|^2)^{\frac{m}{\kappa}} f(z) \in L^2(D, \omega_{\frac{m}{\kappa}}).$$

(7.2*b*) $$(b)\ \phi(\lambda) = (1-\lambda^2)^{\frac{m}{\kappa}} f(i\lambda).$$

Define now on D the function ψ' by

$$\psi'(z) = (1-|z|^2)^{\frac{m}{\kappa'}}(1+z^2)^{\frac{m}{\kappa}-\frac{m}{\kappa'}} f(z).$$

We show now $\psi' \in \mathcal{H}_{\frac{m}{\kappa'}}$. Clearly, ψ' is of the form (4.6*b*) for $E_0 = \dfrac{m}{\kappa'}$. It remain to check that $\psi' \in L^2(D, \omega_{\frac{m}{\kappa'}})$. For that purpose, note that

$$\psi'(z) = \left(\frac{1-|z|^2}{1+z^2}\right)^{\frac{m}{\kappa'}-\frac{m}{\kappa}} \psi(z),$$

so that

$$|\psi'(z)| = \left\|\frac{1-|z|^2}{1+z^2}\right\|^{\frac{m}{\kappa'}-\frac{m}{\kappa}} |\psi(z)|.$$

But, since $\dfrac{m}{\kappa'} - \dfrac{m}{\kappa} \geq 0$, and $1-|z|^2 \leq |1+z^2|$, for $z \in D$, the result follows. We conclude $\psi' \in \mathcal{H}_{\frac{m}{\kappa'}}$ and one sees readily that $(I_{\kappa'}^{-1}\psi') = \phi$. Hence $\phi \in \mathcal{D}_{\kappa'}$.

(ii) This follows directly from the observation that $\psi_n \equiv (1-|z|^2)^{\frac{m}{\kappa}} z^n$, $n \in \mathbb{N}$ are in $\mathcal{H}_{\frac{m}{\kappa}}$. They are the eigenfunctions of the generator of the compact subgroup. Clearly

$$(I_\kappa^{-1}\psi_n)(\lambda) = (1-\lambda^2)^{\frac{m}{\kappa}}\, \lambda^n (i)^n \in\; L^2\left((-1,1), \frac{2d\lambda}{1-\lambda^2}\right).$$

(iii) It follows from (i) and (ii) that $(1-\lambda^2)\lambda^n$ is in $\bigcup_{\kappa>0} \mathcal{D}_\kappa$, for all $n \in \mathbb{N}$. Those polynomials are clearly dense in $L^2\left((-1,1), \frac{2d\lambda}{1-\lambda^2}\right)$. □

We now turn to the limit in (3.2). First we consider for $(k,v) \in \mathcal{P}^{1,1} = K \otimes_s V$,

$$\Pi_\kappa(k,v) = \exp_{SU(1,1)} \kappa v\, k \equiv \begin{pmatrix} \alpha(\kappa) & \beta(\kappa) \\ \bar{\beta}(\kappa) & \bar{\alpha}(\kappa) \end{pmatrix}. \tag{7.3}$$

Using (2.8) one sees that for

$$v = a^0 e_{50} + a^1 e_{51},\ (a^0, a^1) \in \mathbb{R}^2, \tag{7.4a}$$

and

$$k = \exp \varphi\, e_{01}, \tag{7.4b}$$

we have

$$\alpha(\kappa) = \cosh\frac{\varphi}{2} + \frac{i}{2}\kappa\left(a^0\cosh\frac{\varphi}{2} - a^1 \sinh\frac{\varphi}{2}\right) + \mathcal{O}(\kappa^2), \tag{7.5a}$$

$$\beta(\kappa) = -i\sinh\frac{\varphi}{2} - \frac{\kappa}{2}\left(a^1\cosh\frac{\varphi}{2} - a^0 \sinh\frac{\varphi}{2}\right) + \mathcal{O}(\kappa^2). \tag{7.5b}$$

Hence

$$\alpha(\kappa)^2 + \beta(\kappa)^2 = 1 + i\kappa a^0 + \mathcal{O}(\kappa^2), \tag{7.6a}$$

$$\alpha(\kappa)\bar{\beta}(\kappa) + \bar{\alpha}(\kappa)\beta(\kappa) = -\kappa a^1 + \mathcal{O}(\kappa^2). \tag{7.6b}$$

For later use we introduce

$$A(\kappa) \equiv \frac{\overline{\alpha(\kappa)}^2 + \overline{\beta(\kappa)}^2}{\alpha(\kappa)^2 + \beta(\kappa)^2} = 1 - 2i\kappa a^0 + \mathcal{O}(\kappa^2), \tag{7.7a}$$

$$B(\kappa) \equiv \frac{\alpha(\kappa)\overline{\beta(\kappa)} + \overline{\alpha(\kappa)}\beta(\kappa)}{\alpha(\kappa)^2 + \beta(\kappa)^2} = -\kappa a^1 + \mathcal{O}(\kappa^2). \tag{7.7b}$$

Note that all errors depend on $(k,v) \in \mathcal{P}^{1,1}$, and are easily controlled uniformily on compacta.

Now let $\phi \in \hat{\mathcal{D}} \equiv \bigcup_{\frac{m}{\kappa}\geq 1} \mathcal{D}_\kappa$. Then $\phi \in \mathcal{D}_\kappa$ for all κ sufficiently small (Lemma 7.1 (i)). Hence, for such κ

$$(I_\kappa\phi)(z) = (1-|z|^2)^{\frac{m}{\kappa}} \frac{\phi(-iz)}{(1+z^2)^{\frac{m}{\kappa}}} \tag{7.8}$$

(see (7.2)). Therefore, using (4.7) and (7.1) we find after some calculation

$$\begin{aligned} &(I_\kappa^{-1}U_{\frac{m}{\kappa}}(\Pi_\kappa(k,v))I_\kappa\phi)(\lambda) \\ &= \left[\alpha(\kappa)^2+\beta(\kappa)^2\right]^{-\frac{m}{\kappa}} \left[\frac{1-\lambda^2}{1-2iB(\kappa)\lambda - A(\kappa)\lambda^2}\right]^{\frac{m}{\kappa}} \phi\left(-i\frac{i\overline{\alpha(\kappa)}\lambda-\beta(\kappa)}{-i\overline{\beta(\kappa)}\lambda+\alpha(\kappa)}\right), \end{aligned} \tag{7.9}$$

where $A(\kappa)$ and $B(\kappa)$ are defined in (7.7). Using (7.5)-(7.7) it is then very easy to see that

$$\lim_{\kappa\to 0}(I_\kappa^{-1}U_{\frac{m}{\kappa}}(\Pi_\kappa(k,v)I_\kappa\phi)(\lambda) = (U^m(a;\exp\varphi\, e_{01})\phi)(\lambda) \tag{7.10}$$

$\forall\lambda \in (-1,1)$. In order to prove that the family $(\mathcal{H}_{\frac{m}{\kappa}}, U_{\frac{m}{\kappa}}, SU(1,1))$ contracts to $(\mathcal{H}^m, U^m, \mathcal{P}^{1,1})$ we have to prove (7.10) holds also as a strong limit in $\mathcal{H}$. For that purpose we first need control over the prefactor in (7.9). We introduce the notation

$$Y_\kappa(\lambda) = \left[\frac{1-\lambda^2}{1-2iB(\kappa)\lambda - A(\kappa)\lambda^2}\right]. \tag{7.11}$$

LEMMA 7.2. — *Let $Z_\kappa(\lambda) = [\alpha(\kappa)^2+\beta(\kappa)^2]^{-\frac{m}{\kappa}}[Y_\kappa(\lambda)]^{\frac{m}{\kappa}}$. Then there exists a constant C so that $\forall\lambda \in (-1,1)$, for all κ sufficiently small*

$$|Z_\kappa(\lambda)| \leq C. \tag{7.12}$$

Proof. — We first recall from (4.7) and (7.1) that $Z_\kappa(\lambda)$ can be rewritten as

$$Z_\kappa(\lambda) = \left\{\frac{1-\lambda^2}{[1+(\Pi_\kappa(k,v)^{-1}\cdot i\lambda)^2][-\overline{\beta(\kappa)}i\lambda+\alpha(\kappa)]^2}\right\}^{\frac{m}{\kappa}}. \tag{7.13}$$

The expression in the denominator is for each κ a polynomial of second order in λ :

$$P(\kappa,\lambda) \equiv [1+(\Pi_\kappa(k,v)^{-1}\cdot i\lambda)^2][-\overline{\beta(\kappa)}i\lambda+\alpha(\kappa)]^2. \tag{7.14}$$

Since, for $|\lambda| < 1$ we know $[-\bar{\beta}(\kappa)i\lambda + \alpha(\kappa)] \neq 0$, the two zeros $\lambda_\pm(\kappa)$ of $P(\kappa, \lambda)$ satisfy

$$1 = |\Pi_\kappa(k, v)^{-1} \cdot i\lambda_\pm(\kappa)|. \tag{7.15}$$

Now since the unit circle is an orbit of $SU(1,1)$, we can conclude

$$|\lambda_\pm(\kappa)| = 1. \tag{7.16}$$

A simple calculation using (7.7) yields

$$\lambda_\pm(\kappa) = \pm 1 - i\kappa[a^1 \mp a^0] + \mathcal{O}(\kappa^2). \tag{7.17}$$

Now, since $|\alpha(\kappa)^2 + \beta(\kappa)^2| > |\alpha(\kappa)|^2 - |\beta(\kappa)|^2 = 1$, and since for κ sufficiently small (7.7a) implies that $|A(\kappa)| > 1 - C'\kappa^2$, for some $C' > 0$, we have

$$|Z_\kappa(\lambda)| \leq C \left| \frac{(1-\lambda)(-1-\lambda)}{(\lambda_+(\kappa) - \lambda)(\lambda_-(\kappa) - \lambda)} \right|^{\frac{m}{\kappa}} \tag{7.18}$$

for some $C > 0$. For $\lambda \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, the expression in the right hand side of (7.18) is bounded uniformly. For $\lambda \in \left(\frac{1}{2}, 1\right)$, $(1-\lambda) \leq |\lambda_+(\kappa) - \lambda)|$ and for $\lambda \in \left(-1, -\frac{1}{2}\right)$, $|-1-\lambda| < |\lambda_-(\kappa) - \lambda)$ since $|\lambda_\pm(\kappa)| = 1$. The result then follows easily from (7.18). □

Remark. — The precise information on the location of $\lambda_\pm$ that we used is crucial in obtaining (7.12). Indeed, if in (7.11) we only knew the coefficients $A(\kappa)$ and $B(\kappa)$ behaved as in (7.7), then (7.12) could not be guaranteed to hold.

DEFINITION 7.3. —

$$\mathcal{D} = \Big\{ \phi \in \widehat{\mathcal{D}} \equiv \bigcup_{\frac{m}{\kappa} \geq 1} \mathcal{D}_\kappa \subset L^2\left((-1,1);\ \frac{2d\lambda}{1-\lambda^2}\right) \mid \exists\, \delta > 0$$

$$\text{so that } \phi'(\lambda) = \frac{\phi(\lambda)}{(1-\lambda^2)^\delta} \text{ has a bounded analytic extension to } D \Big\}.$$

Note that $\mathcal{D}$ is dense in $\mathcal{H}$ since $(1-\lambda^2)\lambda^n \in \mathcal{D}, \forall n \geq 0$. We can now formulate the central result of this paper.

THEOREM 7.4. — The representation $(\mathcal{H}^m, U^m, \mathcal{P}^{1,1})$ is a contraction of the family $(\mathcal{H}_{\frac{m}{\kappa}}, U_{\frac{m}{\kappa}}, SU(1,1))$, with $\frac{m}{\kappa} \geq 1$ a half-integer.

Proof. — We have to verify (3.2) is satisfied. We take $\mathcal{D}$ as defined in Definition 7.3 and I_κ is the inverse of I_κ^{-1} in (7.1) defined on $\mathcal{D}_\kappa = (\mathrm{Im} I_\kappa^{-1}) \cap L^2\left((-1,1), \frac{2d\lambda}{1-\lambda^2}\right)$. Now each ϕ in $\mathcal{D}$ belong to some $\mathcal{D}_{\kappa_0}$ and, by Lemma 7.1 $\phi \in \mathcal{D}_\kappa$, $\forall \kappa \geq \kappa_0$. This takes care of the first statement in Definition 3.2(i). To prove the second part of Definition 3.2(i), we need to show that $I_\kappa^{-1} U_\kappa(\Pi_\kappa(k,v) I_\kappa \phi$, with I_κ^{-1} defined in (7.1), belongs to $\mathcal{H}_m$. We prove this simultaneously with (3.2*b*) as follows. For $\phi \in \mathcal{D}$, write

$$
\begin{aligned}
&|(I_\kappa^{-1} U_{\frac{m}{\kappa}}(\Pi_\kappa(k,v)) I_\kappa \phi)(\lambda) - (U^m(a;k)\phi)(\lambda)| \\
&\quad \leq |Y_\kappa(\lambda)^{\frac{m}{\kappa}-\delta} [\alpha(\kappa)^2 + \beta(\kappa)^2]^{-(\frac{m}{\kappa}-\delta)} (1-\lambda^2)^\delta (-\overline{\beta(\kappa)} i\lambda + \alpha(\kappa))^{-2\delta}| \\
&\qquad \times \left\{ \frac{\phi(-i(\Pi_\kappa(k,v)^{-1} \cdot i\lambda))}{[1 + (\Pi_\kappa(k,v)^{-1} \cdot i\lambda)^2]^\delta} - \frac{\phi(-i(k^{-1} \cdot i\lambda))}{[1 + (k^{-1} \cdot (i\lambda))^2]^\delta} \right\} \\
&+ \Bigg| Y_\kappa(\lambda)^{\frac{m}{\kappa}-\delta} [\alpha(\kappa)^2 + \beta(\kappa)^2]^{-(\frac{m}{\kappa}-\delta)} \frac{(1-\lambda^2)^\delta}{(1 + (k^{-1} \cdot (i\lambda))^2)^\delta} (-\overline{\beta(\kappa)} i\lambda + \alpha(\kappa))^{-2\delta} \\
&\quad - \exp - \left\{ \mathrm{im} \left(\frac{1+\lambda^2}{1-\lambda^2} \right) a^0 + \mathrm{im} \frac{2\lambda}{1-\lambda^2} a^1 \right\} \Bigg| |\phi(-i(k^{-1} \cdot i\lambda))| .
\end{aligned}
\tag{7.19}
$$

The first term in the right hand side of (7.19) is bounded by $C(1-\lambda^2)^\delta$ in view of Lemma 7.2 and the assumptions on ϕ. Since in addition it converges to zero pointwise, the Lebesgue dominated convergence theorem assures it converges in $L^2\left((-1,1), \frac{2d\lambda}{1-\lambda^2}\right)$. Similarly the second term in (7.19) converges pointwise to zero and is bounded by $C|\phi(-i(k^{-1} \cdot i\lambda))|$ which is in $L^2\left((-1,1), \frac{2d\lambda}{1-\lambda^2}\right)$. Hence

$$||I_\kappa^{-1} U_{\frac{m}{\kappa}}(\Pi_\kappa(k,v)) I_\kappa \phi - U^m(a,k)\phi||_{\mathcal{H}} \to 0$$

which proves both (3.2*b*) and the second part of Definition 3.2(i). □

Remark. — The definition 3.2 refers to an a priori arbitrary choice of the family I_κ. Our analysis of the geometry of the coadjoint orbits in section 5 shows that the choice in (7.1) arises naturally. On the other hand, the crucial injectivity of I_κ^{-1} follows from the *Kählerian* character of the orbits $\mathcal{O}^+_{\frac{m}{\kappa}}$ which is also responsible for the fact that the $\mathcal{H}_{\frac{m}{\kappa}}$ are Hilbert subspaces of $L^2(\mathcal{O}^+_{\frac{m}{\kappa}}, \omega_{\frac{m}{\kappa}})$. The limiting orbit $\mathcal{O}^m$ of $\mathcal{P}^{1,1}$ is no longer *Kählerian* which explains why $\mathcal{H}^m$ is no longer realized as a Hilbert subspace of $L^2(\mathcal{O}^m)$.

This remark is at the origin of most of the technical difficulties encountered in this paper, as already pointed out in [DBE]. There it was proven that one can identify the $\mathcal{O}^+_{\frac{m}{\kappa}}$ with $\mathcal{O}^m$; it was then shown in which way the $\mathcal{H}_{\frac{m}{\kappa}}$, seen as subspaces of $\mathcal{H}^m \equiv L^2(\mathcal{O}^m)$, leave $\mathcal{H}^m$as κ tends to zero.

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BIBLIOGRAPHY

[AAG] S. T. ALI, J. P. ANTOINE, and J.P. GAZEAU, De Sitter to Poincaré contraction and relativistic coherent states, Ann. Inst. H. Poincaré, 52 (1990), 83-111.

[DBE] S. De BIÈVRE and A. El GRADECHI, Quantum mechanics and coherent states on the Anti-de Sitter spacetime and their Poincaré contraction, Ann. Inst. H. Poincaré, 57 (1992), 403-428.

[D] A. H. DOOLEY, Contractions of Lie groups and applications to analysis, in : Topics in modern harmonic analysis, Vol.I, 483-515 (Instituto Nazionale di Alta Matematica Francesco SEVERI, Roma 1983).

[DR1] A. H. DOOLEY and J. W. RICE, Contractions of rotation groups and their representations, Math. Proc. Camb. Phil. Soc., 94 (1983), 509-517.

[DR2] A. H. DOOLEY and J. W. RICE, On contractions of semisimple Lie groups, Trans. Amer. Math. Soc., 289 (1985), 185-202.

[E] A. El GRADECHI, Théories classique et quantique sur l'espace-temps Anti-de Sitter et leurs limites à courbure nulle, Thèse de Doctorat, Université Paris 7, décembre 1991.

[EDB] A. El GRADECHI and S. De BIÈVRE, Phase space quantum mechanics on the Anti-de Sitter spacetime and its Poincaré contraction, preprint 1992.

[Fr] C. FRONSDAL, Elementary particles in a curved space, Rev. Mod. Phys., 37 (1965), 221-224.

[GH] J. P. GAZEAU and V. HUSSIN, Poincaré Contraction of $SU(1,1)$ Fock-Bargmann Structure, J. of Physics A, Math. Gen., 25 (1992), 1549-1573

[Gi] R. GILMORE, Lie groups, Lie algebras and some of their Applications, Wiley, New York, 1974.

[IW] E. INÖNÜ and E. P. WIGNER, On the contraction of groups and their representations, Proc. Nat. Acad. Sci. U. S., 39 (1953), 510-524.

[Ki] A. KIRILLOV, Eléments de la Théorie des Représentations, Editions Mir, Moscou, 1974.

[Ko] B. KOSTANT, Quantization and unitary representations, Lecture Notes in Math., 170, Springer Verlag, Berlin, 1970.

[LM] P. LIBERMANN and C. M. MARLE, Symplectic Geometry and Analytical Mechanics, D. Reidel Publishing Company, 1987.

[Ma] G. W. MACKEY, On the analogy between semisimple Lie groups and certain related semi-direct product groups, in : Lie Groups and Their Representations, I. M. Gelfand (Ed.), 339-364 (Akadémiai Kiadó, Budapest 1975).

[MN] J. MICKELSSON and J. NIEDERLE, Contractions of representations of the Sitter groups, Commun. Math. Phys., 27 (1972), 167-180.

[Pe] A. PERELOMOV, Generalized Coherent States and their Applications, Springer Verlag, Berlin, 1986.

[R] P. RENOUARD, Variétés Symplectiques et Quantification, Thèse Orsay, 1969.

[Ra] J.H. RAWNSLEY, Representations of a semi-direct product by quantization, Math. Proc. Camb. Phil. Soc., 78 (1975), 345-350.

[Sa] J. SALETAN, Contraction of Lie groups, J. Math. Phys., 2 (1961), 1-21.

[Wo] N.M.J. WOODHOUSE, Geometric Quantization, Clarendon Press, Oxford, 1980.

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