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# INFINITESIMAL CONJUGACIES AND WEIL-PETERSSON METRIC

by A. FATHI and L. FLAMINIO

## 0. Introduction.

Let $M$ be a closed manifold $M$ and denote by $\mathcal{M}M$ the space of $C^\infty$ Riemannian metrics on $M$. For two Riemannian metrics $g_1$ and $g_2$ in $\mathcal{M}M$ we define their *intersection* (or *geodesic stretch*) as in [CF].

Let $v$ be a $g_1$-unit length vector and let $\gamma(t)$ be $g_1$-geodesic with initial velocity $v$ parameterized by $g_1$-arclength. For a fixed $t > 0$, define $\varphi(v,t)$ as the lower bound of $g_2$-lengths of paths homotopic to $\gamma[0,t]$ with endpoints fixed. Denote by $\mathrm{Liou}_{g_1}$ the Liouville measure obtained from $g_1$ on the tangent unit bundle $S_{g_1}(M)$, note that we do not normalize this measure as a probability measure. The intersection $i(g_2, g_1)$ is defined by

$$i(g_2,g_1) = \lim_{t\to\infty} \frac{1}{t} \int_{S_{g_1}M} \varphi(v,t)\, d\,\mathrm{Liou}_{g_1}(v).$$

For surfaces of genus at least 2, the intersection for metrics of negative curvature coincides, up to a universal constant, with the intersection of their Liouville currents as defined by Bonahon [Bo]. Thus it is symmetric and it depends only on the orbit of the metrics under $\mathrm{Diff}_0(M)$, the group of diffeomorphisms of $M$ isotopic to the identity.

Before Bonahon, Thurston gave a definition of $i(g_1, g_2)$ for $g_1$ and $g_2$ metrics of constant negative curvature $-1$ on a surface [Wo]. He observed

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that the map $g_2 \mapsto i(g_1, g_2)$ has a unique critical point on Teichmüller space at $g_1$. Moreover the second derivative of the above map at the critical point is a (strictly) positive definite form and thus defines a Riemannian metric on Teichmüller space. Wolpert in [Wo] showed that this metric on Teichmüller space is proportional to the Weil-Petersson metric which is defined in the following way. The tangent space to $\mathcal{M}M$ at $g$ is the space of sections of the symmetric bilinear forms on $M$. Thus the formula

$$\langle S, T\rangle_{WP} = \frac{1}{2}\int_M g^{ij} g^{kl} S_{ik} T_{jl}\, d\,\mathrm{Vol}_g$$

defines the Weil-Petersson metric on $\mathcal{M}M$. The tangent space to Teichmüller space at a point, represented by a metric $g$ in $\mathcal{M}M$, can be identified with the orthogonal complement in the tangent space $T_g\mathcal{M}M$ of the subspace tangent to the orbit of $g$ under the action of $\mathrm{Diff}_0 M$ on $\mathcal{M}M$. Due to its invariance under that action, the Weil-Petersson metric is well defined on Teichmüller space.

In this paper we study the infinitesimal version of the Morse-Anosov-Gromov conjugacy of geodesic flows for manifolds of negative curvature. We apply this study to compute derivatives of the intersection. This, together with some elementary theory of the representations of $SL(2, \mathbb{R})$, yields another proof of Wolpert's theorem which does not use Complex Analysis. Our proof is closer in spirit to the variational point of view on Teichmüller space described by Tromba in [Tr].

## 1. Background.

Let $M$ be a compact manifold. Denote by $SM$ the bundle of oriented directions on $M$, i.e. $SM = \{v \in TM \mid v \neq 0\}/\sim$, where $v \sim v'$ if and only if $\exists c > 0, v = cv'$.

Given a Riemannian metric $g$ on $M$ there is an identification of $SM$ with the unit tangent bundle $S_gM = \{v \in TM | g(v,v) = 1\}$. Thus the geodesic flow $g^t$ on $S_gM$ determined by $g$ can also be considered as a flow on $SM$. With abuse of notation we will also denote this flow by $g^t$.

Generalizing a definition of Bonahon [Bo], we define, for $g$ of strictly negative curvature, the space of geodesic currents $\mathcal{C}$ on $M$ as the space of positive transverse invariant measures to the $g^t$-orbit foliation of $SM$ endowed with the weak topology. The fact that $\mathcal{C}$ is independent of the choice of the metric of strictly negative curvature $g$ is due to the following theorem ([An], [Gr] and [Mo]) :

THEOREM 1.1 (Morse-Anosov-Gromov). — *If $g_0$ and $g_1$ are two Riemannian metrics of strictly negative curvature on $M$, there exists a homeomorphism $h : SM \to SM$ homotopic to the identity which sends orbits of $g_0^t$ to orbits of $g_1^t$ bijectively. Moreover, if $h_1$ and $h_2$ are two such homeomorphisms then there exists a real valued function $t(v)$ on $SM$ such that $\forall v \in SM, h_1^{-1} \circ h_2(v) = g_0^{t(v)}(v)$. We shall call such an homeomorphism a $(g_0, g_1)$-Morse correspondence.*

The space of geodesic currents $\mathcal{C}$ is in bijection with the set of positive finite measures on $SM$ invariant under the geodesic flow $g^t$. The correspondence is given in the following way : if $\mu$ is a geodesic current, $T$ is a small transversal to the geodesic flow $g^t$ in $SM$ and $dt$ is the measure on the orbits of the geodesic flow obtained from the parameterization $g^t$, the measure on $SM$ that corresponds to $\mu$ is given locally by the product $\mu \otimes dt$. We will denote this measure on $SM$ by $\Phi_g(\mu)$.

In particular, the Liouville measure obtained from $g$ defines a current that we will denote by $\lambda_g$, or just by $g$, if there is no ambiguity. Closed orbits of $g^t$ in $SM$ define geodesic currents which we will call the Dirac geodesic currents. They are in bijection with homotopy classes of oriented closed curves in $M$, since for metrics of strictly negative curvature there exists a unique closed geodesic in every homotopy class of curves. If $\alpha$ is an oriented homotopy class of closed curves, we will denote the corresponding current by $\lambda_\alpha$ or just by $\alpha$, if there is no ambiguity. The following theorem is a form of the Anosov closing lemma for Anosov flows [An].

THEOREM 1.2. — *Multiples of the Dirac geodesic currents are dense in the space of geodesic currents.*

We define the *intersection* $i(g, \mu)$ of the Riemannian metric of negative curvature $g$ with the geodesic current $\mu$ as the total mass of $SM$ for the measure $\Phi_g(\mu)$. The following theorem is clear from the definition.

THEOREM 1.3. — *If $g$ is a Riemannian metric with negative curvature and $\alpha$ is an oriented homotopy class of curves in $M$, we have :*

$$i(g, \alpha) = l_g(\alpha).$$

An equivalent definition of $i(g, \mu)$ is given by the following theorem :

THEOREM 1.4. — *Let $g_1$ and $g_2$ be two Riemannian metrics of negative curvature on $M$. For $v \in SM$ let $\gamma(t)$ be a $g_1$-geodesic with initial velocity $v$ parameterized by $g_1$-arclength. For a fixed $t > 0$, define $\varphi(v,t)$ to be the infimum of $g_2$-lengths of paths homotopic in $M$ to $\gamma[0,t]$ with endpoints fixed. For every geodesic current $\mu$, we have*

$$i(g_2,\mu) = \lim_{t\to\infty} \frac{1}{t} \int_{SM} \varphi(v,t)\, d\Phi_{g_1}(\mu)(v).$$

*Proof.* — Without loss of generality (see the construction in [Gr], [Gh] page 78, or the lemma on page 187 in [LM]), we can assume that the conjugacy $h$ from the geodesic flow of $g_1$ to the geodesic flow of $g_2$ is at least $C^1$ along the orbits of the geodesic flow of $g_1$. For $v \in S_{g_1}M$ define $\psi_t(v)$ by $h(g_1^t(v)) = g_2^{\psi_t(v)}(h(v))$ and let $\theta(v) = \frac{d}{dt}\psi_t(v)_{t=0}$. From the definition it is not difficult to see that $(h^{-1})_*\Phi_{g_2}(\mu) = \theta\Phi_{g_1}(\mu)$. It follows that $i(g_2,\mu) = \int_{SM}\theta(v)\,d\Phi_{g_1}(\mu)(v)$. Since the measure $\Phi_{g_1}(\mu)$ is invariant under the geodesic flow of $g_1$, for every $t > 0$ we have $\displaystyle\int_{SM}\theta(v)\,d\Phi_{g_1}(\mu)(v) = \frac{1}{t}\int_{SM}\int_0^t \theta(g_1^s v)\,ds\,d\Phi_{g_1}(\mu)(v)$. If $p : SM \to M$ is the projection, note that $\int_0^t \theta(g_1^s v)\,ds$ is the $g_2$-length of the $g_2$-geodesic $s \mapsto phg_1^s v, 0 \le s \le t$, and $\varphi(v,t)$ is the $g_2$-length of the $g_2$-geodesic homotopic to $s \mapsto pg_1^s v$, $0 \le s \le t$ with fixed endpoints. Since $h$ is homotopic to the identity, we can find a homotopy $h_u$, $0 \le u \le 1$ which is $C^1$ with respect to $u$. By the compactness of $M$, the supremum $K$ over $v \in SM$ of the $g_2$-length of the $C^1$ paths $u \mapsto ph_u v, 0 \le u \le 1$ is finite. For every $v \in SM$ and every $t > 0$ we have $|\varphi(v,t) - \int_0^t \theta(g_1^s v)\,ds| \le 2K$. □

From the above theorem, it follows that the intersection of $g_1$ with the Liouville current of $g_2$ coincides with the intersection of $g_1$ and $g_2$ as defined in the introduction.

THEOREM 1.5. — *Let $\mathcal{M}_-M$ be the space of Riemannian metrics of negative curvature on $M$. The map*

$$(g,\mu) \in \mathcal{M}_-M \times \mathcal{C} \mapsto i(g,\mu) \in \mathbb{R}$$

*is jointly continuous if $\mathcal{M}_-M \times \mathcal{C}$ is given the product topology of the uniform topology on the first factor and the weak topology on the second factor.*

*Proof.* — Observe that for $g$ fixed $\mu \mapsto i(g,\mu)$ is continuous by definition of the weak topology on measures. On the other hand if $g_1$ is a Riemannian metric such that $C^{-2}g_1 \leq g \leq C^2 g_1$ then from the previous theorem $\forall \mu \in \mathcal{C}, C^{-1}i(g_1,\mu) \leq i(g,\mu) \leq Ci(g_1,\mu)$. □

## 2. Infinitesimal Morse correspondences.

We will now give more details on Morse correspondences. Throughout this section $g$ will denote a smooth Riemannian metric of negative curvature on a compact manifold $M$ and $g_\alpha$ will be a smooth one-parameter family of smooth metrics on $M$ of negative curvature with $g_0 = g$. *We will identify the bundle of oriented directions $SM$ with the tangent unit bundle $S_gM$.* The generator of the geodesic flow of $g$ on $SM$ will be denoted by $X$.

For a proof of the following theorem see [MML], Appendix A, page 591.

THEOREM 2.1. — *There exists a one-parameter family $h_\alpha$ of $(g, g_\alpha)$-Morse correspondences such that the map $\alpha \mapsto h_\alpha$is smooth with values in the Banach manifold of continuous maps $SM \to SM$. The tangent to the curve $h_\alpha$ is a smooth curve of continuous vector fields $\Xi_\alpha$ on $SM$. Two different smooth choices $h'_\alpha$ and $h''_\alpha$ of $(g, g_\alpha)$-Morse correspondence yield tangent vector fields $\Xi'_\alpha$ and $\Xi''_\alpha$ which at $\alpha = 0$ differ by a continuous multiple of the generator $X$ of the flow $g^t$.*

*Notation* 2.2. — The tangent bundle $T(S_gM)$ to $S_gM$ has a Whitney sum decomposition $T(S_gM) = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V}$ is the vertical subbundle i.e. the subbundle tangent to the fibers of the projection $p : S_gM \to M$, and $\mathcal{H}$ is the horizontal subbundle for the Levi-Civita connection of the metric $g$. The subbundle $\mathcal{H}$ is the kernel of the connection map $K : T(S_gM) \to T(M)$. The restriction to the fiber $\mathcal{V}_v$ of the connection map $K$ is the canonical identification of the $\mathcal{V}_v$ with the $g$-orthogonal complement of $v$ in $T_{p(v)}(M)$. With respect to the standard lift $\bar{g}$ of the metric $g$ to $S_gM$ this decomposition is orthogonal and the projection $p : S_gM \to M$ is a Riemannian submersion.

The vector $X(v)$ is, by definition, the horizontal lift of $v$. We denote by $\mathcal{X}^\perp$ the subbundle of $T(S_gM)$ of vectors $\bar{g}$-orthogonal to $X$. Thus $\mathcal{X}(v)^\perp$ is the subspace of $T_v(S_gM)$ spanned by $\mathcal{V}$ and the horizontal lift of the $g$-orthogonal complement $v^\perp$ of $v$.

*Remark* 2.3. — From Theorem 2.1, it follows that the $\mathcal{X}^\perp$ component of the vector field $\Xi_0$ is independent of the choice of a smooth curve $h_\alpha$ of $(g, g_\alpha)$-Morse correspondences.

*Remark* 2.4. — The $(g, g_1)$-Morse correspondence, as we have seen, is not unique. For a sufficiently small $\varepsilon$, if $g_1$ is in a sufficiently small $C^2$-neighborhood of $g$ we can choose the correspondence $h$ so that the footpoint of $h(v)$ belongs to the hypersurface of points $\{\exp_g X \mid X \perp v,\ \|X\| < \varepsilon\}$. This choice makes the $(g, g_1)$-Morse correspondence unique for every $g_1$ in a $C^2$-neighborhood of $g$.

*Remark* 2.5. — If $g_\alpha$, $|\alpha| < \alpha_0$, is a smooth curve of $C^\infty$ negatively curved metrics on $M$ and the $(g, g_\alpha)$-Morse correspondence $h_\alpha$ is chosen as in the previous remark, then the derivative $\Xi$ of $h_\alpha$ with respect to $\alpha$ at $\alpha = 0$ satisfies $Tp(\Xi(v)) \perp v$ for all $v \in S_gM$. Thus, by Remark 2.3, it coincides with the $\mathcal{X}^\perp$ component of the vector field $\Xi'$ for any other choice $h'_\alpha$ of $(g, g_\alpha)$-Morse correspondences.

DEFINITION 2.6 (Infinitesimal Morse correspondence). — *Let $g_\alpha$, $|\alpha| < \alpha_0$, be a smooth curve of $C^\infty$ negatively curved metrics on $M$ and let $h_\alpha$ be the $(g, g_\alpha)$-Morse correspondence chosen as in Remark 2.4. We call "infinitesimal Morse correspondence at $g$ for the curve $g_\alpha$" the vector field $\Xi$ given by the derivative of $h_\alpha$ with respect to $\alpha$ at $\alpha = 0$.*

Our goal now is to find the relationship between the curve $g_\alpha$ and the infinitesimal Morse correspondence at $g$ for the curve $g_\alpha$. We denote with $\nabla^\alpha$ the Levi-Civita connection of $g_\alpha$. For $g = g_0$ we will denote its Levi-Civita connection simply by $\nabla$ and $R$ will be its curvature tensor. Remark that $\partial_\alpha \nabla^\alpha$ is a (1,2) tensor field.

PROPOSITION 2.7. — *Let $\Xi$ be the infinitesimal Morse correspondence at $g$ for the curve $g_\alpha$ and let $\xi = Tp(\Xi)$ be the projection of $\Xi$ in $TM$ along a unit speed $g$-geodesic $\gamma(t)$. Then $\xi$ satisfies the equation*

$$(*) \qquad \nabla^2_{\gamma'}\xi + R(\xi, \gamma')\gamma' + \Gamma_{\gamma'}\gamma' - \gamma' g(\gamma', \Gamma_{\gamma'}\gamma') = 0,$$

*where we have set $\gamma'(t) = \dfrac{d}{dt}\gamma(t)$ and $\Gamma = \partial_\alpha \nabla^\alpha|_{\alpha=0}$. Equivalently, with $S = \partial_\alpha g_\alpha|_{\alpha=0}$,*

$$(**) \ \nabla^2_{\gamma'}\xi + R(\xi, \gamma')\gamma' = -\nabla_{\gamma'} S^\sharp(\gamma') + \frac{1}{2}(\nabla S(\gamma', \gamma'))^\sharp + \frac{1}{2}\nabla_{\gamma'} S(\gamma', \gamma')\gamma'.$$

*Moreover $\xi$ is the unique bounded solution of $(*)$ satisfying $g(\xi, \gamma') = 0$ along $\gamma(t)$. The vertical component of $\Xi$ is given by $\nabla_{\gamma'}\xi$.*

*Proof.* — Denote by $\tilde{M}$ the universal cover of $M$ and, to simplify notation, let $g_\alpha$ also denote the lift to $\tilde{M}$ of the metric $g_\alpha$ on $M$. Let $\gamma(t)$

be a unit speed $g$-geodesic and let $(\alpha, t) \in ]-\alpha_0, \alpha_0[\times\mathbb{R} \mapsto \gamma_\alpha(t) \in \tilde{M}$ be a smooth map satisfying the following conditions :

1) for each $\alpha$ the map $t \in \mathbb{R} \mapsto \gamma_\alpha(t)$ is a geodesic for $g_\alpha$ and $\gamma_0 = \gamma$;
2) for each $\alpha$ and $t$ the point $\gamma_\alpha(t)$ belongs the hypersurface of points $\{\exp_g X \mid X \perp \gamma'(t), \|X\| < \varepsilon\}$.

Note that the geodesics $\gamma_\alpha$ are not parametrized by arc-length.

The map $(\alpha, t) \mapsto \gamma_\alpha(t)$ is jointly differentiable in $\alpha$ and $t$. This follows from Theorem 2.1 and standard facts on ordinary differential equations. Define $\xi(\alpha, t) = \partial_\alpha \gamma_\alpha(t)$.

In the computation below we should pull back the Levi-Civita connections on $TM$ to $]-\alpha_0, \alpha_0[\times\mathbb{R}$ using the map $(\alpha, t) \mapsto \gamma_\alpha(t)$ see [Kl], pages 46–47. As customary, in order to minimize notations we will proceed as if the map $(\alpha, t) \mapsto \gamma_\alpha(t)$ were an embedding.

Let $V$ be a smooth vector field on $\tilde{M}$ with compact support. From the equation of the geodesics $\nabla^\alpha_{\gamma'_\alpha} \gamma'_\alpha(t)$ is a scalar multiple of $\gamma'_\alpha(t)$. Thus for every $\alpha$ we have

(1)
$$\int_{-\infty}^{\infty} g_\alpha[V(\gamma_\alpha(t)), \nabla^\alpha_{\gamma'_\alpha} \gamma'_\alpha(t)]\, dt$$
$$= \int_{-\infty}^{\infty} g_\alpha[V(\gamma_\alpha(t)), \gamma'_\alpha(t)] g_\alpha[\gamma'_\alpha(t), \nabla^\alpha_{\gamma'_\alpha} \gamma'_\alpha(t)] g_\alpha[\gamma'_\alpha(t), \gamma'_\alpha(t)]^{-1}\, dt.$$

Differentiating equation (1) with respect to $\alpha$, at $\alpha = 0$, we obtain for the left hand side

(2)
$$\int_{-\infty}^{\infty} \{g(\nabla_\xi V, \nabla_{\gamma'}\gamma') + g(V, \nabla_\xi \nabla_{\gamma'_\alpha} \gamma'_\alpha|_{\alpha=0}) + S(V, \nabla_{\gamma'}\gamma') + g(V, \Gamma_{\gamma'}\gamma')\}\, dt$$

where for clarity we have suppressed the dependence on $t$.

Since $\xi$ and $\gamma'_\alpha$ commute we have that $\nabla_\xi \nabla_{\gamma'_\alpha} = \nabla_{\gamma'_\alpha} \nabla_\xi + R(\xi, \gamma'_\alpha)$. Using this and the fact that $\nabla_{\gamma'}\gamma' = 0$ the above expression becomes

$$\text{(3)} \qquad \int_{-\infty}^{\infty} \{g(V, \nabla_{\gamma'_\alpha} \nabla_\xi \gamma'_\alpha|_{\alpha=0}) + g(V, R(\xi, \gamma')\gamma') + g(V, \Gamma_{\gamma'}\gamma')\}\, dt.$$

Since $\nabla_\xi \gamma'_\alpha = \nabla_{\gamma'_\alpha} \xi$ we conclude that (2) equals

$$\text{(4)} \qquad \int_{-\infty}^{\infty} \{g(V, \nabla^2_{\gamma'} \xi + R(\xi, \gamma')\gamma' + \Gamma_{\gamma'}\gamma')\}\, dt.$$

In order to evaluate the derivative with respect to $\alpha$ of the right hand side of (1) at $\alpha = 0$, it is convenient to observe that, since $\nabla_{\gamma'}\gamma' = 0$, the only non zero contributions come from the derivative with respect to $\alpha$ of the term $\nabla^{\alpha}_{\gamma'_\alpha}\gamma'_\alpha$. Taking also in account that $g_\alpha(\gamma'_\alpha(t),\gamma'_\alpha(t)) = 1$ when $\alpha = 0$, the derivative with respect to $\alpha$ of the right hand side of (1) at $\alpha = 0$ simplifies as

$$\int_{-\infty}^{\infty} g(V,\gamma')\left\{g(\gamma',\nabla_\xi\nabla_{\gamma'_\alpha}\gamma'_\alpha|_{\alpha=0}) + g(\gamma',\Gamma_{\gamma'}\gamma')\right\}\,dt. \tag{5}$$

As before we can rewrite (5) as

$$\int_{-\infty}^{\infty} g(V,\gamma')g(\gamma',\nabla^2_{\gamma'}\xi + R(\xi,\gamma')\gamma' + \Gamma_{\gamma'}\gamma')\,dt. \tag{6}$$

Now since $\xi$ is orthogonal to $\gamma'$ also $\nabla^2_{\gamma'}\xi$ and $R(\xi,\gamma')\gamma'$ are orthogonal to $\gamma'$ and (6) becomes

$$\int_{-\infty}^{\infty} g(V,\gamma' g(\gamma',\Gamma_{\gamma'}\gamma'))\,dt. \tag{7}$$

Putting together (4) and (7) we have that differentiation of (1) for $\alpha = 0$ yields that for all $V$

$$\int_{-\infty}^{\infty}\left\{g(V,\nabla^2_{\gamma'}\xi + R(\xi,\gamma')\gamma' + \Gamma_{\gamma'}\gamma' - \gamma' g(\gamma',\Gamma_{\gamma'}\gamma'))\right\}\,dt = 0.$$

Thus the field $\xi$ along $\gamma$ satisfies the equation

$$\nabla^2_{\gamma'}\xi + R(\xi,\gamma')\gamma' + \Gamma_{\gamma'}\gamma' - \gamma' g(\gamma',\Gamma_{\gamma'}\gamma') = 0$$

which proves the first statement.

Notice that from the definition of Levi-Civita connection it follows that

$$g(Z,\Gamma_X Y) = \frac{1}{2}[\nabla_X S(Y,Z) + \nabla_Y S(X,Z) - \nabla_Z S(X,Y)].$$

From $g(V,\Gamma_{\gamma'}\gamma') = \nabla_{\gamma'}S(\gamma',V) - \frac{1}{2}\nabla_V S(\gamma',\gamma')$ and $g(\gamma',\Gamma_{\gamma'}\gamma') = \frac{1}{2}\nabla_{\gamma'}S(\gamma',\gamma')$, the above equation becomes

$$\nabla^2_{\gamma'}\xi + R(\xi,\gamma')\gamma' = -\nabla_{\gamma'}S^\sharp(\gamma') + \frac{1}{2}(\nabla S(\gamma',\gamma'))^\sharp + \frac{1}{2}\nabla_{\gamma'}S(\gamma',\gamma')\gamma' \tag{**}$$

proving the second statement.

Since by hypothesis $(M,g)$ is negatively curved, the operator $\xi \in v^\perp \mapsto R(\xi,v)v \in v^\perp$ is negative definite. It follows that Equation (**) has a unique bounded solution which is perpendicular to $\gamma'$ at all times, namely the projection of the infinitesimal Morse correspondence $\Xi$ at $g$ along the curve $g_\alpha$. The vertical component of $\Xi$ is given by

$$\nabla_\xi g(\gamma'_\alpha,\gamma'_\alpha)^{-1/2}\gamma'_\alpha$$

evaluated at $\alpha = 0$. Taking into account that $g(\gamma', \gamma') = 1$, that $\nabla_\xi \gamma'_\alpha = \nabla_{\gamma'_\alpha} \xi$ and that $\nabla_{\gamma'} \xi \perp \gamma'$ one sees that the vertical component of $\Xi$ is given by

$$\nabla_{\gamma'} \xi$$

concluding the proof of the proposition. □

*Remark* 2.8. — As expected the infinitesimal Morse correspondence along the variation $g_\alpha$ depends only on $g$ and $S = \partial_\alpha g_\alpha|_{\alpha=0}$. Thus we can simply speak of the *infinitesimal Morse correspondence at g in the direction of S*.

*Notation* 2.9. — For a negatively curved metric $g$ on $M$, let us introduce a new norm on sections of $\mathcal{X}^\perp$ by setting for a section $\chi$

$$|||\chi|||^2 = \int_{SM} \{g[K\chi(v), K\chi(v)] - g[R(Tp\chi(v), v)v, Tp\chi(v)]\} \, d\,\mathrm{Liou}_g(v),$$

where $K$ is the connection map and $R$ the Riemann curvature tensor of the Levi-Civita connection for $g$.

PROPOSITION 2.10. — *In the hypothesis of Proposition 2.7 we have*

$$|||\Xi|||^2 = -\frac{1}{2} \int_{SM} L_\Xi S(v, v) \, d\,\mathrm{Liou}_g(v)$$

*where $L_\Xi S(v, v)$ denotes the Lie derivative of the function $v \in SM \mapsto S(v, v)$ in the direction of the vector field $\Xi$.*

*Proof.* — Let $v$ be a point in $SM$. Let $\gamma(t)$ be the unit speed geodesic with initial velocity $v$. Define $\gamma_\alpha(t)$ as in the proof of Proposition 2.7, and $\xi(t) = \partial_\alpha \gamma_\alpha(t)|_{\alpha=0}$. Observe that $\xi = Tp(\Xi(\gamma'))$ and $\nabla_{\gamma'} \xi = K\Xi(\gamma')$. We pull back the connection $\nabla$ to $]-\alpha_0, \alpha_0[\times\mathbb{R}$. We have

$$L_\Xi(S(v, v)) = \partial_\alpha \left. \frac{S(\gamma'_\alpha, \gamma'_\alpha)}{\|\gamma'_\alpha\|^2} \right|_{\alpha=0, t=0}. \tag{8}$$

Since $\partial_\alpha \|\gamma'_\alpha\|^2|_{\alpha=0} = 2g(\nabla_{\partial/\partial\alpha} \gamma'_\alpha, \gamma'_\alpha)_\alpha = 2g(\nabla_{\partial/\partial t} \xi, \gamma') = 2\partial_t(g(\xi, \gamma')) - 2g(\xi, \nabla_{\partial/\partial t} \gamma')$ and since $\xi \perp \gamma'$ and $\nabla_{\partial/\partial t} \gamma' = \nabla_{\gamma'} \gamma' = 0$ by the equation of geodesics, we have that $\partial_\alpha \|\gamma'_\alpha\|^2 = 0$ at $\alpha = 0$. Thus

$$L_\Xi(S(v, v)) = \partial_\alpha S(\gamma'_\alpha, \gamma'_\alpha)|_{\alpha=0, t=0}. \tag{9}$$

From

$$\partial_\alpha S(\gamma'_\alpha, \gamma'_\alpha) = (\nabla_{\partial/\partial_\alpha} S)(\gamma'_\alpha, \gamma'_\alpha) + 2S(\nabla_{\partial/\partial_\alpha} \gamma'_\alpha, \gamma'_\alpha),$$

using $\nabla_{\partial/\partial_\alpha}\gamma'_\alpha|_{\alpha=0} = \nabla_{\gamma'}\xi$ and $\nabla_{\gamma'}\gamma' = 0$, we obtain

(10)
$$\begin{aligned}\partial_\alpha S(\gamma'_\alpha,\gamma'_\alpha)|_{\alpha=0} &= (\nabla_\xi S)(\gamma',\gamma') + 2S(\nabla_{\gamma'}\xi,\gamma')\\ &= (\nabla_\xi S)(\gamma',\gamma') - 2(\nabla_{\gamma'}S)(\xi,\gamma') - 2S(\xi,\nabla_{\gamma'}\gamma')\\ &\qquad + 2\partial_t(S(\xi,\gamma'))\\ &= (\nabla_\xi S)(\gamma',\gamma') - 2(\nabla_{\gamma'}S)(\xi,\gamma') + 2\partial_t(S(Tp(\Xi(\gamma')),\gamma')).\end{aligned}$$

From equation $(**)$ we see that

$$\begin{aligned}(\nabla_\xi S)(\gamma',\gamma') - 2(\nabla_{\gamma'}S)(\xi,\gamma') &= 2[g(\xi,\nabla^2_{\gamma'}\xi) + g(R(\xi,\gamma')\gamma',\xi))]\\ &= 2[\partial_t g(\xi,\nabla_{\gamma'}\xi) - g(\nabla_{\gamma'}\xi,\nabla_{\gamma'}\xi)\\ &\qquad + g(R(\xi,\gamma')\gamma',\xi))]\end{aligned} \tag{11}$$

and using (11) in (10) we obtain

$$\begin{aligned}\frac{1}{2}\partial_\alpha S(\gamma'_\alpha,\gamma'_\alpha)_{\alpha=0} &= -g(\nabla_{\gamma'}\xi,\nabla_{\gamma'}\xi) + g(R(\xi,\gamma')\gamma',\xi))\\ &\qquad + \partial_t[S(Tp(\Xi(\gamma')),\gamma') + g(Tp(\Xi(\gamma')),K\Xi(\gamma'))].\end{aligned}$$

Thus from (9)

(12)
$$\begin{aligned}\frac{1}{2}L_\Xi(S(v,v)) &= -g(K\Xi(v),K\Xi(v)) + g(R(Tp(\Xi(v)),v)v,Tp(\Xi(v))))\\ &\qquad + L_X[S(Tp(\Xi(v)),v) + g(Tp(\Xi(v)),K\Xi(v))].\end{aligned}$$

Integrating (12) over $SM$, the term $L_X[S(Tp(\Xi(v)),v) + g(Tp(\Xi(v)), K\Xi(v))]$ integrates to zero, since the geodesic flow preserves the Liouville measure. The proposition is proved. □

## 3. Surfaces.

In the case of surfaces the equation for the infinitesimal Morse correspondence can be simplified further.

*Notation.* — Let $(M,g)$ be an oriented Riemannian surface of negative curvature. As before we identify $SM$ with the unit tangent bundle $S_gM$.

Let $J$ be the complex structure determined by $g$ and the given orientation. Then $T(SM)$ has a trivialization $(X, X^\perp, \Theta)$ where

1) $X(v)$ is the horizontal lift of $v$, i.e. the generator of the geodesic flow of $g$;

2) $X^{\perp}(v)$ is the horizontal lift of $Jv$;

3) $\Theta = \frac{d}{d\theta}R_\theta$, where $R_\theta(v)$ is the rotation of $v$ by an angle $\theta$, oriented so that $J = R_{\pi/2}$.

We denote by $\kappa$ the Gaussian curvature of $g$.

PROPOSITION 3.1. — *Let $S$ be a symmetric tensor field. Then the infinitesimal Morse correspondence $\Xi$ at $g$ in the direction of $S$ for the variation $g_\alpha$ is given by*

$$\Xi = fX^{\perp} + (L_X f)\Theta$$

*where $f$ is the unique bounded solution to the equation*

$$(L_X^2 f + f\kappa \circ p)(v) = -L_X(S(v, Jv)) + \frac{1}{2}L_{X^{\perp}}(S(v,v)). \tag{13}$$

*Proof.* — Let $\gamma(t)$ be a $g$-geodesic, $v_t = \gamma'(t)$. Let $\xi(t) = p^*\Xi(v_t)$. Then $\xi(t) = f(v_t)Jv_t$. Since $\nabla_{v_t}Jv_t = 0$ , equation (**) becomes

$$(L_X^2 f(v_t))Jv_t + \kappa \circ p(v_t)f(v_t)Jv_t$$
$$= -\nabla_{v_t}S^\sharp(v_t) + \frac{1}{2}(\nabla S(v_t,v_t))^\sharp + \frac{1}{2}\nabla_{v_t}S(v_t,v_t)v_t.$$

Taking the $g$-inner product with $Jv_t$ we obtain the equation for $f$

$$L_X^2 f(v_t) + \kappa \circ p(v_t)f(v_t) = -\nabla_{v_t}S(v_t, Jv_t) + \frac{1}{2}\nabla_{Jv_t}S(v_t,v_t).$$

Since $v_t$ and $Jv_t$ are parallel along $\gamma(t)$, $\nabla_{v_t}S(v_t,Jv_t) = L_X S(v_t, Jv_t))$. Now from the relation between parallel transport and covariant derivative

$$\nabla_{Jv_t}S(v_t,v_t) = \frac{d}{ds}S(P^s v_t, P^s v_t)|_{s=0}$$

where $P^s$ is the parallel transport along the geodesic with initial velocity $Jv_t$. But $P^s v_t$ is also the $\Psi^s_{X^\perp}$, where $\Psi^s_{X^\perp}$ is the flow generated on $SM$ by $X^\perp$. Hence $\nabla_{v_t}S(v_t,Jv_t) = L_{X^\perp}(S(v,v))$, concluding the proof. $\square$

## 4. Riemann surfaces.

We need to introduce some generalities on metrics of constant negative curvature. Refer to [Ta] and [La] for details. In this section $g$ is a Riemannian metric of constant negative curvature $-1$. Thus the vector fields $(X, X^\perp, \Theta)$ satisfy

$$[X,\Theta] = -X^\perp \qquad [X^\perp,\Theta] = X \qquad [X,X^\perp] = -\Theta$$

and thus are generators of an action of $SL(2,\mathbf{R}) \equiv SU(1,1)$. The corresponding embedding of $su(1,1)$ in the vector fields on $SM$ is given by

$$\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \mapsto X \qquad \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix} \mapsto X^{\perp} \qquad \begin{pmatrix} -i/2 & 0 \\ 0 & i/2 \end{pmatrix} \mapsto \Theta.$$

In fact every metric of constant curvature on a compact surface $M$ induces an identification of $M$ with $\Gamma\backslash SU(1,1)/K$, where $\Gamma$ is a discrete group of $SU(1,1)$ determined up to conjugacy and $K \equiv U(1)$ is the subgroup of diagonal matrices. With respect to this identification the unit tangent bundle of $M$ rests identified with $\Gamma\backslash SU(1,1)$ and the above action with the right action of $SU(1,1)$ on $\Gamma\backslash SU(1,1)$. Furthermore, the lift of the Liouville measure of $g$ under the covering map $SU(1,1) \to \Gamma\backslash SU(1,1)$ is a Haar measure on $SU(1,1)$. Thus $\mathrm{Liou}_g$ is invariant under the action of $SU(1,1)$ on $SM \equiv \Gamma\backslash SU(1,1)$ and $L^2(SM, \mathrm{Liou}_g)$ decomposes as an orthogonal sum of irreducible unitary representations of $SU(1,1)$. The occurring representations are of two types, indexed by the eigenvalues of the Casimir operator : the principal series and the discrete series. Cyclic vectors for the irreducible representation of the principal series are the lifts to $SM$ of eigenfunctions of the Laplacian on $M$. Cyclic vectors for the irreducible representations of discrete series are the holomorphic sections of the powers $\tau^m M$ of the canonical line bundle $\tau M$ and their complex conjugates. These, as all symmetric tensors, yield functions on $SM$ by evaluation along the diagonal.

We introduce the complex vector fields on $SM$ given by $\eta^+ = (X - iX^{\perp})/2$ and $\eta^- = (X + iX^{\perp})/2$. The derivation $\eta^+$ is the formal adjoint of $-\eta^-$. The holomorphic (resp. anti-holomorphic) sections of $\tau^m M$ (resp. $\bar{\tau}^m M$) are given by the kernel of $\eta^-$ (resp. $\eta^+$) [GK]. They are also called holomorphic (resp. anti-holomorphic) differentials. In particular the holomorphic sections of $\tau^2 M$ are called holomorphic quadratic differentials. Complex conjugation is an anti-linear automorphism of $L^2(SM, \mathrm{Liou}_g)$. It sends holomorphic sections of $\tau^m M$ to anti-holomorphic sections of $\bar{\tau}^m M$, where $\bar{\tau}^m M$ is the bundle of differential of type $(d\bar{z})^m$. We denote by $H_Q$ (resp. $H_{\bar{Q}}$) the irreducible representation generated by a holomorphic differential $Q$ (resp. anti-holomorphic differential $\bar{Q}$).

LEMMA 4.1. — *For a given $m$, all irreducible representations $H_Q$ generated by holomorphic sections $Q$ of $\tau^m M$ are equivalent. If $Q$ is such a section, for every $h \in H_Q$ we have $\langle \bar{h}, h \rangle = 0$ and $\|h\| = \sqrt{2}\|\Re h\| = \sqrt{2}\|\Im h\|$.*

*Proof.* — The first statement is a standard fact of the Representation

Theory of $SL(2,\mathbf{R})$ (see [Ta] or [La]). Now, for $n \in \mathbf{Z}$, let $H_n$ be the subspace of $L^2(SM, \mathrm{Liou}_g)$ of eigenvectors with eigenvalue $ni$ for the operator $\Theta$. From the commutation relation of $\eta^+$, $\eta^-$ and $\Theta$ it can be easily seen that $\eta^\pm : H_n \to H_{n\pm1}$. For a given holomorphic section $Q$ of $\tau^m M$, an orthogonal basis for the irreducible representation $H_Q$ containing $Q$ is given by $({\eta^+}^k Q)_{k\geq 0}$. Thus $H_Q \subset \bigoplus_{n\geq m} H_n$. Since $\overline{Q} \in H_{-m}$ it follows that $\overline{Q}$ is orthogonal to $H_Q$ and thus that $H_Q \perp H_{\overline{Q}}$.

Since $\Im h = (h - \bar{h})/2i$ and $\Re h = (h + \bar{h})/2$ we have $\|\Im h\|^2 = \langle \Im h, \Im h\rangle = \langle h - \bar{h}, h - \bar{h}\rangle/4 = (\|h\|^2 + \|\bar{h}\|^2 - 2\Re\langle \bar{h}, h\rangle)/4 = \|h\|^2/2$. Similarly $\|\Re h\|^2 = \|h\|^2/2$. □

For proofs of the following two lemmata see, for example, the paper of Fischer and Tromba [FT], pages 336–337.

LEMMA 4.2. — *Let $M$ be a compact connected oriented surface and $g_\alpha$, (for $|\alpha| < \alpha_0$) a smooth curve of $C^\infty$ Riemannian metrics of curvature $-1$ with $g_0 = g$. The symmetric tensor $S = \partial_\alpha g_\alpha|_{\alpha=0}$ decomposes uniquely as a sum*

$$S = L_V g + S_{TT},$$

*where $V$ is a $C^\infty$ vector field on $M$ and $S_{TT}$ is a trace free divergence free symmetric tensor.*

LEMMA 4.3. — *A trace free divergence free symmetric 2-tensor field on a compact connected oriented Riemannian surface of curvature $-1$ is the real part of a holomorphic quadratic differential.*

*Remark* 4.4. — By 4.2 and 4.3, if $g_\alpha$, (for $|\alpha| < \alpha_0$) is a smooth curve of $C^\infty$ Riemannian metrics of curvature $-1$ with $g_0 = g$ and $S$ is the symmetric tensor field $\partial_\alpha g_\alpha$ at $\alpha = 0$, we can write $S = L_V g + \Re Q$ with $V$ is a $C^\infty$ vector field on $M$ and $Q$ holomorphic quadratic differential. The linearity of the equation (13) allows us to split its solution as a sum or the solutions for $S = L_V g$ and $S = \Re Q$.

We will now study the solution of equation (13) for $S = \Re Q$.

PROPOSITION 4.5. — *Let $S = \Re Q$ be a symmetric tensor field which is the real part of a holomorphic quadratic differential $Q$. Then the infinitesimal Morse correspondence $\Xi$ at $g$ in the direction of $S$ is given by $\Xi = fX^\perp + (L_X f)\Theta$ where $f = \Im F$ is the imaginary part of the the unique* *bounded solution $F$ to the equation*

$$L_X^2 F - F = \frac{1}{2}\eta^+ Q. \tag{14}$$

*Proof.* — Since $Q$ is a holomorphic quadratic differential we have $Q(v, Jv) = iQ(v,v)$ and $L_{\eta^-}Q(v,v) = 0$. From $X = \eta^+ + \eta^-$ and $\eta^- = (X + iX^\perp)/2$ we obtain $L_X Q(v,v) = -iL_{X^\perp}Q(v,v) = L_{\eta^+}Q(v,v)$. The right hand side of equation (13)

$$\begin{aligned}
-L_X(\Re Q(v,Jv)) + \frac{1}{2}L_{X^\perp}(\Re Q(v,v)) &= -\Re L_X(Q(v,Jv)) + \frac{1}{2}\Re L_{X^\perp}(Q(v,v)) \\
&= -\Re i L_X(Q(v,v)) + \frac{1}{2}\Re i L_X(Q(v,v)) \\
&= -\frac{1}{2}\Re i L_X(Q(v,v)) = \frac{1}{2}\Im L_X(Q(v,v)) = \frac{1}{2}\Im L_{\eta^+}(Q(v,v)).
\end{aligned}$$

This concludes our proof. □

PROPOSITION 4.6. — *There is a constant $\gamma$ such that for every closed surface $M$ with Riemannian metric $g$ of curvature $-1$, and every quadratic differential $Q$ on $M$, the infinitesimal Morse correspondence $\Xi$ at $g$ in the direction of $\Re Q$ satisfies*

$$|||\Xi||| = \gamma ||Q||.$$

*Proof.* — Since $\Xi = (\Re F)X^\perp + \Re(L_X F)\Theta$ and the curvature is constant $-1$, we have $|||\Xi|||^2 = ||\Re F||^2 + ||\Re L_X F||^2$. The linearity of equation (14) implies that its solution $F$ belongs to the same irreducible subspace $H_Q$ of $L^2(SM, \mathrm{Liou}_g)$ to which the $Q$ belongs. Since all irreducible representation $H_Q$ generated by holomorphic quadratic differentials $Q$ are equivalent, we have $||F|| = \alpha||Q||$ and $||L_X F|| = \beta||Q||$ for some constants $\alpha$ and $\beta$ which do not depend either on $M$ or on the quadratic differential $Q$. Thus, using the second part of Lemma 4.1, we conclude that $|||\Xi||| = \gamma||Q||$ for some universal constant $\gamma$. □

In the remainder of this section we compute $\gamma$.

An irreducible representation of $SU(1,1)$ equivalent to the irreducible subspace $H_Q$ of $L^2(SM, \mathrm{Liou}_g)$ to which $Q$ belongs is the representation of $SU(1,1)$ on the Hilbert space $H = L^2(\mathbf{D}, 3\pi^{-1}(1-r^2)^2\, dA)$, where $dA$ is the Euclidean area, given by (see [La])

$$T_g f(z) = f\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)(\bar{b}z+\bar{a})^{-4} \quad \text{where} \quad g^{-1} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}.$$

The quadratic differential $Q \in L^2(SM, \mathrm{Liou}_g)$ is sent by the equivalence to the constant $||Q|| \in H$. An orthonormal basis for $H$ is given by $\begin{pmatrix} k+3 \\ 3 \end{pmatrix}^{1/2} z^k$. Thus analytic vectors are dense in $H$. Recall that the vector fields $X$, $X^\perp$, $\Theta$ embed in $su(1,1)$ by

$$X \mapsto \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \qquad X^\perp \mapsto \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix} \qquad \Theta \mapsto \begin{pmatrix} -i/2 & 0 \\ 0 & i/2 \end{pmatrix}.$$

Thus the derivations $L_X$, $L_{X^\perp}$, $L_X$, $L_\Theta$, $L_{\eta^+}$, $L_{\eta^-}$ are sent to the operators

$$X(F) = 2zF + \frac{1}{2}(z^2-1)F' \qquad X^\perp(F) = 2izF + \frac{i}{2}(1+z^2)F'$$

$$\Theta(F) = 2iF + izF' \qquad \eta^+(F) = 2zF + \frac{1}{2}z^2F' \qquad \eta^-(F) = -\frac{1}{2}F'$$

acting on analytic vectors $F$.

LEMMA 4.7. — *In the above equivalence of representations $\eta^+Q$ is identified with $2||Q||z$ and the unique bounded solution $F$ of equation (14) corresponds to the function $z \mapsto \dfrac{||Q||z}{3(z^2-1)}$.*

*Proof.* — Without loss of generality we can assume that $||Q|| = 1$. Since $Q$ is identified with the constant 1 in $H$ , $\eta^+Q$ is identified with $\eta^+(1) = 2z$. Thus the solution $F$ is identified with a solution of the equation

$$(X^2-1)g(z) = z$$

or of the equation

$$\frac{1}{4}(z^2-1)^2g'' + \frac{5}{2}z(z^2-1)g' + (5z^2-2)g = z.$$

The general solution of the above equation is

$$g(z) = \frac{z}{3(z^2-1)} + \frac{c_1}{(1-z)(1+z)^3} + \frac{c_2}{(1-z)^3(1+z)}$$

and the only solution square integrable with respect to the measure $3\pi^{-1}(1-r^2)^2\,dA$ is obtained for $c_1 = c_2 = 0$. □

PROPOSITION 4.8. — *Under the hypothesis of Proposition 4.4 we have $|||\Xi|||^2 = \dfrac{1}{24}||Q||^2$.*

*Proof.* — Again without loss of generality we can assume that $||Q|| = 1$. From equation (14) we obtain that

$$\bar{F}L_X^2F - |F|^2 = \frac{1}{2}\bar{F}L_{\eta^+}Q$$

and integrating by parts on $SM$ we obtain

$$\|F\|^2 + \|L_X F\|^2 = -\frac{1}{2}\langle F, L_{\eta^+} Q\rangle.$$

Thus from Lemma 4.1 and the definition of $\Xi$ we also have

$$|||\Xi|||^2 = \|\Re F\|^2 + \|\Re L_X F\|^2 = \frac{1}{2}(\|F\|^2 + \|L_X F\|^2) = -\frac{1}{4}\langle F, L_{\eta^+} Q\rangle.$$

By the previous Lemma 4.7

$$\begin{aligned}\langle F, L_{\eta^+} Q\rangle_{H_Q} &= \left\langle \frac{z}{3(z^2-1)}, 2z\right\rangle_H = -\frac{2}{3}\left\langle \sum_{n=0}^{\infty} z^{2n+1}, z\right\rangle_H \\ &= -\frac{2}{3}\|z\|_H^2 = -\frac{2}{3}\left(\frac{4}{3}\right) = -\frac{1}{6}.\end{aligned}$$

Thus

$$|||\Xi|||^2 = \frac{1}{24}.$$

This concludes the proof since $\|Q\| = 1$. □

## 5. Wolpert's Theorem : The second variation of $i$ is the Weil-Petersson metric.

LEMMA 5.1. — *Let $g_\alpha$ be a smooth curve of negatively curved Riemannian metrics on the closed manifold $M$. For every geodesic current $\mu$ the function $\alpha \mapsto i(g_\alpha, \mu)$ is smooth. Moreover all its derivatives are jointly continuous in $\alpha$ and $\mu$.*

*Proof.* — By [LMM] Appendix A page 591, we can find a $(g, g_\alpha)$-Morse correspondence $h_\alpha$, such that, if we write $h_\alpha(g^t v) = g_\alpha^{\psi_\alpha(t,v)} h_\alpha(v)$, then $\partial_t \psi_\alpha(t, v)$ exists. Moreover, if we set $\theta_\alpha(v) = \partial_t \psi_\alpha(t, v)_{t=0}$, then the derivatives $\partial_\alpha^k \theta_\alpha(v)$ exist for all $k$ and are all jointly continuous in $\alpha$ and $v$. But as in the proof of Theorem 1.4, we have $i(g_\alpha, \mu) = \int_{SM} \theta_\alpha(v)\, d\Phi_g(\mu)(v)$. □

PROPOSITION 5.2. — *Let $(M, g)$ be a negatively curved Riemannian manifold and let $g_\alpha$ be a smooth curve of Riemannian metrics on $M$ with $g_0 = g$. Let $S_\alpha = \partial_\alpha g_\alpha$. Let $\Xi$ be the infinitesimal Morse correspondence for $g_\alpha$ at $\alpha = 0$ . Let $\mu$ be a geodesic current. Then*

$$\partial_\alpha i(g_\alpha, \mu)|_{\alpha=0} = \frac{1}{2}\int_{SM} S(v, v)\, d\Phi_g(\mu)(v)$$

*and*

(15)

$$\partial_\alpha^2 i(g_\alpha,\mu)|_{\alpha=0} = \frac{1}{2}\int_{SM} \partial_\alpha S_\alpha(v,v)|_{\alpha=0}\, d\Phi_g(\mu)(v) + \frac{1}{2}\int_{SM} L_\Xi S(v,v)\, d\Phi_g(\mu)(v) - \frac{1}{4}\int_{SM} S(v,v)^2\, d\Phi_g(\mu)(v),$$

*where $L_\Xi S(v,v)$ is the Lie derivative of the function $v \mapsto S(v,v)$ on $SM$ in the direction of $\Xi$.*

*Proof.* — By the density of the Dirac currents (Theorem 1.2) and Lemma 5.1, we can assume that $\mu$ is a Dirac current given by the oriented homotopy class of the closed $g$-geodesic $\gamma$. For each $\alpha$ let $\gamma_\alpha$ be a closed $g_\alpha$-geodesic belonging to the homotopy class of $\gamma$. We can choose a parameterization $t \mapsto \gamma_\alpha(t)$, $0 \le t \le \ell$, such that $\gamma_\alpha(t)$ is jointly smooth in $\alpha$ and $t$. For $v \in TM$, we set $\|v\|_\alpha = (g_\alpha(v,v))^{1/2}$ and to simplify notation $\|v\| = \|v\|_0$.

We have :

$$\partial_\alpha \|\dot\gamma_\alpha\|_\alpha = \frac{1}{2}\frac{S_\alpha(\dot\gamma_\alpha,\dot\gamma_\alpha)}{\|\dot\gamma\|_\alpha} + \partial_\beta\|\dot\gamma_\beta\|_\alpha|_{\beta=\alpha}. \tag{16}$$

Since $\gamma_\alpha$ is a $g_\alpha$-geodesic the second term integrates to 0, the first derivative of the length $\ell_\alpha$ of $\gamma_\alpha$ is given by

$$\frac{d\ell_\alpha}{d\alpha} = \frac{1}{2}\int_0^\ell \frac{S_\alpha(\dot\gamma_\alpha,\dot\gamma_\alpha)}{\|\dot\gamma_\alpha\|_\alpha}\, dt. \tag{17}$$

Since $\ell_\alpha$ is equal to $i(g_\alpha,[\gamma])$, this proves the first formula of the Proposition.

The second derivative of $\ell_\alpha$ at $\alpha = 0$ is given, since $\gamma_0 = \gamma$, by

(18)

$$\left.\frac{d^2\ell_\alpha}{d\alpha^2}\right|_{\alpha=0} = \frac{1}{2}\int_0^\ell \frac{1}{\|\dot\gamma\|}(\partial_\alpha S_\alpha(\dot\gamma,\dot\gamma))|_{\alpha=0}\, dt + \frac{1}{2}\int_0^\ell \frac{1}{\|\dot\gamma\|}\left.\frac{\partial S(\dot\gamma_\alpha,\dot\gamma_\alpha)}{\partial\alpha}\right|_{\alpha=0} dt - \frac{1}{2}\int_0^\ell \frac{S(\dot\gamma,\dot\gamma)}{\|\dot\gamma\|^2}\left(\frac{\partial\|\dot\gamma_\alpha\|_\alpha}{\partial\alpha}\right)_{\alpha=0} dt.$$

In order to simplify (18), we will assume that the parameterization $t \mapsto \gamma_\alpha(t)$ further satisfies the following conditions :

a) the $g$-geodesic $\gamma$ is parameterized by $g$-arclength;

b) the vector field $X = \partial_\alpha\gamma_\alpha|_{\alpha=0}$ along $\gamma$ satisfies $g(X,\dot\gamma) = 0$.

From the second condition we obtain that $X$ is the projection under the map $Tp : T(SM) \to TM$ of the infinitesimal Morse correspondence $\Xi$.

From
$$S(\dot\gamma_\alpha,\dot\gamma_\alpha)=\|\dot\gamma_\alpha\|^2 S\Big(\frac{\dot\gamma_\alpha}{\|\dot\gamma_\alpha\|},\frac{\dot\gamma_\alpha}{\|\dot\gamma_\alpha\|}\Big),$$
differentiating with respect to $\alpha$, we obtain at $\alpha=0$
$$\frac{\partial S(\dot\gamma_\alpha,\dot\gamma_\alpha)}{\partial\alpha}\bigg|_{\alpha=0}=\|\dot\gamma\|^2 L_\Xi S(\dot\gamma,\dot\gamma)+S(\dot\gamma,\dot\gamma)\frac{\partial g(\dot\gamma_\alpha,\dot\gamma_\alpha)}{\partial\alpha}\bigg|_{\alpha=0}. \tag{19}$$
Using the fact that $\gamma$ is a $g$-geodesic parameterized by $g$-arclength and $g(X,\dot\gamma)=0$, we also have
$$\frac{\partial g(\dot\gamma_\alpha,\dot\gamma_\alpha)}{\partial\alpha}\bigg|_{\alpha=0}=2g(\nabla_X\dot\gamma,\dot\gamma)=2g(\nabla_{\dot\gamma}X,\dot\gamma)=\frac{dg(X,\dot\gamma)}{dt}=0. \tag{20}$$
Using (20) and the fact that $\|\dot\gamma\|=1$, equation (19) becomes
$$\frac{\partial S(\dot\gamma_\alpha,\dot\gamma_\alpha)}{\partial\alpha}\bigg|_{\alpha=0}=L_\Xi S(\dot\gamma,\dot\gamma). \tag{21}$$
Furthermore from (16) and (20) we also have
$$\frac{\partial\|\dot\gamma_\alpha\|_\alpha}{\partial\alpha}\bigg|_{\alpha=0}=\frac12 S(\dot\gamma,\dot\gamma).$$
Finally equation (18) becomes
$$\frac{d^2\ell_\alpha}{d\alpha^2}\bigg|_{\alpha=0}=\frac12\int_0^\ell \partial_\alpha S_\alpha(\dot\gamma,\dot\gamma)|_{\alpha=0}\,dt+\frac12\int_0^\ell L_\Xi S(\dot\gamma,\dot\gamma)\,dt-\frac14\int_0^\ell S(\dot\gamma,\dot\gamma)^2\,dt.$$
This concludes the proof. □

When the geodesic current $\mu$ is given by the Liouville measure induced by $g$ the formula given by Proposition 5.2 simplifies further especially if the metrics $g_\alpha$ assign the same volume to $M$.

PROPOSITION 5.3. — *Let $g_\alpha$ be a smooth curve of negatively curved Riemannian metrics with constant volume on the closed manifold $M$ with $g_0=g$. Let $S=\partial_\alpha g_\alpha|_{\alpha=0}$ and define $H$ by $S(v,w)=g(Hv,w)$. Then*
$$\frac{d^2 i(g_\alpha,g)}{d\alpha^2}\bigg|_{\alpha=0}=-|||\Xi|||^2+\frac{s_{n-1}}{2n(n+2)}\int_M\Big[(n+1)\operatorname{Tr}H^2-\frac{n+3}{2}(\operatorname{Tr}H)^2\Big]\,d\operatorname{Vol}(g)$$
*where $s_{n-1}$ is the volume of the $(n-1)$-dimensional unit sphere in Euclidean space and $\Xi$ is the infinitesimal conjugacy along $g_\alpha$ at $g$.*

*Proof.* — Set $S_\alpha=\partial_\alpha g_\alpha$ and define $H_\alpha$ by $S_\alpha(v,w)=g_\alpha(H_\alpha v,w)$. From
$$\partial_\alpha d\operatorname{Vol}(g_\alpha)=\frac12\operatorname{Tr}(H_\alpha)\,d\operatorname{Vol}(g_\alpha),$$

differentiating with respect to $\alpha$ one more time and integrating over $M$, we obtain

$$\frac{d^2\,\mathrm{Vol}(g_\alpha)}{d\alpha^2} = \frac{1}{2}\int_M [\mathrm{Tr}(\partial_\alpha H_\alpha) + \frac{1}{2}\mathrm{Tr}(H_\alpha)^2]\, d\,\mathrm{Vol}(g_\alpha). \tag{22}$$

Since $\partial_\alpha S_\alpha(v,w) = g_\alpha\big((H_\alpha^2 + \partial_\alpha H_\alpha)v, w\big)$ we have $\mathrm{Tr}\,\partial_\alpha H_\alpha = \mathrm{Tr}_{g_\alpha}\partial_\alpha S_\alpha - \mathrm{Tr}\,H_\alpha^2$ and equation (22) becomes

$$\frac{d^2\,\mathrm{Vol}(g_\alpha)}{d\alpha^2} = \frac{1}{2}\int_M [\mathrm{Tr}_{g_\alpha}(\partial_\alpha S_\alpha) - \mathrm{Tr}(H_\alpha^2) + \frac{1}{2}\mathrm{Tr}(H_\alpha)^2]\, d\,\mathrm{Vol}(g_\alpha).$$

Since we assume that $\mathrm{Vol}(g_\alpha)$ is constant , it follows that

$$\int_M \mathrm{Tr}_{g_\alpha}(\partial_\alpha S_\alpha) d\,\mathrm{Vol}(g_\alpha) = \int_M [\mathrm{Tr}(H_\alpha^2) - \frac{1}{2}\mathrm{Tr}(H_\alpha)^2]\, d\,\mathrm{Vol}(g_\alpha). \tag{23}$$

Integration over the fibers of the bundle $S_g M = SM \to M$ gives

(24)

$$\int_{SM} \partial_\alpha S_\alpha|_{\alpha=0}(v,v)\, d\,\mathrm{Liou}_g(v) = \frac{s_{n-1}}{n}\int_M \mathrm{Tr}_g(\partial_\alpha S_\alpha\|_{\alpha=0})\, d\,\mathrm{Vol}(g) \quad \text{and}$$

$$\int_{SM} S(v,v)^2\, d\,\mathrm{Liou}_g(v) = \frac{s_{n-1}}{n(n+2)}\int_M [2\,\mathrm{Tr}\,H^2 + (\mathrm{Tr}\,H)^2] d\,\mathrm{Vol}(g).$$

From Propositions 2.7, 2.10, (24) and (23) we obtain :

$$\begin{aligned}\left.\frac{d^2 i(g_\alpha, g)}{d\alpha^2}\right\|_{\alpha=0} &= \frac{s_{n-1}}{2n}\int_M [\mathrm{Tr}(H^2) - \frac{1}{2}\mathrm{Tr}(H)^2]\, d\,\mathrm{Vol}(g) + \\ &- |||\Xi|||^2 - \frac{1}{4}\frac{s_{n-1}}{n(n+2)}\int_M [2\,\mathrm{Tr}\,H^2 + (\mathrm{Tr}\,H)^2] d\,\mathrm{Vol}(g),\end{aligned}$$

and we are done. □

For next theorem see [Wo] Theorem 3.4 on page 152, Wolpert denotes by $A$ the intersection $i$.

THEOREM 5.4 (Wolpert). — *Let $(M,g)$ be an oriented surface of constant negative curvature -1 and let $g_\alpha$ be a smooth curve of Riemannian metrics on $M$ of constant negative curvature with $g_0 = g$. For $S_\alpha = \partial_\alpha g_\alpha|_{\alpha=0}$, we write $S = L_V g + \Re Q$ as in Lemmata 4.2-4.3 where $Q$ is a holomorphic quadratic differential and $V$ is a smooth vector field on $M$. Then*

$$\left.\frac{d^2 i(g_\alpha, g)}{d\alpha^2}\right|_{\alpha=0} = \frac{1}{3}\|Q\|^2 = \frac{2}{3}\|\Re Q\|^2 = \frac{2\pi}{3}\|Q\|^2_{WP},$$

*where $\|.\|_{WP}$ is the norm induced from the Weil-Petersson metric on quadratic differentials.*

*Proof.* — Since the intersection $i(g_\alpha, g)$ is invariant under the action of $\mathrm{Diff}_0(M)$ on each of its arguments, it is sufficient to consider the case $S = \Re Q$. Let $J$ be the complex structure determined by $g$ and the orientation of $M$. As before, let $H$ be defined by $S(v,w) = g(Hv,w)$. If $v$ is a $g$-unit vector in the tangent space to $M$ the matrix of $H$ in the orthonormal basis $(v, Jv)$ is

$$\begin{pmatrix} S(v,v) & S(v,Jv) \\ S(v,Jv) & S(Jv,Jv) \end{pmatrix}.$$

For $S = \Re Q$, from $Q(Jv,Jv) = -Q(v,v)$ and $Q(v,Jv) = iQ(v,v)$, we have

$$\begin{aligned} \mathrm{Tr}(H^2) &= S(v,v)^2 + S(Jv,Jv)^2 + 2S(v,Jv)^2 \\ &= 2(\Re Q(v,v))^2 + 2(\Im Q(v,v))^2 = 2|Q(v,v)|^2. \end{aligned} \tag{25}$$

Integrating over $M$

$$\int_M \mathrm{Tr}(H^2)(x)\, d\,\mathrm{Vol}(g)(x) = \frac{1}{2\pi}\int_{SM} \mathrm{Tr}(H^2)(pv)\, d\,\mathrm{Liou}_g(v) = \frac{1}{\pi}\|Q\|^2.$$

From $Q(Jv,Jv) = -Q(v,v)$, we also have $\mathrm{Tr}(H) = \Re\,\mathrm{Tr}_g Q = 0$. From Proposition 4.8 $|||\Xi|||^2 = \dfrac{1}{24}\|Q\|^2$. Using the previous proposition we obtain

$$\left.\frac{d^2 i(g_\alpha, g)}{d\alpha^2}\right|_{\alpha=0} = -\frac{1}{24}\|Q\|^2 + \frac{3}{8}\|Ql\|^2 = \frac{10}{3}\|Q\|^2 = \frac{2}{3}\|\Re Q\|^2,$$

concluding the proof of the first 2 equalities.

To finish the proof, let us recall the definition of the Weil-Petersson metric on quadratic differential. If we denote by $\|.\|_g$ the norm induced by $g$ on the tensor bundle $\otimes^2_{\mathbb{C}} T^*M$, we have $\|Q\|^2_{WP} = \int_M \|Q_x\|^2_g\, d\,\mathrm{Vol}(g)(x)$. Now, since the fibers of $\otimes^2_{\mathbb{C}} T^*M$ are of dimension 1 over $\mathbb{C}$, we have $|Q(v,v)| = \|Q_x\|_g$ for every $g$-unit tangent vector $v \in T_xM$. Remark that by (25) this coincides with the formula we gave in the introduction. □

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