ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 43, nº 1 (1993), p. 125-142

<http://www.numdam.org/item?id=AIF_1993__43_1_125_0>

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Ann. Inst. Fourier, Grenoble
43, 1 (1993), 125-142

AN EXTENSION OF THE NEWTON-PUISEUX POLYGON CONSTRUCTION TO GIVE SOLUTIONS OF PFAFFIAN FORMS

by José CANO

0. Introduction.

We say that a formal power series $A(X,Y) = \sum A_{ij} X^i Y^j \in \mathbb{C}[[X,Y]]$ has s-Gevrey index ($\infty \geq s \geq 1$) if

$$\beta_{s-1}(A) = \sum \frac{A_{ij}}{(i+j)!^{s-1}} X^i Y^j \in \mathbb{C}\{X,Y\}.$$

Let us observe that 1-Gevrey index means convergence. Consider the formal Pfaffian form $\omega = A(X,Y)dX + B(X,Y)dY$, $A, B \in \mathbb{C}[[X,Y]]$. The form ω has s-Gevrey index if A and B have s-Gevrey index. The parametrization $(x(t), y(t)) \in \mathbb{C}[[t]]^2$ has s-Gevrey index if either $x(t) = 0$ or it is irreducible, in the sense of [18], and equivalent to one of the type $(t^n, y_1(t))$ such that $y_1(t)$ has s-Gevrey index.

A formal solution of (ω=0) through the origin is a parametrization $(x(t), y(t)) \in \mathbb{C}[[t]]^2 \setminus \mathbb{C}^2$ such that $x(0) = y(0) = 0$ and

$$A(x(t), y(t))x'(t) + B(x(t), y(t))y'(t) = 0.$$

C. Camacho and P. Sad [3] proved that if A and B are holomorphic then there is a convergent solution of (ω=0). Their method uses : first

Partially supported by the D.G.I.C.Y.T.
Key words : Pfaffian form – Integral curve – Gevrey solutions.
A.M.S. Classification : 34A20 – 58A17 – 34E05.

an index theorem for complex singular foliations and second the reduction of singularities of Pfaffian forms studied among others by Bendixon [1], Poincaré [13], Seidenberg [17], Mattei and Moussu [12], and F. Cano [4].

In this paper we construct a s-Gevrey index solution of $(\omega{=}0)$ if ω has s-Gevrey index. By one hand this extends the result of Camacho and Sad (case $s = 1$), by the other, our construction based on the Newton Polygon for differential equations (following a method suggested by Briot and Bouquet [2], Fine [6] [7], Ritt [16], and Ince [8]) is rather elementary and gives the solution by an algorithm.

1. The Newton polygon.

Let ω be a formal Pfaffian form as above. The differential polynomial associated to ω is given by $f_\omega = A_\omega(X,Y) + B_\omega(X,Y)\,Y'$, where $A_\omega, B_\omega \in \mathbb{C}[[X,Y]]$.

We will say that $f \in \mathcal{C}_q$, $q \in \mathbb{N}$, if we have that

$$f = A(X,Y) + B(X,Y)\,Y' = \sum_{(\alpha,\beta)} M_{\alpha\beta}(f),$$

where $M_{\alpha\beta}(f) = A_{\alpha\beta}X^\alpha Y^\beta + B_{\alpha\beta}X^{\alpha+1}Y^{\beta-1}Y'$, $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are complex numbers such that $B_{(\alpha,0)} = 0$ for all α, and the index (α,β) runs over the set $\left(\frac{1}{q}\mathbb{Z}\right)_{(\geq -1)} \times \mathbb{N}$ where $\left(\frac{1}{q}\mathbb{Z}\right)_{(\geq -1)} = \{r \geq -1 \mid q \cdot r \in \mathbb{Z}\}$.

A Puiseux's series $z \in \mathbb{C}[[X^{1/n}]]$ with $\operatorname{ord}_X(z) > 0$ is a solution of $(f{=}0)$ if and only if $f[z] = A(X,z) + B(X,z)\dfrac{\partial z}{\partial X} = 0$. The series z is a solution of $(f_\omega = 0)$ if and only if the pair $(t^n, z(t^n))$ is a solution of $(\omega{=}0)$. We also say that $(X{=}0)$ is a solution of $(f_\omega = 0)$ if the pair $(0,t)$ is a solution of $(\omega{=}0)$ (i.e. $B_\omega(0,Y) = 0$).

Let $\Delta(f) = \{(\alpha,\beta) \mid M_{\alpha\beta}(f) \neq 0\}$. As usual we define the *Newton polygon* $N(f)$ of f with respect to (X,Y) to be the convex envelope of $\bigcup_{P\in\Delta(f)} (P + \mathbb{R}_0^2)$, $\mathbb{R}_0 = \{r \in \mathbb{R} \mid r \geq 0\}$.

For any $\mu \in \mathbb{Q}_+$ let $L(f,\mu)$ be the only straight line with slope $-1/\mu$ which intersects $N(f)$ exactly either on a side or on a vertex. Consider

$$\Phi_{(f,\mu)}(C) = \sum_{(\alpha,\beta)\in L(f,\mu)} (A_{\alpha\beta} + \mu B_{\alpha\beta})C^\beta$$

$$In^{(\mu)}(f) = \sum_{(\alpha,\beta)\in L(f,\mu)} M_{\alpha\beta}(f).$$

A simple computation over the least degree terms of $f[z]$ proves the following result :

LEMMA 1. — *Let* $z = c_0 X^{\mu_0} + \text{h.d.t.}$ *(higher degree terms) be a solution of* $(f{=}0)$*, then we have that* $\Phi_{(f,\mu_0)}(c_0) = 0$.

2. Changing the variable.

For each $\mu \in \mathbb{Q}_+$ and $c \in \mathbb{C}$ define the differential polynomial $f[cX^\mu + Y]$ by

$$f[cX^\mu + Y] = f(X, cX^\mu + Y, \mu c X^{\mu-1} + Y').$$

Then the following properties hold :

2.1. The series $z = \sum_{i=1}^{\infty} c_i X^{\mu_i}$ is a solution of $(f{=}0)$ if and only if $\sum_{i=2}^{\infty} c_i X^{\mu_i}$ is a solution of $(f[c_1 X^{\mu_1} + Y]{=}0)$. In particular, $z = c_1 X^{\mu_1}$ is a solution of $(f{=}0)$ if and only if $(Y{=}0)$ is a solution of $(f[c_1 X^{\mu_1} + Y]{=}0)$.

2.2. If $\Delta(f) \subset \left(\frac{1}{q}\mathbb{Z}\right)_{(\geq -1)} \times \mathbb{N}$ then there is an integer $q' > 0$ such that $\Delta(f[cX^\mu + Y]) \subset \left(\frac{1}{q'}\mathbb{Z}\right)_{(\geq -1)} \times \mathbb{N}$. In particular, the Newton Polygon $N(f[cX^\mu + Y])$ has a finite number of sides and vertices.

2.3. If $\mu' < \mu$ then :

$$\begin{aligned} L(f, \mu') &= L(f[cX^\mu + Y], \mu') \\ In^{(\mu')}(f) &= In^{(\mu')}(f[cX^\mu + Y]) \\ \Phi_{(f,\mu')}(C) &= \Phi_{(f[cX^\mu+Y],\mu')}(C). \end{aligned}$$

Thus if $Q(f, \mu)$ is the highest point of $N(f) \cap L(f, \mu)$, the portions of $N(f)$ and $N(f[cX^\mu + Y])$ higher than $Q(f, \mu)$ are equal (see the picture of the example). Moreover $L(f, \mu) = L(f[cX^\mu + Y], \mu)$ and if we write

$$In^{(\mu)}(f) = A_0 X^\alpha + \sum_{i=1}^{t} A_i X^{\alpha_i} Y^i + B_i X^{\alpha_i + 1} Y^{i-1} Y', \qquad \alpha_i = \alpha - i\mu,$$

then we have that for any $c \in \mathbb{C}$

$$In^{(\mu)}(f[cX^\mu + Y]) = A_0(c) X^\alpha + \sum_{i=1}^{t} A_i(c) X^{\alpha_i} Y^i + B_i(c) X^{\alpha_i+1} Y^{i-1} Y'.$$

If we write $\Phi_{(f,\mu)}(C) = \alpha(C) + \mu C\beta(C)$ where $\beta(C) = \sum_{i=1}^{t} B_i C^{i-1}$ and $\alpha(C) = \sum_{i=0}^{t} A_i C^i$ then we have that

$$
(1) \qquad \begin{aligned}
A_0(c) &= \Phi(c) \\
A_j(c) &= \frac{1}{j!}\,\Phi^{(j)}(c) - \mu\frac{1}{(j-1)!}\beta^{(j-1)}(c) \quad & j = 1,\ldots,t \\
B_j(c) &= \frac{1}{(j-1)!}\,\beta^{(j-1)}(c) & j = 1,\ldots,t
\end{aligned}
$$

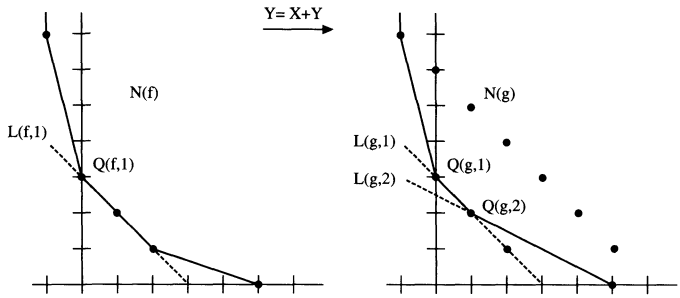
where $\Phi^{(j)}(c) = \dfrac{\partial^j \Phi}{\partial C^j}(c)$ and $\beta^{(j)}(c) = \dfrac{\partial^j \beta}{\partial C^j}(c)$. Specially $A_t = A_t(c)$, $B_t = B_t(c)$, and $A_0(c) = 0$ if and only if $\Phi(c) = 0$.

3. The algorithm.

The basic idea of Fine [6], [7] modified by Ince [8], in order to find a solution of $(f_\omega{=}0)$, is to extend to differential equations the classical Puiseux's construction for algebraic curves in the following way. Assume that neither $(Y{=}0)$ nor $(X{=}0)$ are solutions of $(f_\omega{=}0)$. Then there is a side of $N(f_\omega)$ with slope $-1/\mu_0$ such that $\Phi_{(f_\omega,\mu_0)}(C)$ has a root $c_0 \neq 0$. Consider $g = f_\omega[c_0 X^{\mu_0} + Y]$. If $(Y{=}0)$ is a solution of $(g{=}0)$, then $c_0 X^{\mu_0}$ is a solution of $(f_\omega{=}0)$ and we are done. Otherwise, look for a side of $N(g)$ of slope $-1/\mu_1$, with $\mu_1 > \mu_0$, such that $\Phi_{(g,\mu_1)}(C)$ has a root $c_1 \neq 0$. Continue in this way.

This algorithm, does not work in general. Actually, it may happen even with the restrictive conditions imposed by Ince [8] that the polynomials $\Phi_{(g,\mu_1)}(C)$ with $\mu_1 > \mu_0$ do not have any non zero root, as the following example shows :

Example. — Let $f = Y^6Y' + XY^2Y' + XY^2 - 3X^2YY' - X^2Y + 2X^3Y' + X^5$ and let L be the side with slope -1, then $\Phi_{(f,1)}(C) = C(C-1)^2$. The only nonzero root is $C = 1$. Put $g = f[X+Y] = (X+Y)^6(1+Y') + XY^2Y' - X^2YY' + 2XY^2 + X^5$. The only side of $N(g)$ with slope bigger than -1 has slope $-1/2$ and $\Phi_{(g,2)} = 1 + (2 + 2(-1))C^2 = 1$. Thus we can not continue the procedure. Moreover, if $\mu > 1$ we have that $\Phi_{(g,\mu)}$ does not have non zero roots, then by the Lemma 1, it does not exist any solution $z = cX +$ h.d.t. with $c \neq 0$.



Now we shall give a criterion to choose at each step the "correct" side of the Newton Polygon and the "correct" root of the associated polynomial in order to avoid situations as in the example. Then the procedure will always continue, unless we arrive at a situation in which $(Y{=}0)$ is a solution. In this way we shall produce a formal solution of $(f_\omega{=}0)$.

Let L be a side of $N(f)$ with slope $-1/\mu$ and denote by $(a(L), b(L))$ the highest vertex and $(a'(L), b'(L))$ the lowest vertex on L. The side L is *good* if and only if the following properties holds :

$$(L,a)_f := B_{(a(L),b(L))} \neq 0 \text{ and } \frac{-A_{(a(L),b(L))}}{B_{(a(L),b(L))}} \notin \mathbb{Q}_{(\geq\mu)} = \{r \in \mathbb{Q} \mid r \geq \mu\}$$

$$(L,b)_f := A_{(a'(L),b'(L))} + \mu B_{(a'(L),b'(L))} \neq 0.$$

We say that L is the *principal side* of f if $(Y{=}0)$ is not a solution of $(f{=}0)$ and L is the good side of $N(f)$ with biggest slope.

LEMMA 2. — *Assume that $(Y{=}0)$ is not a solution of $(f{=}0)$ and there is a side L of $N(f)$ with slope $-1/\mu$ satisfying $(L,a)_f$. Then the principal side of f exists and it has slope greater or equal than $-1/\mu$.*

Proof. — Let $L_0 = L, \ldots, L_k$ be all the sides of $N(f)$ with slopes $-1/\mu < -1/\mu_1 < \ldots < -1/\mu_k$ greater or equal than $-1/\mu$. Then $(L_0, a)_f$ and $(L_k, b)_f$ are true. If $i = 0, \ldots, k-1$ we have that

$$\text{not } (L_i, b)_f \Longrightarrow (L_{i+1}, a)_f$$

(remember that $(a'(L_i), b'(L_i)) = (a(L_{i+1}), b(L_{i+1}))$). Hence, there is at least a good side L_i. □

COROLLARY 1. — *Assume that L is the principal side of f with slope $-1/\mu$, then one of the following properties holds :*

(i) $B_{(a'(L),b'(L))} = 0$

(ii) $B_{(a'(L),b'(L))} \neq 0$ and $\dfrac{-A_{(a'(L),b'(L))}}{B_{(a'(L),b'(L))}} \in \mathbb{Q}_{(>\mu)} = \{r \in \mathbb{Q} \mid r > \mu\}$.

Proof. — If L is the side of $N(f)$ with biggest slope then (i) is true ($(Y{=}0)$ is not a solution of $(f{=}0)$). If it is not so, let L' be the side of $N(f)$ with highest vertex equal to the lowest vertex of L. Then (L', a) is not true by the lemma and the principality of L, thus our result follows because

$$\text{(i) or (ii)} \Longleftrightarrow (L, b)_f \text{ and not } (L', a)_f.$$

□

COROLLARY 2. — *Let f_ω be the differential polynomial associated to ω and assume that $(X{=}0)$ and $(Y{=}0)$ are not solution of $(f_\omega{=}0)$, then f_ω has principal side.*

Proof. — If $(X{=}0)$ is not a solution of $(f_\omega{=}0)$, then $(-1,\beta)$ is a vertex of $N(f_\omega)$ for some β, and $A_{(-1,\beta)} = 0$. Thus, if L the side of $N(f_\omega)$ whose highest vertex is $(-1,\beta)$, then $(L,a)_{f_\omega}$ is true. □

PROPOSITION 1. — *Assume that f has principal side L with slope $-1/\mu$. Then it is possible to choose effectively a root $c_0 \neq 0$ of $\Phi_{(f,\mu)}$ such that if $g = f[c_0X^\mu + Y]$ then one of the following properties holds :*

(i) *$(Y{=}0)$ is a solution of $(g = 0)$.*

(ii) *g has a principal side with slope bigger than $-1/\mu$.*

Proof. — Let $h = In^{(\mu)}(f)$ and write

$$h = A_0X^\alpha + \sum_{i=k}^{t} A_iX^{\alpha_i}Y^i + B_iX^{\alpha_i+1}Y^{i-1}Y'; \qquad \alpha_i = \alpha - i\mu,$$

where $k \le t$, $(A_k, B_k) \neq (0,0)$ and $(A_t, B_t) \neq (0,0)$.

Let $\mathcal{R} = \{\gamma_1, \ldots, \gamma_s\}$ be the set of all nonzero roots of $\Phi_{(f,\mu)}(C)$ ($\mathcal{R}$ is not empty because L is good). By 2.3 we can write

$$\begin{aligned} In^{(\mu)}(f[\gamma_jX^\mu + Y]) &= h[\gamma_jX^\mu + Y] \\ &= \sum_{i=k_j}^{t} A_i(\gamma_j)X^{\alpha_i}Y^i + B_i(\gamma_j)X^{\alpha_i+1}Y^{i-1}Y' \end{aligned}$$

where either $A_{k_j}(\gamma_j)$ or $B_{k_j}(\gamma_j)$ is different from zero.

We choose c_0 as any root γ_{j_0} which satisfies the following property :

(P) $B_{k_{j_0}}(\gamma_{j_0}) \neq 0$ and $\dfrac{-A_{k_{j_0}}(\gamma_{j_0})}{B_{k_{j_0}}(\gamma_{j_0})} \notin \mathbb{Q}_{(>\mu)} = \{r \in \mathbb{Q} \mid r > \mu\}$.

If $(Y{=}0)$ is not a solution of $(g = 0)$ then (ii) holds. In fact, let L' be the side of $N(g)$ whose highest vertex is $(\alpha_{k_{j_0}}, k_{j_0}) = (\alpha', \beta')$. Then the slope $-1/\mu'$ of L' is bigger than $-1/\mu$. Moreover,

$$M_{\alpha'\beta'}(g) = A_{k_{j_0}}(c_0)X^{\alpha'}Y^{\beta'} + B_{k_{j_0}}(c_0)X^{\alpha'+1}Y^{\beta'-1}Y'.$$

Since $\mu' > \mu$, the property (P) implies that $(L', a)_g$ is true and we can apply the Lemma 2.

It remains only to prove that there is a root of $\Phi_{(f,\mu)}(C)$ satisfying (P). In order to do this, we shall reason by contradiction, assuming that (P) is not satisfied for any γ_j. Then for all index j we have that (a) and $\{(b.1)$ or $(b.2)\}$ hold, where :

(a) $\qquad A_i(\gamma_j) = B_i(\gamma_j) = 0 \qquad i = 1, 2, \ldots, k_j - 1.$

$(b.1)$ $\qquad B_{k_j}(\gamma_j) = 0$ and $A_{k_j}(\gamma_j) \neq 0.$

$(b.2)$ $\qquad B_{k_j}(\gamma_j) \neq 0$ and $\dfrac{-A_{k_j}(\gamma_j)}{B_{k_j}(\gamma_j)} \in \mathbb{Q}_{(>\mu)}.$

Write as in 2.3 :

$$(2)\qquad \begin{aligned}\Phi(C) = \Phi_{(f,\mu)}(C) &= A_0 + \sum_{i=k}^{t}(A_i + \mu B_i)C^i \\ &= (A_t + \mu B_t)C^r(C - \gamma_1)^{e_1} \cdots (C - \gamma_s)^{e_s} = \alpha(C) + \mu C\beta(C).\end{aligned}$$

By (1), the properties (a), $(b.1)$ and $(b.2)$ are respectively equivalent to :

$(a)'$ $\qquad \Phi^{(i)}(\gamma_j) = 0$ and $\beta^{(i-1)}(\gamma_j) = 0 \quad i = 1, \ldots, k_j - 1.$

$(b.1)'$ $\qquad \beta^{(k_j-1)}(\gamma_j) = 0$ and $\Phi^{(k_j)}(\gamma_j) \neq 0.$

$(b.2)'$ $\qquad \beta^{(k_j-1)}(\gamma_j) \neq 0$ and $\dfrac{-1}{k_j}\dfrac{\Phi^{(k_j)}(\gamma_j)}{\beta^{(k_j-1)}(\gamma_j)} \in \mathbb{Q}_+.$

In particular $\Phi^{(k_j)}(\gamma_j) \neq 0$ for all $j = 1, \ldots, s$ and thus $k_j = e_j$. Let us assume that $(a)'$ and $(b.1)'$ are satisfied for $j = 1, \ldots, l$. Then we have the following properties :

I. $\beta(\gamma_j) = \beta^{(1)}(\gamma_j) = \cdots = \beta^{(e_j-2)}(\gamma_j) = 0$ if $j = 1, \ldots, s$.

II. $\beta^{(e_j-1)}(\gamma_j) = 0$ if $j = 1, \ldots, l$.

III. For each $j = l + 1, \ldots, s$ there is a $q_j \in \mathbb{Q}_+$ such that

$$\beta^{(e_j-1)}(\gamma_j) = (e_j - 1)!(-q_j)(A_t + \mu B_t)\gamma_j^r \prod_{\substack{i=1 \\ i\neq j}}^{s}(\gamma_j - \gamma_i)^{e_i}.$$

Now we shall distinguish two possibilities : either $A_0 = 0$ or $A_0 \neq 0$.

– Assume that $A_0 = 0$. Then $r = k \geq 1$. Write

$$\beta(C) = \sum_{i=k}^{t} B_i C^{i-1}. \tag{3}$$

By the properties I, II, and III, we can write $\beta(C)$ as follows :
(4)

$$\begin{aligned}\beta(C) =& B_t C^{k-1} \prod_{i=1}^{s} (C - \gamma_i)^{e_i} \\ &+ \sum_{i=l+1}^{s} (-q_i)\gamma_i (A_t + \mu B_t) C^{k-1} (C - \gamma_i)^{e_i - 1} \prod_{\substack{j=1 \\ j \neq i}}^{s} (C - \gamma_j)^{e_j}.\end{aligned}$$

Looking at the coefficients of C^{k-1} in (4) we have that

$$B_k = \Big(\prod_{i=1}^{s} (-\gamma_i)^{e_i}\Big) \Big\{ B_t + \Big(\sum_{i=l+1}^{s} q_i\Big)(A_t + \mu B_t) \Big\} \tag{5}$$

and by (2)

$$A_k + \mu B_k = \Big(\prod_{i=1}^{s} (-\gamma_i)^{e_i}\Big) \Big(A_t + \mu B_t\Big). \tag{6}$$

Since L is the principal side, by Corollary 1 we have

$$B_k = 0 \quad \text{or} \quad \Big\{ B_k \neq 0 \,\text{and}\, \frac{-A_k}{B_k} \in \mathbb{Q}_{(>\mu)} \Big\} \tag{7}$$

and also $(L, a)_f$ and $(L, b)_f$ hold.

Assume that $B_k = 0$. By (5) we have that

$$\Big(\frac{A_t}{B_t} + \mu\Big)\Big(\sum_{i=l+1}^{s} q_i\Big) = -1 \Longrightarrow \frac{-A_t}{B_t} \in \mathbb{Q}_{(>\mu)}$$

in contradiction with $(L, a)_f$. Then necessarily $B_k \neq 0$. Dividing (6) over (5) we obtain :

$$\frac{A_k}{B_k} + \mu = \frac{1}{\Big(\sum_{i=l+1}^{s} q_i\Big) + \frac{1}{\left(\frac{A_t}{B_t} + \mu\right)}} \Longrightarrow \frac{-A_k}{B_k} \notin \mathbb{Q}_{(>\mu)}$$

in contradiction with the property (7).

– Assume that $A_0 \neq 0$. Then $r = 0$ and hence $e_1 + \cdots + e_s = t$ in view of (2). Now, the properties I, II, and III define entirely $\beta(C)$ and we have that

$$\beta(C) = \sum_{i=l+1}^{s} (-q_i)(A_t + \mu B_t)(C - \gamma_1)^{e_1} \cdots (C - \gamma_i)^{e_i - 1} \cdots (C - \gamma_s)^{e_s}$$

hence

$$B_t = (A_t + \mu B_t)\Big(\sum_{i=l+1}^{s} (-q_i)\Big) \Longrightarrow \frac{-A_t}{B_t} \in \mathbb{Q}_{(>\mu)}$$

in contradiction with $(L,a)_f$. □

Remark 1. — Our algorithm is then defined by applying repeatedly the above Proposition 1.

Remark 2. — To use the algorithm in the practice it is only necessary to select c_0 at each step with the property (P), and this is easy to do.

4. The formal solution.

Here we prove that the above algorithm produces a formal Puiseux's solution of $(f_\omega=0)$. In the next paragraph we shall prove that this solution is in fact a convergent one if f_ω is convergent.

THEOREM 1. — *Let f_ω be the differential polynomial associated to ω and assume that neither $(X=0)$ nor $(Y=0)$ are solutions of $(f_\omega=0)$. Let L be the principal side of f_ω (it exits by Corollary 2) and let $-1/\mu$ be the slope of L. Then the above algorithm provides a solution of $(f_\omega=0)$ of the type*

$$z = \sum_{i=p}^{\infty} c_i X^{i/q} \quad \textit{where} \quad c_p \neq 0 \quad \textit{and} \quad \mu = p/q.$$

Proof. — Applying repeatedly the Proposition 1 we can obtain $z = \sum_{i=0}^{\infty} c_i X^{\mu_i}$, with $\mu_i < \mu_{i+1}$, such that if we write $f_0 = f_\omega$, $f_{i+1} = f_i[c_i X^{\mu_i} + Y]$ then one of the following statements holds :

(i) f_i has a principal side L_i with slope $-1/\mu_i$ and $c_i \neq 0$ is a root of $\Phi_{(f_i,\mu_i)}(C)$ such that f_{i+1} has either $(Y=0)$ by solution or has a principal side with slope $-1/\mu_{i+1} > -1/\mu_i$.

(ii) f_i has not a principal side, $(Y=0)$ is a solution of $(f_i=0)$, and the coefficients c_j are equal to zero for all $j \geq i$.

It remains to show that z is a solution of $(f_\omega=0)$ and it has the form described in the theorem. In order to do this, we can suppose that all the coefficients c_i are different from zero; otherwise by (ii) only a finite number of c_i are different from zero and then the theorem is done.

Let $Q_i = (a_i, b_i)$ be the highest vertex on $L_i = L(f_i, \mu_i)$. By 2.3 we have that Q_i is a vertex of $N(f_{i+1})$. Since the slope of L_{i+1} is bigger than the slope of L_i, then $b_{i+1} \leq b_i$. Hence there is an index i_0 such that $b_i = b_{i_0}$, $\forall i \geq i_0$. This implies that $Q_i = Q_{i_0}$ for $i \geq i_0$. Call this point $Q_{i_0} = (a, b)$ the *pivot point* of f with respect to z. Let us observe that $b \geq 1$ because c_i is a root of $\Phi_{(f_i,\mu_i)}(C)$ for all index.

Assume that $b = 1$. Then we have that

$$In^{(\mu_i)}(f_i) = A^{(i)}_{(a,1)} X^a Y + B^{(i)}_{(a,1)} X^{a+1} Y' + A^{(i)}_{(a+\mu_i,0)} X^{a+\mu_i}, \qquad \forall i \geq i_0.$$

Since $Q_i = (a, 1)$ for all $i \geq i_0$, we have that $A^{(i)}_{(a,1)} = A^{(i_0)}_{(a,1)} = A^{(0)}$ and $B^{(i)}_{(a,1)} = B^{(i_0)}_{(a,1)} = B^{(0)}$ for all $i \geq i_0$. Then we obtain that

$$\Phi_{(f_i,\mu_i)}(C) = (A^{(0)} + \mu_i B^{(0)})C + A^{(i)}_{(a+\mu_i,0)}, \qquad i \geq i_0.$$

Since the statement (i) holds, we have the following formulae

$$c_i = \frac{-1}{A^{(0)} + \mu_i B^{(0)}} \cdot A^{(i)}_{(a+\mu_i,0)}, \qquad i \geq i_0. \tag{8}$$

Let us observe that $L(f_{i_0}, \mu_{i_0})$ is a principal side of f_{i_0}, then $B^{(0)} \neq 0$ and $A^{(0)} + \mu_i B^{(0)} \neq 0$ for $i \geq i_0$. Moreover, since $c_i \neq 0$, we have that $A^{(i)}_{(a+\mu_i,0)} \neq 0$ and $B^{(i)}_{(a,1)} \neq 0$ for $i \geq i_0$. If $f_i \in \mathcal{C}_q$ (see §1 for the definition of $\mathcal{C}_q$) and $i \geq i_0$ then $a, a + \mu_i \in \frac{1}{q}\mathbb{Z}$, so $\mu_i \in \frac{1}{q}\mathbb{N}$ and $f_{i+1} = f_i[c_i X^{\mu_i} + Y] \in \mathcal{C}_q$. Thus if $f_{i_0} \in \mathcal{C}_{q'}$, we have that $\mu_i \in \frac{1}{q'}\mathbb{N}$ for all $i \geq i_0$; so there exists $q \in \mathbb{N}$ such that $\mu_i \in \frac{1}{q}\mathbb{N}$ for all index. By other hand, looking at the Newton Polygon of f_{i+1} we have that

$$\mathrm{ord}_X(f_\omega[c_1 X^{\mu_1} + \cdots + c_i X^{\mu_i}]) = \mathrm{ord}_X(f_{i+1}[0]) \geq a + \mu_i, \quad i \geq i_0.$$

Since $\mu_i < \mu_{i+1}$ and $\mu_i \in \frac{1}{q}\mathbb{N}$, then $\lim \mu_i = \infty$ and $f[z] = 0$.

Assume that $b \geq 2$. Consider the curve

$$g = \frac{\partial^{b-1} f_\omega}{\partial Y^{b-2} \partial Y'},$$

and denote by $g_0 = g$, $g_{i+1} = g_i[c_i X^{\mu_i} + Y]$. By the chain rule we have that

$$g_i = \frac{\partial^{b-1} f_i}{\partial Y^{b-2} \partial Y'}, \qquad i \geq 0, \tag{9}$$

thus

$$\Delta(g_i) \subseteq \varphi(\Delta(f_i)) \cap \{(\alpha, \beta) \in \mathbb{R}^2 \mid \beta \geq 0\},$$

where $\varphi(\alpha,\beta) = (\alpha+1,\beta-b+1)$. Since $L(f_i,\mu_i)$ is a principal side of f_i we have that the coefficient of $X^{a+1}Y^{b-1}Y'$ of f_i is different from zero if $i \geq i_0$. By (9) we have that $(a+1,1) \in \Delta(g_i)$ if $i \geq i_0$. Moreover, the point $(a+1,1) \in L(g_i,\mu_i)$ for $i \geq i_0$. This implies that the coefficient of $X^{a+\mu_i+1}Y^0$ of g_i is different from zero (otherwise $(a+1,1) \notin L(g_{i+1},\mu_{i+1})$ because we are assuming that $c_i \neq 0$ for all index). As above we prove that $\mu_i \in \dfrac{1}{q}\mathbb{N}$ for all index, thus $\lim \mu_i = \infty$, and looking at the Newton Polygons of f_i and g_i we obtain that $f[z]=0$ and $g[z]=0$. □

5. Convergence.

Let $A(X,Y) \in \mathbb{C}[[X^{1/q},Y]]$, $q \in \mathbb{N}$. We will say that A is "convergent" if $A(T^q,Y) \in \mathbb{C}\{T,Y\}$. Let $f = A(X,Y)+B(X,Y)Y' \in \mathcal{C}_q$; we will say that f is "convergent" if A and B are convergent.

Let $q \in \mathbb{N}\setminus\{0\}$, if we write $\Lambda_q(f) = T^{q-1}A(T^q,Y) + \dfrac{1}{q}B(T^q,Y)Y'$, then we have :

5.1. If $z = \sum_{i=1}^{\infty} c_i X^{i/q}$ is a solution of f then $z' = \sum_{i=1}^{\infty} c_i T^i$ is a solution of $\Lambda_q(f)$.

5.2. Let $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\varphi(x,y) = (qx+q-1,y)$ then $\varphi(N(f)) = N(\Lambda_q(f))$ (the sides correspond bijectively and their slopes are multiplied by $1/q$).

5.3. For all $\mu \in \mathbb{Q}_+$ we have that

$$\Phi_{(f,\mu)}(C) = \Phi_{(\Lambda_q(f),q\cdot\mu)}(C).$$

5.4. For all side L of $N(f)$ we have that

$$(L,a)_f \Longleftrightarrow (\varphi(L),a)_{\Lambda_q(f)}$$
$$(L,b)_f \Longleftrightarrow (\varphi(L),b)_{\Lambda_q(f)}\ .$$

5.5. For all $c \in \mathbb{C}$ and $\mu \in \mathbb{Q}_+$ we have that

$$\Lambda_q\big(f[cX^\mu + Y]\big) = \Lambda_q(f)[cT^{q\mu}+Y].$$

THEOREM 2. — *Let f be the differential polynomial associated to ω, assume that ω is convergent and that neither $(X=0)$ nor $(Y=0)$ are solutions of $(f=0)$. Then the solution z constructed in Theorem 1 is convergent.*

Proof. — Take the notation as in the proof of Theorem 1. We know that there is an integer $q \in \mathbb{N}$ such that $z \in \mathbb{C}[[X^{1/q}]]$ and $f_i \in \mathcal{C}_q$, for all i. In view of the above properties (5.1) to (5.5) we can assume without loss of generality that $q = 1$ and hence we can write

$$z = \sum_{i=1}^{\infty} c_i X^i.$$

Consider the pivot point of f with respect to z, $Q = (a, b)$. If $b \geq 2$, by the proof of Theorem 1, z is a solution of a convergent curve. Thus it is a convergent series by Puiseux's Theorem. The case $b = 0$ is not possible because $f[z] = 0$. Assume that the pivot point $Q = (a, 1)$ and that it is reached at the index i_0. Then the coefficient of $X^{a+1}Y^0Y'$ of $f_{i+1} = f[c_1X + \cdots + c_iX^i + Y]$ is fixed and different from zero if $i \geq i_0$, name it by $B^{(0)}$. Moreover $B^{(0)}$ is the coefficient of $X^{a+1}Y^0Y'$ of $f[z+Y]$. By the chain rule we have that

$$\frac{\partial f}{\partial Y'}[z+Y] = \frac{\partial(f[z+Y])}{\partial Y'} \qquad \text{and} \qquad \frac{\partial f}{\partial Y}[z+Y] = \frac{\partial(f[z+Y])}{\partial Y},$$

then $\operatorname{ord}_X\Big(\dfrac{\partial f}{\partial Y'}[z]\Big) = a + 1$ and $\operatorname{ord}_X\Big(\dfrac{\partial f}{\partial Y}[z]\Big) \geq a$. Thus, the differential operator along the solution z

$$L(f, z) = \frac{\partial f}{\partial Y}[z] + \Big(\frac{\partial f}{\partial Y'}[z]\Big) \cdot \partial$$

is different from zero with regular singularity at the origin. This implies that the solution z is convergent (see Malgrange [9]).

In the following lines we are going to give a self-contained and elementary proof of this result. By other hand, this proof will be worthy in §6.

Assume that the pivot point, $Q = (a, 1)$, it is reached at the index i_0; name by $A^{(0)}$ and $B^{(0)}$ the coefficient of $X^aYY'^0$ and $X^{a+1}Y^0Y'$ of f_i, $i \geq i_0$, respectively. If $c_i \neq 0$ and $i \geq i_0$ then $L(f_i, i)$ is a principal side of f_i; thus $B^{(0)} \neq 0$ and $A^{(0)} + j \cdot B^{(0)} \neq 0$ for $j \geq i$. If we assume that $c_{i_0} \neq 0$ then we have that the formulae (8) holds for all $i \geq i_0$, that is

$$c_i = \frac{-1}{A^{(0)} + i \cdot B^{(0)}} \cdot \operatorname{Coeff}_{X^{a+i}Y^0Y'^0}(f_i), \qquad i \geq i_0, \tag{10}$$

where $\operatorname{Coeff}_{X^{a+i}Y^0Y'^0}(f_i)$ means the coefficient of $X^{a+i}Y^0Y'^0$ of $f_i = f[c_1X + \cdots + c_{i-1}X^{\mu_{i-1}} + Y]$. Write

$$f_{i_0} = \sum_{(\alpha,\beta)\in N_1} A_{\alpha\beta}X^\alpha Y^\beta + \sum_{(\alpha,\beta)\in N_2} B_{\alpha\beta}X^{\alpha+1}Y^{\beta-1}Y',$$

where $N_1 = N(f_{i_0}) \bigcap \mathbb{N}^2$ and $N_2 = N(f_{i_0}) \bigcap \{(\alpha, \beta) \in \mathbb{Z}^2 \mid \alpha \geq -1, \beta \geq 1\}$. For each index $i \geq i_0$ consider the following series over the indeterminates $\{C_j\}_{j\geq i_0}, \{T_{\alpha\beta}\}_{(\alpha,\beta)\in N_1 \cup N_2}, X$, Y, and Y' :

$$H_{i+1}^{(1)} = \sum_{(\alpha,\beta)\in N_1} T_{\alpha\beta} X^{\alpha} (C_{i_0} X^{i_0} + \cdots + C_i X^i + Y)^{\beta},$$

$$H_{i+1}^{(2)} = \sum_{(\alpha,\beta)\in N_2} T_{\alpha\beta} X^{\alpha+1} (C_{i_0} X^{i_0} + \cdots + C_i X^i + Y)^{\beta-1} \cdot (i_0 C_{i_0} X^{i_0 - 1} + \cdots + i C_i X^{i-1} + Y'),$$

$$H_{i+1}^{(3)} = \sum_{(\alpha,\beta)\in N_2} T_{\alpha\beta} X^{\alpha+1} (C_{i_0} X^{i_0} + \cdots + C_i X^i + Y)^{\beta-1} \cdot (C_{i_0} X^{i_0 - 1} + \cdots + C_i X^{i-1} + Y').$$

We have that $H_{i+1}^{(j)} \in \mathbb{N}[C_{i_0}, \ldots, C_i, \{T_{\alpha\beta}\}_{(\alpha,\beta)\in N_j}][[X, Y, Y']]$, $j = 1, 2, 3$ and $N_3 = N_2$. Obviously, the following formulae holds :

(11)

$$f_{i+1} = H_{i+1}^{(1)}(c_{i_0}, \ldots, c_i, \{A_{\alpha\beta}\}_{(\alpha,\beta)\in N_1}) + H_{i+1}^{(2)}(c_{i_0}, \ldots, c_i, \{B_{\alpha\beta}\}_{(\alpha,\beta)\in N_2}), \quad i \geq i_0 \ .$$

If we consider

$$P_{i+1}^{(j)}(C_{i_0}, \ldots, C_i, \{T_{\alpha\beta}\}) = \mathrm{Coeff}_{X^{a+i+1} Y^0 Y'^0}(H_{i+1}^{(j)}), \quad j = 1, 2, 3,$$

then $P_{i+1}^{(j)} \in \mathbb{N}[C_{i_0}, \ldots, C_i, \{T_{\alpha\beta}\}_{(\alpha,\beta)\in N_j}]$, and the coefficients of $P_{i+1}^{(2)}$ are smaller than the corresponding ones of the polynomial $i \cdot P_{i+1}^{(3)}$. By (11) we have that

(12)

$$\mathrm{Coeff}_{X^{a+i} Y^0 Y'^0}(f_i) = P_i^{(1)}(c_{i_0}, \ldots, c_{i-1}, \{A_{\alpha\beta}\}) + P_i^{(2)}(c_{i_0}, \ldots, c_i, \{B_{\alpha\beta}\}), \quad i > i_0 \ .$$

Since $A^{(0)} + iB^{(0)} \neq 0$ for $i \geq i_0$ and $B^{(0)} \neq 0$, there is a constant $k \in \mathbb{N}$ such that $\left| \dfrac{i}{A^{(0)} + iB^{(0)}} \right| < k$ for all $i \geq i_0$. Define $c'_{i_0}, c'_{i_0+1}, \ldots$ as follows :

$c'_{i_0} = |c_{i_0}|$,

$c'_i = k\{P_i^{(1)}(c'_{i_0}, \ldots, c'_{i-1}, \{|A_{\alpha\beta}|\}) + P_i^{(3)}(c'_{i_0}, \ldots, c'_{i-1}, \{|B_{\alpha\beta}|\})\}$, $\forall i > i_0$.

Let us prove that $|c_i| \leq c'_i$, $\forall i \geq i_0$. It is true for $i = i_0$. Suppose that $i > i_0$ and that $|c_j| \leq c'_j$, for $j < i$, then we have that

$$\begin{aligned}
|c_i| &\leq \frac{P_i^{(1)}(|c_{i_0}|, \ldots, |c_{i-1}|, \{|A_{\alpha\beta}|\}) + P_i^{(2)}(|c_{i_0}|, \ldots, |c_{i-1}|, \{|B_{\alpha\beta}|\})}{|A^{(0)} + iB^{(0)}|} \\
&\leq \frac{P_i^{(1)}(|c_{i_0}|, \ldots, |c_{i-1}|, \{|A_{\alpha\beta}|\}) + iP_i^{(3)}(|c_{i_0}|, \ldots, |c_{i-1}|, \{|B_{\alpha\beta}|\})}{|A^{(0)} + iB^{(0)}|} \\
&\leq k\{P_i^{(1)}(c'_{i_0}, \ldots, c'_{i-1}, \{|A_{\alpha\beta}|\}) + P_i^{(3)}(c'_{i_0}, \ldots, c'_{i-1}, \{|B_{\alpha\beta}|\})\} = c'_i \ .
\end{aligned}$$

Now, let $\Phi(X,W) \in \mathbb{C}\{X,W\}$ be given by

$$\Phi(X,W) = X \cdot \sum_{(\alpha,\beta)\in N_1'} |A_{\alpha\beta}| X^{\alpha} W^{\beta} + \sum_{(\alpha,\beta)\in N_2'} |B_{\alpha\beta}| X^{\alpha+1} W^{\beta},$$

where $N_j' = N_j \setminus \{(a,1),(a+i_0,0)\}$ (note that $\Phi(X,W)$ is convergent because f is so). Put $\Psi(X,W) = -X^{a+1}W + |c_{i_0}|X^{a+i_0+1} + k\cdot\Phi(X,W)$ (let us observe that $\Psi(X,W) \neq 0$). We have that

$$\mathrm{Coeff}_{X^{a+i+1}}\Big(\Phi(X,\sum_{i=i_0}^{\infty} c_i' X^i)\Big) = \mathrm{Coeff}_{X^{a+i+1}}\big(\Phi(X, c_{i_0}' X + \cdots + c_{i-1}' X^{i-1})\big).$$

And thus $z_1 = \sum_{i=i_0}^{\infty} c_i' X^i$ is a solution of $\Psi(X,W)$. Since $\Psi(X,W)$ is convergent then z_1 is convergent and so, z is convergent. $\square$

6. Solutions of s-Gevrey index.

We have proved that a convergent first order and first degree ordinary differential equation has at least one convergent solution, but may be it has also formal non convergent solutions. It is well known that a series solution of a convergent ordinary differential equation has certain Gevrey index (see Maillet [10], Mahler [9], Ramis [14], [15], and Malgrange [11]). In a forthcoming paper [5], we prove that the ring of Gevrey power series is, in some sense, closed for ordinary differential equations. Now we are going to prove that the ring of formal power series with fixed s-Gevrey index has in common with the ring of convergent power series the property of any first order and first degree differential equation with coefficients into the ring has a solution in this ring.

THEOREM 3. — *Assume that the Pfaffian form $\omega = A(X,Y)dX + B(X,Y)dY$ has s-Gevrey index ($s \geq 1$). Then there is at least a solution of $(\omega{=}0)$ with s-Gevrey index.*

Proof. — Let $z = \sum_{i=1}^{\infty} c_i X^{i/q}$ be the solution constructed in Theorem 1 for $f_\omega = A(X,Y) + B(X,Y)Y'$. Assume that the greater common divisor of the set $\{q\} \cup \{i \mid c_i \neq 0\}$ is one. Then the solution $(t^q, z(t^q))$ of $(\omega{=}0)$ has s-Gevrey index if $y(X) = z(X^q) \in \mathbb{C}[[X]]$ has s-Gevrey index. In the proof of Theorem 2 we have found a series $z_1(X) \in \mathbb{C}[[X]]$ which is a solution of a formal curve $\Psi(X,W)$ and $y(X)$ is majored by $z_1(X)$. If

$A(X,Y)$ and $B(X,Y)$ has s-Gevrey index then $\Psi(X,W)$ has also s-Gevrey index. Thus the result follows from the following proposition.

PROPOSITION 2. — *Let $h(X,Y)=\sum A_{\alpha\beta}X^\alpha Y^\beta$ be a formal series which has s-Gevrey index, $s\geq 1$, and $y=\sum_{i=1}^{\infty} c_i X^{i/q}$ a solution of h, then we have that*

$$\sum_{i=1}^{\infty}\frac{c_i}{(i!)^{s-1}}T^i\in\mathbb{C}\{T\}.$$

Proof. — The following assertions are easy to prove :

(i) $\beta_{s-1}(h)\in\mathbb{C}\{X,Y\}\Longleftrightarrow\sum\frac{A_{\alpha,\beta}}{\alpha!^{s-1}\beta!^{s-1}}X^\alpha Y^\beta\in\mathbb{C}\{X,Y\}$.

(ii) $\beta_{s-1}(h)\in\mathbb{C}\{X,Y\}\Longrightarrow\beta_{s-1}(h(T^q,Y))\in\mathbb{C}\{T,Y\}$, $q\in\mathbb{N}$.

(iii) $\beta_{s-1}(h)\in\mathbb{C}\{X,Y\}\Longrightarrow\beta_{s-1}(h(X,cX^j+Y))\in\mathbb{C}\{X,Y\};j\in\mathbb{N}\backslash\{0\}$.

Then we can suppose that $y(X)=\sum_{i=1}^{\infty}c_iX^i$ is a solution of $h\in\mathbb{C}[[X,Y]]$ such that $c_1=\cdots=c_{k-1}=0$, $c_k\neq 0$. Let $Q=(a,b)$ the pivot point (see proof of Th.1) of h with respect to y. We may assume that $b=1$, otherwise $y(X)$ is a solution of the s-Gevrey curve $\dfrac{\partial^{b-1}h}{\partial Y^{b-1}}$ having the ordinate of the pivot point equal to one. Then we can also suppose that $k>(a+4)^2$. If we multiply h by $1/A^{(0)}$, by (10) and (12) we have that

$$c_i=-P_i^{(1)}(c_1,\ldots,c_{i-1},\{A_{\alpha\beta}\}_{(\alpha,\beta)\in N_1}),\qquad i\geq k,\ N_1=N(h)\cap\mathbb{N}^2.$$

Looking at the polynomial $P_i^{(1)}$ we have that

$$P_i^{(1)}=\sum B_{(\alpha,\beta,d_k,\ldots,d_{i-1})}T_{\alpha\beta}C_k^{d_k}\cdots C_{i-1}^{d_{i-1}},$$

with the coefficients $B_{(\alpha,\beta,d_k,\ldots,d_{i-1})}\geq 0$, and if $B_{(\alpha,\beta,d_k,\ldots,d_{i-1})}\neq 0$ then we have that

$$\begin{aligned}a+i&=\alpha+kd_k+\cdots+(i-1)d_{i-1}\\ \beta&=d_k+\cdots+d_{i-1}.\end{aligned}\tag{13}$$

Define $c'_1,c'_2,\ldots$ by

$$c'_1=\cdots=c'_{k-1}=0;\quad c'_k=|c_k|/k!^{s-1}$$

$$c'_i=P_i^{(1)}\Big(c'_k,\ldots,c'_{i-1},\Big\{\frac{|A_{\alpha\beta}|}{[\alpha-a]!^{s-1}(\beta)!^{s-1}}\Big\}\Big),\quad i>k,$$

where $[\alpha-a]=\alpha-a$ if $\alpha>a$ and 1 in other case. We have that $\sum_{i=k}^{\infty}c'_iX^i$ is convergent because it is a solution of the following convergent equation :

$$-X^aY+\frac{|c_k|}{k!^{s-1}}X^{a+k}+\sum_{(\alpha,\beta)\in N'_1}\frac{|A_{\alpha\beta}|}{[\alpha-a]!^{s-1}(\beta)!^{s-1}}X^\alpha Y^\beta=0,$$

where $N_1' = N_1 \setminus \{(a,1),(a+k,0)\}$.

In order to prove by induction that $\frac{|c_i|}{i!^{s-1}} \leq c_i'$, $i \geq k$ it is only necessary to prove that

$$B_{(\alpha,\beta,d_k,\ldots,d_{i-1})} \neq 0 \Longrightarrow i! \geq [\alpha - a]!\,(\beta)!\,k!^{\,d_k}\cdots(i-1)!^{\,d_{i-1}}. \tag{14}$$

Let $B_{(\alpha,\beta,d_k,\ldots,d_{i-1})} \neq 0$ and let $d_{j_1},\ldots,d_{j_r}$, with $j_1 < \cdots < j_r$, be the exponents different from zero. If $\alpha > a$ then by (13) we have that

$$\begin{aligned} i! &\geq 1{\cdot}2\cdots(\alpha-a)(\alpha-a+1)\cdots(\alpha-a+\beta) \\ &\quad \cdot\{(m+1)\cdots(m+j_1-1)\}^{d_{j_1}}\cdots\{(m+1)\cdots(m+j_r-1)\}^{d_{j_r}} \end{aligned} \tag{15}$$

where $m = \alpha - a \geq 1$ and then $(m+1)\cdots(m+j_i-1) \geq j_i!$ holds for $i = 1,\ldots,r$. So (15) implies (14). If $\alpha \leq a$ then we have that

$$\begin{aligned} i! \geq (\beta)!\,&\{(\beta+1)\cdots(\beta+j_1-1)\}^{d_{j_1}}\cdots\{(\beta+1)\cdots(\beta+j_{r-1}-1)\}^{d_{j_{r-1}}} \\ &\cdot\{(\beta+1)\cdots(\beta+j_r-1)\}^{d_{j_r}-1}\cdot\frac{(m+1)\cdots(m+j_r-1)}{(i+1)\cdots(i+a)} \end{aligned}$$

where $m = a+i-j_r+1$. Since $\alpha \leq a$ then by (13) we have that $\beta \geq 2$ and then $m \geq j_1 \geq k \geq (a+4)^2$. Since $i+a = m+j_r-1$, it is only necessary to prove that $\forall a \geq 4$ and $\forall m,n \geq a^2$ then we have that

$$\frac{(m+1)\cdot\cdots\cdot(m+n)}{(m+n-a+1)\cdots(m+n)} = (m+1)\cdots(m+n-a) \geq (n+1)! \tag{16}$$

We will prove (16) by induction over n. If $n = a^2$ then the left hand of (16) is always greater or equal than $(a^2+1)\cdots(2a^2-a)$. Then we have to prove that

$$(a^2+1)\cdots(2a^2-a) \geq (a^2+1)! \qquad \forall a \geq 4 \tag{17}$$

which is true for $a = 4$, assume that (17) is true for $4,\ldots,a$. In order to prove that it is true for $a+1$ we have to prove that

$$(a^2+2a+2)\cdots(2a^2+3a+2) \geq (a^2+2a+2)!$$

By the induction hypothesis over a, this is equivalent to

$$(2a^2-a+1)\cdots(2a^2+3a+2) \geq (a^2+1)(a^2+2)^2\cdots(a^2+2a+1)^2(a^2+2a+2)$$

which is obviously true. We finish the induction over n by pointing out that $m+n+1-a \geq n+2$ provided that $m,n \geq (a+4)^2$. $\square$

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Manuscrit reçu le 24 octobre 1991,
révisé le 12 mars 1992.

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