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HARMONIC ANALYSIS OF SPHERICAL FUNCTIONS ON $SU(1,1)$

by Y. BENYAMINI (*) and Y. WEIT

Introduction.

Harmonic functions in $\mathbf{R}^n$ can be defined in two ways. They are the solutions of the differential equation $\Delta f = 0$, where Δ is the Laplacian, and they are the functions that satisfy the mean value property, i.e. the functions that satisfy the convolution equation $f * \mu = f$ for every radial probability measure on $\mathbf{R}^n$.

The first definition extends from $\mathbf{R}^n$ to general Riemannian manifolds where the Laplacian is replaced by the Laplace-Beltrami operator.

We shall be more interested in this article with the extensions of the second definition. If H is a topological group, or more generally a homogeneous space of a topological group, we can consider solutions of convolution equations of the form $f * \mu = f$ for certain families of measures μ on H.

This point of view was taken by Choquet and Deny [CD], who studied bounded solutions of the equation $f * \mu = f$ when μ is a probability measure on a locally compact abelian group. They gave necessary and sufficient conditions on μ so that the only bounded solution to the equation will be the constant functions.

In his work on Poisson boundaries for semi-simple Lie groups, Furstenberg [Fu1], [Fu2] studied similar equations on these groups and on their homogeneous spaces G/K (where K is a maximal compact

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subgroup). Given a probability measure μ on G, Furstenberg called a bounded function f on G, μ-harmonic if it satisfies the convolution equation $f * \mu = f$.

The case where f and μ actually «live» on G/K is of special interest. If μ also happens to be *spherical*, i.e. invariant under the left as well as the right action of K, and absolutely continuous, Furstenberg proved [Fu2], Theorem 5, that the only bounded functions f on G/K satisfying the equation $f * \mu = f$, are the harmonic functions, i.e. those functions that are annihilated by the Laplace-Beltrami operator on G/K.

Furstenberg's methods (and interests) are probabilistic, and his main tool is the theory of martingales. Thus his methods do not extend to the study of the equation $f * \mu = f$ when μ is not a probability measure.

In this article we propose to study such equations using methods of Harmonic Analysis, i.e. by using the ideal theory of the group algebra.

Although our results should hold for more general Lie groups we limit ourselves to the very concrete case where $G = SU(1.1)$, where G/K is the usual unit disk in the complex plane. There are two reasons for doing this. First, it allows us to give direct and complete proofs, without using the general theory of Lie groups. As this special case is important in a variety of areas of analysis, we hope that this detailed exposition will be helpful to the non-specialist in representation theory, who might find it difficult to find his way in the literature. Secondly, this concrete setup allows us to treat some classical problems in function theory, and to obtain characterizations of harmonic and holomorphic functions in the unit disk.

We now describe the content of the article in more detail.

After we introduce the necessary notation and terminology, the preliminaries section is used to give reasonably complete and direct proofs of the basic properties of the spherical functions and of the Fourier transform that we need. This is possible because we restrict ourselves to $SU(1, 1)$, where things are very explicit and concrete.

In the first section we study ideals in $L^1(K\backslash G/K)$, which in our special case is just the algebra of radial functions on the unit disk, integrable with respect to the conformally invariant measure, and with the convolution induced from $SU(1,1)$. This is a commutative semi-simple algebra, and thus its ideals can be studied in terms of the Fourier (Gelfand) transform. The main problem here is to find an analogue of Wiener's general Tauberian theorem, that is to find conditions on the Fourier transforms of the elements of an ideal I that will ensure that it is all of $L^1(K\backslash G/K)$. This problem was considered by Ehrenpreis and Mautner [EM1], [EM2], who have noted that it is not enough that the Fourier transforms of the elements of I will not have a common zero. We formulate a conjecture in this direction, but can prove only special cases. Other cases have been proved in [EM1], and our proofs borrow heavily from the ideas and methods of [EM1].

In sections 2-4 we apply the results of section 1 to various concrete problems. In section 2, we give conditions that ensure that a given radial measure μ is mixing, i.e. $\mu^n * f \to 0$ for every $f \in L^1(K\backslash G/K)$ with zero integral. In particular we prove that every radial *probability* measure $\mu \neq \delta_0$ is mixing.

In section 3, we prove the result of Furstenberg mentioned above, that when $\mu \neq \delta_0$ is a radial probability measure then the only bounded solutions to the equation $f * \mu = f$ on the unit disk are the harmonic functions. The Fourier analysis approach indicates that this result should be true not only for probability measures, but for every measure μ whose Fourier transform takes the value 1 only for $s = 0$ or 1, and we conjecture that this is indeed the case. Our results on the ideal theory of $L^1(K\backslash G/K)$, however, are not good enough to prove this in general and we prove it only when μ has a compact support.

In the final section, 4, we characterize holomorphic functions in the unit disk. For example we give a « two circles theorem » for most pairs r_1, r_2 (the precise condition is given in section 4) : If f is a measurable function on the unit disk satisfying $|f(z)| \leqslant c(1-|z|^2)^{-1}$ there, and if $\int_{g(\gamma_i)} f(z)\,dz = 0$ for all $g = SU(1,1)$, where γ_i is the central circle of radius r_i, then f is holomorphic.

This result was announced in [A], but the proof there is not complete. The intricate ideal theory in $L^1(K\backslash G/K)$ was not noticed, and the author assumed that the non-vanishing of the Fourier transforms suffices to imply that the ideal is all of $L^1(K\backslash G/K)$.

0. Preliminaries.

We shall use standard notation and terminology. For more details, see [EM1], [L] and [S].

We denote by $SU(1,1)$ the group G of 2×2 complex matrices $g = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$ where $|\alpha|^2 - |\beta|^2 = 1$. Each such g is identified with the conformal automorphism $g(z) = (\alpha z + \beta)/(\bar\beta z + \bar\alpha)$ of the unit disk $\mathbf{D}$.

We denote by K the subgroup of rotations in G, and by A the subgroup of all matrices of the form $g_\zeta = \begin{pmatrix} \cos h\zeta & \sin h\zeta \\ \sin h\zeta & \cos h\zeta \end{pmatrix}$. Every g in G, $g \neq 1$, has a unique representation as a product $g = kul$, where $k, l \in K$ and $u \in A$. A function f on G is called *spherical* if $f(g) = f(u)$ in the above representation, or, equivalently, $f(g) = f(kgl)$ for all $g \in G$ and $k, l \in K$. When we identify G/K with the unit disk, functions on $\mathbf{D}$ are identified with functions on G satisfying $f(g) = f(gk)$ for all $g \in G$ and $k \in K$, and the spherical functions are identified with the radial functions on $\mathbf{D}$.

More generally we shall consider (m,n)-spherical functions on G. These are the functions that satisfy $f(kgl) = k^m l^n f(g)$ for all $g \in G$ and $k, l \in K$. Every function (or measure) on G has a formal representation $f = \Sigma f_{n,m}$, where $f_{n,m}$ is (m,n)-spherical. $(m,0)$- or $(0,n)$-spherical function will be called left m-radial, or right n-radial functions respectively.

We denote by dg the Haar measure on G, normalized so that it induces the conformally invariant measure $d\lambda = (1-|z|^2)^{-2}\, dx\, dy$ on $\mathbf{D}$.

Denote by $\mathbf{T}$ the unit circle with the normalized Lebesgue measure. By abuse of notation we shall sometimes write θ instead of $e^{i\theta}$ for points of $\mathbf{T}$. For each fixed $g \in G$ and each complex number s we define an operator $U(g,s)$ on $L^2(\mathbf{T})$ by the formula

$$(U(g,s)a)(\theta) = |g'(\theta)|^s a(g\theta) \quad \text{for} \quad a \in L^2(\mathbf{T})$$

and this defines, for each s, a representation of G by bounded operators on $L^2(\mathbf{T})$. This representation is unitary when $\operatorname{Re}(s) = 1/2$.

The *zonal spherical functions* are the functions

$$\varphi(g,s) = \langle U(g,s)1,1\rangle = \int |g'(\theta)|^s\, d\theta = \int |\beta e^{i\theta}+\alpha|^{-2s}\, d\theta$$

where $g = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$.

$L^1(G)$ is a noncommutative Banach algebra under convolution. We shall be interested in its subalgebra $L^1(K\backslash G/K)$ of spherical functions. $L^1(K\backslash G/K)$ is a *commutative* algebra (see [L], p. 21). It is a semi-simple Banach algebra, and its maximal space is the strip $\mathscr{R} = \{s : 0 \leqslant \operatorname{Re}(s) \leqslant 1\}$. The Fourier (or Gelfand) transform of $f \in L^1(K\backslash G/K)$ is given by $F(s) = \hat f(s) = \int f(g)\varphi(g,s)\, dg$ for $s \in \mathscr{R}$, (compare with [L], Thm. 7, page 60). For each fixed $s \in \mathscr{R}$ this formula indeed gives a multiplicative functional because U is a representation. (The integral is finite because φ is bounded on G when $s \in \mathscr{R}$, see below).

The following two lemmas summarize the basic properties of the zonal spherical functions and the Fourier transform. For proofs of these well known facts, see e.g. [S].

LEMMA 0.1. — 1) *Each zonal spherical function $\varphi(\cdot,s)$ is a spherical function on G.*

2) *The relations $\varphi(g,s) = \varphi(g,1-s) = \bar\varphi(g,\bar s)$ hold for all $g \in G$ and all s.*

3) *For each fixed $g \in G$, $\varphi(g,s)$ is an entire function of the complex variable s, and it is of exponential type 2ζ, when $g = kul$, and $u = g_\zeta$.*

4) *If $s \in \mathscr{R}$ then $|\varphi(g,s)| \leqslant 1$ for all $g \in G$, with equality iff g is in K, or $s = 0$ or 1. (For $s \notin \mathscr{R}$, $\varphi(\cdot,s)$ is unbounded.)*

5) *For each fixed g, and every strip $T = \{s : a \leqslant \operatorname{Re}(s) \leqslant b\}$, $\varphi(g,s) \to 0$ uniformly as $|s| \to \infty$ in T.*

LEMMA 0.2. — 1) *For each $f \in L^1(K\backslash G/K)$, its Fourier transform $\hat f(s)$ is continuous and bounded in the strip $\mathscr{R}$ and is analytic in the interior of $\mathscr{R}$. If f has compact support, or more generally if μ is a radial measure on G of compact support, then $\hat f$ (respectively $\hat\mu$) is an entire function of exponential type.*

2) $|\hat{f}(s)| \leqslant \|f\|$ *for all* $s \in \mathscr{R}$ *and* $f \in L^1(K\backslash G/K)$.

3) $\hat{f}(s) = \hat{f}(1-s)$ *for all* $s \in \mathscr{R}$.

4) $\hat{f}(s) \to 0$ *when* $|s| \to \infty$, $s \in \mathscr{R}$. *In fact* $\hat{\mu}(s) \to 0$ *for all spherical measures* μ *with no atom at* $1 \in G$.

More generally we shall consider the other components of the matrix of $U(g,s)$, namely $\varphi_{m,n}(g,s) = \langle U(g,s)e^{\mathrm{i}m\theta}, e^{\mathrm{i}n\theta}\rangle$. These, too, are entire functions of exponential type for each fixed g. We shall also consider the operator valued Fourier transform $\mathscr{F}(\mu,s) = \int U(g,s)\,d\mu(g)$ for measures μ of compact support. For each such μ, this is an operator valued entire function of exponential type.

The L^2 theory of the Fourier transform is completely analogous to the theory on the real line: There is a measure δs on the line $L = 1/2 + i\mathbf{R}$ (the Plancherel measure) so that the Fourier transform is an isometry of $L^2(K\backslash G/K)$ onto $L^2(L,\delta s)$ i.e. $\int_G |f(g)|^2\,dg = \int_L |\hat{f}(s)|^2\,\delta s$.

To define the Schwartz class, $\mathscr{S}$, on G, we consider the Laplace-Beltrami operator Δ on G (restricted to spherical functions). Identifying spherical functions with functions on $\mathbf{R}^+$ by writing $f(\zeta)$ for $f(g_\zeta)$, Δ is given by the formula $\Delta = d/dy(y(1+y))\,d/dy$, where $y = \sin h^2\zeta$.

One checks directly that $(\Delta f)\hat{}(s) = s(1-s)\,\hat{f}(s)$.

The space $\mathscr{S}$ is then the space of all infinitely differentiable functions $f(\zeta)$ so that for all n and m the semi-norms $\sup|\zeta^n \sin h2\zeta\,\Delta^m f(\zeta)|$ and $\sup|\Delta^m f(\zeta)$ are finite.

The Fourier transform is an isomorphism of $\mathscr{S}$ onto the space $\hat{\mathscr{S}}$ of all functions $F(s)$, infinitely differentiable on the strip $\mathscr{R}$, analytic in its interior and satisfying $F(s) = F(1-s)$ there, for which the semi-norms $\sup|d^m/ds^m(s^n(1-s)^n F(s))|$ are finite.

It follows from the formula $(\Delta f)\hat{}(s) = s(1-s)\hat{f}(s)$, that if f is a smooth function with compact support, then $s^n\hat{f}\,(s) \to 0$ uniformly in every strip. $\{a \leqslant \mathrm{Re}(s) \leqslant b\}$, as $|s| \to \infty$.

We finish this section with the following crucial lemma. It shows that *radial* measures behave in a very special way under convolution.

LEMMA 0.3. — *Let μ and ν be two finite measures on the unit disk* $\mathbf{D}$, *with no atom at* 0. *Then $\mu * \nu$ is absolutely continuous.*

Proof. — Denote by τ_r the normalized Lebesgue measure on the central circle of radius r, and we start with an explicit computation of $\tau_r * \tau_s$.

Let B_t be the central disk of radius t. Then

$$\tau_s * \tau_r(B_t) = \int \tau_r(x^{-1}B_t)\, d\tau_s(x).$$

When x runs through the circle of radius s, so does x^{-1}, and thus $\tau_r(x^{-1}B_t)$ depends only on s, and not on x, and is just the normalized arc length measure of the intersection of the central circle of radius r with the disk $x^{-1}B_t$.

In computing this length, there are t_1, t_2 depending on r and s, so that if $t < t_1$ the circle and the disk B_t do not intersect at all, and $\tau_r * \tau_s(B_t) = 0$. If $t > t_2$ the disk contains the whole circle of radius r, and $\tau_s * \tau_s(B_t) = 1$. If $t_1 \leqslant t \leqslant t_2$ then the cosine law shows that $\tau_r * \tau_s(B_t)$ is given by

$$2 \arccos \{[s^2 - t^2 + r^2(1 - t^2s^2)][2rs\ \ (1-t^2)(1-s^2t^2)]^{-1}\}.$$

Thus $\tau_r * \tau_s(B_t)$ is a differentiable function of t (except at t_1 and t_2) which ends the proof in this case.

The general case follows from this special case. Indeed there are finite measures α and β on $(0,1)$ so that $\mu = \int \tau_r\, d\alpha(r)$ and $\nu = \int \tau_s\, d\beta(s)$. But then $\mu * \nu = \int \tau_r * \tau_s\, d\alpha(r)\, d\beta(s)$, and the result follows from the previous discussion.

1. Ideals in $L^1(K\backslash G/K)$.

Our goal in this section will be to find conditions on an ideal I in $L^1(K\backslash G/K)$ so that it is either equal to all of $L^1(K\backslash G/K)$, or to $L^1_0(K\backslash G/K)$, its maximal ideal of all functions whose integral is zero. (By *ideal* we shall mean a *closed* ideal, unless stated otherwise.) A necessary condition is that the Fourier transforms of the elements of I have no common zeros (or that the only common zeros are $s = 0,1$

respectively). Indeed, otherwise I is contained in the ideal of all functions whose Fourier transform vanishes in this set of common zeros. It was, however, already discovered by Ehernpreis and Mautner that this is not enough, and that some restriction on the rate of decay of the Fourier transforms at infinity is also necessary.

Motivated by the results of Ehrenpreis and Mautner, we conjecture that the correct condition is the following :

CONJECTURE. — *Let I be an ideal in $L^1(K\backslash G/K)$ so that the following conditions hold :*

1) *For every $s \in \mathscr{R}$, there is an $f \in I$, so that $\hat{f}(s) \neq 0$.*

2) *There is an $f \in I$ whose Fourier transform does not decay too fast on the line $1/2 + i\mathbf{R}$, in the sense that*

$$\limsup_{|t|\to\infty} |\hat{f}(1/2+it)\exp(\lambda e^{|t|})| > 0 \quad \text{for every } \lambda > 0 .$$

Then $I = L^1(K\backslash G/K)$.

If instead of (1) *I satisfies :*

1′) *For every $s \in \mathscr{R}$, $s \neq 0,1$, there is an $f \in I$ so that $\hat{f}(s) \neq 0$ and every $g \in I$ satisfies $\hat{g}(1) = \hat{g}(0) = 0$.*

Then $I = L^1_0(K\backslash G/K)$.

Unfortunately we cannot prove this conjecture, and in this section we prove it only with some additional conditions on the ideal I.

Denote by $\mathscr{R}(\delta)$ the strip $-\delta \leqslant \mathrm{Re}\,(s) \leqslant 1+\delta$. Let $A_0(\delta)$ denote the space of all functions continuous in $\mathscr{R}(\delta)$ and analytic in its interior which satisfy $\lim_{|s|\to\infty} f(s) = 0$, *and* $f(s) = f(1-s)$ for all s in $\mathscr{R}(\delta)$, with the supremum norm.

The main result of this section is then the following :

THEOREM 1.1. — *Let I be an ideal in $L^1(K\backslash G/K)$, which has a set of generators whose Fourier transforms belong to $A_0(\delta)$ for some $\delta > 0$. Assume that I contains a function whose Fourier transform extends analytically to $\mathscr{R}(\delta)$, and does not decay too fast on the line $1/2 + i\mathbf{R}$, i.e.* $\limsup_{|t|\to\infty} |\hat{f}(1/2+it)\exp(\lambda e^{|t|})| > 0$ *for every $\lambda > 0$.*

If I satisfies

(1) *For every $s \in \mathscr{R}(\delta)$, there is an $f \in I$, so that $\hat{f}(s) \neq 0$. Then $I = L^1(K\backslash G/K)$.*

If instead of (1), *I satisfies*

(1′) *For every $s \in \mathscr{R}(\delta)$, $s \neq 0,1$, there is an $f \in I$ so that $\hat{f}(s) \neq 0$ and every $g \in I$ satisfies $\hat{g}(1) = \hat{g}(0) = 0$.*

Then $I = L_0^1(K\backslash G/K)$.

To prove the theorem we first study ideals in $A_0(\delta)$. Let ψ_δ be the conformal mapping of the strip $\mathscr{R}(\delta)$ onto the unit disk $\mathbf{D}$, which satisfies the relations $\psi_\delta(1-s) = -\psi_\delta(s)$, and takes the line $1/2 + i\mathbf{R}$ onto the segment between i and $-i$. More specifically we have

$$z = \psi_\delta(s) = i(1-e^{\pi i(2s-1)/(2+4\delta)})/(1+e^{\pi i(2s-1)/(2+4\delta)})$$

and

$$s = \psi_\delta^{-1}(z) = 1/2 + \frac{1+2\delta}{\pi i}\log\frac{1+iz}{1-iz}.$$

Let $A(\mathbf{D})$ denote the algebra of all functions analytic in $\mathbf{D}$, and continuous in its closure. Composition with ψ_δ identifies $A_0(\delta)$ with $A_0(\mathbf{D})$, the subspace of $A(\mathbf{D})$ of all functions satisfying $f(i) = f(-i) = 0$, and $f(z) = f(-z)$ for all $z \in \mathbf{D}$.

The ideal theory of $A(\mathbf{D})$ is known completely. Indeed by the theorem of Beurling and Rudin (see [Ho] pp. 82-89), every such ideal is of the form $\{gF\colon g \in A(\mathbf{D}) \text{ and } g|_K = 0\}$ for some closed set K of measure zero in the unit circle and some inner function F.

This also gives a complete description of the ideals of $A_0(\mathbf{D})$ as the following simple lemma shows. (See also Theorem 3.1 in [H].)

LEMMA 1.2. – *For every ideal I in $A_0(\mathbf{D})$, there is an ideal J in $A(\mathbf{D})$, so that $I = J \cap A_0(\mathbf{D})$.*

Proof. – The functions $e_n(z) = (z^2+1)/(z^2+1+1/n)$ belong to $A_0(\mathbf{D})$, $|e_n(z)| \leqslant 2$ on $\mathbf{D}$, and $|e_n(z)| \to 1$ uniformly on compact subsets of $\bar{\mathbf{D}}$ disjoint from $\pm i$.

Let J be the ideal generated by I in $A(\mathbf{D})$. Fixing any f in $J \cap A_0(\mathbf{D})$, we need to show that $f \in I$. Find functions $f_{n,i} \in I$ and $g_{n,i} \in A(\mathbf{D})$, so that the sequence $f_n = \Sigma_i f_{n,i} g_{n,i}$ converges to f. Averaging, we can assume that $g_{n,i}(z) = g_{n,i}(-z)$. Given any $\varepsilon > 0$, fix n so that

$\|f-f_n\| < \varepsilon$, a neigborhood U of $\pm i$ so that $|f| < \varepsilon$ in U, and an m so that $|1-e_m(z)| < \varepsilon$ on $\bar{\mathbf{D}}\setminus U$. One sees then easily that $\|f-f_n e_m\| < C\varepsilon$, where C is a constant that depends only on f. But $f_n e_m \in I$, because each $g_{n,i}e_m$ belongs to $A_0(\mathbf{D})$. As ε was arbitrary, the result follows.

It follows from Lemma 1.2 and the Beurling-Rudin Theorem (see the last corollary on page 88 of [Ho]) that if the functions of an ideal I in $A_0(\mathbf{D})$ have no common zero other then $\pm i$, and if I contains a function whose decay at $\pm i$ is less than exponential, then $I = A_0(\mathbf{D})$.

We can now explain the necessity of the decay condition (2) above (see [EM2], Lemma 7.3). The inner function $F(z) = \exp[(z^2-1)/(z^2+1)]$, satisfies $F(z) = F(-z)$, and decays to zero exponentially fast as $z \to \pm i$ along the imaginary axis. The principal ideal in $A_0(\mathbf{D})$ generated by $(z^2+1)F(z)$ has the form $I = \{gF\colon g \in A_0(\mathbf{D})\}$, and its elements have no common zeros other then $\pm i$. As all the functions in I decay very fast along the imaginary axis, $I \neq A_0(\mathbf{D})$. Indeed I does not contain z^2+1 for example.

Composing $(z^2+1)F(z)$ and the ideal it generates with the conformal mapping ψ_0 of $\mathcal{R}$ onto $\mathbf{D}$, (i.e. we take $\delta = 0$) the image of I is an ideal J in $A_0(0)$ which is generated by a function $G(s)$ which never vanishes in the strip $\mathcal{R}$. Yet it is not all of $A_0(0)$, and it does not contain, for example, the image $H(s)$ of z^2+1 under composition with ψ_0. Direct computations shown that both G and H belong to $\hat{\mathcal{S}}$, hence to $(L^1(K\backslash G/K))^\wedge$. Writing $H = \hat{h}$, and $G = \hat{g}$, the principal ideal in $L^1(K\backslash G/K)$ generated by g cannot contain h, because convergence in $L^1(K\backslash G/K)$ implies uniform convergence of the Fourier transforms.

The rate of decay of G above at infinity is indeed too fast, and one verifies easily that it does not satisfy condition (2) of the conjecture. Indeed, condition (2) is the translation of the Beurling-Rudin condition to $A_0(0)$, as one sees by composition with ψ_0, and is thus necessary.

Let $\delta > 0$. We shall say that a function F *decays double exponentially* in $\mathcal{R}(\delta)$, if there is a $\lambda > 0$ so that $F(s)\exp[-\lambda(s-1/2)^2] \in A_0(\delta)$.

LEMMA 1.3. — *Fix* $\delta > 0$. *The set of all functions* f *in* $\mathcal{S}$ (resp. with zero integral) *whose Fourier transform extends analytically to* $\mathcal{R}(\delta)$ *and decays double exponentially there is dense in* $L^1(K\backslash G/K)$ (resp. in $L^1_0(K\backslash G/K)$).

Proof. — We shall show that, in fact, these functions are dense even in $\mathscr{S}$ (resp. in the space of functions in $\mathscr{S}$ whose integral is zero) in the $\mathscr{S}$-topology.

As the smooth functions with compact support are dense in $\mathscr{S}$, we need to show that if g is a smooth function with compact support and if $G = \hat{g}$, then G can be approximated in $\hat{\mathscr{S}}$ by Fourier transforms of functions that decay double exponentially in $\mathscr{R}(\delta)$. As g has compact support, G is an entire function of exponential type, thus the functions $F_m(s) = G(s)\exp[(s-1/2)^2/m]$ belong to $A_0(\delta)$ for every m. They decay double exponentially in $\mathscr{R}(\delta)$, and are the Fourier transforms of smooth functions. Fix now n, and write

$$|s^n(F_m(s)-G(s))| = |G(s)s^n|\,|1-\exp[(s-1/2)^2/m|.$$

The first factor converges to zero as $|s| \to \infty$ in $\mathscr{R}(\delta)$, because g is smooth, and the second is bounded and converges uniformly to zero (as $m \to \infty$) on compact subsets of $\mathscr{R}(\delta)$. Thus $s^n(F_m(s)-G(s))$ converges uniformly to zero in all of $\mathscr{R}(\delta)$ (as $m \to \infty$). As these are analytic functions, Cauchy's formula shows that the same is true for all their derivatives in the smaller strip $\mathscr{R}$. Thus F_m converge to G in $\hat{\mathscr{S}}$.

If the integral of g is zero, then $G(0) = 0$, and the same will also hold for the F_m's.

The next lemma is the key to transferring results on closure in the $A_0(\delta)$ topology to the $\mathscr{S}$ topology, hence also to the $L^1(K\backslash G/K)$ topology.

LEMMA 1.4. — *Let $\delta > 0$. Fix functions f_n and f whose Fourier transforms F_n and F, respectively, belong to $A_0(\delta)$. If there is a $\lambda > 0$ so that $F(s)\exp[-\lambda(s-1/2)^2] \in A_0(\delta)$, and so that the sequence F_n converges to $F(s)\exp[-\lambda(s-1/2)^2]$ in the $A_0(\delta)$ topology, then $F_n(s)\exp[\lambda(s-1/2)^2]$ converge to F in the $\hat{\mathscr{S}}$ topology.*

Proof. — Using Cauchy's formula again, it is enough to show that for each fixed m, the sequence $s^m(F_n(s)\exp[\lambda(s-1/2)^2] - F(s))$ converges uniformly to zero in the larger strip $\mathscr{R}(\delta)$ as $n \to \infty$. But this is just the product of the bounded function $s^m\exp[\lambda(s-1/2)^2]$ and the sequence $F_n(s) - F(s)\exp[-\lambda(s-1/2)^2]$ that converges uniformly to zero by the hypothesis.

Proof of theorem 1.1. — Let (g_i) be a set of generators for I, so that their Fourier transforms, G_i, belong to $A_0(\delta)$. Assume also that g_0 satisfies $\limsup_{|t|\to\infty} |\hat{g}_0(1/2+it)\exp(\lambda e^{|t|})| > 0$ for every $\lambda > 0$. By the

Beurling-Rudin Theorem, the algebraic ideal generated by the G_i's is dense in $A_0(\delta)$. Fix any function f whose Fourier transform F decays double exponentially in $\mathscr{R}(\delta)$, say, $F(s)\exp[-\lambda(s-1/2)^2] \in A_0(\delta)$. In particular there are functions (F_n) in the algebraic ideal generated by the G_i's, so that $F_n \to F(s)\exp[-\lambda(s-1/2)^2]$ in $A_0(\delta)$. By Lemma 1.4, the functions $F_n(s)\exp[\lambda(s-1/2)^2]$ converge to F in the $\hat{\mathscr{S}}$ topology, and they all belong to $\hat{I}$! As f was arbitrary, Lemma 1.3 proves the Theorem under hypothesis (1). The proof under hypothesis (1′) is similar.

All of our applications of Theorem 1.1 will be to ideals that contain functions of compact support, in fact they will be generated by these functions. In this case the condition on the decay of the Fourier transform is automatic, and we obtain

COROLLARY 1.5. – *Let I be an ideal in $L^1(K\backslash G/K)$, which has a set of generators whose Fourier transforms belong to $A_0(\delta)$ for some $\delta > 0$. Assume that I contains a nonzero function with compact support.*

If I satisfies

(1) *For every $s \in \mathscr{R}(\delta)$, there is an $f \in I$, so that $\hat{f}(s) \neq 0$.*
Then $I = L^1(K\backslash G/K)$.

If instead of (1), *I satisfies*

(1′) *For every $s \in \mathscr{R}(\delta)$, $s \neq 0,1$, there is an $f \in I$ so that $\hat{f}(s) \neq 0$ and every $g \in I$ satisfies $\hat{g}(1) = \hat{g}(0) = 0$.*
Then $I = L^1_0(K\backslash G/K)$.

Proof. – This follows from Theorem 1.1, and the fact that the Fourier transform of a function with compact support is an entire function of exponential type. By a theorem of Carlson (see section 5.8, page 185 in [T]), such a function cannot even decay exponentially fast on any line.

Remarks. – 1) The main ideas in the proof of Theorem 1.1 follow ideas from [EM1]. Our main new idea is the use of the ideal theory of $A(\mathbf{D})$.

The strongest result in [EM1] in the direction of the conjecture is their Theorem 7 :

Assume that $\hat{f}$ has a bounded second derivative in the closed strip $\mathscr{R}$ and that it does not vanish there. If both $\hat{f}(s)^{-1}\exp(-s^{4n})$ and its second derivative are bounded in $\mathscr{R}$, for some integer $n > 0$, then the principal ideal generated by f is all of $L^1(K\backslash G/K)$.

See also their Proposition 5.1.

2) The algebra $L^1(K\backslash G/K)$, seems to be related to the weighted convolution algebra $L^1(\mathbf{R}, e^{|x|/2})$. The maximal space of this algebra is also a strip of unit width in the complex plane. The principal ideals in $L^1(\mathbf{R}, e^{|x|/2})$ generated by functions whose Fourier transform does not vanish in the strip are completely determined by the rate of decay of the Fourier transform at infinity. (See [K], [G] and [H]). Although there does not seem to be a formal relation between the algebras (and in particular one can check that there are functions in $L^1(\mathbf{R}, e^{|x|/2})$ whose Fourier transform is not a Fourier transform of a $L^1(K\backslash G/K)$ function), we conjecture that a similar result should be true in $L^1(K\backslash G/K)$.

2. Mixing.

Recall that a measure on a locally compact abelian group H is called mixing, if $\mu^n * f \to 0$ for every $f \in L_0^1(H)$. If e is the identity in H then δ_e is certainly not mixing, but usually there are other non-mixing probability measures.

Similarly, if μ is a radial measure on the disk $\mathbf{D}$, we shall say that it is mixing if $\mu^n * f \to 0$ for every $f \in L_0^1(K\backslash G/K)$. It turns out that radial measures behave in a different way than general measures. In this section we use the ideal theory of $L^1(K\backslash G/K)$ and ideas from [Fo], [RW] and [KT], to show that radial measures are usually mixing.

THEOREM 2.1. – *Let μ be a radial probability measure on $\mathbf{D}$, so that $\mu \neq \delta_0$. Then μ is mixing.*

Proof. – Let $J = \{f \in L_0^1(K\backslash G/K) : \mu^n * f \to 0\}$. J is a closed ideal in $L^1(K\backslash G/K)$, contained in $L_0^1(K\backslash G/K)$, and we need to show that, in fact $J = L_0^1(K\backslash G/K)$.

Without loss of generality, we can assume that μ is absolutely continuous. Indeed, by Lemma 0.3 $\nu = \mu * \mu$ is, and it suffices to show that ν is mixing, because it then follows that $\mu^{2n} * f \to 0$ for every f in $L_0^1(K\backslash G/K)$, hence also $\mu^{2n+1} * f = \mu * (\mu^{2n} * f) \to 0$, i.e. μ is mixing too.

Given a compact radial set $A \subset \mathbf{D}$, we denote by μ_A the normalized restriction of μ to A, i.e. $\mu_A(B) = \mu(A)^{-1}\mu(A \cap B)$. The same proof as that of Lemma 1 of [Fo] (which uses Lemma 2.1 of [FW]), shows that $(\mu-\mu_A) * f \in J$ for every $f \in L_0^1(K\backslash G/K)$, hence also $f_{A,B} = (\mu_A - \mu_B) * f \in J$ for all compact radial sets A and B, and all $f \in L_0^1(K\backslash G/K)$.

Let $I \subset J$ be the ideal generated by those $f_{A,B}$'s where the function $f \in L_0^1(K\backslash G/K)$ has a compact support.

We claim that the Fourier transforms of functions in I have no common zero other than $s = 0$ and $s = 1$.

To see this fix such an s. The function $\varphi(\cdot, s)$, considered as a function on the disk is a real analytic function. (This follows from the explicit formula for φ, or, alternatively, from the fact that it is an eigenfunction of the Laplace-Beltrami operator, an elliptic operator with real analytic coefficients.) As a real analytic function $\varphi(\cdot, s)$ cannot be constant on a set of positive measure. Thus there are $r_1 \neq r_2$ in the support of μ so that $\varphi(r_1, s) \neq \varphi(r_2, s)$. As $\varphi(\cdot, s)$ is a continuous radial function this implies that there are compact radial sets A and B, containing r_1 and r_2 respectively, so that $\mu(A)^{-1} \int_A \varphi(g,s)\, d\mu \neq \mu(B)^{-1} \int_B \varphi(g,s)\, d\mu$, i.e. $\hat{\mu}_A(s) \neq \hat{\mu}_B(s)$.

But this means that $(\mu_A - \mu_B)\hat{}(s) \neq 0$ hence also $\hat{f}_{A,B}(s) \neq 0$ for any $f \in L_0^1(K\backslash G/K)$ with compact support, with $\hat{f}(s) \neq 0$.

Corollary 1.5 now applies to the ideal I, and shows that it is all of $L_0^1(K\backslash G/K)$. As I is contained in J, the theorem follows.

Remark 1. – The theorem was formulated in the more difficult case when μ is a probability measure, but it certainly holds for every measure $\mu \neq \delta_0$ with $\|\mu\| = 1$. Indeed, we can assume again that μ is absolutely continuous, hence if μ does not have a constant sign, its spectral radius as an element of $L^1(K\backslash G/K)$, given by $r = \sup\{|\hat{\mu}(s)| : s \in \mathscr{R}\}$ is strictly less then one. (Indeed, $\hat{\mu}$ is a continuous function on $\mathscr{R}$, which, by (4) and (5) of Lemma 0.1 takes its values in the open unit disk, and whose limit as $|s| \to \infty$ is 0.)

As $r = \lim \|\mu^n\|^{1/n}$ it follows from $r < 1$ that $\|\mu^n\| \to 0$, which is stronger than just mixing.

We shall say that a radial measure μ is power bounded if $\|\mu^n\|$ is a bounded sequence. By the uniform boundeness principle this is certainly a necessary condition for a measure to be mixing. Another necessary condition is that $|\hat{\mu}(s)| \neq 1$ for every $s \in \mathscr{R}$, with $s \neq 0,1$.

Otherwise, if $|\hat{\mu}(s)| = 1$ for some s, then taking an $f \in L_0^1(K\backslash G/K)$ with $\hat{f}(s) \neq 0$, we see that $(\mu^n * f)\hat{\ }(s) = \hat{\mu}(s)^n \hat{f}(s)$ does not converge to zero. We conjecture that these two conditions are also sufficient, but can prove it only with some additional conditions on μ.

We first formulate as a separate lemma the basic ingredient from the ideal theory that we shall need.

LEMMA 2.2. — *Let μ be a radial measure of compact support on the unit disk* $\mathbf{D}$, *and assume there is a $\delta > 0$, so that $\hat{\mu}(s) \neq 1$ for every $s \in \mathscr{R}(\delta)$, with $s \neq 0,1$. Let J be the closed ideal in $L_0^1(K\backslash G/K)$, generated by $\{(\mu - \delta_0) * h : h \in L^1(K\backslash G/K)\}$. Then $J \supseteq L_0^1(K\backslash G/K)$.*

Proof. — Restricting ourselves to functions h of compact support only, the functions $(\mu - \delta_0) * h$ also have compact supports and their Fourier transforms have no common zero in $\mathscr{R}(\delta)$ other than $s = 0,1$ (because $\hat{\mu}(s) \neq 1$ for all $s \in \mathscr{R}(\delta)$, $s \neq 0,1$). By Corollary 1.5, there are two possibilities : If $\mu(\mathbf{D}) = 1$ then $\hat{\mu}(0) = \hat{\mu}(1) = 1$, and the Fourier transforms of all these functions vanish at $s = 0,1$, hence the ideal they generate is $L_0^1(K\backslash G/K)$. If $\mu(\mathbf{D}) \neq 1$, the Fourier transforms have no common zero in all of $\mathscr{R}$, hence the ideal they generate is all of $L^1(K\backslash G/K)$.

Remark 2. — If the Conjecture of section 1 holds, we do not have to assume in Lemma 2.2 that μ has compact support, and it is enough to take $\delta = 0$, i.e. to assume that $\hat{\mu}(s) \neq 1$ in $\mathscr{R}$. We only have to make sure that $(\hat{\mu}(s) - 1)$ does not decay too fast on the line $1/2 + i\mathbf{R}$. This last condition can be roughly interpreted as saying that μ is « different enough » from δ_0.

Before we formulate our next result, note that radial measures on $\mathbf{D}$ can be considered as operators on $L_0^1(K\backslash G/K)$, acting by convolution. This identification gives an algebraic and isometric isomorphism of the algebra of radial measures on $\mathbf{D}$ into the algebra of bounded linear operator on $L_0^1(K\backslash G/K)$.

THEOREM 2.3. — *Let μ be a power bounded radial measure of compact support on the unit disk* $\mathbf{D}$, *and assume there is a $\delta > 0$, so that $|\hat{\mu}(s)| \neq 1$ for every $s \in \mathscr{R}(\delta)$, with $s \neq 0,1$. Then μ is mixing.*

Proof. — Multiplying by a number of unit modulus, we can assume that $\hat{\mu}(0) \geqslant 0$. We also continue to assume that μ is absolutely continuous.

By Lemma 2.2 the closed ideal generated by $\{(\mu - \delta_0) * h : h \in L^1(K\backslash G/K)\}$, contains $L^1_0(K\backslash G/K)$. It follows that it is enough to show that $\mu^n * (\mu - \delta_0) * h \to 0$ for every $h \in L^1(K\backslash G/K)$.

By a theorem of Katznelson and Tzafriri ([KT], Theorem 1, and the remark after the proof of Theorem 2), this will follow once we check that the spectrum of μ, when considered as an operator on $L^1(K\backslash G/K)$, intersects the unit circle at most at the single point 1. (Note that we not only obtain that μ is mixing, but the stronger result that $\|\mu^n * (\mu - \delta_0)\| \to 0)$.

Let $\mathscr{A}$ be the algebra generated by adding δ_0 to $L^1(K\backslash G/K)$. $\mathscr{A}$ is a commutative Banach algebra with a unit, and its maximal ideal space $\mathscr{M}(\mathscr{A})$, is the one point compactification of $\mathscr{R}$, the maximal ideal space of $L^1(K\backslash G/K)$.

Fix $\lambda \neq 1$ on the unit circle. To show that $\mu - \lambda\delta_0$ is invertible as an operator on $L^1(K\backslash G/K)$, we show that it is invertible in $\mathscr{A}$. But this follows from the fact that its Gelfand transform never vanishes on $\mathscr{M}(\mathscr{A})$, indeed, as $\hat{\mu}(s) \to 0$ when $|s| \to \infty$ in $\mathscr{R}$, and as $\hat{\mu} \neq \lambda$ there, there is an $\varepsilon > 0$, so that $|(\mu - \lambda\delta_0)\hat{}(s)| \geqslant \varepsilon$ for every $s \in \mathscr{R}$. But $\mathscr{R}$ is dense in $\mathscr{M}(\mathscr{A})$.

Remark 3. — By Remark 2, the same proof shows that if the Conjecture of section 1 holds, the theorem remains true without the assumption that μ has compact support, and it is enough to assume that $|\hat{\mu}(s)| \neq 1$ in $\mathscr{R}$. As we can always assume that μ is absolutely continuous, we do not need to worry about the rate of decay of $(\hat{\mu}(s) - 1)$. As $\hat{\mu}$ converges to 0, $(\hat{\mu}(s) - 1)$ does not decay at all.

3. Characterization of harmonic functions.

Let f be an harmonic function on the unit disk $\mathbf{D}$. For any $g \in G$, $f \circ g$ is also harmonic. It follows that its value at zero is the average of its values on any circle centered at zero. It follows that if μ is a radial measure on $\mathbf{D}$ with $\mu(\mathbf{D}) = 1$ then

$$(*) \qquad \int_G f(gh)\, d\mu(h) = f(g) \quad \text{for all} \quad g \in G$$

whenever f is a bounded harmonic function on $\mathbf{D}$, (i.e. $f \in L_\infty(\mathbf{D}) = L_\infty(G/K))$.

We shall be interested in the converse question. Given a radial measure μ as above, under what additional conditions on μ the only functions $f \in L_\infty(\mathbf{D})$ satisfying $(*)$ are harmonic?

This question was treated by Furstenberg in a more general context. In [Fu1], [Fu2] Furstenberg considered measures and functions on G/K where G is a semisimple Lie group with finite center, and K a maximal compact subgroup. He showed ([Fu2] Theorem 5, p. 370), that when μ is an absolutely continuous radial probability measure on G/K every $f \in L_\infty(G/K)$ satisfying $(*)$, is necessarily harmonic.

As Furstenberg uses probabilistic methods, the assumption that μ is a probability measure, is essential to his method. (In our context where $G = SU(1,1)$, the absolute continuity of the radial measure μ on $\mathbf{D}$ is not essential, as follows from Lemma 0.3.)

The results of the previous section allow us to give some easy answers to these converse problems.

Writing $d\nu(g) = d\mu(g^{-1})$ equation $(*)$ becomes $f * \nu = f$, and we are thus led to the study of convolution equations.

THEOREM 3.1. – *Let $\mu \neq \delta_0$ be a radial measure on the unit disk $\mathbf{D}$ with $\mu(\mathbf{D}) = 1$, and let $f \in L_\infty(\mathbf{D})$ satisfy the convolution equation $f * \mu = f$. If μ satisfies one of the following conditions, then f is necessarily harmonic:*

1) *μ is a probability measure.*

2) *μ has compact support, and there is a $\delta > 0$ so that $\hat{\mu}(s) \neq 1$ for every $s \in \mathscr{R}(\delta)$, $s \neq 0,1$.*

Proof. – We first claim that if the equation $f * \mu = f$ has a non-harmonic solution then it also has a radial non-constant solution. To see this note that if f is a solution, so is $f_g(z) = f(gz)$ for every $g \in G$, and so is the radial average $F_g = \int_K f_g(kz)\,dk$. But if f is non harmonic, there is a circle with respect to which it does not satisfy the mean value property, i.e. there is a $g \in G$ so that $F_g(z) = \int_K f_g(kz)\,dk \neq f_g(0) = F_g(0)$. It follows that F_g is a radial non constant solution of the equation.

We can thus assume that f is a radial solution of the equation $f * \mu = f$, and we need to show that f is constant.

Assume first that (1) holds, i.e. μ is a probability measure. (This is the case covered by Furstenberg's Theorem.) For every $h \in L^1_0(K\backslash G/K)$, and for every n, f also satisfies the convolution equation $h * \mu^n * f = h * f$. By Theorem 2.1 μ is mixing, so $h * \mu^n \to 0$, and the left hand side converges to zero. It follows that $h * f = 0$ for all $h \in L^1_0(K\backslash G/K)$, which means that f is constant.

If (2) holds, Lemma 2.2 implies that the closed ideal J generated by $\{(\mu-\delta_0)*h : h \in L^1(K\backslash G/K)\}$ is all of $L^1_0(K\backslash G/K)$. But $k * f = 0$ for all $k \in J$, hence also for all $k \in L^1_0(K \backslash G/K)$, and f is constant.

The assumption in the Theorem that $\hat{\mu}(s) \neq 1$ for all $s \in \mathcal{R}$, with $s \neq 0,1$, is clearly necessary. Indeed if $\hat{\mu}(s) = 1$ then $\mu * \varphi(\cdot, s) = \varphi(\cdot, s)$, and as $s \in \mathcal{R}$, $\varphi(\cdot, s)$ is a bounded radial function, and it is non constant if $s \neq 0,1$. (Thus, hidden behind case (1) of the theorem is the fact that when $\mu \neq \delta_0$ is a probability measure, then $\hat{\mu}(s) \neq 1$ in $\mathcal{R}$ except for $s = 0,1$. This is indeed the case as (4) of Lemma 0.1 implies.) We conjecture that this is the only necessary assumption on μ.

More precisely we formulate :

CONJECTURE. — *Let* $\mu \neq \delta_0$ *be a radial measure on the unit disk* $\mathbf{D}$, *so that* $\mu(\mathbf{D}) = 1$ and $\hat{\mu}(s) \neq 1$ *for all* $s \neq 0,1$ in $\mathcal{R}$. *If* f *is a bounded radial function on* $\mathbf{D}$ *satisfying* $f * \mu = f$ *then* f *is constant.*

This will follow if the conjecture in section 1 holds. Indeed, by Remark 2 of section 2, if the conjecture of section 1 holds then the closed ideal J in $L^1_0(K\backslash G/K)$ generated by $\{(\mu-\delta_0)*h : h \in L^1(K\backslash G/K)\}$ is all of $L^1_0(K\backslash G/K)$. (By considering $\mu * \mu$ instead of μ, we can assume that μ is absolutely continuous. Thus $\hat{\mu}(s) - 1$ does not even converge to zero, and the decay condition is trivially satisfied.) The proof proceeds now as in case (2) above.

The use of ideal theory rather then the probabilistic methods of [Fu1], [Fu2] also allows us to characterize harmonic functions as those satisfying a *family* of convolution equations. Thus the same arguments as in the proof of Lemma 2.2 and Theorem 3.1 give

THEOREM 3.2. — *Let* $\mathcal{M}$ *be a family of radial measures of compact support on* $\mathbf{D}$, *and assume there is a* $\delta > 0$ *so that for every* s *in* $\mathcal{R}(\delta)$, $s \neq 0,1$, *there is a* $\mu \in \mathcal{M}$ *with* $\hat{\mu}(s) \neq 1$. *If* f *is a bounded measurable function on* $\mathbf{D}$, *so that* $f * \mu = f$ *for every* $\mu \in \mathcal{M}$, *then* f *is harmonic.*

Once again, if the conjecture of section 1 is true, then one can take $\delta = 0$, and drop the assumption that the measures have compact support. We can always assume that the decay condition on $\hat{\mu}(s) - 1$ is satisfied by replacing the measures by their powers if necessary.

4. Characterization of holomorphic functions.

In this section we characterize holomorphic functions in the unit disk $\mathbf{D}$. We use ideas from [A] to translate these problems to problems of harmonic analysis on $SU(1,1)$. We then use structure of the ideals in $L^1(K\backslash G/K)$ to improve some of the results of [A], and to fill a gap in one of the proofs there.

Let $\mathcal{M}$ be a set of left K-invariant measures on G of compact support. For each $f \in L^1(\mathbf{D})$ (which we identify with the right K-invariant functions in $L^1(G)$), and every $\mu \in \mathcal{M}$, $\mu * f$ is a spherical function. Denote by $I(\mathcal{M})$ the closed ideal in $L^1(K\backslash G/K)$ generated by $\{\mu * f : f \in L^1(\mathbf{D}), \mu \in \mathcal{M}\}$.

The ideal theory in $L^1(K\backslash G/K)$ will be used in the following theorem on the uniqueness of the solution of a system of convolution equations.

THEOREM 4.1. — *Let $\mathcal{M}$ be a set of left K-invariant measures on G of compact support, and assume that there is a $\delta > 0$ so that the operator valued Fourier transforms $\mathcal{F}(\mu, s)$, where $\mu \in \mathcal{M}$, do not have a common zero in $\mathcal{R}(\delta)$. Let F be a bounded right K-invariant function on G. If F satisfies the system of convolution equations*

$$F * \mu = 0 \quad \textit{for every} \quad \mu \in \mathcal{M}$$

then $F = 0$.

Proof. — It is enough to show that $I(\mathcal{M}) = L^1(K\backslash G/K)$. Indeed, if f is in $L^1(\mathbf{D})$ and $\mu \in \mathcal{M}$, then $F * (\mu * f) = (F * \mu) * f = 0$, hence $F * h = 0$ for every $h \in I(\mathcal{M})$. If $I(\mathcal{M}) = L^1(K\backslash G/K)$, this holds for every $h \in L^1(K\backslash G/K)$. But if $F \neq 0$, there is an h in $L^1(K\backslash G/K)$, with $F * h \neq 0$. Just take a radial smooth approximation of the Dirac measure at $0 \in \mathbf{D}$.

For each $s \in \mathcal{R}(\delta)$, we can find $\mu \in \mathcal{M}$ and a function $h \in L^1(\mathbf{D})$, of compact support so that $(\mu * h)\hat{}(s) \neq 0$. Indeed, there is a $\mu \in \mathcal{M}$, so that $\mathcal{F}(\mu, s) \neq 0$. As μ is left K-invariant, there is an n so that

$\mathcal{F}(\mu_n, s) \neq 0$, where μ_n is the $(0,n)$-radial component of μ. Take now h to be any $(n,0)$-radial function of compact support so that $\mathcal{F}(h,s) \neq 0$. Then $(\mu * h)\hat{}(s) = \mathcal{F}(\mu_n, s)\mathcal{F}(h,s) \neq 0$.

The functions $\mu * h$, are then functions of compact support in $I(\mathcal{M})$, and their Fourier transforms do not have a common zero in $\mathcal{R}(\delta)$. By Corollary 1.5, $I(\mathcal{M})$, which contains all these functions, is all of $L^1(K\backslash G/K)$. By the remarks at the beginning of the proof, this completes the proof of the theorem.

We shall need some notation. Recall that $d\lambda = \rho\, dx\, dy$ is the conformally invariant measure on $\mathbf{D}$, where $\rho(z) = (1-|z|^2)^{-2}$. We denote by $\partial\bar{z}$ the differential operator $\rho^{-1}\,\partial/\partial\bar{z}$.

Given $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in G$, and $z \in \mathbf{D}$, put $r(g,z) = |-\bar{\beta}z+\alpha|^{-2}$.

If ϕ is any smooth function of compact support on G, and f is a measurable locally integrable function on $\mathbf{D}$, put

$$(R_\phi f)(z) = \int \phi(g) r(g,z) f(g^{-1}z)\, dg.$$

$(R_\phi f)(z)$ is a smooth function in $L^1_{\mathrm{loc}}(\mathbf{D})$, and if ϕ_n is a smooth approximate identity on G then $(R_{\phi_n} f) \to f$ in the L^1 norm on any compact subset of $\mathbf{D}$. If γ is a smooth simple closed curve in $\mathbf{D}$, then a change of variables in the integral shows that $\int_{g(\gamma)} f(z)\, dz = \int_\gamma f(g^{-1}z) r(g,z)\, dz$. It follows that if f in $L^1_{\mathrm{loc}}(\mathbf{D})$ satisfies

$$\int_{g(\gamma)} f(z)\, dz = 0, \quad \text{for almost all} \quad g \in \mathrm{G},$$

then the same is true for $(R_\phi f)(z)$.

The main ingredient we use from [A] is the following estimate:

LEMMA 4.2. — *Let $f \in L^1_{\mathrm{loc}}(\mathbf{D})$ and let ϕ be a smooth function of compact support on G. Then $|\partial/\partial\bar{z}(R_\phi f)(z)| \leqslant (1-|z|^2)^{-1}F(z)$ for every $z \in \mathbf{D}$, where $F(z) = \int_G \psi(z)|f(g^{-1}z)|dg$ and ψ is a bounded non-negative function of compact support on G, that depends only on ϕ.*

In particular, if f satisfies $|f(z)| \leqslant c(1-|z|^2)^{-1}$ in $\mathbf{D}$ then $\bar{\partial}(R_\phi f)$ is a bounded function on $\mathbf{D}$.

(Lemma 3.1 in [A] is formulated differently, but this is the main estimate in its proof.)

We are now ready for the main result of this chapter. Let γ_i be a family of smooth closed simple curves in $\mathbf{D}$, and let Δ_i be their interiors. We denote by μ_i the left K-invariant measure on G, $d\mu_i(g) = \chi_i(g^{-1})\,dg$, where χ_i is the indicator function of Δ_i, considered as a function on G. We denote by $\mathscr{F}(\mu_i, s)$ their (operator valued) Fourier transforms.

THEOREM 4.3. — *With the notation as above, assume that there is a* $\delta > 0$ *so that* $\mathscr{F}(\mu_i, s)$ *do not have a common zero in* $\mathscr{R}(\delta)$. *Let* f *be a measurable function on* $\mathbf{D}$ *satisfying* $|f(z)| \leqslant c(1-|z|^2)^{-1}$ *for every* $z \in \mathbf{D}$. *If*

$$(*) \qquad \int_{g(\gamma_i)} f(z)\,dz = 0, \quad \textit{for all } i \textit{ and almost all } g \in G,$$

then f *coincides almost everywhere in* $\mathbf{D}$ *with a holomorphic function.*

Proof. — As f can be approximated by functions of the form $R_\phi f$, it suffices to prove that $R_\phi f$ is holomorphic for every smooth function of compact support ϕ on G.

Thus fix such a ϕ, and put $h = R_\phi f$. By the remarks before Lemma 4.2, h also satisfies $(*)$. By Green's Theorem it then follows that for each fixed i and each $g \in G$

$$\begin{aligned} 0 &= \int_{g(\gamma_i)} h(z)\,dz = \iint_{g(\Delta_i)} \partial h/\partial \bar{z}\,dx\,dy \\ &= \iint_{g(\Delta_i)} \bar{\partial} h(z)\,d\lambda(z) = \iint_D \bar{\partial} h(z)\chi_i(g^{-1}z)\,d\lambda(z) \\ &= \int_G \bar{\partial} h(x)\chi_i(g^{-1}x)\,dx = (\bar{\partial} h) * \mu_i(g) \end{aligned}$$

where the last integral identifies integration of functions on $\mathbf{D}$ with respect to λ with integration of the lifted function on G with respect to the Haar measure.

By Lemma 4.2 $\bar{\partial} h$ is a bounded function on $\mathbf{D}$, and by Theorem 4.1 it follows that it must be identically zero. But by the definition of $\bar{\partial}$ it follows that $\partial h/\partial \bar{z} \equiv 0$, i.e. h is holomorphic as required.

Remarks. – (1) The usefulness of the theorem is limited by the necessity to find the zeros of $\mathscr{F}(\mu_i, s)$. This is a very strong limitation as the only curves for which we know how to compute the Fourier transform are circles (see [A], p. 177) :

If χ_r, $0 < r < 1$, is the indicator function of the central disk of radius r, then χ_r is a radial function, and its Fourier transform $\hat{\chi}_r$ is $2\pi r^2(1-r)^{is-4}(1+r)^{-is}J_r(s)$, where $J_r(s) = F(2-is, 3/2, 3, -4r(1-r)^{-2})$, and F is the hypergeometric function.

In fact it is known that if γ is not real analytic then $\mathscr{F}(\mu,s)$ does not have any zeros in the whole complex plane. Even for real analytic curves it is an open problem if the only curve fo which $\mathscr{F}(\mu,s)$ has zeros is the circle. (For details on these matters and their relation to the Pompeiu problem, see [BY], [BZ] and [Z].) It follows that for many types of curves one does not need a family of curves, and one curve suffices to characterize holomorphic functions.

(2) Theorem 4 (2) in [A], claims that if γ_i, $i = 1,2$ are circles of radii r_i so that J_{r_i} do not have a common zero in the strip $0 < \mathrm{Re}(s) < 1$ and f satisfies the growth condition and $(*)$ as in Theorem 4.3, then f is holomorphic. The strict inequalities in the definition of the strip seem to be a misprint, as the maximal ideal space of $L^1(K\backslash G/K)$ is the closed strip.

More seriously, the proof in [A] does not take into account the intricate structure of the ideals in $L^1(K\backslash G/K)$, and assumes that the only condition necessary for an ideal to coincide with all of $L^1(K\backslash G/K)$ is that the Fourier transforms of its elements have no common zero in the maximal ideal space.

Our theorem requires that the J_{r_i} do not have a common zero in $\mathscr{R}(\delta)$ for some $\delta > 0$. We do not know whether having no common zeros in $\mathscr{R}$ suffices, but we conjecture that this is so. This will follow, for example if the Conjecture in section 1 holds, or if Theorem 7 in [EM1], quoted in Remark 1 at the end of section 1 holds for non-principal ideals.

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