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ON THE DISTRIBUTION OF SCATTERING POLES FOR PERTURBATIONS OF THE LAPLACIAN

by Georgi VODEV (*)

1. Introduction.

In this note we study the distribution of the scattering poles associated to second order differential operators of the form

$$G = c(x)^{-1} \left(- \sum_{i,j=1}^n \partial_{x_i} (g_{ij}(x) \partial_{x_j}) + \sum_{j=1}^n b_j(x) \partial_{x_j} + a(x) \right)$$

in \mathbb{R}^n , $n \geq 3$, odd, where the coefficients are such that the following conditions are fulfilled :

- (i) The operator G admits a selfadjoint realization, which will be again denoted by G , in the Hilbert space $H = L^2(\mathbb{R}^n; c(x) dx)$ with domain $D(G)$;
- (ii) There exists a constant $\rho_0 > 0$ so that for any $u \in D(G)$ such that $u = 0$ for $|x| \leq \rho_0$ we have $u \in H^2(\mathbb{R}^n)$ and $Gu = -\Delta u$, Δ being the Laplacian in \mathbb{R}^n ;
- (iii) G is positively definite, i.e. $(Gu, u)_H \geq 0$, $\forall u \in D(G)$.

In what follows $\|\cdot\|$ will denote the norm in $\mathfrak{L}(H, H)$, the space of all linear bounded operators acting from H into H . It is easy to see by (i) and (iii) that the resolvent $R(z) = (G - z^2)^{-1} \in \mathfrak{L}(H, H)$ is well defined and holomorphic in $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$, and

$$(1.1) \quad \|R(z)\| \leq C(\text{Im } z)^{-2} \quad \text{for} \quad \text{Im } z > 0.$$

Choose a function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi = 1$ for $|x| \leq \rho_0 + 1$ and set $R_\chi(z) = \chi R(z) \chi$ for $z \in \mathbb{C}_+$. When

- (iv) $R_\chi(z_0)$ is a compact operator in $\mathfrak{L}(H, H)$ for some $z_0 \in \mathbb{C}_+$,

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it is well known that the cutoff resolvent $R_\chi(z)$ admits a meromorphic continuation from \mathbb{C}_+ to the entire complex plane \mathbb{C} (see the analysis in the next section). The poles of this continuation are known as scattering poles or resonances and in our case they all are in $\bar{\mathbb{C}}_-$, where $\mathbb{C}_- = \{z \in \mathbb{C} : \text{Im } z < 0\}$. Note that if (iv) holds for at least one z_0 , it holds for all z_0 . Let $\{\lambda_j\}$ be the poles of $R_\chi(z)$, repeated according to multiplicity, and set

$$N(r) = \# \{\lambda_j : |\lambda_j| \leq r\}.$$

When the operator G is elliptic, in [8] and [14] (see also [13]) it is proved (without assuming (iii)) that

$$(1.2) \quad N(r) \leq Cr^n + C.$$

It also follows from the analysis in [8] and [14] that for hypoelliptic operators, i.e. when we have the estimates

$$(1.3) \quad \|f\|_{s+2\delta} \leq C_s(\|Gf\|_s + \|f\|_s), \quad \forall s \geq 0, \forall f \in D(G), Gf \in H^s,$$

where $0 < \delta < 1$ and $\|\cdot\|_s$ denotes the norm in the usual Sobolev space H^s , (again without assuming (iii)) the number of the poles satisfies the bound

$$(1.4) \quad N(r) \leq Cr^{n/\delta} + C.$$

Note that (1.3) implies (iv) at once. By (1.4) one actually concludes that the less regular the operator G is, the worse bound for $N(r)$ one has. In this work we show that outside a conic neighbourhood of the real axis the number of the scattering poles satisfies a much better estimate than (1.4) no matter how regular the operator G is. It actually has a bound of the type (1.2). To be more precise, given any ε , $0 < \varepsilon \ll 1$, set $\Lambda_\varepsilon = \{z \in \mathbb{C} : \varepsilon \leq \arg z \leq \pi - \varepsilon\}$ and

$$N(\varepsilon, r) = \# \{\lambda_j : |\lambda_j| \leq r, -\lambda_j \in \Lambda_\varepsilon\}.$$

Our main result is the following :

THEOREM 1. — *Assume (i)-(iv) fulfilled. Then for any ε , $0 < \varepsilon \ll 1$, there exists a constant $C_\varepsilon > 0$ so that*

$$(1.5) \quad N(\varepsilon, r) \leq C_\varepsilon r^n + C_\varepsilon.$$

The estimate (1.5) shows that to study the counting function $N(r)$ modulo terms $O(r^n)$ for positively definite selfadjoint hypoelliptic operators it suffices to study the number of the scattering poles in a conic ε -neighbourhood of the real axis for any small $\varepsilon > 0$.

The idea for the proofs of polynomial bounds of the scattering poles originates from Melrose [4] (see also [2], [5], [11], [12], [13], [14], [17]). One first needs to find an entire family of compact operators, $K(z)$, so that $(1 - K(z))R_\chi(z)$ is an entire operator-valued function and $1 - K(z)$ is invertible for at least one $z \in \mathbb{C}$. Thus one concludes that the poles of $R_\chi(z)$, with multiplicity, are among the poles of $(1 - K(z))^{-1}$ and hence among the zeros of an entire function $h(z) = \det(1 - K(z)^p)$, where $p \geq 1$ is an integer taken so that $K(z)^p$ is trace class. Thus the problem is reduced to obtaining suitable estimates for $|h(z)|$.

To prove (1.5) we need to find a family $K(z)$ as above so that $(1 - K(z))^{-1}$ can be expressed in terms of $R(z)$ for $z \in \mathbb{C}_+$ (see (2.5)), and $K(z) - K(-z)$ is trace class for any $z \in \mathbb{C}$. This enables us to characterize the poles of $R_\chi(z)$ in \mathbb{C}_- , with multiplicity, as zeros of a function $h(z)$, defined and holomorphic in \mathbb{C}_- , such that for any $\gamma > 0$ there exists a constant $C_\gamma > 0$ so that

$$(1.6) \quad |h(-z)| \leq C_\gamma \exp(C_\gamma |z|^n) \quad \text{for} \quad \text{Im } z \geq \gamma.$$

Then, we derive (1.5) from (1.6) and a classical result due to Carleman (see Lemma 2).

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2. Representation of the cutoff resolvent.

Denote by G_0 the selfadjoint realization of $-\Delta$ in the Hilbert space $H_0 = L^2(\mathbb{R}^n)$ and let $R_0(z)$ denote the outgoing resolvent of $-\Delta - z^2$, $z \in \mathbb{C}$. Then $R_0(z) = (G_0 - z^2)^{-1} \in \mathfrak{L}(H_0, H_0)$ for $z \in \mathbb{C}_+$ and as is well-known the kernel of $R_0(z)$ is given in terms of Hankel's functions by

$$(2.1) \quad R_0(z)(x, y) = (i/4)(z/2\pi|x - y|)^{(n-2)/2} H_{(n-2)/2}^{(1)}(z|x - y|).$$

It is easy to see that $\chi R_0(z)\chi \in \mathfrak{L}(H_0, H_0)$ for all $z \in \mathbb{C}$ and it forms an entire family of compact pseudodifferential operators of order -2 . Using this together with the assumption (iv) we shall build the meromorphic continuation of the cutoff resolvent of G . Set $Q = G - G_0$ and fix a $z_0 \in \mathbb{C}_+$. Clearly, for all $z \in \mathbb{C}_+$ we have

$$(2.2) \quad R(z) = R_0(z) + R(z)QR_0(z)$$

and

$$(2.3) \quad R(z) = R(z_0) + (z^2 - z_0^2)R(z)R(z_0).$$

Combining (2.2) and (2.3) yields

$$R(z)(1 - (z^2 - z_0^2)QR_0(z)R(z_0)) = R(z_0) + (z^2 - z_0^2)R_0(z)R(z_0)$$

for $z \in \mathbb{C}_+$. Multiplying the both sides of this identity by χ , since $Q = \chi Q$, we get

$$(2.4) \quad R_\chi(z)(1 - K(z)) = R_\chi(z_0) + K_1(z) \quad \text{for } z \in \mathbb{C}_+,$$

where

$$K(z) = (z^2 - z_0^2)QR_0(z)R(z_0)\chi$$

$$K_1(z) = (z^2 - z_0^2)\chi R_0(z)R(z_0)\chi.$$

Moreover, since $R(z)$ is well defined in \mathbb{C}_+ , it is easy to see by (2.4) that $1 - K(z)$ is invertible in $\mathfrak{L}(H, H)$ for all $z \in \mathbb{C}_+$ and

$$(2.5) \quad (1 - K(z))^{-1} = 1 + (z^2 - z_0^2)QR_0(z)(R_0(z_0) + R_0(z_0)QR(z))\chi$$

for $z \in \mathbb{C}_+$. Now, since $R_0(z)$ and $R(z)$ are holomorphic in \mathbb{C}_+ with values in $\mathfrak{L}(H, H)$ and since $QR_0(z) = QR_0(z_0)(1 + (z^2 - z_0^2)R_0(z))$ for $z \in \mathbb{C}_+$, we deduce from (2.5) that $(1 - K(z))^{-1}$ is holomorphic in \mathbb{C}_+ with values in $\mathfrak{L}(H, H)$. Moreover, by (1.1), which clearly holds with $R(z)$ replaced by $R_0(z)$ as well, for any $\gamma > 0$ there exists a constant $C_\gamma > 0$ so that

$$(2.6) \quad \|(1 - K(z))^{-1}\| \leq C_\gamma(1 + |z|)^4 \quad \text{for } \operatorname{Im} z \geq \gamma.$$

Now let us see that the operator-valued functions $K(z)$ and $K_1(z)$, defined in \mathbb{C}_+ , extend analytically to the entire \mathbb{C} with values in the compact operators in $\mathfrak{L}(H, H)$. We shall consider $K(z)$ only, since $K_1(z)$ is treated similarly. Using that $R(z_0) = R_0(z_0) + R_0(z_0)QR(z_0)$ it is easy to see that

$$(2.7) \quad K(z) = (z^2 - z_0^2)QR_0(z)R_0(z_0)\chi(1 + QR_\chi(z_0))$$

for $z \in \mathbb{C}_+$. Choose functions $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^n)$ such that $\chi_1 = 1$ on $\operatorname{supp} Q$, $\chi_2 = 1$ on $\operatorname{supp} \chi_1$ and $\chi = 1$ on $\operatorname{supp} \chi_2$. After a standard computation (2.7) takes the form

$$(2.8) \quad K(z) = (z^2 - z_0^2)QR_0(z)R_0(z_0)\chi K_2 + (z^2 - z_0^2)QR_0(z)\chi K_3 R_\chi(z_0)$$

for $z \in \mathbb{C}_+$, where

$$K_2 = 1 + [\chi_2, G_0]R_0(z_0)[\chi_1, G_0]R_0(z_0)QR_\chi(z_0),$$

$$K_3 = \chi_1 R_0(z_0)Q + \chi_2 R_0(z_0)[\chi_1, G_0]R_0(z_0)Q.$$

Here $[\cdot, \cdot]$ denotes the comutator. Clearly, we have $K_2, K_3 \in \mathfrak{L}(H, H)$. Further on, by a similar computation, for $z \in \mathbb{C}_+$, one obtains

$$(2.9) \quad (z^2 - z_0^2)QR_0(z)R_0(z_0)\chi$$

$$= (K_4 + (z^2 - z_0^2)K_5)\chi R_0(z)\chi - K_4\chi R_0(z_0)\chi$$

and

$$(2.10) \quad QR_0(z)\chi = (K_4 + (z^2 - z_0^2)K_5)\chi R_0(z)\chi + K_5,$$

where

$$K_4 = QR_0(z_0)[G_0, \chi_1]R_0(z_0)[G_0, \chi_2],$$

$$K_5 = QR_0(z_0)\chi_1 + QR_0(z_0)[G_0, \chi_1]R_0(z_0)\chi_2.$$

Clearly, $K_4, K_5 \in \mathfrak{L}(H, H)$. Thus, by (2.8)-(2.10) we deduce

$$(2.11) \quad K(z) = K_6(z)\chi R_0(z)\chi K_2 + K_7(z)\chi R_0(z)\chi K_8 + K_9(z)$$

for $z \in \mathbb{C}_+$, where

$$K_6(z) = K_4 + (z^2 - z_0^2)K_5,$$

$$K_7(z) = (z^2 - z_0^2)K_6(z),$$

$$K_8 = K_3R_\chi(z_0),$$

$$K_9(z) = -K_4\chi R_0(z_0)\chi K_2 + (z^2 - z_0^2)K_5K_3R_\chi(z_0).$$

Clearly, these four operators are analytic $\mathfrak{L}(H, H)$ -valued functions. Now, since $\chi R_0(z)\chi$ forms an entire family of compact operators and by (iv) so does $K_9(z)$, by (2.11) we can extend $K(z)$ analytically to the entire \mathbb{C} . Then, since $K(z_0) = 0$, by Fredholm theorem, $(1 - K(z))^{-1}$ is a meromorphic $\mathfrak{L}(H, H)$ -valued function on \mathbb{C} . Thus, by (2.4) we obtain the desired meromorphic continuation of $R_\chi(z)$. Moreover, clearly the poles of this continuation coincide, with multiplicity, with the poles of $(1 - K(z))^{-1}$. Thus, since $1 - K(z)$ is invertible for $z \in \mathbb{C}_+$, we have that all the poles are in $\bar{\mathbb{C}}_-$. Now, for $z \in \mathbb{C}_+$, we have

$$(2.12) \quad 1 - K(-z) = (1 - K(z))(1 - T(z)),$$

where

$$T(z) = (1 - K(z))^{-1}(K(-z) - K(z)).$$

By (2.11) we have

$$(2.13) \quad T(z) = T_1(z)\chi S(z)\chi K_2 + T_2(z)\chi S(z)\chi K_8$$

where

$$\begin{aligned} S(z) &= R_0(-z) - R_0(z) \\ T_1(z) &= (1 - K(z))^{-1} K_6(z) \\ T_2(z) &= (1 - K(z))^{-1} K_7(z). \end{aligned}$$

By (2.6), for any $\gamma > 0$, we get

$$(2.14) \quad \|T_j(z)\| \leq C_\gamma(1 + |z|)^8 \quad \text{for} \quad \text{Im } z \geq \gamma, \quad j = 1, 2.$$

On the other hand, by (2.1) and the well known properties of the Hankel functions, we have the following formula for the kernel of $S(z)$:

$$\begin{aligned} (2.15) \quad S(z)(x, y) &= (i/2)(z/2\pi|x-y|)^{(n-2)/2} J_{(n-2)/2}(z|x-y|) \\ &= (i/2)(2\pi)^{-n+1} z^{n-2} \int_{\mathbb{S}^{n-1}} \exp(iz\langle x-y, w \rangle) dw, \quad x, y \in \mathbb{R}^n, \end{aligned}$$

where \mathbb{S}^{n-1} denotes the unit sphere in \mathbb{R}^n . Denote by $\tilde{S}(z)$ the operator with kernel given by the integral above. Now it is easy to see by (2.15) that $\chi S(z)\chi$ forms an entire family of trace class operators in $\mathfrak{L}(H, H)$. Hence, by (2.13), $T(z)$ is holomorphic in \mathbb{C}_+ with values in the trace class operators in $\mathfrak{L}(H, H)$. Now, by (2.12) it is easy to see that $1 - T(z)$ is invertible in $\mathfrak{L}(H, H)$ for those $z \in \mathbb{C}_+$ for which so is $1 - K(-z)$, and then we have

$$(2.16) \quad (1 - K(-z))^{-1} = (1 - T(z))^{-1}(1 - K(z))^{-1}.$$

Since $(1 - K(z))^{-1}$ is holomorphic in \mathbb{C}_+ , by (2.16) we conclude that the poles of $(1 - K(-z))^{-1}$ lying in \mathbb{C}_+ , with multiplicity, coincide with the poles of $(1 - T(z))^{-1}$. Introduce the function

$$h(z) = \det(1 - T(z)),$$

which is well defined and holomorphic in \mathbb{C}_+ . Now, by the above analysis we conclude that if $\lambda_j, \lambda_j \in \mathbb{C}_-$, is a scattering pole, then $-\lambda_j$ is a zero of $h(z)$ with the corresponding multiplicity. Thus we can characterize the scattering poles as zeros of $h(-z)$, $z \in \mathbb{C}_-$. Notice that the fact that $T(z)$ is trace class does not depend on whether (iv) is

fulfilled or not. Hence the function $h(z)$ is always defined, under the conditions (i)-(iii), and holomorphic in \mathbb{C}_+ . Now we are going to study the distribution of the zeros of $h(z)$ without assuming (iv). Note that in general the zeros of $h(z)$ may accumulate at points on the real axis. Let $\{z_j\} \subset \mathbb{C}_+$ be the zeros of $h(z)$, repeated according to multiplicity, and given $0 < \varepsilon, \delta \ll 1, r \gg 1$, set

$$N(\varepsilon, \delta, r) = \#\{z_j : \delta \leq |z_j| \leq r, z_j \in \Lambda_\varepsilon\}.$$

We have the following :

THEOREM 2. — *Assume (i)-(iii) fulfilled. Then, for any ε, δ, r as above there exists a constant $C_{\varepsilon, \delta} > 0$, independent of r , so that*

$$(2.17) \quad N(\varepsilon, \delta, r) \leq C_{\varepsilon, \delta} r^n \quad \text{for} \quad r \geq 1.$$

When (iv) is fulfilled the number of the scattering poles in $\{z \in \mathbb{C} : |z| \leq \delta\}$ is finite for any $\delta > 0$, and hence (1.5) is obtained as an immediate consequence of (2.17).

3. Proof of Theorem 2.

We start with the following :

LEMMA 1. — *Under the assumptions (i)-(iii), for any $\gamma > 0$ there exists a constant $C_\gamma > 0$ so that*

$$(3.1) \quad |h(z)| \leq C_\gamma \exp(C_\gamma |z|^n) \quad \text{for} \quad \text{Im } z \geq \gamma.$$

Proof. — The estimate (3.1) is established in the same way as in [13] (see also [17]). Here we shall sketch the proof. Given a compact operator A , $\mu_j(A)$ will denote the characteristic values of A , i.e. the eigenvalues of $(A^*A)^{1/2}$, repeated according to multiplicity and ordered to form a nonincreasing sequence. First, recall some well known properties of $\mu_j(A)$:

$$(3.2) \quad \mu_j(A) \leq \|A\|, \quad \forall j,$$

$$(3.3) \quad \{\mu_j(AB), \mu_j(BA)\} \leq \mu_j(A)\|B\|, \quad \forall j,$$

$$(3.4) \quad \mu_j\left(\sum_{i=1}^k A_i\right) \leq \sum_{i=1}^k \mu_{j_k}(A_i), \quad \forall j,$$

where $j_k \sim [j/k]$, $[a]$ denotes the integer part of a . By (2.13)-(2.15) and (3.2)-(3.4) it is easy to see that

$$(3.5) \quad \mu_j(T(z)) \leq C_\gamma(1+|z|)^{n+6}\mu_{j_2}(\chi\tilde{S}(z)\chi) \quad \text{for} \quad \text{Im } z \geq \gamma.$$

On the other hand, clearly we have

$$(3.6) \quad \|\chi\tilde{S}(z)\chi\| \leq C \exp(C|z|), \quad \forall z \in \mathbb{C}.$$

Combining (3.5) and (3.6) yields

$$(3.7) \quad \mu_j(T(z)) \leq C_\gamma \exp(C|z|), \quad \forall j, \quad \text{for} \quad \text{Im } z \geq \gamma.$$

Further on, we shall show that there exists a constant $C > 0$ so that

$$(3.8) \quad \mu_j(\chi\tilde{S}(z)\chi) \leq Ce^{-|z|j^{-n/(n-1)}} \quad \text{if} \quad j \geq C|z|^{n-1}, \quad \forall z \in \mathbb{C}.$$

This is actually proved in [13], but for the sake of completeness we shall repeat the key points. The key observation is the representation

$$(3.9) \quad \tilde{S}(z) = S_1(z)S_2(z),$$

where $S_1(z)$ is the operator with kernel $S_1(z)(x, w) = \exp(iz\langle x, w \rangle)$, $S_2(z)$ is the operator with kernel $S_2(z)(w, x) = \exp(-iz\langle x, w \rangle)$, $x \in \mathbb{R}^n$, $w \in \mathbb{S}^{n-1}$. Then, using (3.3) and (3.9) we have

$$(3.10) \quad \mu_j(\chi\tilde{S}(z)\chi) \leq \|\chi S_1(z)\|_1 \|(1-\Delta_w)^m S_2(z)\chi\|_2 \mu_j((1-\Delta_w)^{-m}), \quad \forall j,$$

for any integer $m \geq 1$, where Δ_w denotes the Laplace-Beltrami operator on \mathbb{S}^{n-1} , $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the norms in $\mathcal{L}(L^2(\mathbb{S}^{n-1}), L^2(\mathbb{R}^n))$ and $\mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{S}^{n-1}))$, respectively. On the other hand, we have with a constant $C > 0$,

$$(3.11) \quad \mu_j((1-\Delta_w)^{-m}) \leq C^m j^{-2m/l},$$

where $l = \dim \mathbb{S}^{n-1} = n - 1$, and

$$(3.12) \quad \|\chi S_1(z)\|_1 \leq C \exp(C|z|),$$

$$(3.13) \quad \begin{aligned} \|(1-\Delta_w)^m S_2(z)\chi\|_2 &\leq C \sup_{x, w} |\chi(x)(1-\Delta_w)^m(e^{-z\langle x, w \rangle})| \\ &\leq C^{2m+1}(|z|^{2m} + (2m)^{2m})e^{C|z|}. \end{aligned}$$

Thus, by (3.10)-(3.13),

$$(3.14) \quad \mu_j(e^{|z|}\chi\tilde{S}(z)\chi) \leq C^{2m+1}(|z|^{2m} + (2m)^{2m})e^{C|z|j^{-2m/l}},$$

with a new constant $C > 0$. Now, (3.8) is an easy consequence of (3.14) (see [13], [17]).

Thus, by (3.5) and (3.8), we have

$$(3.15) \quad \mu_j(T(z)) \leq C_\gamma j^{-n/(n-1)} \quad \text{if } j \geq C|z|^{n-1}, \quad \text{for } \text{Im } z \geq \gamma,$$

with new constants C_γ , $C > 0$. Now, it is a straightforward calculation that (3.7) and (3.15) together with Weyl's convexity estimate imply (3.1) (see [13], [17]. The proof of Lemma 1 is completed.

To derive (2.17) from (3.1), instead of Jensen's inequality, we shall use the following classical result (see [9], Section 3, Carleman's theorem).

LEMMA 2. — Given $r > r_0 > 0$, set $\Omega = \{z \in \mathbb{C} : r_0 \leq |z| \leq r, \text{Im } z \geq 0\}$. Let $f(z)$ be a function holomorphic in Ω and let $r_1 \exp(i\varphi_1), r_2 \exp(i\varphi_2), \dots, r_k \exp(i\varphi_k)$ be the zeros of $f(z)$ in Ω repeated according to multiplicity. Then,

$$\begin{aligned} \sum_{j=1}^k (r_j^{-1} - r_j r^{-2}) \sin \varphi_j &= (\pi r)^{-1} \int_0^\pi \log |f(re^{i\varphi})| \sin \varphi \, d\varphi \\ &\quad + (2\pi)^{-1} \int_{r_0}^r (t^{-2} - r^{-2}) \log |f(t)f(-t)| \, dt \\ &\quad - (\pi r_0)^{-1} \int_0^\pi \log |f(r_0 e^{i\varphi})| \sin \varphi \, d\varphi. \end{aligned}$$

Note that each term in the sum above is ≥ 0 . Fix $\varepsilon, \delta, 0 < \varepsilon, \delta \ll 1$, and let $r > 6$. Let

$$z_1 = r_1 \exp(i\varphi_1), \quad z_2 = r_2 \exp(i\varphi_2), \dots, z_k = r_k \exp(i\varphi_k)$$

be the zeros of $h(z)$, repeated according to multiplicity, satisfying the conditions: $3 \leq r_j \leq r/2$; $\varepsilon \leq \varphi_j \leq \pi - \varepsilon$. Clearly,

$$(3.16) \quad N(\varepsilon, \delta, r/2) \leq k + N(\varepsilon, \delta, 3).$$

Set $f(z) = h(z + i\gamma)$ where $\gamma = \sin \varepsilon$. Clearly, $f(z)$ is holomorphic in $\bar{\mathbb{C}}_+$ and by (3.1) we have

$$(3.17) \quad |f(z)| \leq C_\varepsilon \exp(C_\varepsilon |z|^n), \quad \forall z \in \bar{\mathbb{C}}_+.$$

Moreover, $z'_j = z_j - i\gamma, j = 1, \dots, k$, are zeros of $f(z)$. Set $r'_j = |z'_j|$ and $\varphi'_j = \arg z'_j$. It is easy to check that $2 \leq r'_j \leq 2r/3$ and $\sin \varphi'_j \geq$

$2^{-1} \sin \varepsilon$, $j = 1, \dots, k$. Hence

$$(3.18) \quad (r_j'^{-1} - r_j' r^{-2}) \sin \varphi_j' \geq (5\gamma/12)r^{-1}, \quad j = 1, \dots, k.$$

Now, applying Lemma 2 to $f(z)$ with $r_0 = 2$ and using (3.17) and (3.18), we get

$$\begin{aligned} (5\gamma/12)r^{-1}k &\leq \sum_{j=1}^k (r_j'^{-1} - r_j' r^{-2}) \sin \varphi_j' \\ &\leq (\pi r)^{-1} \int_0^\pi \log |f(re^{i\varphi})| \sin \varphi \, d\varphi \\ &\quad + (2\pi)^{-1} \int_2^r (t^{-2} - r^{-2}) \log |f(t)f(-t)| \, dt + C_\varepsilon \\ &\leq C'_\varepsilon r^{-1}(r^n + 1) + C'_\varepsilon \int_2^r t^{-2}(t^n + 1) \, dt + C_\varepsilon \\ &\leq C''_\varepsilon r^{-1}(r^n + 1). \end{aligned}$$

Hence

$$(3.19) \quad k \leq (12/5\gamma)C''_\varepsilon(r^n + 1).$$

Now (2.17) follows from (3.16) and (3.19) at once.

BIBLIOGRAPHY

- [1] C. BARDOS, J. GUILLOT, J. RALSTON, La relation de Poisson pour l'équation des ondes dans un ouvert non borné. Application à la théorie de la diffusion, *Commun. Partial Differ. Equations*, 7 (1982), 905-958.
- [2] A. INTISSAR, A polynomial bound on the number of scattering poles for a potential in even dimensional space \mathbb{R}^n , *Commun. Partial Differ. Equations*, 11 (1986), 367-396.
- [3] P. D. LAX, R. S. PHILLIPS, *Scattering Theory*, Academic Press, 1967.
- [4] R. B. MELROSE, Polynomial bounds on the number of scattering poles, *J. Func. Anal.*, 53 (1983), 287-303.
- [5] R. B. MELROSE, Polynomial bounds on the distribution of poles in scattering by an obstacle, *Journées « Équations aux dérivées partielles »*, Saint-Jean-de-Monts (1984).
- [6] R. B. MELROSE, Weyl asymptotics for the phase in obstacle scattering, *Commun. Partial Differ. Equations*, 13 (1988), 1431-1439.
- [7] J. SJÖSTRAND, Geometric bounds on the number of resonances for semiclassical problems, *Duke Math. J.*, 60 (1990), 1-57.

- [8] J. SJÖSTRAND, M. ZWORSKI, Complex scaling and the distribution of scattering poles, *J. Amer. Math. Soc.*, (1991), 729-769.
- [9] E. C. TITCHMARSH, *The Theory of Functions*, Oxford University Press, 1968.
- [10] B. VAINBERG, *Asymptotic Methods in Equations of Mathematical Physics*, Gordon and Breach Sci. Publ., 1988.
- [11] G. VODEV, Polynomial bounds on the number of scattering poles for symmetric systems, *Ann. Inst. Henri Poincaré (Physique théorique)*, 54 (1991), 199-208.
- [12] G. VODEV, Polynomial bounds on the number of scattering poles for metric perturbations of the Laplacian in \mathbb{R}^n , $n \geq 3$, odd, *Osaka J. Math.*, 28 (1991), 441-449.
- [13] G. VODEV, Sharp polynomial bounds on the number of scattering poles for metric perturbation of the Laplacian in \mathbb{R}^n , *Math. Ann.*, 291 (1991), 39-49.
- [14] G. VODEV, Sharp bounds on the number of scattering poles for perturbations of the Laplacian, *Commun. Math. Phys.*, 145 (1992), to appear.
- [15] M. ZWORSKI, Distribution of poles for scattering in the real line, *J. Func. Anal.*, 73 (1987), 227-296.
- [16] M. ZWORSKI, Sharp polynomial bounds on the number of scattering poles of radial potentials, *J. Func. Anal.*, 82 (1989), 370-403.
- [17] M. ZWORSKI, Sharp polynomial bounds on the number of scattering poles, *Duke Math. J.*, 59 (1989), 311-323.

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