ALLEN HATCHER

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SOME EXAMPLES OF ESSENTIAL LAMINATIONS IN 3-MANIFOLDS

by Allen HATCHER

Incompressible surfaces have long been a major tool in 3-dimensional topology, and more recently foliations without Reeb components have also begun to play an important role, particularly through the work of Gabai. Laminations fill the spectrum between surfaces and foliations, and the correct generalization of incompressibility for surfaces and the absence of Reeb components for foliations seems to be the following : a lamination $L \subset M^3$ without D^2, S^2, or $\mathbf{R}P^2$ leaves is *essential* if (1) the boundary leaves of L are incompressible, ∂-incompressible, and end-incompressible in $M - L$, *i.e.* there are no properly embedded essential disks, half-disks, of half-planes in $M - L$, or more precisely, in the completion of $M - L$ with respect to a path-metric; and (2) L contains no Reeb components , *i.e.* sublaminations of the usual Reeb foliation of a solid torus or solid half-torus. See the foundational paper [GO] for more details. The new condition of end-incompressibility amounts to requiring that non-compact leaves have no "infinite folds". For the extreme cases of surfaces and foliations this holds automatically – which may account for the relatively late discovery (ca. 1986, by the author) of this rather natural condition.

The hope is that most irreducible 3-manifolds are *laminated*, *i.e.* contain an essential lamination. The first piece of evidence supporting this is the following simple construction which shows that in one sense "almost all" closed 3-manifolds are laminated. Viewing closed 3-manifolds as obtained by Dehn filling on surface bundles with a single boundary

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torus, the generic case is pseudo-Anosov monodromy. Then the suspension of either the stable or unstable lamination of the monodromy is an essential lamination in the surface bundle, disjoint from the boundary, and it is not hard to see, using results in [GO], that this lamination stays essential after all but a $\mathbf{Z} \cup \{\infty\}$, or in favorable cases all but one, of the $\mathbf{Q} \cup \{\infty\}$ possible Dehn fillings.

Novikov's theorem implies that 3-manifolds with finite π_1 are non-laminated. This also follows from a result in [GO] that the universal cover of a closed laminated 3-manifold is $\mathbf{R}^3$. Some of the infinite π_1 Seifert-fibered manifolds with base S^2 and three exceptional fibers are also known to be non-laminated [B], [JN]. No non-Seifert non-laminated irreducible manifolds have yet been found, though we shall describe some strong candidates at the end of this paper.

The main purpose of this paper is to give two fairly simple constructions of essential laminations, constructions which are rather special but nevertheless lend a little more support to the idea that most 3-manifolds are laminated. Our first construction, which was strongly motivated by [FH1] and turns out also to be related to Gabai's foliation constructions, yields essential laminations L_σ in an arbitrary 2-bridge knot exterior, with ∂L_σ consisting of parallel curves of slope σ in the torus boundary. The novel feature is that by varying certain parameters in the construction of L_σ, the boundary slope σ can be made to range over an interval of real values. This is in strong contrast to the situation for incompressible surfaces where the set of boundary slopes is always finite [H]. For most 2-bridge knots σ ranges over all of $\mathbf{R}$ in fact. The laminations L_σ with σ rational remain essential after Dehn surgery of slope σ, so we obtain in this way many laminated non-Haken manifolds, since it was shown in [HT] that all but finitely many Dehn surgeries on a 2-bridge knot yield non-Haken irreducible manifolds.

Combining these examples with a much more general and subtle construction of Gabai [G], one can deduce that even in the exceptional cases when the boundary slopes of the L_σ's do not range over all of $\mathbf{R}$, all but finitely many Dehn surgeries on a non-torus 2-bridge knot yield laminated manifolds. Based on the example of 2-bridge knots, one might then conjecture, loosely, that *most surgeries on all knots and all surgeries on most knots yield laminated manifolds*, and similarly for links. This is reminiscent of the situation for hyperbolic Dehn surgery.

Our second construction gives essential laminations F_σ in an arbitrary punctured-torus bundle, again with boundary consisting of parallel curves of slope σ ranging over intervals. The laminations F_σ are in fact foliations, which makes verifying their essentiality both before and after Dehn filling considerably simpler. In the main case of pseudo-Anosov monodromy the manifolds produced by the Dehn fillings for the rational boundary slopes of the $F_\sigma' s$ were already known to be laminated, by the suspension construction described earlier. Nevertheless, the construction of the F_σ's, which is quite different from the other constructions of essential laminations, gives further evidence that essential laminations exist in great abundance. This construction has subsequently been generalized by R. Roberts to bundles with fiber a once-punctured compact orientable surface of arbitrary positive genus.

1. 2-bridge knots.

A simple example, the figure-eight knot $K_{2/5}$, will suffice to give the essential idea. The laminations here will be obtained by grafting together two measured laminations. As in [HT] and [FH1], these measured laminations will be described as finite sequences of 1-dimensional laminations in level 4-punctured spheres, each obtained from the previous one by passing through saddles.

Figure 1 shows the first measured lamination. It starts with two bands of parallel arcs of slope ∞, of thickness $\alpha+1$ and 1. Two batches of saddles of thickness γ and $1-\gamma$ change this configuration to two bands of parallel arcs of slope 0 and one band of slope ∞. Next, two batches of saddles of thickness δ and $\alpha-\delta$ yield two bands of slope 0 and one band of slope $1/2$, at least if $\delta \leq 1$ and $\alpha - \delta \leq 1$. If these inequalities fail to hold, we can use thinner batches of saddles to reduce the thickness of the slope ∞ band to $\alpha - 1$, creating a slope $1/2$ band of thickness 1, (retaining the two slope 0 bands of thickness 1), then repeat this process until all of the slope ∞ band is replaced by the slope $1/2$ band. Finally, we reverse the first step, with slope $1/2$ now instead of slope ∞ (a Dehn twist about a slope 0 circle achieves this), to produce two bands of slope $1/2$, of thickness $\alpha+1$, and 1.

The second measured lamination is simpler : take two bands of slope $1/2$ and thickness 1, and change them to two bands of slope $2/5$ and thickness 1 by two batches of saddles of thickness η and $1-\eta$. This is the same as the first step in the preceding construction, but with $\alpha = 0$ and a linear change of coordinates in the 4-punctured sphere to change slopes ∞ and 0 to $1/2$ and $2/5$.

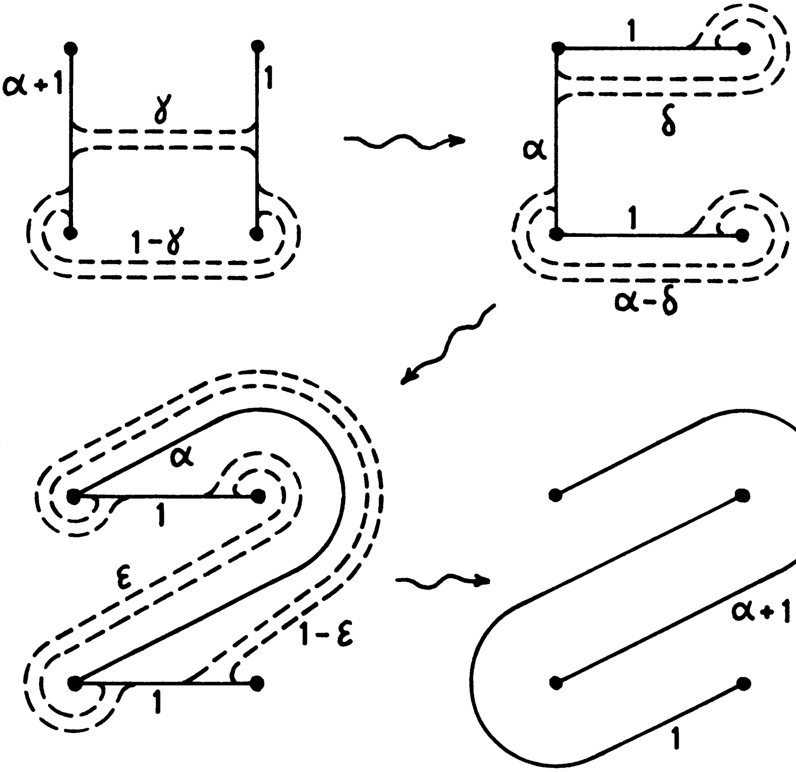

Figure 1

These two measured laminations are grafted together at their common slope 1/2 configuration by identifying the two slope 1/2 bands in the first lamination with the two slope 1/2 bands in the second lamination, using identifications which preserve the transverse measure up to scalar multiplication. The result is a lamination L_α in the exterior of $K_{2/5}$. (The values of the parameters $\gamma, \delta, \varepsilon, \eta$ are not important, and could be fixed in advance.) L_α meets the peripheral torus transversely in a lamination ∂L_α transverse to meridians, consisting simply of parallel curves (circles or lines) since it inherits a well-defined projective class of transverse measures from the measures on the two pieces which make up L_α.

In the limiting case $\alpha = 0$ the second picture in Figure 1 is skipped, and the lamination L_0 is measured; L_0 is in fact just a thickening of the incompressible Seifert surface for $K_{2/5}$. In the setting of [HT], L_0 corresponds to the edgepath with successive vertices 1/0, 0/1, 1/2, 2/5 in the strip of triangles shown in Figure 2.

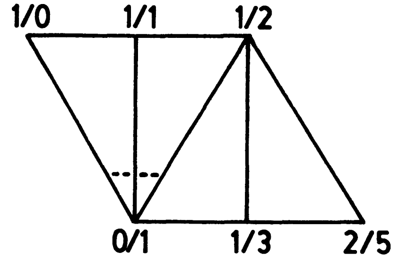

Figure 2

For $\alpha > 0$ the edgepath for L_α does not pass through the vertex $0/1$ but takes the "shortcut" indicated by the dotted line.

The ∂-slope of L_α can be computed by measuring the twisting of the leaves around the four punctures as we go from one picture in Figure 1 to the next, seeing what proportion of the total weight at each puncture passes across a fixed reference direction, say the direction radiating outward from the centerpoint of each picture. Clockwise rotation counts positive (right-hand twisting), counterclockwise rotation negative. Reading around the four punctures in clockwise order starting at the upper right puncture, we find for the transition from the first picture to the second the twists $\gamma - 1$, $1 - \gamma$, $(\gamma - 1)/(\alpha + 1)$, and $(1 - \gamma)/(\alpha + 1)$ for a net twist of 0. For the second transition the twists are δ, $\alpha - \delta$, $(\delta - \alpha)/(\alpha + 1)$, and $(\alpha - \delta)/(\alpha + 1)$ for a net twist of α. For the third transition we have ε, $1 - \varepsilon$, $\varepsilon/(\alpha + 1)$, and $(1 - \varepsilon)/(\alpha + 1)$ for a net twist of $1 + (\alpha + 1)^{-1}$. The total is thus $1 + \alpha + (\alpha + 1)^{-1}$. When $\alpha = 0$ this is 2, and as α increases it increases to $+\infty$. The rest of the lamination, going from slope $1/2$ to slope $2/5$, is the same as the thickened Seifert surface, L_0. Since L_0 has ∂-slope 0, the ∂-slope of L_α ranges over the interval $[0, \infty)$ as α goes from 0 to ∞.

Now we generalize the construction to an arbitrary 2-bridge knot $K_{p/q}$. As in [HT], incompressible surfaces in the complement of $K_{p/q}$ correspond to minimal edgepaths from $1/0$ to p/q in the strip of triangles shown in Figure 5 of [HT], consisting of k "wedges", the i^{th} wedge containing a_i triangles where the a_i's are the terms in the standard continued fraction expansion of p/q. (Figure 2 above is the case $k = 2$, $a_1 = a_2 = 2$.) Without loss of generality we may assume $a_k > 1$ (otherwise the last wedge can be combined with the next-to-last) and also $0 < p/q < 1/2$, hence $a_1 > 1$. Consider a minimal edgepath which turns across two triangles at some vertex p_i/q_i. At this vertex we can "shortcut" the edgepath and construct a lamination L_α just as in the figure-eight example. This involves applying a homeomorphism $\varphi \in GL_2(\mathbf{Z})$ of the 4-punctured sphere taking the slopes $1/0$, $0/1$, and $1/2$ in Figure 1 to the slopes of the endpoints of the two edges of the edgepath entering and leaving p_i/q_i. If p_i/q_i lies on the lower edge of the strip of triangles, as in our previous example, we can take $\varphi \in SL_2(\mathbf{Z})$, preserving orientations, and then as α goes from 0 to $+\infty$, the ∂-slope of L_α goes from its value for the thickened incompressible surface L_0 to $+\infty$. On the other hand if p_i/q_i is on the upper edge of the strip, φ must reverse orientations, so the ∂-slope of L_α goes from its value for L_0 to $-\infty$. For example, for $L_{2/5}$ we could use the same edgepath as before and shortcut at the vertex $1/2$ to produce laminations with ∂-slopes ranging over $(-\infty, 0)$.

More generally, if a minimal edgepath turns across an even number of triangles at a vertex p_i/q_i, we can again construct a corresponding lamination L_α by simply repeating the transition from the second to the third picture in Figure 1, once for each pair of triangles crossed at p_i/q_i. The boundary behavior of this L_α is just as in the previous case, varying to $\pm\infty$.

PROPOSITION 1. — *Each lamination L_α constructed in this way is essential, as is its completion L'_α in the manifold obtained by the Dehn surgery associated to the ∂-slope of L_α.*

Proof. — We show such a lamination L_α is fully carried by a branched surface Σ which is essential in the exterior of $K_{p/q}$. Then by [GO] L_α is essential. As in [HT] and [FH1], Σ is made by stacking up pieces of branched surface, the piece shown in Figure 3 in [HT] for edges not involved in the shortcut, and the pieces Σ_A and Σ_D shown in Figure 3.1 in [FH1], Σ_A for the two shortened edges and Σ_D for the shortcut itself. In addition, if α is large enough to require several steps to get from the second to the third picture in Figure 1 above, then we also use pieces obtained by enlarging Σ_D by extending the slope 0 or slope $1/2$ arc to go all the way from the top to the bottom of the piece.

To see that Σ is essential there are several things to check. First, in each complementary region of Σ the horizontal boundary is incompressible and ∂-incompressible. See Figures 21 in [HT] and 6.2 in [FH1] for the case of complementary regions not involving enlarged Σ_D's. For the case of enlarged Σ_D's it is easy to see that the complementary region is again a product $D^2 \times 1$ as in (b) and (c) of Figure 6.2 in [FH1].

Next we must see that Σ has no disks or half-disks of contact. The former cannot occur since the singular locus of Σ contains no circles. A half-disk of contact would give a null-homotopy in $S^3 - K_{p/q}$ of a singular arc of Σ. Such an arc has slope (in its level 4-punctured sphere) equal to that of some intermediate vertex in the strip of triangles for $K_{p/q}$. This arc then is an essential arc on an incompressible surface in $S^3 - K_{p/q}$ corresponding to a minimal edgepath through that vertex (the top or bottom border of the strip, for example). Hence the arc cannot be trivial in the complement of $K_{p/q}$.

To complete the verification that Σ is essential it remains to check that Σ has no Reeb components. A Reeb component bounded by a torus is impossible since Σ carries no closed surfaces. As for a Reeb component bounded by an annulus, a small meridian of $K_{p/q}$ would be a circle passing through this annulus transversely, and also passing through each product complementary component of Σ from one side to the other. This cannot happen in a Reeb component.

Now let M be obtained by Dehn surgery on $K_{p/q}$ with rational slope equal to the ∂-slope of some L_α. M is obtained from the exterior of $K_{p/q}$ by filling in a solid torus T, and we extend Σ to a branched surface $\Sigma' \subset M$ by filling in sheets in T as follows. Σ meets ∂T in a train track which is transverse to meridians and has all its complementary regions digons. As we move into T through parallel copies of ∂T we enlarge this train track by taking a copy of one boundary arc of each digon and moving this new arc across the digon, always staying transverse to meridians. At the same time we extend L_α by having some sheets follow this new section of Σ', so the extended L_α is fully carried by the partially extended Σ'. After this has been done for each digon in turn, we have returned to the original train track in a parallel copy of ∂T. Continuing to move through parallel copies of ∂T, we split this train track along arcs to eliminate all its branching points, in such a way that the partially extended L_α is still fully carried. This yields a train track consisting of disjoint circles, which we cap off with disks in T to finish the construction of Σ', fully carrying the completion $L'_\alpha \subset M$ of L_α.

The complementary regions of Σ' contained in T are just $D^2 \times I$'s, while the other complementary regions are obtained from complementary regions of Σ in the exterior of $K_{p/q}$ by pinching boundary digons. So all complementary regions of Σ' in M have incompressible, ∂-incompressible horizontal boundary. By the same argument as before, using meridians of $K_{p/q}$, Σ' has no Reeb components.

Ruling out disks of contact for Σ' is a little more involved. Suppose D is a disk of contact for Σ'. If ∂D lies in the boundary of a $D^2 \times I$ complementary component of Σ', then D together with one of the disks of $D^2 \times \partial I$ gives a sphere carried by Σ'. This means Σ carries a genus zero surface. This is a measured lamination, with the same measure at each of the four punctures in Figure 1, so this surface is carried by a branched subsurface of Σ obtained by setting $\alpha = 0$. This is one of the incompressible branched surfaces of [HT]. These carry genus zero surfaces only when $K_{p/q}$ is the torus knot $K_{1/q}$ and the genus zero surface is an annulus of positive ∂-slope $2q$. But for $K_{1/q}$ the L_α's have negative ∂-slope (see Proposition 2 below), and we have reached a contradiction.

Now suppose ∂D lies in the boundary of a solid torus complementary component S of Σ', as in Figure 21 of [HT] (with the dashed lines now replaced by solid lines, according to our construction of Σ'). There are two subcases depending on whether there are one or two circles of cusp points in ∂S. If there is one circle, then the complementary annulus of this circle in ∂S together with two parallel copies of D gives a sphere carried by Σ', which leads to a contradiction as before. If there are two cusp circles in ∂S, they must both bound disks of contact since the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ of 180 degree rotations of the 4-punctured sphere acts on Σ and hence, we may assume, on Σ', interchanging the two cusp circles. These two disks of contact together with an annulus in ∂S between their boundary circles give a sphere carried by Σ'. This may only be immersed, with double curves where the two disks of contact intersect each other, but the usual innermost disk argument leads to an embedded sphere carried by Σ'. Again this yields a contradiction.

Thus Σ', and hence L'_α, is essential in M.

PROPOSITION 2. — *For the 2-bridge knot $K_{p/q}$ the set of ∂-slopes of the essential laminations L_α is all of $\mathbf{R}$ except in the following cases :*

(a) *When $k = 1$ (so $p = 1$ and $K_{1/q}$ is a torus knot), it is $(-\infty, 0]$;*

(b) *When $k = 2$ and a_1 and a_2 have opposite parity, it is $(-\infty, 0] \cup [4n, \infty)$, where $2n$ is the even a_i.*

(c) *When $k = 3$ and all three a_i's are odd, it is $(-\infty, -4-4n] \cup [0, \infty)$, where $2n + 1 = \min\{a_1, a_3\}$.*

Proof. — To obtain all ∂-slopes it suffices to find a minimal edgepath which turns across an even number of triangles at vertices on both the upper and lower borders of the strip of triangles. Figure 3 illustrates how to do this when $k \geq 4$. Namely, start across the top of the first wedge, then cut down between the second and third wedges. If a_3 is odd, continue the rest of the way across the bottom of the strip, while if a_3 is even, come immediately back to the top of the strip and continue the rest of the way across the top.

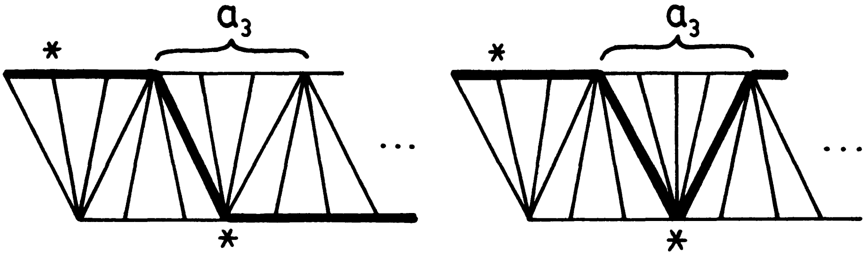

Figure 3

The cases $k \leq 3$ we leave for the reader. □

2. Punctured-torus bundles.

Let $M_\varphi \to S^1$ be a bundle with fiber T, a torus with an open disk removed, and monodromy $\varphi \in SL_2(\mathbf{Z})$. We shall construct foliations F_σ in M_φ by specifying how they intersect the fibers, and to do this it is convenient to look in the cover $\widetilde{T}$ which is $\mathbf{R}^2$ minus a neighborhood of $\mathbf{Z}^2$. Here we start with the thickened train track shown in Figure 4.

We assign weights to the various sections of this track as shown by the numbers located within the thickening. Outside the region pictured, assign weights so that rightward translation by one unit multiplies weights by 2, while upward translation by one unit multiplies weights by the parameter $\mu > 0$. These weights determine a measured lamination $\widetilde{L}(\mu, t)$ in $\widetilde{T}$. (Ignore the dotted line for now). $\widetilde{L}(\mu, t)$ can easily be chosen to be invariant under deck transformations, with the measure multiplied by scalars. $\widetilde{L}(\mu, t)$ then covers a lamination $L(\mu, t)$ in T. By collapsing the complement of $L(\mu, t)$ in the usual way, we can convert $L(\mu, t)$ into a foliation $F(\mu, t)$, transverse to ∂T, with a single saddle singularity.

When $t = 0$, $L(\mu, 0)$ has a single closed leaf, a circle of slope 0, and all other leaves form a band of parallel half-lines starting at ∂T and spiralling

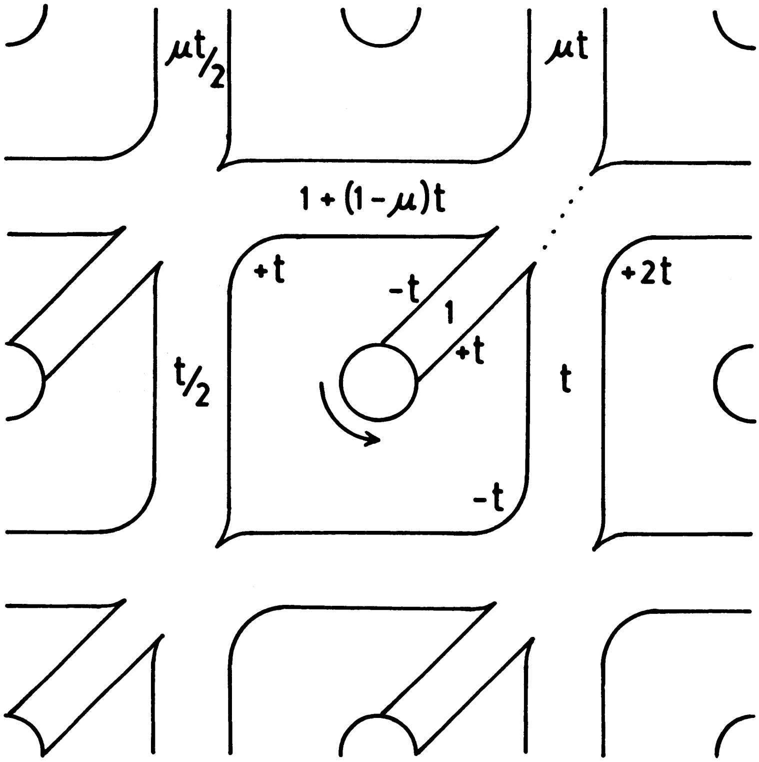

Figure 4

in to the circle leaf from below. If we choose $(\mu, t) = (\mu_1, t_1)$ satisfying $2+(1-\mu)t = \mu t$ (which can be solved either for μ or t) then the lamination $\widetilde{L}(\mu_1, t_1)$ splits open along the translates of the dotted line in Figure 4 and $L(\mu_1, t_1)$ again has a single closed leaf, this time a circle of slope 1, with all other leaves spiralling in to this circle from below. (One can check that $1 + (1 - \mu_1)t_1 > 0$, so $\widetilde{L}(\mu_1, t_1)$ is defined.)

Note that if we fix μ_1 and let t increase we are transferring sheets of leaves of thickness t from one pair of opposite sides of the complementary quadrilateral of $L(\mu, t)$ to the other pair of opposite sides, as indicated in Figure 4. (We are thinking of ∂T as a vertex when we regard the complementary region as a quadrilateral.) This means we can construct a measured lamination in $\widetilde{T} \times I$ covering a lamination in $T \times I$ which meets $T \times 0$ in $L(\mu_1, 0)$ and $T \times 1$ in $L(\mu_1, t_1)$ with a layer of saddles (critical points of index one for the projection onto the I factor of $T \times I$) achieving the "surgery" of $L(\mu_1, 0)$ to $L(\mu_1, t_1)$. The foliation version of this lamination in $T \times I$ we call $F_{t_1}\{(1,0),(1,1)\}$, the vectors $(1,0)$ and $(1,1)$ corresponding to the limit cycles of the boundary foliations $F(\mu_1, 0)$ in $T \times 0$ and $F(\mu_1, t_1)$ in $T \times 1$, oriented by the direction of spiralling. This $F_{t_1}\{(1,0),(1,1)\}$ meets each slice $T \times s$ in a foliation transverse to $\partial T \times s$, with a single saddle singularity.

More generally, applying a suitable linear map in $SL_2(\mathbf{Z})$to T, we have foliations $F_{t_1}\{(p,q),(r,s)\}$ in $T \times I$ meeting $T \times 0$ and $T \times 1$ in foliations (each with a single saddle singularity) which spiral in from the right to oriented limit cycles with slopes q/p and s/r, provided $ps - qr = 1$.

For the given monodromy $\varphi \in SL_2(\mathbf{Z})$ choose a sequence $(p_i, q_i) \in \mathbf{Z}^2$, $i = 0, \ldots, n$ with $p_i q_{i+1} - p_{i+1} q_i = 1$ and $(p_n, q_n) = \varphi(p_0, q_0)$. (There are infinitely many such sequences; they correspond to edgepaths in the diagram of $SL_2(\mathbf{Z})$, Figure 4 in [HT], from q_0/p_0 to q_n/p_n which go around the diagram an even number of times – to take care of the ambiguity that (p_n, q_n) and $(-p_n, -q_n)$ both have slope q_n/p_n.) The corresponding blocks $F_{t_i}\{(p_i, q_i),(p_{i+1}, q_{i+1})\}$ can then be stacked end-to-end to form a foliation F_σ of M_φ. Fitting two adjacent blocks together is possible since they have the same oriented limit cycle. The gluing is uniquely determined if we require it to preserve the transverse measure at a fixed lift of ∂T to $\widetilde{T}$. (This gluing operation may be clearer if one thinks of the associated laminations obtained by splitting open the saddle singularities.) The foliation ∂F_σ of the torus ∂T then has a transverse measure, so consists of parallel lines or circles.

The foliations F_σ have no compact leaves, for in the block $F_{t_1}\{(1,0),(1,1)\}$ the only compact leaf of $F(\mu_1, 0)$ in $T \times 0$ is the limit cycle, and this is immediately connected by saddles to non-compact leaves in nearby fibers $T \times s$. Since F_σ has no compact leaves, it has no Reeb components, hence is essential.

We remark that breaking F_σ up into the blocks $F_{t_i}\{(p_i, q_i),(p_{i+1}, q_{i+1})\}$ is somewhat artificial, and was done mainly for convenience of exposition. If one looks at how F_σ intersects the various fibers of M_φ one sees foliations on T whose associated laminations (obtained by splitting open the saddle singularity) have leaves starting at ∂T and limiting on a sublamination which is either a circle, of rational slope, or a standard irrational-slope lamination in $\text{int}(T)$. The slopes of these limit sublaminations vary continuously (and monotonically) from fiber to fiber, filling in the intervals between the values q_i/p_i. Each rational limit slope is taken on for a whole interval of fibers, with the other leaves spiralling into the limit cycle from both sides (not just from one side as in the fibers where one block is joined to the next), in varying proportions as one moves across the interval of fibers. By contrast, a given irrational limit slope is taken on only at an isolated fiber. See [HO] for more on these laminations in T, which have transverse affine structures.

Now we consider the question of which ∂-slopes are realized by the foliations F_σ. There are several cases, depending on φ. Suppose first that φ has distinct positive real eigenvalues. We may conjugate φ so that its expanding eigenvectors are in the first and third quadrants, its contracting eigenvectors in the second and fourth quadrants. This is equivalent to the four entries of φ being positive. Let $(p_0, q_0) = (1, 0)$, so $\varphi(p_0, q_0) = (p_n, q_n)$ is also in the first quadrant, and let the sequence (p_i, q_i) lie in the first quadrant and have monotonically increasing slopes. To speak of ∂-slopes for F_σ requires specifying slopes 0 and ∞ in ∂M_φ. It is natural to choose the fibers to have slope ∞, and in the present case we choose the curve in ∂M_φ determined by a first-quadrant eigenvector of φ for slope 0. The ∂-slope of F_σ is a measure of how many leaves of ∂F_σ cross the slope 0 curve. For the block $F_{t_1}\{(1,0),(1,1)\}$ it is clear from Figure 4 that the total ∂-twisting is exactly t_1. Similarly, $F_{t_i}\{(p_i, q_i, (p_{i+1}, q_{i+1})\}$ has ∂-twisting t_i, so the ∂-slope of F_σ is Σt_i, an arbitrary positive number. Arbitrary negative ∂-slopes are realizable by a completely analogous construction, starting with the reflection of Figure 4 across the x-axis, choosing the (p_i, q_i)'s again in the first quadrant, starting with $(0, 1)$ and with monotonically decreasing slope.

The next case is that φ has distinct negative real eigenvalues, *i.e.* the negative of a φ in the previous case. Here the choice of "slope 0" in ∂M_φ is less natural, since a circle traced out in ∂M_φ by an eigendirection intersects the fiber (slope ∞) twice. If we choose slope 0 so that this circle has slope $1/2$, then it is not hard to see that the ∂-slopes of the F_σ's range over $(-\infty, 0) \cup (1, \infty)$.

In the remaining cases the reader can check that the foliations F_σ again have ∂-slopes varying over the complement of a finite closed interval.

In the case that φ has distinct positive real eigenvalues, our construction does not produce a foliation with boundary slope 0. In some cases 0 is a boundary slope of an incompressible, ∂-incompressible surface, and in other cases 0 is not such a ∂-slope; see [FH2]. In the cases that 0 is not a ∂-slope it is natural to conjecture that the closed manifold obtained by slope 0 Dehn filling is non-laminated, since the suspended stable and unstable laminations of φ in M_φ become inessential after this filling. In most cases these closed manifolds seem to have hyperbolic structures. Similarly, in the case that φ has distinct negative eigenvalues one might conjecture that the Dehn-filled manifolds for which the suspended stable and unstable laminations become inessential are non-laminated.

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Allen HATCHER,
Dept. of Mathematics
Cornell University
Ithaca NY 14853–7901 (USA).