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RANDOM WALKS ON FREE PRODUCTS

by M. Gabriella KUHN

1. Introduction.

Let $G = *_{j=1}^{q+1} G_{n_j+1}$ be the free product of $q+1$ $(q+1) > 3$ finite groups each of order $n_j + 1$ and let $\mathcal{G}$ be the Cayley graph of G with respect to the generators $\{a_j\,;\, a_j \in G_{n_j+1}\}_{j=1}^{q+1}$.

We recall that $\mathcal{G}$ is a connected graph with the property that at each vertex V there meet exactly $q+1$ polygons $P_j(V)$ with $n_j + 1$ sides, and any two vertices belonging to the same polygon are connected by an edge.

Identify G (as a set) with $\mathcal{G}$ and consider G acting on the « homogeneous space » $\mathcal{G}$ by left multiplication.

Choose $q+1$ positive numbers $p_1, \ldots, p_{q+1}$ satisfying the condition $\sum_{j=1}^{q+1} p_j = 1$. Let μ be a probability measure which assigns the probability p_j to each copy of $G_{n_j+1}\backslash e$. If we look at $\mathcal{G}$, it is natural to consider *equal* all the vertices belonging to the same polygon. This suggests to make the simplest possible choice for the measure μ.

Set $\mu(x) = \dfrac{p_j}{n_j}$ if $x \in G_{n_j+1}\backslash e (j=1,\ldots,q+1)$ and zero otherwise.

Consider the random walk on $\mathcal{G}$ with law μ. Then the transition probability $p(V) \to (V')$ of moving from a vertex V' to a vertex V is $\dfrac{p_j}{n_j}$ if both V and V' belong to the same polygon P_j and $V \neq V'$.

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Observe that the structure of each factor group G_{n_j+1} is really unimportant for the description of the random walk on $\mathcal{G}$ and the associated Green function G_γ.

On the other hand, G_γ plays a central role in order to understand the operator of right convolution with μ on $\ell^2(G)$ and has been considered by many authors [AK] [CS2] [CT] [T2].

We know that G_γ can be described by means of « elementary » functions, and sometimes this is enough to understand completely its behaviour. Nevertheless the cases which are completely described are still very few :

$q + 1 = 2$ by [CS1] and [T2]; $n_j + 1 = 2$ $\forall j$ and $p_{q+1} \leqslant p_q, \cdots \leqslant p_1$ by [F-TS]; $p_1 = p_2 = \cdots p_{q+1}$ and $n_j + 1 = \mathrm{N}\, \forall j$ by [IP] [T1] (see also [CT]). The last case, $n_j + 1 = N\, \forall j$, is also described in [K] with several choices of the p_j with $p_{q+1} \leqslant p_q \leqslant \cdots \leqslant p_1$.

In this paper we shall give a complete description of the point spectrum of μ in $C^*_{\mathrm{reg}}(G)$ by means of the numbers n_j.

The continuous spectrum sp_c (in $C^*_{\mathrm{reg}}(G)$) will be computed in several cases. In spite of the point spectrum, sp_c depends on the p_j as well as on the numbers n_j. We shall give a necessary condition for sp_c to be connected.

Finally following the aim of [IP] and [F-TS] we shall produce a decomposition of the regular representation of G by means of μ. We shall also prove that this decomposition is into irreducibles exactly when there are not true eigenspaces of μ.

Notation.

G will always denote the free product of $q + 1$ finite groups G_{n_j+1} each of order $n_j + 1$.

Let e denote the group identity. It is convenient to set, for every j

$$\tilde{G}_{n_j+1} = G_{n_j+1} \backslash e .$$

Each x in G, $x \neq e$, may be uniquely represented as a reduced word, as $x = a_{j_1} a_{j_2}, \ldots, a_{j_m}$ where $a_j \in \tilde{G}_{n_j+1}$ and $j_k \neq j_{k+1}$ for $1 \leqslant k \leqslant m - 1$. The length of x, that we shall denote by $|x|$, is the minimum number of elements $a_j \in \{\tilde{G}_{n_j+1}\}_{j=1}^{q+1}$ needed to represent x.

Path distance on $\mathscr{G}$ corresponds to this notion of length.

Let δ_x denote the Kronecker delta at x. Set

$$\mu(x) = \sum_{j=1}^{q+1} p_j \mu_j(x)$$

where

$$\mu_j(x) = \sum \frac{1}{n_j} \delta_{a_j}, \qquad a_j \in \tilde{G}_{n_j+1} \qquad \text{and} \qquad p_j \geqslant 0, \ \sum_{j=1}^{q+1} p_j = 1.$$

Arrange the n_j so that $n_1 \leqslant n_2 \leqslant n_3 \cdots \leqslant n_{q+1}$.

Let C^*_{reg} denote the C^*-algebra generated by the left regular representation of G. Since G is discrete the Kronecker delta $\delta_e(x)$ is an identity (with respect to convolution) in $\ell^2(G)$.

As a consequence, any element T of $C^*_{\text{reg}}(G)$ can be identified with an operator of right convolution on $\ell^2(G)$ by the formula

$$T(f) = T(f * \delta_e) = f * T(\delta_e) = f * t$$

being $t(x) = T(\delta_e)(x)$. Identify μ with the operator T_μ on $\ell^2(G)$ given by

$$T_\mu(f) = f * \mu$$

and let $\operatorname{sp}(\mu)$, $\operatorname{sp}_c(\mu)$, $\operatorname{res}(\mu)$ denote (respectively) the spectrum, the continuous spectrum, the resolvent of T_μ.

Since the walk is symmetric, meaning that $\mu(x^{-1}) = \mu(x)$ for every x in G, the corresponding operator T_μ is self adjoint. Hence we may use the functional calculus to produce the resolution of the identity for T_μ by means of the resolvent $R_\mu(\gamma) = (\gamma - \mu)^{-1}$ of T_μ.

We refer to [DS], Chapter X, for standard facts concernig the functional calculus. Since $R_\mu(\gamma)$ itself is an element of $C^*_{\text{reg}}(G)$, there exists an ℓ^2-function $g_\gamma(x)$ called the resolvent, or *Green function* $G_\gamma(e,x)$ of μ such that

$$R_\mu(\gamma)(f) = f * g_\gamma.$$

For large values of γ, say $|\gamma| > 1$, $g_\gamma(x)$ is given by

$$g_\gamma(x) = \sum_{n=0}^{\infty} \frac{\mu^{*n}(x)}{\gamma^{n+1}}. \tag{2.1}$$

We shall also write $(\gamma-\mu)^{-1}(x)$ for $g_\gamma(x) = R_\mu(\gamma)(\delta_e)(x)$. In general, see [W2] (see also [A] and [S] in the case of a finitely generated free group) we know that $G_\gamma(e,x)$ is an algebraic function of γ for any walk whose law measure μ is finitely supported. In this case however the algebricity of the Green function follows readily from the formulas (3.1), (3.2) and (3.3) of Section 3. If $G_\gamma(e,x)$ satisfies some functional equation, we shall think of taking the analytic continuation $g_\gamma(x)$ to satisfy the analogue equation, whenever this is possible. Keeping this in mind, we shall calculate the spectral measure $E(\sigma)(\delta_e,\delta_e)$ associated with T_μ. Fix $x \in G$ and integrate 2.1 term by term to get

$$\frac{1}{2\pi i}\int_\Gamma g_\gamma(x)\, d\gamma = \delta_e(x)$$

whenever Γ is a smooth curve around all the singularities of the analytic function $R_\mu(\gamma)(\delta_e)(x)$.

If we let now Γ shrink around $\mathrm{sp}(\mu)$ we get

$$(2.2)\quad \delta_e(x) = -\frac{1}{\pi}\int_{\mathrm{sp}_c(\mu)} \mathrm{Im}\, g_\sigma(x)\, d\sigma + \sum_{j\in \mathrm{sp}(\mu)\setminus \mathrm{sp}_c(\mu)} P_j(x)$$

where

$$\mathrm{Im}\, g_\sigma(x) = \lim_{\varepsilon\to 0^+} \{(\sigma+i\varepsilon-\mu)^{-1}(x) - (\sigma-i\varepsilon-\mu)^{-1}(x)\}$$

and $P_j(x)$ are mutually orthogonal projections onto the ℓ^2 eigenspaces of μ (corresponding to the poles m_j of $g_\gamma(x)$). We refer to section 4 for a more detailed description of $g_\sigma(x)$.

The spectral measure $E(\sigma)(\delta_e,\delta_e)$ is nothing but the positive measure obtained by letting $x = e$ in (2.2). Let us simply write $dm(\sigma)$ for it, then

$$dm(\sigma) = -\frac{1}{\pi}\mathrm{Im}\, g_\sigma(e)\, d\sigma + \sum_{j\in \mathrm{sp}(\mu)\setminus \mathrm{sp}_c(\mu)} \operatorname*{Res}_{\gamma=m_j} g_\gamma(e)\delta_{m_j}.$$

In the next section we shall see that the poles of $g_\gamma(x)$ are the same as the poles of $g_\gamma(e)$ and we shall compute the continuous and the discrete spectrum of μ.

3. Computation of sp (μ).

Identify $\mathcal{G}$, as a set, with G and think of G as a state space. The random walk on G with law μ is exactly the walk described in the introduction, if we let $\{p(x,y) = \mu(x^{-1}y)\}_{x,y\in G}$ assign the one-step transition probabilities. The geometry of $\mathcal{G}$ leads to the following considerations. Suppose that $\{x_0, x_1, \dots, x_n)$ is a path from e to x, that is, a sequence of points $x_0, x_1, \dots, x_n$ with $x_0 = e$, $x_n = x$ and $p(x_j, x_{j+1}) > 0$ for $0 \leqslant j \leqslant n-1$. Suppose that $x = a_{j_1}a_{j_2}, \dots, a_{j_m}$ is the reduced expression for x. Then *at least one* of the x_j must be equal to a_{j_1}. Keeping in mind that the walk is also invariant with respect to the left action of G, one can describe more precisely the Green function $g_\gamma(x)$. The earliest description was given in [DM] in the case of G equal to the free group, later, independently, many people discovered analogue formulas for free products of finite groups (see [CS2] [T2] and also [AK] [ML] [F-TS] [W1]). Hence we may assume that it is well known that $g_\gamma(x)$ may be written as a scalar multiple of a function $h_\gamma(x)$ satisfying

$$(3.1)\quad \begin{aligned} h_\gamma(e) &= 1 \\ h_\gamma(xy) &= h_\gamma(x)\cdot h_\gamma(y) \quad \text{whenever } |xy| = |x| + |y| \\ h_\gamma(z_1) &= h_\gamma(z_2) \quad \text{if both } z_1 \text{ and } z_2 \text{ belong to } \tilde{G}_{n_j+1}. \end{aligned}$$

We recall that, for any function satisfiing (3.1), we can easily compute the ℓ^p norm (see [F-TS] or [T2]). In fact, if $h_\gamma(a_j)$ denotes the (constant) value of h_γ on $\tilde{G}_{n_j+1}$, then h_γ belongs to ℓ^p if and only if

$$\sum_{j=1}^{q+1} \frac{n_j|h_\gamma(a_j)|^p}{1 + n_j|h_\gamma(a_j)|^p} < 1 .$$

When this happens we have

$$\|h_\gamma\|_p^{-p} = 1 - \sum_{j=1}^{q+1} \frac{n_j|h_\gamma(a_j)|^p}{1 + n_j|h_\gamma(a_j)|^p} .$$

If we set

$$g_\gamma(e) = \frac{1}{2w}$$

then $h_\gamma(x)$ may be written as an analytic function of w. In particular, if $a_j \in \tilde{G}_{n_j+1}$, then

$$(3.2) \qquad h(a_j) = \xi_j^{\pm} = \frac{\left\{\pm\sqrt{z_j^2 + \frac{4p_j^2}{n_j}} - z_j\right\}}{2p_j}$$

being

$$z = z_j(w) = 2w - p_j\left(\frac{n_j - 1}{n_j}\right)$$

for a suitable choice of the sign in the above square root.

We shall simply write ξ_j whenever the choice of the sign of the square root is not specified. We recall that, for any fixed x, the function $\gamma : \to g_\gamma(x)$ is analytic, and equal to the Green function $G_\gamma(e,x)$ for large values of γ. Taking the analytic continuation of (3.2), after some calculations we get

i) $$\gamma = 2w + \sum_{j=1}^{q+1} p_j \xi_j$$

ii) $$p_j\left(\xi_j - \frac{\xi_j^{-1}}{n_j}\right) = p_j\left(\frac{n_j - 1}{n_j}\right) - 2w$$

iii) $$(3.3) \qquad \|g_\gamma\|_p^{-p} = |2w|^p \cdot \left\{1 - \sum_{j=1}^{q+1} \frac{n_j |\xi_j|^p}{1 + n_j |\xi_j|^p}\right\}.$$

Furthermore, if we turn γ into a function of w, we have

$$(3.4) \qquad \frac{1}{2}\frac{d\gamma}{dw} = 1 - \sum_{j=1}^{q+1} \frac{n_j |\xi_j|^2}{1 + n_j |\xi_j|^2} = \|g_\gamma\|_2^{-2}$$

whenever w is real, different from 0, and such that the corresponding value of γ belongs to $R \backslash \mathrm{sp}(\mu)$.

Formulas above can be found in [T2] but can also be deduced directly from the results of [F-TS].

Let us consider first the poles of $g_\gamma(x)$. The following quantity will play a central role in the description of sp (μ).

Call

$$\frac{p_m^2}{n_m} = \max_{1 \leqslant j \leqslant q+1} \frac{p_j^2}{n_j}$$

and let ξ_m be the corresponding value for $h(a_m)$.

THEOREM. – *Let* μ *as above. Then the function* $g_\gamma(x)$ *has a pole if and only if* μ *has a nontrivial* ℓ^2 *eigenspace and this happens if and only if at least one of the following conditions hold :*

$$(1) \qquad \sum_{j=1}^{q+1} \frac{1}{n_j + 1} < 1$$

$$(2) \qquad \frac{1}{n_1 + 1} > \sum_{j=2}^{q+1} \frac{1}{n_j + 1}.$$

Proof. – The ℓ^2 eigenspaces of μ are in one to one correspondence with the poles of $g_\gamma(e)$, which are the same as the poles of $g_\gamma(x)$. In fact, suppose that $g_\gamma(x)$ has a pole for $w = w_0$.

Suppose first that $w_0 \neq 0$. Then $w_0 = \infty$. We shall consider only the case $w_0 = +\infty$, being the other virtually the same.

By (3.3) exactly one of the ξ_j must have a pole too. Also, the choice of the sign for ξ_j in (3.2) must be « – » while, for $k \neq j$, must be « + ». Suppose that $j \neq m$. Then we have

$$\lim_{w \to +\infty} |\xi_m^+ \xi_j^-| = \frac{p_m}{n_m} \frac{1}{p_j}.$$

Let us consider now the subgroup G_m generated by G_{n_m+1} and G_{n_j+1}. It can be easily seen that the above condition implies that

$$\sum_{x \in G_m} \|g_\gamma(x)\|^2 = +\infty$$

for w sufficently large and this a contradiction, since for these values of w $g_\gamma(x)$ must be in ℓ^2. So that the only possibility is that ξ_m has a pole. In this case, write a_m (respectively a_j) to denote any element of $\tilde{G}_{n_m} + 1$ (respectively of $\tilde{G}_{n_j+1}$), then a limit argument shows that

$$(3.5) \quad g_\gamma(x) = -\frac{1}{p_m} \cdot \prod_{i=1}^{s} \left(\frac{-p_{j_i}}{n_{j_i} p_m} \right)$$

if $x = a_m(a_{j_1}a_m)(a_{j_2}a_m), \ldots, (a_{j_s}a_m)$ and $|x| = 2s + 1$

0 otherwise.

In particular, $g_{\gamma}(x)$ is finite for every x.

Hence the only possibility to get a pole for $g_\gamma(x)$ is $w = 0$. Since for complex values of $\gamma = \gamma(w)$, $g_\gamma(x)$ belongs to ℓ^2, by (3.4) we must have

$$1 - \sum_{j=1}^{q+1} \frac{n_j |\xi_j(0)|^2}{1 + n_j |\xi_j(0)|^2} \geqslant 0. \tag{3.6}$$

Now, $|\xi_j(0)| = 1$ or $|\xi_j(0)| = \frac{1}{n_j}$ according with the choice « + » or « − » in (3.2).

Looking at formula (3.6), a moment's reflection shows that no more then *one* sign + is allowed for the ξ_j. Since $n_1 \leqslant n_2 \cdots \leqslant n_{q+1}$, this choice is possible only for ξ_1. Suppose first that ξ_1 has been chosen with the sign « + ». The corresponding curve $\gamma(w)$ is given by

$$\begin{aligned} \gamma_1(w) &= 2w + \sum_{j=2}^{q+1} p_j \xi_j^- + p_1 \xi_1^+ \\ &= p_1 \left(\frac{n_1 - 1}{n_1} \right) + \sum_{j=2}^{q+1} p_j \xi_j^- - p_1 \xi_1^- \end{aligned} \tag{3.7}$$

and

$$\gamma_1(0) = p_1 - \sum_{j=2}^{q+1} \frac{p_j}{n_j} = \gamma_1$$

$$\frac{1}{2} \gamma'(0) = \frac{1}{n_1 + 1} - \sum_{j=2}^{q+1} \frac{1}{n_j + 1}.$$

Suppose now that condition 2) holds. Then, in a neighbourhood of $w = 0$, the function above, associated with the choice of signs « + », ..., « − » gives a resolvent set for γ.

Again, the functional calculus says that

$$dm(\gamma_1) = \lim_{\varepsilon \to 0^+} i\varepsilon \, g_{\gamma_1 + i\varepsilon}(e).$$

Looking at w as a function of γ we can see that

$$\operatorname*{Res}_{\gamma = \gamma_1} g(e) = \lim_{\varepsilon \to 0^+} i\varepsilon \, g_{\gamma_1 + i\varepsilon}(e) = dm(\gamma_1) \tag{3.8}$$

$$\lim_{\varepsilon \to 0^+} \frac{i\varepsilon}{2w(\gamma_1 + i\varepsilon)} = \frac{1}{2} \gamma_1'(0) = \frac{1}{n_1 + 1} - \sum_{j=2}^{q+1} \frac{1}{n_j + 1} > 0$$

hence μ has a nontrivial eigenspace that will be described in the next section. If condition 2 does not hold, suppose first that

$$\frac{1}{n_1+1} < \sum_{j=2}^{q+1} \frac{1}{n_j+1}$$

then it is clear that the function $\gamma_1(w)$ cannot give rise to a resolvent set in a neighbourhood of $w = 0$ so that we can ignore this case.

Finally, suppose that

$$\frac{1}{n_1+1} = \sum_{j=2}^{q+1} \frac{1}{n_j+1}.$$

In this case the limit in (3.8) is zero, hence there are no ℓ^2 eigenspaces corresponding to γ_1.

Let us turn to the choice of signs in (3.6). Suppose now that all the ξ_j have been chosen with the same sign « − ».

Corresponding to this choice we have $\gamma(w)$ given by

$$\begin{aligned} \gamma_0(w) &= 2w + \sum_{j=1}^{q+1} p_j \xi_j^- \\ \gamma(0) &= -\sum_{j=1}^{q+1} \frac{p_j}{n_j} = \gamma_0 \\ \frac{1}{2}\gamma'(0) &= 1 - \sum_{j=1}^{q+1} \frac{1}{n_j+1}. \end{aligned}$$

Arguing as before we can see that, if condition 1 holds then μ has a nontrivial ℓ^2 eigenspace, while, when condition 1 does not hold then $dm(\gamma_0) = 0$. (Actually, a quick check of the behaviour of $\gamma_0(w)$ shows that, when $\gamma_0'(0) < 0$, then γ_0 belongs to res (μ).)

Conversely, if μ has an ℓ^2 eigenspace, then $g_\gamma(e)$ must have a pole. We have seen that, in this case, either $\gamma(w) = \gamma_1(w)$ or $\gamma(w) = \gamma_0(w)$ and a pole may exist if and only if at least condition 1 or 2 hold. □

We shall now investigate the continuous spectrum of μ.

It is clear from (3.3) and (3.4) that, if we want to investigate the ℓ^2 spectrum of μ, we have to consider γ as a function of w and we must check the derivative for all the possible choices of signs for the ξ_j. This will be done in Theorem 3 and Theorem 4 for some special choices of the p_j and of the n_j.

We want to consider first the case $g_\gamma(e) \neq 0$ and let $\gamma = \tilde{\gamma} \in \operatorname{res}(\mu)$.

Then there exists a choice of signs in (3.2) and $w = w_0 \in R$ such that $\gamma(w_0) = \tilde{\gamma}$ and, for w in a neighbourhood of w_0, $\gamma(w) \in \operatorname{res}(\mu)$ and

$$\gamma(w) = 2w + \sum_j p_j \xi_j(w)$$

$$\gamma'(w_0) > 0 .$$

For these values of γ, we have

$$g_\gamma(x) = g_\gamma(e) \cdot h_\gamma(x) = \frac{1}{2w} \cdot h_\gamma(x) .$$

Suppose now

$$\gamma_p \in \operatorname{res}(\mu) \qquad \text{and} \qquad g_{\gamma_p}(e) = 0 .$$

By definition, this may happen only if there exists w_0 such that, for $w = w_0$ the function $w(\gamma)$ has a pole at $\gamma = \gamma_p$. Arguing as in the first part of the proof of Theorem 1, we can conclude that, in this case, exactly ξ_m has a pole and $g_{\gamma_p}(x)$ has the expression given in (3.5).

Furthermore, since for any $a \in \tilde{G}_{n_{n_m^+}1+1}$ we have

$$g_{\gamma_p} * (\gamma_p - \mu)(a) = 0$$

condition 3.3 i) becomes

$$\gamma_p - p_m\left(\frac{n_m - 1}{n_m}\right) = \frac{p_m}{n_m} \cdot \frac{1}{\xi_m} + \sum_{j \neq m} p_j \xi_j$$

thus, letting $w \to w_0$, we can see that

$$\gamma_p = p_m\left(\frac{n_m - 1}{n_m}\right).$$

Observe that, in this case, we have

$$\|g_{\gamma_p}\|_2^2 = n_m p_m^2 \sum_{s=0}^{\infty} \left(\sum_{j=2}^{q+1} \frac{p_j^2}{n_j} \frac{n_m}{p_m^2} \right)^s .$$

Hence $\gamma_p \in \operatorname{res}(\mu)$ and $g_{\gamma_p}(e) = 0$ implies that

$$\frac{p_m^2}{n_m} > \sum_{j \neq m} \frac{p_j^2}{n_j} .$$

Conversely, a quick calculation shows that, if the above condition holds, then the function given in (3.5) satisfies the condition

$$g_\gamma * \left(p_m \left(\frac{n_m - 1}{n_m} \right) - \mu \right) = \delta_e$$

and hence $\gamma = p_m \left(\frac{n_m - 1}{n_m} \right)$ belongs to $\operatorname{res}(\mu)$ and $g_\gamma(e) = 0$.

We are now ready to state a necessary condition for sp_c to be connected.

THEOREM 2. — *Suppose that continuous spectrum of* μ *is connected then*

$$\frac{p_m^2}{n_m} < \sum_{j \neq m} \frac{p_j^2}{n_j}. \tag{3.9}$$

Proof. — It is clear that, for $w \to +\infty$, the best possible choice in order to have $\gamma'(w)$ positive is

$$\gamma_+(w) = 2w + \sum_{j=1}^{q+1} p_j \xi_j^+$$

while, for $w \to -\infty$, it turns into

$$\gamma_0 = 2w + \sum_{j=1}^{q+1} p_j \xi_j^- .$$

The behaviour of the two above curves is very easy to check : γ_+ is convex and has a positive minimum, say ρ_+, while γ_0 is concave and has a maximum, say ρ_0, which is surely negative when $\gamma_0'(0)$ is not positive. As noted in Theorem 1, this occurs when

$$\sum_{j=1}^{q+1} \frac{1}{n_j + 1} \geqslant 1 .$$

In general, we cannot ensure that ρ_0 is a negative number. In any case, the continuous spectrum of μ is contained in the interval $[\rho_0, \rho_+]$. Any other curve $\gamma(w)$ having positive derivative for some w, gives rise to a « hole » in the above interval, which disconnects $\mathrm{sp}(\mu)$.

Since condition (3.9) ensures that the curves

$$\gamma_m(w) = 2w + \sum_{j \neq m} p_j \xi_j^- + p_m \xi_m^+ \quad \text{for } w < 0$$

$$\gamma_m(w) = 2w + \sum_{j \neq m} p_j \xi_j^+ + p_m \xi_m^- \quad \text{for } w > 0$$

have positive derivative for $|w|$ sufficiently large, we get the result. □

The next theorem provides a sufficient condition for the connectedness of sp (μ) when the probabilities are choosen in a *reasonable* way with respect to the orders of the groups: the following condition says essentially that we must assign *small* probabilities to *small* groups.

Recall that $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_{q+1}$. Choose the numbers p_j in such a way that

$$\frac{p_k^2}{n_k} = \frac{p_j^2}{n_j} \quad \text{for every } k \text{ and } j \tag{3.10}$$

then we have the following

THEOREM 3. – *Suppose that the above condition* (3.10) *holds. Then, if*

$$n_{q+1} \leqslant q$$

sp (μ) *consists of exactly one interval.*

Proof. – Observe first that, since $n_{q+1} \leqslant q$, the point spectrum does not occur. Hence we have to prove that the curves γ_+ and γ_0 considered in Theorem 2 are the only possible choices in order to have $\gamma'(w)$ positive. Recall that condition (3.10) implies that

$$p\left(\frac{n_1-1}{n_1}\right) \leqslant p_2\left(\frac{n_2-1}{n_2}\right) \leqslant \cdots \leqslant p_{q+1}\left(\frac{n_{q+1}-1}{n_{q+1}}\right)$$

and set

$$\begin{aligned} \mathrm{I}_0 &= \left(-\infty, \frac{p_1}{2}\left(\frac{n_1-1}{n_1}\right)\right] \\ \mathrm{I}_k &= \left(\frac{p_k}{2}\left(\frac{n_k-1}{n_k}\right), \frac{p_{k+1}}{2}\left(\frac{n_{k+1}-1}{n_{k+1}}\right)\right], \qquad 1 \leqslant k \leqslant q \\ \mathrm{I}_{q+1} &= \left(\frac{p_{q+1}}{2}\left(\frac{n_{q+1}-1}{n_{q+1}}\right), +\infty\right). \end{aligned}$$

We have

$$\gamma'(w) = -(q-1) + \sum_{j=1}^{q+1} \frac{\pm z_j}{\sqrt{z_j^2 + \dfrac{4p_j^2}{n_j}}} \tag{3.11}$$

so that $\gamma'(w)$ is negative whenever at least two terms in the above summation are negative. We shall consider first the best possible choice of sign in every I_k $0 \leqslant k \leqslant q+1$. Hence we have to consider first

$$\gamma_0(w) = 2w - \frac{1}{2}\sum_{j=1}^{q+1}\left(\sqrt{z_j^2+\frac{4p_j^2}{n_j}}+z_j\right) = 2w + \sum_{j=1}^{q+1} p_j\xi_j^- \quad \text{when } w \in I_0$$

$$\gamma_k(w) = 2w - \frac{1}{2}\sum_{j=k+1}^{q+1}\left(\sqrt{z_j^2+\frac{4p_j^2}{n_j}}+z_j\right) + \frac{1}{2}\sum_{j=1}^{k}\left(\sqrt{z_j^2+\frac{4p_j^2}{n_j}}-z_j\right)$$

$$= 2w + \sum_{j=1}^{k-1} p_j\xi_j^+ + \sum_{j=k}^{q+1} p_j\xi_j^- \quad \text{when } w \in \mathrm{I}_k$$

$$\gamma_1(w) = 2w + \frac{1}{2}\sum_{j=1}^{q+1}\left(\sqrt{z_j^2+\frac{4p_j^2}{n_j}}-z_j\right) = 2w + \sum_{j=1}^{q+1} p_j\xi_j^+ \quad \text{when } w \in I_{q+1}.$$

It is clear that, whenever $\gamma_k'(w)$ is negative in I_k, no other curve may give rise to a resolvent set for $w \in I_k$.

Let us start with I_0.

We know that $\gamma_0(w)$ gives a resolvent set for w sufficiently small. Furthermore, since $n_{q+1} \leqslant q$, $\gamma_0'(0)$ is negative and this implies that no curve can give a resolvent set for $0 \leqslant w \leqslant p_1\left(\frac{n_1-1}{n_1}\right)$. Also, since $|z_j| \geqslant |z_1|$ for $w \leqslant p_1\left(\frac{n_1-1}{n_1}\right)$, we can see by (3.11) that the only possible choice, different from γ_0, is given by

$$\gamma^1 = 2w + \sum_{j=2}^{q+1} p_j\xi_j^- + p_1\xi_1^+.$$

A quick check of $\frac{d}{dw}|\xi_1^+\xi_j^-|$ shows that $|\xi_1^+\xi_j^-|$ is decreasing for negative values of w. In particular

$$(3.12) \qquad |\xi_1^+\xi_j^-(w)| \geqslant |\xi_1^+\xi_j^-(0)| = \frac{1}{n_j} \quad \text{for } w \leqslant 0.$$

Consider now the subset A of G consisting of all words of the type

$$(3.13) \qquad x = (a_1a_{j_1})(a_1a_{j_2}), \ldots, (a_1a_{j_s})$$

where a_j denotes any element of G_{n_j+1} and $|x| = 2s$.

Since

$$\sum_{x\in A}|g_\gamma(x)|^2=\frac{1}{4w^2}\sum_{s=0}^{+\infty}\left(\sum_{j=2}^{q+1}n_1n_j|\xi_1^+\xi_j^-|^2\right)^s$$

we see that condition (3.12) and the choice of n_{q+1} greater then q, imply that, for $w\leqslant 0$, the above sum is infinite being

$$\sum_{j=2}^{q+1}n_1n_j|\xi_1^+\xi_j^-|^2\geqslant\sum_{j=2}^{q+1}\frac{n_1}{n_j}\geqslant 1.$$

Hence γ_0 is the only curve giving a resolvent set in I_0.

Let us consider now γ_k in I_k for $1\leqslant k\leqslant q$. It is obvious that, in I_k, the largest possible value for the quantity $|z_j|$ is $p_{q+1}\left(\frac{n_{q+1}-1}{n_{q+1}}\right)$.

Hence, since the quantities $\frac{p_j^2}{n_j}$ are all equal, for $w\in I_k$ we get

$$\gamma_k'(w)=-(q-1)+\sum_{j=1}^{q+1}\frac{|z_j|}{\sqrt{z_j^2+4\frac{p_j^2}{n_j}}}\leqslant-(q-1)+(q+1)\frac{p_{q+1}\left(\frac{n_{q+1}-1}{n_{q+1}}\right)}{p_{q+1}\left(\frac{n_{q+1}+1}{n_{q+1}}\right)}$$

again, the choice of n_{q+1} implies that the right hand side of the above inequality is negative. Finally, let us consider I_{q+1}. This time we have that the smallest of the $|z_j|$ is $|z_{q+1}|=z_{q+1}$. Hence we must consider again the curve γ_q. Observe that $|\xi_j^+\xi_{q+1}^-|$ is increasing for $w\geqslant p_{q+1}\left(\frac{n_{q+1}-1}{n_{q+1}}\right)$ and

$$\left|\xi_j^+\xi_{q+1}^-\left(\frac{p_{q+1}}{2}\left(\frac{n_{q+1}-1}{n_{q+1}}\right)\right)\right|=\frac{1}{\sqrt{n_{q+1}}}\xi_j^+\left(\frac{p_{q+1}}{2}\left(\frac{n_{q+1}-1}{n_{q+1}}\right)\right)$$

$$=\frac{\sqrt{\left(p_{q+1}\left(\frac{n_{q+1}-1}{n_{q+1}}\right)-p_j\left(\frac{n_j-1}{n_j}\right)\right)^2+\frac{4p_j^2}{n_j}}-\left(p_{q+1}\left(\frac{n_{q+1}-1}{n_{q+1}}\right)-p_j\left(\frac{n_j-1}{n_j}\right)\right)}{2p_j\sqrt{n_{q+1}}}$$

$$\geqslant\frac{\sqrt{\left(p_{q+1}\left(\frac{n_{q+1}-1}{n_{q+1}}\right)\right)^2+\frac{4p_j^2}{n_j}}-p_{q+1}\left(\frac{n_{q+1}-1}{n_{q+1}}\right)}{2p_j\sqrt{n_{q+1}}}=\frac{1}{\sqrt{n_{q+1}}}\frac{p_{q+1}}{p_jn_{q+1}}$$

being $\frac{p^2}{n_j}=\frac{p_{q+1}^2}{n_{q+1}}$ for every j.

If we replace a_1 with a_{q+1} in (3.13), a similar argument shows that

$$\|g_\gamma(x)\|_2^2 \geqslant \sum_{s=0}^{\infty}\left(\sum_{j=1}^{q}\frac{1}{n_{q+1}}\right)^s = +\infty$$

under our assumption. □

The last theorem of this section considers a sort of *unreasonable* situation, completely opposite to that of Theorem 3.

THEOREM 4. – *Suppose that* $p_1 = p_2 = \cdots = p_{q+1} = p$.

Suppose also that

(3.14) $n_1 = n_2$ *and, for every* k, *with* $3 \leqslant k \leqslant q+1$, $n_k \leqslant \sum_{j<k} n_j$.

Then the continuous spectrum of μ *consists of exactly one component.*

Proof. – It is convenient to denote by $x_{i,k,s}$ any word having the reduced form similar to that of condition (3.13): set

$$x_{j,k,s} = (a_j a_k)(a_j a_k), \ldots, (a_j a_k) \qquad \text{and } |x| = 2s$$

where a_j (respectively a_k) denotes any element of $\tilde{G}_{n_j+1}$ (respectively $\tilde{G}_{n_k+1}$). As before, we shall show that only two of the curves of 3.3 i) have positive derivative.

Suppose now that $w \leqslant 0$ and set

$$\gamma^k(w) = 2w + \sum_{j\neq k} p\xi_j^- + p\xi_k^+ .$$

It is obvious that, being $n_1 = n_2$, both γ^1 and γ^2 cannot give rise to a resolvent set. Let us consider now γ^k with $k \geqslant 2$.

A short calculation shows that the derivative of $n_k|\xi_k^+|^2$ with respect to n_k is positive when $2w$ is less then $p\left(\frac{n_k+1}{n_k}\right)$. Recall that $\left(\frac{n_1-1}{n_1}\right) \leqslant \left(\frac{n_2-1}{n_2}\right) \leqslant \cdots \leqslant \left(\frac{n_{q+1}}{n_{q+1}}\right)$. Hence, for $w \leqslant \frac{p}{2}\frac{n_1+1}{n_1}$, we have

$$n_1|\xi_1^-|^2 n_k|\xi_j^+|^2 \geqslant n_1|\xi_1^-|^2 n_1|\xi_1^+|^2 = 1$$

which implies that

$$\sum_{s=0}^{\infty} |g_\gamma(x_{1,k,s})|^2 = +\infty .$$

Observe that it is essential to have $n_1 = n_2$. We shall produce an example where γ^1 gives rise to a resolvent set for negative w, providing that n_1 and n_2 are far enough apart.

From the above considerations it is also clear, that, for

$$0 \leqslant w \leqslant \frac{p}{2}\left(\frac{n_1 - 1}{n_1}\right),$$

no curve give a resolvent set for γ. So that the first curves to be considerd are, as well as in Theorem 3, the

$$\gamma_k = 2w + \sum_{j \leqslant k} p\xi_j^+ + \sum_{j \geqslant k+1} p\xi_j^-$$

for $w \in \left(\frac{p}{2}\left(\frac{n_k - 1}{n_k}\right), \frac{p}{2}\left(\frac{n_{k+1} - 1}{n_{k+1}}\right)\right] = I_k, \ (1 \leqslant k \leqslant q)$.

Again nor γ_1 or γ_2 can give a resolvent set. If we look at the derivative of $|\xi_j^\pm|$ with respect to n_j, we see that, for positive values of w, $|\xi_j^\pm|$ is a decreasing function of n_j.

Hence, for $k \geqslant 2$ and $w \in I_k$ we have:

$$(3.15) \qquad |\xi_{k+1}^- \xi_j^+| \geqslant |\xi_{k+1}^- \xi_{k+1}^+| = \frac{1}{n_{k+1}} \qquad \text{for every } j \leqslant k+1 .$$

If we restrict our attention to the words $x_{1,k+1,s}, x_{2,k+1,s}, \ldots, (x_{k,k+1,s}$ we see that the ℓ^2 norm of $g_\gamma(x)$ is greater or equal to

$$\sum_{l=0}^{\infty} \left(\sum_{j=1}^{k} \frac{n_j}{n_{k+1}} \right)^l$$

which is infinite under our assumptions.

Finally, the above considerations show that, also for

$$x \geqslant \frac{p}{2}\left(\frac{n_{q+1} - 1}{n_{q+1}}\right)$$

the only curve giving a resolvent set is $\gamma^+ = 2w + \sum_{j=1}^{q+1} p\xi_j^+$. □

Remark. — Observe that, if $n_1 = 1 < q \leqslant n_2 \leqslant n_3, \ldots, n_{q+1}$, the continuous spectrum of μ consists of at least two components. The curve disconnecting $\mathrm{sp}(\mu)$ is γ^1 which has positive derivative at the point $2w_q = -\dfrac{3q+1}{2q(q+1)}$.

3. The representations.

This section is devoted to the description of the measure $dm(\sigma)$ and of the unitary irreducible representations.

We shall first describe the eigenspaces corresponding to the points γ_0 $\left(\text{when } \sum_{j=1}^{q+1} \frac{1}{n_j+1} < 1\right)$ and γ_1 $\left(\text{when } \frac{1}{n_1+1} > \sum_{j=2}^{q+1} \frac{1}{n_j+1}\right)$.

The corresponding representations will be square integrable and hence *reducible* see [CF-T].

Identify functions defined on G with functions defined on $\mathscr{G}$. Say that a polygon P is of type j if it corresponds to a left coset of G_{n_j+1} in G. We shall also write $\mathscr{P}_j$ for these polygons. Let $\mathscr{N}^0$ consist of all complex valued functions f, defined on $\mathscr{G}$, which have zero average over each polygon. It is easy to verify that $\mathscr{N}^0$ is an eigenspace for the operator induced on $\mathscr{G}$ by right convolution with μ. If f is such a function we have $f * \mu = \gamma_0 f$.

Let $\mathscr{N}_0 = \mathscr{N}^0 \cap \ell^2(G)$.

Let $\mathscr{N}^j$ $(j=1,\ldots,q+1)$ consist of all complex valued functions on $\mathscr{G}$ which are constant on the polygons of type j and have zero average over all the other polygons. Analogously, $\mathscr{N}^j$ are all eigenspaces of μ.

Set $\mathscr{N}_j = \mathscr{N}^j \cap \ell^2(G)$.

We have the following

THEOREM 5.

$$\mathscr{N}_0 \neq \{0\} \quad \textit{if and only if} \quad \sum_{j=1}^{q+1} \frac{1}{n_j+1} < 1$$

$$\mathscr{N}_1 \neq \{0\} \quad \textit{if and only if} \quad \sum_{j=2}^{q+1} \frac{1}{n_j+1} < \frac{1}{n_1+1} \tag{4.1}$$

$$\mathscr{N}_j = \{0\} \quad \textit{for all the other values of } j.$$

Moreover, if we think of $\mathscr{N}_j$ $(j=0,1)$ *as subrepresentation of the regular representation of* G, *their continuous dimension is respectively* $1 - \sum_{j=1}^{q+1} \frac{1}{n_j+1}$ *and* $\frac{1}{n_1+1} - \sum_{j=2}^{q+1} \frac{1}{n_j+1}$.

Proof. – Let us consider first $\mathcal{N}_0$. Suppose that $f \neq 0$ is an element of $\mathcal{N}_0$. Since $\mathcal{N}_j$ $(j=0,\dots,q+1)$ are all invariant by the G action on $\mathcal{G}$, we may always suppose that $f(e) \neq 0$.

We shall take averages of the values of f in order to obtain another element f_0 of $\mathcal{N}_0$ whose ℓ^2 norm can be easily computed. Start from the polygons leaving from the identity.

Let $f_0(a_j)$ be the average of the values of f over all the vertices of $\mathcal{P}_j$ different from the identity. Hence $f_0(a_j) = -\dfrac{f(e)}{n_j}$. Let now $f_0(a_j a_k)$ be the average of the values of f over *all* the verteces at distance two from e which belong to a polygon of type k meeting $\mathcal{P}_j$.

Hence

$$
\begin{aligned}
f_0(a_j a_k) &= \frac{1}{n_j}\frac{1}{n_k} \sum_{\{a_j \in \tilde{G}_{n_j+1}\}} \sum_{\{a_k \in \tilde{G}_{n_k+1}\}} f(a_j a_k) \\
&= \frac{1}{n_j}\frac{1}{n_k} \sum_{\{a_j \in \tilde{G}_{n_j+1}\}} - f(a_j) = + \frac{f(e)}{n_j n_k}.
\end{aligned}
$$

Repeat the same reasoning for the verteces at distance $n \geqslant 3$ from the identity : then

$$
f_0(a_{i_1} a_{i_2}, \dots, a_{i_k}) = \frac{1}{(n_{i_1} n_{i_2}, \dots, n_{i_k})} \sum_{\{a_{i_1} \in \tilde{G}_{n_{i_1}+1}\}} \sum_{\{a_{i_2} \in \tilde{G}_{n_{i_2}+1}\}} \cdots \sum_{\{a_{i_k} \in \tilde{G}_{n_{i_k}+1}\}} f(a_{i_1} a_{i_2}, \dots, a_{i_k}).
$$

If we define

$$
\begin{aligned}
&\Phi(e) = 1 \\
(4.2) \qquad &\Phi(a_j) = -\frac{1}{n_j} \qquad \text{for every } a_j \in \tilde{G}_{n_j+1} \\
&\Phi(xy) = \Phi(x)\Phi(y) \qquad \text{whenever } |xy| = |x| + |y|
\end{aligned}
$$

then $f_0(x) = f(e)\Phi(x)$. By Schwartz inequality Φ belongs to $\ell^2(G)$. On the other hand, Φ satisfies a resolvent-like condition so that we have :

$$
\|\Phi\|_2^2 = \left(1 - \sum_{j=1}^{q+1} \frac{n_j |\Phi(a_j)|^2}{1 + n_j |\Phi(a_j)|^2}\right)^{-1} = \left(1 - \sum_{j=1}^{q+1} \frac{1}{n_j + 1}\right)^{-1}.
$$

Hence Φ belongs to ℓ^2 if and only if $\sum_{j=1}^{q+1} \frac{1}{n_j+1} < 1$. We shall now see that, when this occurs, $\mathcal{N}_0$ is the whole eigenspace corresponding to γ_0. Let

$$\varphi_0 = \left(1 - \sum_{j=1}^{q+1} \frac{n}{n_j+1}\right)\Phi .$$

The functional calculus allows us to recover the orthogonal projection F onto the subspace corresponding to γ_0 by means of $g_\gamma(x)$. In particular $F(g) = g * \Phi_F$ for a suitable positive definite function Φ_F and

$$\langle F(\delta_e), \delta_x \rangle = \Phi_F(x) = \frac{1}{2\pi i} \int_C g_\gamma(x)\, d\gamma$$

where C is a smooth curve around the point γ_0. Observe that γ, as a function of w, is given by the curve $\gamma_0(w)$ considered in Theorem 1 and hence $g_\gamma(a_j) = \xi_j^-$ for every j. If we let C shrink around γ_0, we get :

$$\begin{aligned}\Phi_F(x) &= \frac{1}{2\pi i} \operatorname*{Res}_{\gamma=\gamma_0} g_\gamma(e)\, \Phi(x) \\ &= \lim_{\gamma\to\gamma_0} (\gamma-\gamma_0) \frac{1}{2w(\gamma)} \Phi(x) = \frac{1}{2} \left.\frac{1}{\frac{dw}{d\gamma}}\right|_{\gamma=\gamma_0} \Phi(x) \\ &= \frac{1}{2}\left(\frac{d\gamma_0(w)}{dw}\right)_{w=0} \Phi(x) = \left(1 - \sum_{j=1}^{q+1} \frac{n}{n_j+1}\right)\Phi(x) = \varphi_0(x).\end{aligned}$$

So that, when $\sum_{j=1}^{q+1} \frac{1}{n_j+1} < 1$, φ_0 is an idempotent of $C^*_{\mathrm{reg}}(G)$.

On the other hand, it is obvious that φ_0 is an element of $\mathcal{N}_0$ and hence any other γ_0-eigenfunction of μ must also lie in $\mathcal{N}_0$.

Let us turn to the $\mathcal{N}_j$ for $j \geqslant 1$. Suppose that $f \neq 0 \in \mathcal{N}_j$. Then f is not identically zero on the polygons $\{\mathcal{P}_j\}$. As before, we may assume that $f(e) \neq 0$.

If we repeat the construction above, again we get a new function f_j which is still in $\mathcal{N}_j$.

Again, the ℓ^2 norm of f_j can be easily computed. Consider the functions defined by the rule:

$$\begin{aligned} \Phi_j(e) &= 1 \\ \Phi_j(a_j) &= 1 && \text{where } a_j \in G_{n_j+1} \\ \Phi_j(a_k) &= -\frac{1}{n_k} && \text{if } a_k \in \tilde{G}_{n_k+1} \text{ and } k \neq j \\ \Phi_j(xy) &= \Phi_j(x)\Phi_j(y) && \text{if } |xy| = |x| + |y| \end{aligned}$$

then $f_j = f(e)\Phi_j$.

If we compute now the ℓ^2 norm of Φ_j, we see that this is infinite unless $j = 1$. In other words, the constant value is possible only on the *smallest* polygons. Arguing as before, we can also see that $\mathcal{N}_1$ is nonzero if and only if

$$\sum_{j=2}^{q+1} \frac{1}{n_j+1} < \frac{1}{n_1+1}.$$

In this case the orthogonal projection φ_1 onto $\mathcal{N}_1$ is recovered by considering the function $\gamma_1(w)$ and has the following expression:

$$\varphi_1 = \left(\frac{1}{n_1+1} - \sum_{j=2}^{q+1} \frac{1}{n_j+1}\right)\Phi_1.$$

The final assertion (see e.g. [KS] for the definition of continuous dimension) is a consequence of fact that the continuous dimension of the representations corresponding to γ_j $(j=0,1)$ is nothing but the value that the functions φ_j $(j=0,1)$ take at the identity. □

Let us consider now $\sigma \in \mathrm{sp}_c(\mu)$. Let γ be a complex number with $\mathrm{Re}\,\gamma = \sigma$. Suppose that σ is not a branch point for $g_\gamma(x)$: we have seen in Theorem 1 that $w(\gamma)$ is far from zero when γ tends to σ. Also, $g_{\sigma \pm i_0}(x)$ is finite for every x and, being $g_\gamma(x)$ analytic in the upper half plane, we may ensure that $g_{\sigma \pm i_0}(x)$ are continuous functions of σ when σ is an interior point of $\mathrm{sp}_c(\mu)$. Finally, arguing as in [S], we may deduce that $g_{\sigma + i_0}(e) = g_{\sigma - i_0}(e)$ implies tha σ is a branch point for $g_\gamma(e)$.

Let S denote the set of branch points of $g_\gamma(e)$. Since $g_\gamma(e)$ is an algebraic function, S is finite.

For any $\sigma \in \mathrm{sp}_c(\mu)\backslash S$ define

$$\varphi_\sigma(x) = \frac{g_{\sigma+i0}(x) - g_{\sigma-i0}(x)}{g_{\sigma+i0}(e) - g_{\sigma-i0}(e)}$$

and

$$dm(\sigma) = -\frac{1}{\pi}(g_{\sigma+i0}(e) - g_{\sigma-i0}(e))\, d\sigma.$$

Then the functional calculus says that

$$\delta_e(x) = \varphi_0(x) + \varphi_1(x) + \int_{\mathrm{sp}_c(\mu)} \varphi_\sigma(x)\, dm(\sigma)$$

where φ_0(respectively φ_1) is identically zero if γ_0 (respectively γ_1) does not belong to the point spectrum of μ.

In fact, all the functions φ_σ involved, are two sided eigenfunctions of μ (with eigenvalue σ) and the above sum is an orthogonal sum.

Using the functional calculus again one can argue as in [S] to see that $-\frac{1}{\pi}\{g_{\sigma+i0}(x) - g_{\sigma-i0}(x)\}$ is positive definite for $\sigma \in \mathrm{sp}_c(\mu)$, hence $\varphi_\sigma(x)$ is positive definite for $\sigma \in \mathrm{sp}_c(\mu)\backslash S$.

Corresponding to any $\varphi_\sigma(\sigma \in \mathrm{sp}_c(\mu)\backslash S)$ we may associate a continuous unitary representation of G, say π_σ.

When $\sigma \neq \gamma_i$ $i = 0, 1$ then the corresponding π_σ is realized in a standard Hilbert space $\mathscr{H}_\sigma$, which can be thought to be completion of the space of left translates of φ_σ. For any finitely supported functions f and g we have:

$$f \mapsto f_\sigma = f * \varphi_\sigma, \qquad \pi_\sigma(x) f_\sigma = (\delta_x * f)_\sigma$$
$$(f_\sigma, g_\sigma)_\sigma = (f * \varphi_\sigma, g)$$

(,) denotes the inner product in $\ell^2(G)$ and $(,)_\sigma$ the one in H_σ. Also, we have

$$(f,g) = \int_{\mathrm{sp}(\mu)} (f * \varphi_\sigma, g)\, dm\sigma = (f,g) = (f * \varphi_0, g) + (f * \varphi_1, g) + \int_{\mathrm{sp}_c(\mu)} (f_\sigma, g_\sigma)_\sigma\, dm\sigma.$$

Let $\sigma \in \mathrm{sp}(\mu)\backslash\{\gamma_0, \gamma_1\}$ and let $g_\gamma(x)$ be equal to $(\gamma-\mu)^{-1}(x)$ at $\gamma = \sigma + i\varepsilon$, so that $g_\gamma(e) = \frac{1}{2w(\gamma)}$. In [S] it is proved that if $\lim_{\varepsilon\to 0^+} w(\gamma) \neq \lim_{\varepsilon\to 0^-} w(\gamma) \neq 0 \neq \infty$ then the corresponding representation π_σ is irreducible.

The same arguments used in [S] also apply to our case. Namely, we have the following

THEOREM 6. — *Suppose that* $\sigma \in \mathrm{sp}(\mu)\backslash\{S \cup \{\gamma_0, \gamma_1\}\}$. *Then the corresponding representation* π_σ *on* H_σ *is irreducible.*

Sketch of the proof. — 1) Let $Q(\sigma) = \{\psi \in H_\sigma : \pi_\sigma(\mu)\psi = \sigma\psi\}$. Observe that φ_σ belongs to $Q(\sigma)$ and recall that, if $Q(\sigma)$ is one dimensional, then π_σ is irreducible.

2) Let Q_σ be the orthogonal projection onto H_σ, the functional calculus says that

$$Q_\sigma = \lim_{\varepsilon \to 0^+} i\varepsilon(\sigma + i\varepsilon - \pi_\sigma(\mu))^{-1}.$$

3) Observe that Q_σ can be computed for large values of ε and then take the analytic continuation.

Let $\sigma' = \sigma + i\varepsilon$ and $g_{\sigma'} = (\sigma + i\varepsilon - \mu)^{-1}$. Then for large values of ε we have

$$[\sigma + i\varepsilon - \pi_\sigma(\mu)]^{-1} = \pi_\sigma\{(\sigma + i\varepsilon - \mu)^{-1}\}$$

hence

$$(4.3)\quad (Q_\sigma(\delta_x * \varphi_\sigma), \delta_y * \varphi_\sigma)_\sigma = \lim_{\varepsilon \to 0^+} i\varepsilon(\pi_\sigma\{(\sigma' - \mu)^{-1}\}[\delta_x * \varphi_\sigma], \delta_y * \varphi_\sigma)_\sigma$$
$$= \lim_{\varepsilon \to 0^+} i\varepsilon(g_{\sigma'} * \delta_x * \varphi_\sigma, \delta_y).$$

In order to compute the above limit observe that the right hand side of 4.3 is given by $i\varepsilon \sum_{z \in G} g_{\sigma'}(xz)\varphi_\sigma(zy)$. Since $g_{\sigma'}$ a multiplicative function of (xz) we can use this property providing that $|z| \geqslant |x| + 2$. Hence we shall estimate $\sum_{|z| \geqslant |x| + |y| + 3} g_{\sigma'}(xz)\varphi_\sigma(zy)$.

4) Write $\dfrac{g_{\sigma+i0}(x) - g_{\sigma-i0}(x)}{g_{\sigma+i0}(e) - g_{\sigma-i0}(e)}$ for $\varphi_\sigma(x)$ and compute first $\lim_{\varepsilon \to 0^+} i\varepsilon(g_{\sigma'} * \delta_x * g_{\sigma-i0}, \delta_y)$.

Define vectors $u(x) = (u_1(x), \ldots, u_{q+1}(x))$ $v(x) = (v_1(x), \ldots, v_{q+1}(x))$ as follows :

$$u_j(x) = \sum_t g_{\sigma'}(tx^{-1})g_{\sigma-i0}(t_0^{-1})$$

where the sum is taken over all elements $t \in G$ such that $|t| = |x| + 1$ and the first letter of t does not belong to $\tilde{G}_{n_j+1}$.

$$v_j(x) = \sum_s g_{\sigma-i0}(s^{-1}y)\, g_{\sigma'}(s)$$

where the sum is taken over all s in G such that $|s| = |y| + 1$ and the last letter of s does not belong to $\tilde{G}_{n_j+1}$.

Recall that $g_{\sigma'}(x) = \dfrac{1}{2w(\sigma')} \cdot h_{\sigma'}(x)$, $g_{\sigma - i0}(x) = \dfrac{1}{2w(\sigma - i0)} h_{\sigma - i0}(x)$ and define, for $n = 1, 2, \dots, (q+1)$ by $(q+1)$ matrices $A^{(n)}$ by the rule $A^{(n)}_{j,k} = \sum_{|t|=n} h_{\sigma'}(t) h_{\sigma - i0}(t)$ where the sum is taken over all elements t of length n such that the first letter is an element of $\tilde{G}_{n_j+1}$ the last is an element of $\tilde{G}_{n_k+1}$. Define also a transition matrix T letting

$$T_{j,k} = \begin{cases} 0 & \text{if } j = k \\ n_j \xi'_j \xi_j & \text{if } j \neq k, \quad j, k = 1, \dots, q+1 \end{cases}$$

where $\xi'_j = \xi_j(w(\sigma'))$ and $\xi_j = \xi_j(w(\sigma - i0))$.

Since $A^{(n+1)} = TA^{(n)}$, one can prove that

$$(4.4) \quad (g_{\sigma'} * \delta_x * g_{\sigma - i0}, \delta_y) = \sum_{|t| < 3 + |x| + |y|} g_{\sigma'}(tx^{-1}) g_{\sigma - i0}(t^{-1}y) + \sum_{n=1}^{\infty} v(y) (T^{n-1} A^{(1)}) u(x).$$

5) In order to compute the limit in 4, observe that the first term in the above equality remains bounded as $\varepsilon \to 0^+$, while the second terms is nothing but

$$v(y)(I - T)^{-1} A^{(1)} u(x).$$

The caracteristic polynomial $P_\varepsilon(\alpha)$ of T is given by

$$P_\varepsilon(\alpha) = \left(\prod_{j=1}^{q+1} (\alpha + n_j \xi'_j \xi_j) \right) \cdot \left(1 - \sum_{j=1}^{q+1} \frac{n_j \xi'_j \xi_j}{\alpha + n_j \xi'_j \xi_j} \right).$$

Therefore, as $\varepsilon \to 0^+$ P_ε tends to a polynomial which has 1 as a simple root and this implies that, as $\varepsilon \to 0^+$, limit 4.6 is a product of the form $C(x) \cdot \varphi_\sigma(y)$.

As for the limit of $i\varepsilon(g_{\sigma+i\varepsilon} * \delta_x * g_{\sigma+i0}, \delta_{-y})$ repeat the same reasoning, finding a matrix T which, as $\varepsilon \to 0^+$, converges to a matrix which does not have the eigenvalue one. This implies that

$$\lim_{\varepsilon \to 0^+} i\varepsilon(g_{\sigma+i\varepsilon} * \delta_x * g_{\sigma+i0}, \delta_y) = 0 .$$

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