ANNALES DE L'INSTITUT FOURIER

HARUZO HIDA

On p-adic L-functions of $GL(2) \times GL(2)$ over totally real fields

Annales de l'institut Fourier, tome 41, nº 2 (1991), p. 311-391
<http://www.numdam.org/item?id=AIF_1991__41_2_311_0>

Ann. Inst. Fourier, Grenoble
41, 2 (1991), 311-391

ON p-ADIC L-FUNCTIONS OF $GL(2) \times GL(2)$ OVER TOTALLY REAL FIELDS

by Haruzo HIDA (*)

CONTENT

(*) Supported in part by NSF grant.

Key-words : p-adic L-functions - Modular forms - Rankin product.
A.M.S. Classification : 11F13 - 11F41 - 11F67 - 11F70 - 11F85.

0. Introduction.

In this paper, we shall generalize the result obtained in our previous paper [H3] for $\mathbf{Q}$ to totally real fields F. Namely we will give a p-adic interpolation of the standard L-function $\mathscr{D}(s,\mathbf{f},\mathbf{g})$ of $GL(2) \times GL(2)$ over F regarding all ingredients s and cusp forms $\mathbf{f}$ and $\mathbf{g}$ as variables. Although the idea of dealing with this problem is the same as the one employed in [H3], we encountered new difficulties arising from the fact that the number of variables of our p-adic L-function grows very rapidly according to the degree of F. In fact, our p-adic L-function has at least as many as $2[F:\mathbf{Q}] + 1$ independent variables. The reason for this many number of variables is that the p-adic nearly ordinary Hecke algebra $\mathbf{h}^{\text{n.ord}}$ defined in [H2] is an algebra finite and torsion-free over $\mathbf{Z}_p[[X_1, \ldots, X_r]]$ for $r \geqslant [F:\mathbf{Q}] + 1$ and our p-adic L-function is a p-adic analytic function on the spectrum of the product of two copies of this big algebra. The p-adic continuation of such L-functions along the cyclotomic line (hence of one variable) has already been obtained in a series of works of Panchishkin [P1] and [P2] by a method totally different from ours. Thus the main point of interest in our work is the continuation including the non-abelian variables on the spectrum of the Hecke algebra. Our method of interpolation is a p-adic adaptation of Shimura's way [Sh1] of showing the algebraicity for these special values, which we call the p-adic Rankin-Selberg convolution method. In fact, Shimura went much further and showed algebraicity for $\mathscr{D}(m,\mathbf{f},\mathbf{g})$ in [Sh7] and [Sh8] for each pair of forms $\mathbf{f}$ and $\mathbf{g}$ of mixed weight (i.e. the weight of $\mathbf{f}$ is greater than that of $\mathbf{g}$ for a part of infinite places of F and for the other part, the weight of $\mathbf{f}$ is less than that of $\mathbf{g}$). In this paper, we will show the algebraicity for p-adic L-functions only when the weight of $\mathbf{f}$ is larger than that of $\mathbf{g}$ at every infinite place. Thus the evaluation of our p-adic L-function when $\mathbf{f}$ and $\mathbf{g}$ are of mixed weight is still an open question. This paper is written with an important application to the Iwasawa theory for CM fields in mind which I intend to discuss in subsequent joint articles with J. Tilouine (cf. [HT1-3]). In fact, it seems that the solution of the many variable main conjecture for CM fields might not be so far away and its proof should be based on deformation theory of Galois representations established by Mazur and the theory developed here.

Now let us introduce some technical notation to state our result in a precise form. In fact, we shall use the notation introduced in our previous papers [H1] and [H2] and here we briefly recall them. We fix a rational prime p, algebraic closures $\bar{\mathbf{Q}}$ and $\bar{\mathbf{Q}}_p$ and field embeddings $\iota_p : \bar{\mathbf{Q}} \to \bar{\mathbf{Q}}_p$ and $\iota_\infty : \bar{\mathbf{Q}} \to \mathbf{C}$. Let I be the set of all field embeddings of F into $\bar{\mathbf{Q}}$. We may regard I as the set of infinite places of F via ι_∞ and then the weight of modular forms is a pair of elements (k,w) in the free module $\mathbf{Z}[I]$ generated by embeddings in I. We identify $F_\infty = F \otimes_{\mathbf{Q}} \mathbf{R}$ with $\mathbf{R}^I$ and embed F into $\mathbf{R}^I$ via the diagonal map: $a \mapsto (a^\sigma)_{\sigma \in I}$. Then the identity component $G_{\infty+}$ of $GL_2(F_\infty)$ naturally acts on $\mathscr{Z} = \mathscr{H}^I$ for the Poincaré half plane $\mathscr{H}$. We write $C_{\infty+}$ for the stabilizer in $G_{\infty+}$ of the center point $z_0 = (\sqrt{-1}, \dots, \sqrt{-1})$ in $\mathscr{Z}$. Then for each open compact subgroup S of $GL_2(F_{A_f})$, we denote by $\mathbf{M}_{k,w}(S;\mathbf{C})$ the space of holomorphic modular forms of weight (k,w) with respect to S. Namely $\mathbf{M}_{k,w}(U;\mathbf{C})$ is the space of functions $\mathbf{f} : GL_2(F_A) \to \mathbf{C}$ satisfying the holomorphy condition on $GL_2(F_\infty)$ including cusps and the automorphic condition:

$$\mathbf{f}(\alpha x u) = \mathbf{f}(x) j_{k,w}(u_\infty, z_0)^{-1} \quad \text{for} \quad \alpha \in GL_2(F) \quad \text{and} \quad u \in SC_{\infty+},$$

where $j_{k,w}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = (ad-bc)^{-w}(cz+d)^k$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F_\infty)$ and $z \in \mathscr{Z}$. We write $\mathbf{S}_{k,w}(S;\mathbf{C})$ for the subspace of $\mathbf{M}_{k,w}(S;\mathbf{C})$ consisting of cusp forms. Here we used the convention that $c^s = \prod_{\sigma \in I} c_\sigma^{s_\sigma}$ for $c = (c_\sigma)_{\sigma \in I} \in \mathbf{C}^I$ and $s = \sum_{\sigma \in I} s_\sigma \sigma \in \mathbf{C}[I]$ and we refer to [H1], § 2 for the exact definition of these spaces where the space $\mathbf{S}_{k,w}(S;\mathbf{C})$ was written as $\mathbf{S}_{k,w,I}(S; M_2(F); \mathbf{C})$. Actually we fix a pair of weight (n,v) such that $n \geqslant 0$ (this means $n_\sigma \geqslant 0$ for all σ) and $v \geqslant 0$ and put $k = n + 2t$ and $w = t - v$ for $t = \sum_\sigma \sigma$. To have a non-trivial modular form of weight (k,w), we need to assume that $n + 2v = mt$ for an integer $m \in \mathbf{Z}$. In fact, each irreducible automorphic representation π spanned by forms in $\mathbf{S}_{k,w}(S;\mathbf{C})$ has non-unitary central character which is equal to $|\ |_A^{-m}$ up to finite order characters for the adelic absolute value $|\ |_A$. The twist $\pi^u = \pi \otimes |\ |_A^{m/2}$ is called the unitarization of π. Let us now define the L-function $\mathscr{D}(s,\mathbf{f},\mathbf{g})$. Let $\mathbf{f} \in \mathbf{S}_{k,w}(S;\mathbf{C})$ and $\mathbf{g} \in \mathbf{S}_{\kappa,\omega}(S';\mathbf{C})$ be commun eigenforms of all Hecke operators outside the level of S and S'. Let π and π' be the irreducible automorphic representations spanned by $\mathbf{f}$ and $\mathbf{g}$. Finally let $\mathbf{f}^\circ$ and $\mathbf{g}^\circ$ be the primitive forms associated with π^u and π'^u. Write $\pi^u = \otimes \pi_{\mathfrak{q}}^u$, $\pi'^u = \otimes \pi'^u_{\mathfrak{q}}$, and $\pi_{\mathfrak{q}}^u = \pi(\eta_{\mathfrak{q}}, \eta'_{\mathfrak{q}})$ or $\sigma(\eta_{\mathfrak{q}}, \eta'_{\mathfrak{q}})$, $\pi'^u_{\mathfrak{q}} = \pi(\xi_{\mathfrak{q}}, \xi'_{\mathfrak{q}})$ or $\sigma(\xi_{\mathfrak{q}}, \xi'_{\mathfrak{q}})$, whenever possible, for principal series representations $\pi(\alpha_{\mathfrak{q}}, \beta_{\mathfrak{q}})$ and special representations $\sigma(\alpha_{\mathfrak{q}}, \beta_{\mathfrak{q}})$. Here we used the convention adopted in [H5], § 2 to write down these representations and, for example $\pi(\eta_\mathfrak{q}, \eta'_\mathfrak{q})$ or $\sigma(\eta_\mathfrak{q}, \eta'_\mathfrak{q})$ is the unique infinite dimensional irreducible subquotient of the induced representation of the character: $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \eta_\mathfrak{q}(a)\eta'_\mathfrak{q}(d)\,|ad|_\mathfrak{q}^{1/2}$. Let $\mathfrak{r}$ be the integer ring of F and let $\bar{\omega}_\mathfrak{q}$ be a prime element in the $\mathfrak{q}$-adic completion $\mathfrak{r}_\mathfrak{q}$ of $\mathfrak{r}$ at each prime ideal $\mathfrak{q}$. We may assume that $\eta_\mathfrak{q}\eta'^{-1}_\mathfrak{q}(x) = |x|_\mathfrak{q}^{-1}$ (resp. $\xi_\mathfrak{q}\xi'^{-1}_\mathfrak{q}(x) = |x|_\mathfrak{q}^{-1}$) if $\pi_\mathfrak{q}^u$ (resp. $\pi'^u_\mathfrak{q}$) is special. Then we define the Euler factors $\bar{D}_\mathfrak{q}(X)$, if neither $\pi_\mathfrak{q}^u$ nor $\pi'^u_\mathfrak{q}$ is super cuspidal, by

$$D_\mathfrak{q}(X) = (1-\xi_\mathfrak{q}(\mathfrak{q})\eta_\mathfrak{q}(\mathfrak{q})X)(1-\xi'_\mathfrak{q}(\mathfrak{q})\eta_\mathfrak{q}(\mathfrak{q})X)(1-\xi_\mathfrak{q}(\mathfrak{q})\eta'_\mathfrak{q}(\mathfrak{q})X) \times (1-\xi'_\mathfrak{q}(\mathfrak{q})\eta'_\mathfrak{q}(\mathfrak{q})X),$$

where

$$\eta_\mathfrak{q}(\mathfrak{q}) = \begin{cases} \eta_\mathfrak{q}(\bar{\omega}_\mathfrak{q}) & \text{if } \eta_\mathfrak{q} \text{ is unramified} \\ 0 & \text{if } \eta_\mathfrak{q} \text{ is ramified} \end{cases}$$

$$\eta'_\mathfrak{q}(\mathfrak{q}) = \begin{cases} \eta_\mathfrak{q}(\bar{\omega}_\mathfrak{q}) & \text{if } \eta'_\mathfrak{q} \text{ is unramified and } \pi_\mathfrak{q}^u \text{ is principal} \\ 0 & \text{if either } \eta'_\mathfrak{q} \text{ is ramified or } \pi_\mathfrak{q} \text{ is special} \end{cases}$$

and we define $\xi_\mathfrak{q}(\mathfrak{q})$ and $\xi'_\mathfrak{q}(\mathfrak{q})$ in exactly the same manner for $\pi'^u_\mathfrak{q}$. If either $\pi_\mathfrak{q}^u$ or $\pi'^u_\mathfrak{q}$ is super cuspidal, we simply put $D_\mathfrak{q}(X) = 1$. Then

$$\mathscr{D}(s,\mathbf{f},\mathbf{g}) = \prod_\mathfrak{q} D_\mathfrak{q}(\mathscr{N}(\mathfrak{q})^{-1-s})^{-1} \tag{0.1}$$

and

$$\mathscr{D}_p(s,\mathbf{f},\mathbf{g}) = \prod_{\mathfrak{q}\nmid p} D_\mathfrak{q}(\mathscr{N}(\mathfrak{q})^{-1-s})^{-1}.$$

Thus up to finitely many Euler factors, $\mathscr{D}(s,\mathbf{f},\mathbf{g})$ coincides with the standard L-function $L(s,\pi^u \times \pi'^u)$. Therefore, in general, our p-adic interpolation only yields a possibly meromorphic p-adic L-function for the primitive complex L-function $L(s,\pi^u \times \pi'^u)$. We hope to discuss the problem of p-adic holomorphy for primitive L-functions on a future occasion.

Now let us introduce the notion of p-adic Hecke algebras. Let $\mathfrak{r}$ be the integer ring of F and fix an ideal N prime to p in $\mathfrak{r}$. Then we consider the open compact subgroups:

$$S(p^\alpha) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U \,\middle|\, a \equiv 1 \mod p^\alpha,\ d \equiv 1 \mod Np^\alpha,\ c \equiv 0 \mod Np^\alpha \right\},$$

where U is the standard maximal open compact subgroup of $GL_2(F_{A_f})$. To consider the Hecke algebras, we fix a finite extension $K/\mathbf{Q}_p$ in $\bar{\mathbf{Q}}_p$ containing F^σ for all $\sigma \in I$ and write $\mathcal{O}$ for its p-adic integer ring. Then we write $\mathbf{h}_{k,w}(Np^\alpha;\mathcal{O})$ ($\alpha=1,\ldots,\infty$) for the Hecke algebra for the space $\mathbf{S}_{k,w}(S(p^\alpha);\mathbf{C})$ with coefficients in $\mathcal{O}$ and $\mathbf{h}^{\text{n.ord}}_{k,w}(Np^\alpha;\mathcal{O})$ for its nearly ordinary part. See § 3 in the text for the detailed definition of these algebras. Especially $\mathbf{h}_{k,w}(Np^\infty;\mathcal{O})$ is the natural projective limit of $\mathbf{h}_{k,w}(Np^\alpha;\mathcal{O})$ with respect to α (and hence is a large compact ring) and is known to be independent of the choice of (k,w). Thus we write $\mathbf{h}^{\text{n.ord}}(N;\mathcal{O})$ for $\mathbf{h}^{\text{n.ord}}_{k,w}(Np^\infty;\mathcal{O})$. In $\mathbf{h}^{\text{n.ord}}(N;\mathcal{O})$, we have a natural Hecke operator $\mathbf{T}(y)$ ($y \in F^\times_{A_f}$) corresponding to the double coset action of $S(p^\alpha)\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} S(p^\alpha)$ (in fact, we need to modify the usual Hecke operator to obtain the right definition of $\mathbf{T}(y)$; for its precise definition, see § 3). The central action of $F^\times_A$ on $\mathbf{S}_{k,w}(S(p^\alpha);\mathbf{C})$ gives a group homomorphism : $F^\times_A \to \mathbf{h}^{\text{n.ord}}(N;\mathcal{O})^\times$, which actually factors through the compact quotient :

$$Z(N) = F^\times_A/\overline{F^\times S_F(p^\infty)F^\times_{\infty+}},$$

where $F^\times_{\infty+}$ is the identity component of $F^\times_\infty$ and $S_F(p^\infty) = \bigcap_\alpha S(p^\alpha) \cap F^\times_{A_f}$. We will identify $Z(N)$ with the Galois group of the strict ray class field modulo Np^∞ over F. Similarly, the map $u \mapsto \mathbf{T}(u)$ for $u \in \mathfrak{r}^\times_p$ for $\mathfrak{r}_p = \mathfrak{r} \otimes_Z \mathbf{Z}_p$ is actually a group homomorphism. Thus we have a continuous group homomorphism :

$$\mathbf{G} = \mathbf{G}(N) = Z(N) \times \mathfrak{r}^\times_p \to \mathbf{h}^{\text{n.ord}}(N;\mathcal{O})^\times .$$

Let $\mathcal{O}[[\mathbf{G}]]$ be the continuous group algebra of $\mathbf{G}$. Then by the above homomorphism, $\mathbf{h}^{\text{n.ord}}(N;\mathcal{O})$ becomes naturally an algebra over $\mathcal{O}[[\mathbf{G}]]$. We now fix a decomposition $\mathbf{G} = \mathbf{W} \times \mathbf{G}_{\text{tor}}$ for the maximal finite subgroup $\mathbf{G}_{\text{tor}}$ of $\mathbf{G}$ and $\mathbf{W} \cong \mathbf{Z}^r_p$ $(2[F:\mathbf{Q}] \geqslant r \geqslant [F:\mathbf{Q}]+1)$. Then the continuous group algebra $\mathbf{A} = \mathcal{O}[[\mathbf{W}]]$ is non-canonically isomorphic to the r-variable power series ring over $\mathcal{O}$. Then it is seen in [H2], Th. 2.4 that

(0.2) $\mathbf{h}^{\text{n.ord}}(N;\mathcal{O})$ *is finite and torsion-free over* $\mathbf{A}$.

Let $\mathbf{L}$ be the quotient field of $\mathbf{A}$ and we fix an algebraic closure $\bar{\mathbf{L}}$ of $\mathbf{L}$. We take a pair of primitive (in the sense of Th. 3.4 in the text) $\mathbf{A}$-algebra homomorphisms $\lambda : \mathbf{h}^{\text{n.ord}}(N;\mathcal{O}) \to \bar{\mathbf{L}}$ and $\varphi : \mathbf{h}^{\text{n.ord}}(J;\mathcal{O}) \to \bar{\mathbf{L}}$.

Let $\mathbf{K}$ (resp. $\mathbf{M}$) be a finite extension of $\mathbf{L}$ containing the image of λ (resp. φ). Let $\mathbf{I}$ (resp. $\mathbf{J}$) be the integral closure of $\mathbf{A}$ in $\mathbf{K}$ (resp. $\mathbf{M}$). Extending scalar if necessary, we may assume that $\bar{\mathbf{Q}}_p \cap \mathbf{I} = \mathcal{O}$ *and* $\bar{\mathbf{Q}}_p \cap \mathbf{J} = \mathcal{O}$. Our p-adic L-function is a p-adic meromorphic function on the space $\mathscr{X}(\mathbf{I}) \times \mathscr{X}(\mathbf{J})$, where

$$\mathscr{X}(\mathbf{I}) = \mathrm{Hom}_{\mathcal{O}\text{-alg}}(\mathbf{I},\bar{\mathbf{Q}}_p) = \mathrm{Spec}\,(\mathbf{I})(\bar{\mathbf{Q}}_p).$$

In fact, it is an element $\mathscr{D}$ of the quotient field of $\mathbf{I}\,\hat{\otimes}_{\mathcal{O}}\mathbf{J}$, where $\mathbf{I}\,\hat{\otimes}_{\mathcal{O}}\mathbf{J}$ is the $\mathfrak{m}$-adic completion of $\mathbf{I} \otimes_{\mathcal{O}} \mathbf{J}$ for the unique maximal ideal $\mathfrak{m}$. We consider it as a function on $\mathscr{X}(\mathbf{I}) \times \mathscr{X}(\mathbf{J})$ by

$$\mathscr{D}(P,Q) = P \otimes Q(\mathscr{D}) \quad \text{for} \quad (P,Q) \in \mathscr{X}(\mathbf{I}) \times \mathscr{X}(\mathbf{J})$$

whenever it is well defined. Any point $P : \mathbf{I} \to \mathcal{O}$ in $\mathscr{X}(\mathbf{I})$ is called arithmetic, if for a small neighborhood of 1 in $\mathbf{W}$, P coincides with the character $Z(N) \times \mathfrak{r}_p^\times \ni (z,a) \mapsto \mathcal{N}(z)^m a^v \in \mathcal{O}^\times$ for a pair of weight (n,v) with $n \geqslant 0$, $v \geqslant 0$ and $n + 2v = mt$, where $\mathcal{N} : Z(N) \to Z_p^\times$ is the cyclotomic character given by the Galois action on μ_{p^∞}. Here we used the convention that $a^v = \prod_\sigma a^{\sigma v_\sigma}$. We write $m(P)$, $n(P)$ and $v(P)$ for the corresponding m, n and v for each arithmetic point P. Then

$$\mathbf{G} \ni (z,a) \;\mapsto\; P(\lambda(z,a))\mathcal{N}(z)^{-m}a^{-v}$$

is a finite order character of $Z(N) \times \mathfrak{r}_p^\times$. We write this character as (ψ_P, ψ'_P) for characters ψ_P and ψ'_P of $Z(N)$ and $\mathfrak{r}_p^\times$, respectively. Similarly, we define a pair of finite order characters (χ_Q, χ'_Q) of $Z(N)$ and $\mathfrak{r}_p^\times$ for φ and an arithmetic point Q of $\mathscr{X}(\mathbf{J})$. These characters can be computed explicitly (see § 3). Let $\mathscr{A}(\mathbf{I})$ be the set of arithmetic point of $\mathbf{I}$. For each arithmetic points $(P,Q) \in \mathscr{A}(\mathbf{I}) \times \mathscr{A}(\mathbf{J})$, it is known that $P(\lambda(\mathbf{T}(y)))$ and $Q(\varphi(\mathbf{T}(y)))$ are algebraic and hence can be considered as complex numbers via ι_∞ (see § 3). Moreover there exists a (unique) common eigenforms $\mathbf{f}_P$ (resp. $\mathbf{g}_Q$) in $\mathbf{S}_{k,w}(S(p^\alpha);\mathbf{C})$ for

$$k = n(P) + 2t, \qquad w = t - v(P) \quad (\text{resp. } k = n(Q)+2t, w = t - v(Q))$$

and for suitable α such that

$$(0.3) \quad \mathbf{f}_P | \mathbf{T}(y) = P(\lambda(\mathbf{T}(y)))\mathbf{f}_P \ \textit{and}\ \mathbf{g}_Q | \mathbf{T}(y) = Q(\varphi(\mathbf{T}(y)))\mathbf{g}_Q \quad (\text{see § 3}).$$

By near ordinarity, the automorphic representation attached to $\mathbf{f}_P$ and $\mathbf{g}_P$ are never super cuspidal at places over p. Our p-adic L-function $\mathscr{D}$ interpolates the special values $\mathscr{D}\left(1 + \frac{m(Q) - m(P)}{2}, \mathbf{f}_P, \mathbf{g}_Q^\rho\right)$ for the complex conjugate form $\mathbf{g}_Q^\rho$ of $\mathbf{g}_Q$. We write π^u (resp. π'^u) for the unitarization of the automorphic representation spanned by $\mathbf{f}_P$ (resp. $\mathbf{g}_Q^\rho$) and use the notation introduced above (0.1). Especially $\xi_\mathfrak{p}$, $\xi'_\mathfrak{p}$, $\eta_\mathfrak{p}$, $\eta'_\mathfrak{p}$ are meaningful for primes $\mathfrak{p}$ over p. We also write $W'(\mathbf{f}_P)$ (resp $W'(\mathbf{g}_Q)$) for the prime to p-part of the root number of the standard L-function of π^u (resp. π'^u) (see (4.10 c) for its precise definition). Finally we fix a finite idele d whose ideal is the absolute different $\mathfrak{d}$ of F and put $D = |d|_A^{-1}$. Then our result can be stated as

THEOREM I. — *There exists a unique element $\mathscr{D}$ in the quotient field of $\mathbf{I}\,\hat{\otimes}_\mathcal{O}\mathbf{J}$ satisfying the following interpolation property : Let (P,Q) be a pair of arithmetic points in $\mathscr{A}(\mathbf{I}) \times \mathscr{A}(\mathbf{J})$ satisfying the following two conditions :*

(0.4a) $t \leqslant n(P) - n(Q)$, $n(Q) - n(P) + 2t \leqslant (m(P)-m(Q))t$ *and* $v(Q) \geqslant v(P)$,

(0.4b) χ'_Q *and* Φ'_P *are both induced by finite order Hecke characters of* $F_A^\times/F^\times$ *unramified outside* p,

for which we use the same symbol, and put

$$C(P,Q) = D^{1+m(Q)-m(P)}2^{\{-n(P)-n(Q)+2v(P)-2v(Q)-4t\}} \pi^{\{2v(P)-2v(Q)-n(Q)-3t\}}i^{n(Q)-n(P)} \times \Gamma_F(n(Q)+v(Q)-v(P)+2t)\Gamma_F(v(Q)-v(P)+t),$$

$$W(P,Q) = \chi_{Q_\infty}\chi'_{Q_\infty}\psi_{P_\infty}\psi'_{P_\infty}(-1)\frac{\mathscr{N}_{F/\mathbf{Q}}(J)^{(m(Q)/2)+1}W'(\mathbf{g}_Q)}{\mathscr{N}_{F/\mathbf{Q}}(N)^{(m(P)/2)}W'(\mathbf{f}_P)} \times \prod_{\mathfrak{p}|p}\frac{\xi\xi'(d_\mathfrak{p})|\eta\eta'(d_\mathfrak{p})|G(\xi_\mathfrak{p}^{-1}\psi_\mathfrak{p}'^{-1})G(\xi_\mathfrak{p}'^{-1}\psi'_\mathfrak{p}-1)}{\eta\eta'(d_\mathfrak{p})|\xi\xi'(d_\mathfrak{p})|G(\eta_\mathfrak{p}'^{-1}\psi_\mathfrak{p}'^{-1})}$$

where $G(\alpha_\mathfrak{p})$ is the local Gauss sum which will be defined in § 4 *and*

$$\Gamma_F(s) = \prod_{\sigma\in I}\Gamma(s_\sigma), \qquad \{s\} = \sum_\sigma s_\sigma \in \mathbf{C} \quad \textit{for} \quad s = \sum_{\sigma\in I} s_\sigma\sigma \in \mathbf{C}[I].$$

Then $\mathscr{D}$ is finite at (P,Q) and we have

$$\mathscr{D}(P,Q) = \mathrm{W}(P,Q)\mathrm{C}(P,Q)\mathrm{S}(P)^{-1}\mathrm{E}(P,Q)\frac{\mathscr{D}_p\left(1+\frac{m(Q)-m(P)}{2}, \mathbf{f}_P, \mathbf{g}_Q^\rho\right)}{(\mathbf{f}_P^\circ, \mathbf{f}_P^\circ)},$$

where $\mathbf{f}_P^\circ$ is the primitive form associated with $\pi^u \otimes \psi'_P$, $(\mathbf{f}_P^\circ,\mathbf{f}_P^\circ)$ is the self Petersson inner product of $\mathbf{f}_P^\circ$ and $S(P)$ and $E(P,Q)$ are Euler factors at p which will be defined in Lemma 5.3 *below.*

We can even determine the denominator of $\mathcal{D}$ in the following sense: For any element H in $\mathbf{I}$ which annihilates the congruence module of λ (see § 5 for definition), $(H\otimes 1)\mathcal{D}$ is integral, i.e. is contained in $\mathbf{I}\hat{\otimes}_{\mathcal{O}}\mathbf{J}$, The value $H(P)$ of H may be regarded a p-adic analogue of self Petersson inner product $(\mathbf{f}_P^\circ, \mathbf{f}_P^\circ)$ (see for example [H7]). Since $(H\otimes 1)D$ is an element of $\mathbf{I}\hat{\otimes}_{\mathcal{O}}\mathbf{J}$, which is a finite extension of $\mathcal{O}[[\mathbf{W}\times\mathbf{W}]]$, formally it has $2r$ variables. However, from the fact that $\mathcal{D}_p(s,\mathbf{f}_P\otimes\xi,\mathbf{g}_Q^\rho\otimes\xi^{-1}) = \mathcal{D}_p(s,\mathbf{f}_P,\mathbf{g}_Q^\rho)$ for idele class characters ξ unramified outside p, we see that the number of independent variables of $\mathcal{D}$ is in fact equal to $2[F:\mathbf{Q}]+1$ if the Leopoldt conjecture is true for F.

The root number $W(P,Q)$ and the constant $C(P,Q)$ look complicated but in fact are compatible with the ε-factor at p predicted by the standard conjectures proposed by Coates and Perrin-Riou [Co] (see also [P2]) if one admits the existence of the motives $M(\mathbf{f}_P)$ and $M(\mathbf{g}_Q)$ attached to $\mathbf{f}_P$ and $\mathbf{g}_Q$, which is certainly verifiable in view of the method of Blasius and Rogawski [BR] of constructing Galois representations of $\mathbf{f}_P$ and $\mathbf{g}_Q$ (1). Another method of construction of such Galois representations by Taylor [T] combined with a result of Carayol tells us that the Galois representation of $M(\mathbf{f}_P)\otimes M(\mathbf{g}_Q)^\vee$ satisfies the hypothesis $I(p)$ in [Co], §6, where $M(\mathbf{g}_Q)^\vee$ is the dual of $M(\mathbf{g}_Q)$. In fact, the p-adic Hodge-Tate type of the motive $M(\mathbf{f}_P)$ at $\sigma\in I$ is equal to its Hodge type at σ (considered to be an infinite place) given by $(n_\sigma+1+v_\sigma, v_\sigma)$ for the weight (n,v) of $\mathbf{f}_P$ (e.g. [H5], Prop. 2.3, [H8], Remark 5.2). Hence the weight of $M(\mathbf{f}_P)$ is given by $m(P)+1$. Therefore, up to finitely many Euler factors outside p, we have

$$(*)\qquad \mathcal{D}\Big(1+\frac{m(Q)-m(P)}{2}, \mathbf{f}_P,\ \mathbf{g}_Q^\rho\Big) = L(0, M(\mathbf{f}_P)\otimes M(\mathbf{g}_Q)^\vee).$$

Under the condition (0.4 a), the motive $M(\mathbf{f}_P)\otimes M(\mathbf{g}_Q)^\vee$ is critical in the sense of Deligne [D]. To see the compatibility of our result with the standard conjecture, we may assume that $\psi' = \psi'_P$ is trivial by replacing $\mathbf{f}_P$ by $\mathbf{f}_P\otimes\psi'$ and $\mathbf{g}_Q$ by $\mathbf{g}_Q^\rho\otimes\xi^{-1}$ (this is tantamount to changing point P and Q). Since these conjectures only deal with cyclotomic twists, by assuming that $N=J=1$, we can get to the point much faster, getting rid of the innocuous contribution outside p. Moreover under this assumption, the above equality (*) is exact. For

(1) This remark and the following explanation is added under the request of the referee of the paper. Although this might be clear from the Langlands functoriality and our expression of the L-function, it might help the reader to understand the constant from the motivic side.

the moment, we suppose that χ'_Q is also trivial. Then, we can take

$$c^+(M(\mathbf{f}_P)\otimes M(\mathbf{g}_Q)^\vee) = \Omega(P,Q)\prod_{\mathfrak{p}} G(\chi_{\mathfrak{p}})$$

with

$$D^{1+m(Q)-m(P)}\Omega(P,Q) = (2\pi i)^{\{-2v(P)+2v(Q)+n(Q)+t\}}(-2i)^{\{n(P)+3t\}}\pi^{\{2t\}}(\mathbf{f}_P^\circ, \mathbf{f}_P^\circ)$$

as Deligne's motivic period of $(M(\mathbf{f}_P)\otimes M(\mathbf{g}_Q))^\vee$. The computation to obtain this expression is standard and in fact can be found is many places, for example, [P2], 1, 4, 5.3-4, [H3], Remark 4.6 and [H8], Remarks 5.2 and 8.2. In [Sh1] and in [P2], $v(P)_\sigma$ (resp. $v(Q)_\sigma$) is written as $\frac{k_\sigma - k_0}{2}\left(\text{resp.} \frac{l_\sigma - l_0}{2}\right)$, and $m(P) = k_0$ and $m(Q) = l_0$. When χ'_Q is non-trivial, we find another point Q' such that $n(Q') = n(Q)$, $v(Q') = v(Q)$ and $\chi_{Q'}$ is trivial. Then $\mathbf{g}_Q = \mathbf{g}_{Q'}\otimes\chi'^{-1}$. Thus

$$M(\mathbf{f}_P)\otimes M(\mathbf{g}_Q)^\vee = M(\mathbf{f}_P)\otimes M(\mathbf{g}_{Q'})^\vee(\chi'),$$

where for a motive M, $M(\chi')$ denotes the twist by the Artin motive attached to χ'. Thus we may define

$$c^+(M(\mathbf{f}_P)\otimes M(\mathbf{g}_{Q'})^\vee(\chi')) = \Omega(P,Q)\prod_{\mathfrak{p}} G(\chi_{\mathfrak{p}}\chi'_{\mathfrak{p}})$$

because $\prod_{\mathfrak{p}} G(\chi_{\mathfrak{p}}\chi'^2_{\mathfrak{p}})\,G(\chi'_{\mathfrak{p}^{-1}}) \equiv \prod_{\mathfrak{p}} G(\chi_{\mathfrak{p}}\chi'_{\mathfrak{p}}) \bmod \mathbf{Q}(\chi)^\times$. Since $\xi'_{\mathfrak{p}} = \chi_{\mathfrak{p}}\chi'_{\mathfrak{p}}$, this explains the factor $G(\xi'_{\mathfrak{p}^{-1}}\psi'_{\mathfrak{p}^{-1}})$ in $W(P,Q)$ and powers of $2\pi i$ in $C(P,Q)$. We now concentrate on the following factor which really depends on the cyclotomic twist:

$$(**)\qquad E'(P,Q) = E(P,Q)\prod_{\mathfrak{p}|p}\frac{\xi\xi'(d_{\mathfrak{p}})G(\xi_{\mathfrak{p}}^{-1}\psi'_{\mathfrak{p}^{-1}})G(\xi'_{\mathfrak{p}^{-1}}\psi'_{\mathfrak{p}^{-1}})}{|\xi\xi'(d_{\mathfrak{p}})|},$$

which is identical with the factor defined in [Co], § 6, in view of the Hodge-Tate type of the motive given as above and the Langlands functoriality property of the ε-factor shown in [J] for $L(s,\pi^u\times\pi'^u)$ in the case of non-supercuspidal local representations (see Lemma 5.3 for the exact form of $E(P,Q)$ and also (4.10 c) in the text). For example, if $M(\mathbf{f}_P)$ and $M(\mathbf{g}_Q)$ have good reduction modulo p, i.e. all the characters involved: $\xi_{\mathfrak{p}}$, $\xi'_{\mathfrak{p}}$, $\eta_{\mathfrak{p}}$ and $\eta'_{\mathfrak{p}}$ are unramified for all $\mathfrak{p}|p$, we have

$$E'(P,Q) = \prod_{\mathfrak{p}|p}\frac{(1-\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}}^{-1}(\mathfrak{p}))(1-\alpha'_{\mathfrak{p}}\beta_{\mathfrak{p}}^{-1}(\mathfrak{p}))}{(1-\alpha_{\mathfrak{p}}^{-1}\beta_{\mathfrak{p}}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-1})(1-\alpha'_{\mathfrak{p}^{-1}}\beta_{\mathfrak{p}}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-1})}.$$

By our convention of the parameterization of the representation, the Langlands parameters of π' (resp. π): $\alpha_{\mathfrak{p}}(\mathfrak{p})^{-1}\mathcal{N}(\mathfrak{p})^{-1} = \xi_{\mathfrak{p}}(\mathfrak{p})^{-1}\mathcal{N}(\mathfrak{p})^{-1+m(Q)/2}$ (which is in fact written as $\xi_{\mathfrak{p}}(\bar{\omega})$ in Lemma 5.3) and $\alpha'_{\mathfrak{p}}(\mathfrak{p})^{-1}\mathcal{N}(\mathfrak{p})^{-1} = \xi'_{\mathfrak{p}}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-1+m(Q)/2}$ (resp. $\beta_{\mathfrak{p}}(\mathfrak{p}) = \eta_{\mathfrak{p}}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{m(P)/2}$ and $\beta'_{\mathfrak{p}}(\mathfrak{p}) = \eta_{\mathfrak{p}}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{m(P)/2}$) are eigenvalues of the Frobenius element acting on the l-adic realization of $M(\mathbf{g}_Q)^{\vee}$ (resp. $M(\mathbf{f}_P)$). Thus we see from the Hodge-Tate type of these motives that this expression coincides with the formula of the modification factor in [Co], Lemma 7. It is easy to check that the expression of (**) in Lemma 5.3 in the general p-ramified case gives again the modification factor in [Co], § 6. We will not give the details of the argument since the detailed comparison of ε-factors and the Euler factors at p between motivic side and automorphic side is given in [HT2], § 8 under the assumption that both λ and φ have complex multiplication. This assumption on λ and φ is not at all restrictive because locally at p, the local representation is of this type and these factors only depends on local representations. Especially, the restriction of our p-adic L-function to the cyclotomic line passing through a given point (P,Q) supplies a p-adic L-function attached to the motive $M(\mathbf{f}_P) \otimes M(\mathbf{g}_Q)^{\vee}$ described in [Co], Conjectue A for general motives ; in particular, the restriction gives Panchishkin's p-adic L-function [P1], [P2]. Anyway, noting that

$$\delta = \chi_{Q\infty}\chi'_{Q\infty}\psi_{P\infty}\psi'_{P\infty}(-1)i^{2v(P)+2v(Q)+2n(P)+2n(Q)}$$

is independent of P and Q, we could have written the evaluation formula in the theorem as, when $N = J = 1$,

$$\mathcal{D}(P,Q) = \delta w(P)S(P)^{-1}\frac{E'(P,Q)\Lambda(0,M(P,Q))}{\Omega(P,Q)}$$

where

$$w(P) = \prod_{\mathfrak{p}|p}\frac{|\eta\eta'(d_{\mathfrak{p}})|}{\eta\eta'(d_{\mathfrak{p}})G(\eta_{\mathfrak{p}}'^{-1}\psi_{\mathfrak{p}}'^{-1})}$$

and for $M(P,Q) = M(\mathbf{f}_P) \otimes M(\mathbf{g}_Q)^{\vee}$,

$$\Lambda(0,M(P,Q)) = \Gamma_F(n(Q) + v(Q) - v(P) + 2t) \times \Gamma_F(v(Q) - v(P) + t)L(0,M(P,Q)).$$

We hope this explanation clarifies the nature of the constants in the theorem. When $\mathcal{D}(s,\mathbf{f}_P,\mathbf{g}_Q^{\rho})$ is primitive (i.e., when the conductor of

$\psi_P\chi_Q$ is divisible by NJ), we can get a simple expression as above. However, if not, especially when local representation of either of π and π' is super cuspidal at $\mathfrak{q}|NJ$, the lack of a simple expression of the local ε-factor at $\mathfrak{q}$ causes a little trouble (cf. [H3], Lemma 5.2 (ii)), which is one of the reasons that prevent us from taking this formulation in the theorem besides the fact that the theorem is formulated keeping our later application in [HT2] in mind.

Now we give a brief outline of the paper. In § 1, we summarize results on Fourier expansion and define the adelic q-expansion for complex modular forms. In § 2, we discuss the stability of spaces of modular forms with integral q-expansion coefficients under various Hecke operators. This result will be used to prove in § 3 the duality theorem between Hecke algebras and their spaces of modular forms. This duality is a key to our convolution method. In § 4, we give an exposition of the adelization of the classical Rankin-Selberg method employed in [Sh1], which is a little different from Jacquet's way [J]. In § 5, we restate Theorem I as Theorem 5.2 and deduce it from a crude but more general result (Theorem 5.1), which in turn will be proven in § 10. In § 6, we give an exposition of the computation of q-expansion of Eisenstein series according to Shimura [Sh5] and in § 8, this computation will be incorporated into a definition of p-adic Eisenstein measure which is more adelic than original Katz's definition [K]. In § 7, we summarize definitions and properties of various operators acting on spaces of p-adic and complex modular forms, which are necessary to carry out the computation. In § 9, we discuss the p-adic Rankin-Selberg convolution method in detail.

Notation. – We summarize here adelic notation we will use. The integer ring of F is denoted by $\mathfrak{r}$. We denote by $F_{\mathbf{A}}$ (resp. $\mathbf{A}$) the adele ring of F (resp. $\mathbf{Q}$). We write $F_{\mathbf{A}_f}$ (resp. $\mathbf{A}_f$) for the finite part of $F_{\mathbf{A}}$ (resp. $\mathbf{A}$). Similarly F_∞ denotes the infinite part of $F_{\mathbf{A}}$. Any element $x \in F_{\mathbf{A}}$ (resp. $x \in F_{\mathbf{A}}^\times$) is a sum $x_f + x_\infty$ (resp. a product $x_f x_\infty$) for $x_f \in F_{\mathbf{A}_f}$ and $x_\infty \in F_\infty$. For any $x \in F_{\mathbf{A}}$ and a prime ideal $\mathfrak{q}$ of $\mathfrak{r}$, $x_{\mathfrak{q}}$ is the $\mathfrak{q}$-component of x. For infinite place $\sigma \in I$, we write x_σ for the σ-component of $x \in F_{\mathbf{A}}$. Then we denote by $\mathbf{e}_F : F_{\mathbf{A}}/F \to \mathbf{C}^\times$ the standard additive character such that $\mathbf{e}_F(x_\infty) = \exp\left(2\pi i \sum_\sigma x_\sigma\right)\cdot$ Abusing a little this notation, for any element x and any subset X of $F_{\mathbf{A}}$ or $F_{\mathbf{A}}^\times$ and an ideal N of $\mathfrak{r}$, we write x_N and X_N for the projection of x and X

to $\prod_{\mathfrak{q}|N} F_{\mathfrak{q}}$, where $F_{\mathfrak{q}}$ is the $\mathfrak{q}$-adic completion of F. We also write $\hat{\mathfrak{r}}$ for the product $\prod_{\mathfrak{q}} \mathfrak{r}_{\mathfrak{q}}$ of the $\mathfrak{q}$-adic completions $\mathfrak{r}_{\mathfrak{q}}$ over all prime ideals $\mathfrak{q}$. We extend these conventions to any algebraic group defined over $\mathbf{Q}$ of F we shall use. Especially, we write G for $\mathrm{Res}_{F/\mathbf{Q}}GL(2)$ and G_∞ is the infinite component of $G(\mathbf{A})$ and $G_{\infty+}$ is the connected component of G_∞. Similarly $F^\times_{\infty+}$ is the connected component with identity of F_∞. We also write $G(\mathbf{A})_+$ (resp. $F^\times_{\mathbf{A}+}$) for $G(\mathbf{A}_f)G_{\infty+}$ (resp. $F^\times_{\mathbf{A}f}F^\times_{\infty+}$). We use the notation introduced in [H1] and [H2] throughout the paper with only brief explanation.

1. Fourier expansion of Hilbert modular forms.

Let F be a totally real field and N and M be integral ideals of F. We write $G_{/\mathbf{Q}} = \mathrm{Res}_{F/\mathbf{Q}}(GL(2)_{/F})$. Let $U = GL_2(\hat{\mathfrak{r}})$. We begin with studying the q-expansion of Hilbert modular forms on $G(\mathbf{A}) = GL_2(F_{\mathbf{A}})$ with respect to the various open compact subgroups given by

$$U_0(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in U \,|\, c \in N\hat{\mathfrak{r}}\right\},$$

$$V_1(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in U_0(N) \,|\, d \equiv 1 \bmod N\hat{\mathfrak{r}}\right\},$$

$$U(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in V_1(N) \,|\, a \equiv 1 \bmod N\hat{\mathfrak{r}}\right\},$$

$$U(N,M) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in U(N) \,|\, b \in M\hat{\mathfrak{r}} \quad \text{and} \quad a \equiv d \equiv 1 \bmod MN\hat{\mathfrak{r}}\right\}.$$

Let $U_F(N) = U(N) \cap F^\times_{\mathbf{A}_f}$ and decompose

$$F^\times_{\mathbf{A}} = \bigcup_{i=1}^{h(N)} F^\times a_i U_F(N) F^\times_{\infty+} \quad \text{with} \quad a_i \in F^\times_{\mathbf{A}_f}.$$

Let $(n,v) \in \mathbf{Z}[\mathbf{I}]^2$ be a pair of weights with $n + 2v = mt$ for $t = \sum_\sigma \sigma$. We consider the space of modular forms $\mathbf{M}_{k,w}(U(N);\mathbf{C})$ of weight (k,w) for $k = n + 2t$ and $w = t - v$ (cf. [H1], § 2). We can also decompose

$$G(\mathbf{A}) = \bigcup_{i=1}^{h(N)} G(\mathbf{Q})t_i U(N) G_{\infty+} \quad \text{for} \quad t_i = \begin{pmatrix} a_i^{-1} & 0 \\ 0 & 1\end{pmatrix},$$

where $G_{\infty+}$ is the identity component of $G(\mathbf{R})$. We define several discrete subgroups by

$$E = \{\varepsilon \in \mathfrak{r}^\times \mid \varepsilon \gg 0\}, \qquad \mathfrak{r}^\times(N) = \{\varepsilon \in \mathfrak{r}^\times \mid \varepsilon \equiv 1 \bmod N\},$$

$E(N) = E \cap \mathfrak{r}^\times(N)$, and for a fractional ideal $\mathfrak{a}$,

$$\Gamma(N;\mathfrak{a}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathfrak{r} & \mathfrak{a}^{-1} \\ N\mathfrak{a} & \mathfrak{r} \end{pmatrix} \middle| ad\text{-}bc \in E,\ a \equiv d \bmod N \in E/E(N) \right\},$$

where $\xi \gg 0$ means that $\xi^\sigma > 0$ for all $\sigma \in I$. Then we see

$$t_i E \cdot U(N) t_i^{-1} G_{\infty+} \cap GL_2(F) = \Gamma(N;\mathfrak{a}_i) \quad \text{for} \quad \mathfrak{a}_i = a_i\mathfrak{r}.$$

For any congruence subgroup Γ of the form $\Gamma(N;\mathfrak{a}_i)$, we consider the space $M_{k,w}(\Gamma)$ of modular forms $f : \mathscr{Z} \to \mathbf{C}$ satisfying the following conditions: (i) $f|_{k,w}\gamma(z) = f(\gamma(z))j_{k,w}(\gamma,z)^{-1} = f(z)$ for all $\gamma \in \Gamma$ and (ii) $f|_{k,w}\alpha$ for all $\alpha \in GL_2(F) \cap G_{\infty+}$ has the following type of Fourier expansion:

$$f|_{k,w}\alpha(z) = a(0,f|_{k,w}\alpha) + \sum_{0\iota \ll \xi \in F} a(\xi, f|_{k,w}\alpha)\mathbf{e}_F(\xi z),$$

where $a(\xi,f|_{k,w}\alpha) \in \mathbf{C}$ and $\mathbf{e}_F(\xi z) = \exp(2\pi i \sum_{\sigma \in I} \xi^\sigma z_\sigma)$ and actually ξ runs over totally positive elements in a $\mathbf{Z}$-lattice in F. For each $\mathbf{f} \in \mathbf{M}_{k,w}(U(N);\mathbf{C})$, we define $\mathbf{f}_i(z) \in M_{k,w}(\Gamma(N;\mathfrak{a}_i))$ by

$$\mathbf{f}_i(z) = j_{k,w}(u_\infty, z_0)\mathbf{f}(t_i u_\infty), \quad \left(\mathbf{f}\left(t_i\begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix}\right) = y_\infty^w \mathbf{f}_i(z)\right),$$

where $z_0 = (\sqrt{-1}, \ldots, \sqrt{-1}) \in \mathscr{Z} = \mathscr{H}^I$ and $u_\infty \in GL_2(F_\infty)$ with $u_\infty(z_0) = z$. (Hereafter, we always write $z = x + iy$ for $x \in F_\infty$ and $y \in F_{\infty+}$.) Then, as seen in [H1], (2.6 a), (2.8), this correspondence induces an isomorphism:

$$\mathbf{M}_{k,w}(U(N);\mathbf{C}) = \mathbf{M}_{k,w}(E \cdot U(N);\mathbf{C}) \cong \bigoplus_{i=1}^{h(N)} M_{k,w}(\Gamma(N;\mathfrak{a}_i)).$$

Each element f of $M_{k,w}(\Gamma(N;\mathfrak{a}))$ has the following Fourier expansion:

$$f(z) = a(0,f) + \sum_{0 \ll \xi \in \mathfrak{a}\mathfrak{d}^{-1}} a(\xi,f)\mathbf{e}_F(\xi z) \text{ for the different } \mathfrak{d}.$$

Now we compute $a(\varepsilon\xi, f)$ for $\varepsilon \in E$. By an easy computation, to have a non-trivial f, we need to suppose that

(1.1 a) $\qquad \varepsilon^{k-2w} = \mathscr{N}_{F/\mathbf{Q}}(\varepsilon)^m = 1$ *for all* $\varepsilon \in E(N)$.

We then have $f(\varepsilon z) = \varepsilon^{-w} f(z)$ for $\varepsilon \in E(N)$ and

$$\text{(1.1 b)} \qquad a(\varepsilon\xi, f) = \varepsilon^{w} a(\xi, f) \quad \textit{for all} \quad \varepsilon \in E(N),$$
$$\textit{and} \quad a(0,f) = 0 \quad \textit{unless} \quad w \in \mathbf{Z}\cdot t \quad \textit{for} \quad t = \sum_{\sigma\in I} \sigma .$$

Now take $\mathbf{f} \in \mathbf{M}_{k,w}(U(N); \mathbf{C})$ and consider the Fourier expansion of the corresponding element $\mathbf{f}_i \in M_{k,w}(\Gamma(N; \mathfrak{a}_i))$:

$$\mathbf{f}_i(z) = y_\infty^{-w} \mathbf{f}\left(t_i \begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix}\right) = a(0, \mathbf{f}_i) + \sum_{0 \ll \xi \in \mathfrak{a}\mathfrak{d}^{-1}} a(\xi, \mathbf{f}_i) \mathbf{e}_F(\xi z).$$

We fix throughout the paper a finite idele d such that $d\mathfrak{r} = \mathfrak{d}$ is the absolute different of F. Let Φ be the composite of all F^σ in $\mathbf{Q}$ for all $\sigma \in I$. Let $\mathfrak{r}_\Phi$ be the integer ring of Φ. We write $\mathscr{V}$ for the integer ring or a valuation ring of a finite extension K_0 of Φ. We then assume

(1.2) *In* $\mathscr{V}$, $\mathfrak{a}^\sigma \mathscr{V}$ *is principal for every integral ideal* $\mathfrak{a}$ *of* F *and for any* $\sigma \in I$.

We choose a generator $\{\mathfrak{q}^\sigma\} \in \mathscr{V}$ of $\mathfrak{q}^\sigma \mathscr{V}$ for each prime ideal $\mathfrak{q}$ of $\mathfrak{r}$ and define $\{\mathfrak{a}^v\} \in \mathscr{V}$ for each fractional ideal $\mathfrak{a}$ of F as in [H2], § 1. Then $\{\mathfrak{a}^v\}$ is a generator of $\mathfrak{a}^v \mathscr{V}$. For each $y \in F_{\mathbf{A}+}^\times$, writing $y = \xi a_i^{-1} du$ with $u \in U_F(N) F_{\infty+}^\times$, we define a function $F_{\mathbf{A}+}^\times \ni y \mapsto \mathbf{a}(y,\mathbf{f}) \in \mathbf{C}$ and $F_{\mathbf{A}+}^\times \ni y \mapsto \mathbf{a}_p(y,\mathbf{f}) \in \bar{\mathbf{Q}}_p$ (if $a(\xi, \mathbf{f}_i) \in \bar{\mathbf{Q}}$ for all i) by

(1.3 a) $\mathbf{a}(y,\mathbf{f}) = a(\xi, \mathbf{f}_i)\{y^{-v}\}\xi^v |a_i|_{\mathbf{A}}$ *and*
$\mathbf{a}_p(y,\mathbf{f}) = a(\xi, \mathbf{f}_i) y_p^{-v} \xi^v \mathscr{N}(a_i)^{-1}$
if $y = \xi a_i^{-1} du$ *and* $u \in U_F(N) F_{\infty+}^\times$ *and* $y \in \hat{\mathfrak{r}} F_{\infty+}^\times$,
and otherwise $\mathbf{a}(y,\mathbf{f}) = 0$ *and* $\mathbf{a}_p(y,\mathbf{f}) = 0$,

where $\mathscr{N} : Z(1) \to \mathbf{Q}_p^\times$ is the cyclotomic character such that $\mathscr{N}(y) = y_p^{-t} |y_f|_{\mathbf{A}}^{-1}$ for $y \in F_{\mathbf{A}}^\times$. We then have, by choosing a_i so that $a_{i,p} = a_{i,\infty} = 1$,

$$\text{(1.3 b)} \qquad \mathbf{a}_p(y,\mathbf{f}) = \mathbf{a}(y,\mathbf{f})\{y^v\}(y_p)^{-v} \quad \text{for all} \quad y \in F_{\mathbf{A}+}^\times .$$

If $y = \xi a_i^{-1} du = \xi' a_i^{-1} du'$, then $0 \ll \varepsilon = \xi^{-1}\xi' = u'^{-1} u \in U_F(N)$ and hence $\varepsilon \in E(N)$ and

$$a(\xi', \mathbf{f}_i)\xi'^v = a(\varepsilon\xi, \mathbf{f}_i)\varepsilon^v \xi^v = a(\xi, \mathbf{f}_i)\xi^v \varepsilon^{t-v} \varepsilon^v = a(\xi, \mathbf{f}_i)\xi^v .$$

Thus $\mathbf{a}(y,\mathbf{f})$ (resp. $\mathbf{a}_p(y,\mathbf{f})$) only depends on the cosets of y modulo $U_F(N)$ (resp. $U_F(Np^\infty) = \{u \in U_F(N) \,|\, u_p = 1\}$). Similarly we define a function $F_{\mathbf{A}+}^{\times} \ni y \mapsto \mathbf{a}_0(y,\mathbf{f}) \in \mathbf{C}$ (if $v=[v]t$ with $[v] \in \mathbf{Z}$) by

$$\mathbf{a}_0(y,\mathbf{f}) = a(0,\mathbf{f}_i)|a_i|_{\mathbf{A}}^{-[v]} \quad \text{if} \quad y = \xi a_i^{-1} du \ (u \in U_F(N)F_{\infty+}^{\times}, \xi \in F_+^{\times}),$$

which is a function on $\mathrm{Cl}_F(N) = F_{\mathbf{A}+}^{\times}/F_+^{\times} U_F(N)F_{\infty+}^{\times}$. Define a function $\mathbf{f}_0$ on

$$B(\mathbf{A})_+ = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \middle| y \in F_{\mathbf{A}+}^{\times} \quad \text{and} \quad x \in F_{\mathbf{A}} \right\}$$

by the Fourier series

$$\mathbf{f}_0\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y|_{\mathbf{A}}\{\mathbf{a}_0(yd,\mathbf{f})|y|_{\mathbf{A}}^{-[v]} + \sum_{\xi \gg 0} \mathbf{a}(\xi yd,\mathbf{f})\{(\xi yd)^v\}(\xi y_\infty)^{-v}\mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(\xi x)\}.$$

Then for all i, we have

$$\mathbf{f}_0\left(t_i\begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix}\right) = y_\infty^w\left\{a(0,\mathbf{f}_i) + \sum_{0 \ll \xi \in \mathfrak{a}\mathfrak{d}^{-1}} a(\xi,\mathbf{f}_i)\mathbf{e}_F(\xi z)\right\} = \mathbf{f}\left(t_i\begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix}\right).$$

We see easily from the above expansion that $\mathbf{f}_0$ is left-invariant under $B(\mathbf{Q})_+$ and therefore $\mathbf{f}_0$ coincides with the restriction of $\mathbf{f}$ to $B(\mathbf{A})_+$. Namely we have

THEOREM 1.1. – *Each $\mathbf{f} \in \mathbf{M}_{k,w}(U(N);\mathbf{C})$ has the Fourier expansion of the following type :*

$$\mathbf{f}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y|_{\mathbf{A}}\{\mathbf{a}_0(yd,\mathbf{f})|y|_{\mathbf{A}}^{-[v]} + \sum_{0 \ll \xi \in F^{\times}} \mathbf{a}(\xi yd,\mathbf{f})\{(\xi yd)^v\}(\xi y_\infty)^{-v}\mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(\xi x)\},$$

where $F_{\mathbf{A}+}^{\times} \ni y \mapsto \mathbf{a}_0(y,\mathbf{f})$ is a function invariant under $F_+^{\times} U_F(N)F_{\infty+}^{\times}$ (i.e. it factors through $\mathrm{Cl}_F(N)$) and vanishes identically unless $w \in \mathbf{Z}\cdot t$, and $F_{\mathbf{A}+}^{\times} \ni y \mapsto \mathbf{a}(y,\mathbf{f})$ is a function vanishing outside $\hat{\mathfrak{r}}F_{\infty+}^{\times}$ and depending only on the coset of $y_f U_F(N)$. Moreover, formally replacing $\mathbf{e}_F(i\xi y)\mathbf{e}_F(\xi x)$ by q^ξ and $(\xi y_\infty)^{-v}$ by $(\xi d_p y_p)^{-v}$, we have the q-expansion :

$$\mathbf{f} = \mathcal{N}(y)^{-1}\left\{\mathbf{a}_{0,p}(yd,\mathbf{f}) + \sum_{0 \ll \xi \in F^{\times}} \mathbf{a}_p(\xi yd,\mathbf{f})q^\xi\right\},$$

where $\mathbf{a}_{0,p}(y,\mathbf{f}) = \mathbf{a}_0(y,\mathbf{f})\mathcal{N}(yd^{-1})^{[v]}$.

We call the above expansion as the q-expansion of the p-adic modular form corresponding to $\mathbf{f}$ (if $\mathbf{a}(y,\mathbf{f}) \in \bar{\mathbf{Q}}$). Especially if one specializes y to a_i^{-1}, we can recover the q-expansion of the p-adic modular form $\mathbf{f}_i$ out of the above expansion. To see this, first write the expansion

$$\mathcal{N}(y)^{-1}\left\{\mathbf{a}_{0,p}(yd,\mathbf{f}) + \sum_{0\ll\xi\in F^\times} \mathbf{a}_p(\xi yd,\mathbf{f})q^\xi\right\}$$

as

$$\mathcal{N}(y)^{-1}\{\mathbf{a}_0(yd,\mathbf{f})\mathcal{N}(y)^{[v]} + \sum_{0\ll\xi\in F^\times} \mathbf{a}_p(\xi yd,\mathbf{f})(\xi y_p d_p)^v(\xi y_p d_p)^{-v}q^\xi\}$$

and then replace $\mathcal{N}^{-1}$ by $|\ |_{\mathbf{A}}$ and $(\xi y_p d_p)^{-v}q^\xi$ by $(\xi y_\infty)^{-v}\mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(\xi x)$. In this way, one recovers the Fourier expansion of $\mathbf{f}$. A simple computation yields:

$$\mathcal{N}(d)^{-1}\mathbf{f}(q)\mathbf{g}(q) = |d|_{\mathbf{A}}\mathbf{fg}. \tag{1.4}$$

Here the above formula implies that the formal q-expansion $\mathcal{N}(d)^{-1}\mathbf{f}(q)\mathbf{g}(q)$ corresponds to the complex modular form $|d|_{\mathbf{A}}\mathbf{fg}$. We can formulate the Fourier expansion for modular forms with respect to the group $U(N,M)$ similarly to Theorem 1.1. Since for $\alpha = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha U(MN)\alpha^{-1} = U(N,M)$ and

$$\mathbf{M}_{k,w}(U(MN);\mathbf{C}) \ni \mathbf{f} \mapsto \mathbf{f}|\alpha(x) = \mathbf{f}(x\alpha) \in \mathbf{M}_{k,w}(U(N,M);\mathbf{C})$$

gives an isomorphism: $\mathbf{M}_{k,w}(U(MN);\mathbf{C}) \cong \mathbf{M}_{k,w}(U(N,M);\mathbf{C})$, it is a mere interpretation of Theorem 1.1 and hence we leave it to the readers.

Now we consider the space $\mathbf{N}_{k,w}(S;\mathbf{C})$ of nearly holomorphic modular forms of weight (k,w) (in Shimura's sense) with respect to each open compact subgroup U. The space $\mathbf{N}_{k,w,m}(S;\mathbf{C})$ for $0 \leqslant m \in \mathbf{Z}[I]$ consists of functions $\mathbf{f}: G(\mathbf{A}) \to \mathbf{C}$ satisfying the following properties (cf. [Sh1], [Sh2]):

NH.1. $\mathbf{f}(axu) = j_{k,w}(u_\infty,z_0)^{-1}\mathbf{f}(x)$ *for* $u \in SC_{\infty+}$ *and* $a \in G(\mathbf{Q})$;

NH.2. *For each* $x \in G(\mathbf{A}_f)$, *we define* $\mathbf{f}_x(z) = j_{k,w}(u_\infty,z_0)\mathbf{f}(xu_\infty)$ *for* u_∞ *with* $u_\infty(z_0) = z \in \mathcal{H}^I$.

Then $\mathbf{f}_x(z) = a(0,\mathbf{f}_x)(4\pi y) + \sum_{0\ll\xi\in L(x)} a(\xi,\mathbf{f}_x)(4\pi y)\mathbf{e}_F(\xi z)$ *for a polynomial* $a(\xi,\mathbf{f}_x)(Y)$ *in variable* $(Y_\sigma)_{\sigma\in I}$ *of degree less than* m_σ *in* Y_σ *for each* σ, where $L(x)$ *is a lattice of* F *depending on* x.

We write $\mathbf{f}_i$ *for* $\mathbf{f}_{t_i}$ *and define*

(1.5) *if* $y \in \hat{\mathfrak{r}} F_{\infty+}^\times$ and $y = \xi a_i^{-1} du$ *with* $u \in U_F(N) F_{\infty+}^\times$,

$$\mathbf{a}(y,\mathbf{f})(Y) = \{y^{-v}\}\xi^v |a_i|_{\mathbf{A}} a(\xi,\mathbf{f}_i)(Y),$$
$$\mathbf{a}_P(y,\mathbf{f})(Y) = y_P^{-v} \mathcal{N}^{-1}(a_i)\xi^v a(\xi,\mathbf{f}_i)(Y),$$

and otherwise

$$\mathbf{a}(y,\mathbf{f}) = \mathbf{a}_p(y,\mathbf{f}) = 0.$$

Similarly we define a function $F_{\mathbf{A}}^\times \ni y \mapsto a_0(y,\mathbf{f})(Y) \in \mathbf{C}[Y]$ by

$$\mathbf{a}_0(y,\mathbf{f})(Y) = |a_i|_{\mathbf{A}} a(0,\mathbf{f}_i)(Y) \quad \textit{if} \quad \mathrm{v} \in \mathbf{Z}\cdot t \quad \textit{and} \quad y = \xi a_i^{-1} du$$

($u \in U_F(N) F_{\infty+}^\times, \xi \in F_+^\times$). Then the Fourier expansion of $\mathbf{f}\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right)$ is given by, for $Y = (4\pi y_\infty)^{-1}$,

$$(1.6) \quad |y|_{\mathbf{A}}\{\mathbf{a}_0(yd,\mathbf{f})(Y) y_\infty^{-v} + \sum_{\xi \gg 0} \mathbf{a}(\xi yd,\mathbf{f})(Y)$$
$$\{(\xi yd)^v\}(\xi y_\infty)^{-v}\mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(\xi x)\}.$$

Moreover, formally replacing $\mathbf{e}_F(i\xi y)\mathbf{e}_F(\xi x)$ by q^ξ, $(\xi y_\infty)^{-v}$ by $(\xi d_p y_p)^{-v}$ and $4\pi y_\infty$ by y_p, we have its q-expansion:

$$\mathbf{f} = \mathcal{N}^{-1}(y)\left\{\mathbf{a}_{0,p}(yd,\mathbf{f})((y_p)^{-1}) y_p^{-v} + \sum_{0 \ll \xi \in F^\times} \mathbf{a}_p(\xi yd,\mathbf{f})((y_p)^{-1}) q^\xi\right\},$$

where $\mathbf{a}_{0,p}(y,\mathbf{f})(Y) = \mathbf{a}_0(y,\mathbf{f})(Y)$.

Let S be a subgroup of $U_0(N)$ containing $U(N)$. For any subring A of $\mathbf{C}$ containing $\mathcal{V}$, we define

$$\mathbf{M}_{k,w}(U;A) = \{\mathbf{f} \in \mathbf{M}_{k,w}(U;\mathbf{C}) \mid \mathbf{a}_0(y,\mathbf{f}) \in A, \mathbf{a}(y,\mathbf{f}) \in A\},$$
$$\mathbf{m}_{k,w}(U;A) = \{\mathbf{f} \in \mathbf{M}_{k,w}(U;\mathbf{C}) \mid \mathbf{a}(y,\mathbf{f}) \in A\}$$
$$\mathbf{N}_{k,w,m}(U;A) = \{\mathbf{f} \in \mathbf{N}_{k,w,m}(U;\mathbf{C}) \mid \mathbf{a}(y,\mathbf{f})(Y) \in A[Y], \mathbf{a}_0(y,\mathbf{f})(Y) \in A[Y]\}.$$

Now we introduce differential operators on $\mathcal{Z}$:

$$\delta_\lambda^\sigma = \frac{1}{2\pi i}\left(\frac{\lambda}{2iy_\sigma} + \frac{\partial}{\partial z_\sigma}\right) \quad \text{and} \quad d^\sigma = \frac{1}{2\pi i}\frac{\partial}{\partial z_\sigma}.$$

For each $0 \leqslant r \in \mathbf{Z}[I]$ and $k \in \mathbf{Z}[I]$, we further put

$$(1.7\text{ a}) \qquad \delta_k^r = \left\{\prod_{\sigma \in I}(\delta_{k_\sigma+2r_\sigma-2}^\sigma \cdot \ldots \cdot \delta_{k_\sigma}^\sigma)\right\}$$

and

$$d^r = \left\{ \prod_{\sigma \in I} d^{\sigma r_\sigma} \right\}.$$

Then we see (*cf.* [Sh2], (1.8))

(1.7 b) $\quad \delta_k^r(f|_{k,w}x) = (\delta_k^r f)|_{k+2r,w+r}x \quad$ *for* $\quad x \in GL_2(F_\infty)_+$,

and thus, defining $\delta_k^r \mathbf{f}$ by $\mathbf{a}(y,\delta_k^r\mathbf{f})(Y) = \{y^{-v+r}\}\xi^{v-r}|a_i|_{\mathbf{A}}a(\xi,\delta_k^r\mathbf{f}_i)(Y)$ and $\mathbf{a}_0(y,\delta_k^r\mathbf{f})(Y) = |a_i|_{\mathbf{A}}a(0,\delta_k^r\mathbf{f}_i)(Y)$ if $y = \xi a_i^{-1}du$, we have

(1.8) $\quad \delta_k^r\mathbf{f}(x) = j_{k+2r,w+r}(x_\infty, z_0)^{-1}\delta_k^r(\mathbf{f}(x)\det(x_\infty)^{-w}j(x_\infty,z_0)^k)$.

Note that $U_0(Np^\alpha)/U(Np^\alpha) \cong (\mathfrak{r}/Np^\alpha\mathfrak{r})^\times \times (\mathfrak{r}/Np^\alpha\mathfrak{r})^\times$ via the correspondence: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (d^{-1},a^{-1}d) \in (\mathfrak{r}/Np^\alpha\mathfrak{r})^\times \times (\mathfrak{r}/Np^\alpha\mathfrak{r})^\times$. Thus we can consider, for each pair of characters ψ, ψ' of $(\mathfrak{r}/Np^\alpha\mathfrak{r})^\times$ with values in the subring A of $\mathbf{C}$, a subspace of $\mathbf{N}_{k,w,m}(U(Np^\alpha);A)$ given by

$$\text{(1.9)} \quad \mathbf{M}_{k,w}(U_0(Np^\alpha),\psi',\psi;A) = \mathbf{N}_{k,w,m}(U_0(Np^\alpha),\psi',\psi;A) \cap \mathbf{M}_{k,w}(U(Np^\alpha);A)$$

and

$$\mathbf{N}_{k,w,m}(U_0(Np^\alpha),\psi',\psi;A) = \{\mathbf{f} \in \mathbf{N}_{k,w,m}(U(Np^\alpha);A) \,|\, \mathbf{f}|\langle z,a\rangle = \psi(z)\psi'(a)\mathbf{f}\},$$

where $\mathbf{f}|\langle z,a\rangle(x) = \mathbf{f}(xu)$ for $u \in U_0(Np^\alpha)$ corresponding to (z,a) in $(\mathfrak{r}/Np^\alpha\mathfrak{r})^\times \times (\mathfrak{r}/Np^\alpha\mathfrak{r})^\times$. Note that $\mathbf{N}_{k,w,m}(U_0(Np^\alpha),\psi',\psi;A)$ is reduced to 0 unless ψ factors through $(\mathfrak{r}/Np^\alpha\mathfrak{r})^\times/E$.

PROPOSITION 1.2. — *Suppose that A contains $\mathbf{Q}$. Then δ_k^r sends $\mathbf{N}_{k,w,m}(U;A)$ into $\mathbf{N}_{k+2r,w+r,m+r}(U;A)$. Moreover if $k_\sigma > 2m_\sigma$ for all σ, $\mathbf{f} \in \mathbf{N}_{k,w,m}(U_0(Np^\alpha),\psi',\psi;A)$ can be expressed uniquely*

$$\mathbf{f} = \sum_{0 \leq r \leq m} \delta_{k-2r}^r \mathbf{f}_r \quad \textit{with} \quad \mathbf{f}_r \in \mathbf{M}_{k-2r,w-r}(U_0(Np^\alpha),\psi',\psi;A).$$

This follows from [Sh1], Lemma 4.10 applied to each $\mathbf{f}_i$.

2. Stability of integral forms under the Hecke operators.

Let Φ be the composite of F^σ in $\bar{\mathbf{Q}}$ for all $\sigma \in I$. Let $\mathfrak{r}_\Phi$ be the integer ring of Φ. Let S be a subgroup of $U_0(N)$ containing $U(N)$. In this section, N is an integral ideal of $\mathfrak{r}$ and may have common factors with p. Hereafter we suppose that $\mathscr{V}$ is the valuation ring corresponding to the fixed embedding $\iota_p : \Phi \to \bar{\mathbf{Q}}_p$. Thus we may assume that $\{y^v\} = 1$ whenever $y\mathfrak{r}$ is prime to p. We then define, for any $\mathcal{V}$-algebra A in $\mathbf{C}$,

$$\mathbf{M}_{k,w}(S;A) = \{\mathbf{f} \in \mathbf{M}_{k,w}(S;\mathbf{C}) \mid \mathbf{a}_0(y,\mathbf{f}) \in A \quad \text{and} \quad \mathbf{a}(y,\mathbf{f}) \in A\},$$
$$\mathbf{m}_{k,w}(S;A) = \{\mathbf{f} \in \mathbf{M}_{k,w}(S;\mathbf{C}) \mid \mathbf{a}(y,\mathbf{f}) \in A\}.$$

We can prove the following lemma in exactly the same way as in [H1], Cor. 4.5 using Shimura's Galois action on modular forms [Sh1], Th. 1.5 :

LEMMA 2.1. – *Suppose that $U_0(N) \supset S \supset U(N)$. Let α be a non-negative integer. Then for $S(p^\alpha) = S \cap U(p^\alpha)$ $(0 \leqslant \alpha \in \mathbf{Z})$ and for any finite extension K/Φ in $\mathbf{C}$, we have*

$$\mathbf{M}_{k,w}(S(p^\alpha);K) \cong \mathbf{M}_{k,w}(S(p^\alpha);\Phi) \underset{\Phi}{\otimes} K.$$

By this lemma, for any automorphism $\sigma \in \mathrm{Aut}\,(\mathbf{C}/\Phi)$ and for each $\mathbf{f} \in \mathbf{M}_{k,w}(S;\mathbf{C})$, there exists $\mathbf{f}^\sigma \in \mathbf{M}_{k,w}(S;\mathbf{C})$ such that

(2.1) $\mathbf{a}(y,\mathbf{f}^\sigma) = \mathbf{a}(y,\mathbf{f})^\sigma$ and $\mathbf{a}_0(y,\mathbf{f}^\sigma) = \mathbf{a}_0(y,\mathbf{f})^\sigma$ for all y.

We now study the effect of Hecke operators $T(\varpi)$ on the coefficients $\mathbf{a}$ and $\mathbf{a}_p$ for a prime element ϖ in $\mathfrak{r}_\mathfrak{q}$. Decomposing the double coset $U(N)\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}U(N)$ into a disjoint union $\bigcup_{i} \gamma_i U(N)$, we define $\mathbf{f} \mid T(\varpi)(x) = \sum_i \mathbf{f}(x\gamma_i)$. The operator $T_0(\varpi)$ is defined by $\{\varpi^{-v}\}T(\varpi)$ (see [H1], § 3). We write down the formulas only for $\mathbf{a}_p$ because one can recover the corresponding ones for $\mathbf{a}$ by (1.3 b). A simple computation using the following explicit decomposition :

$$U(N)\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}U(N) = \bigcup_{u \bmod \varpi}\begin{pmatrix}\varpi & u\\ 0 & 1\end{pmatrix}U(N) \cup \begin{pmatrix}1 & 0\\ 0 & \varpi\end{pmatrix}U(N) \quad \text{if} \quad \mathfrak{q} \nmid N$$

and

$$U(N)\begin{pmatrix}\varpi^m & 0\\ 0 & 1\end{pmatrix}U(N) = \bigcup_{u \bmod \varpi^m}\begin{pmatrix}\varpi^m & u\\ 0 & 1\end{pmatrix}U(N) \text{ ([H2], (1.6))} \quad \text{if} \quad \mathfrak{q} \mid N,$$

yields, for $\mathbf{f} \in \mathbf{M}_{k,w}(U(N);\mathbf{C})$,

(2.2 a) *When ϖ is a prime element of $\mathfrak{r}_\mathfrak{q}$ for prime $\mathfrak{q}$ outside N,*

$$\mathbf{a}_p(y,\mathbf{f} \mid T_0(\varpi))$$
$$= \mathbf{a}_p(y\varpi,\mathbf{f})\{\varpi^{-v}\}\varpi_p^v + \mathcal{N}(\mathfrak{q})\{\mathfrak{q}^{-2v}\}\mathbf{a}_p(y\varpi^{-1},\mathbf{f} \mid \langle\mathfrak{q}\rangle)\{\varpi^v\}\varpi_p^{-v},$$
$$\mathbf{a}_{0,p}(y,\mathbf{f} \mid T_0(\varpi))$$
$$= \mathcal{N}(\varpi)^{[v]}\mathbf{a}_{0,p}(y\varpi,\mathbf{f}) + \mathcal{N}(\varpi)^{-[v]}\mathcal{N}(\mathfrak{q})^{1-2[v]}\mathbf{a}_{0,p}(y\varpi^{-1},\mathbf{f} \mid \langle\mathfrak{q}\rangle)$$

(2.2 b) *If ϖ is a prime element of $\mathfrak{r}_\mathfrak{q}$ for prime $\mathfrak{q}$ dividing N,*

$$\mathbf{a}_p(y,\mathbf{f}\,|\,T_0(\varpi^m)) = \mathbf{a}_p(y\varpi^m,\mathbf{f})\{\varpi^{-mv}\}\varpi_p^{mv}, \mathbf{a}_0(y,\mathbf{f}\,|\,T_0(\varpi^m)) = \mathbf{a}_{0,p}(y\varpi^m,\mathbf{f}),$$

and

$$\mathbf{a}_{0,p}(y,\mathbf{f}\,|\,T_0(\varpi)) = \mathcal{N}(\varpi)^{[v]}\mathbf{a}_{0,p}(y\varpi,\mathbf{f}).$$

When $\mathfrak{q}$ is prime to N, we write usually $T(\mathfrak{q})$ and $T_0(\mathfrak{q})$ for $T(\varpi)$ and $T_0(\varpi)$ because these operators do not depend on the choice of ϖ.

Now we consider the action of $T(a,b) = \left[U(N)\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}U(N)\right]$ for $a,\ b \in \mathfrak{r}_N$. Note that $x = \begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}$ normalizes $U(N)$ and $x = b\begin{pmatrix} ab^{-1} & 0 \\ 0 & 1\end{pmatrix}$. We compute the Fourier expansion of $\mathbf{f}\,|\,T(a,1)$ as above and we have

(2.2 c) *For* $a,\ b \in \mathfrak{r}_N{}^\times$, $\mathbf{a}_p(y,\mathbf{f}\,|\,T(a,1)) = \mathbf{a}_p(ya,\mathbf{f})a_p^v$

$\mathbf{a}_{0,p}(y,\mathbf{f}\,|\,T(a,1)) = \mathbf{a}_{0,p}(ya,\mathbf{f})a_p^v$,

$\mathbf{a}_p(y,\mathbf{f}\,|\,T(a,b)) = \mathbf{a}_p(yab^{-1},\mathbf{f}\,|\,b)a_p^v b_p^{-v}$

and $\mathbf{a}_{0,p}(y,\mathbf{f}\,|\,T(a,b)) = \mathbf{a}_0(yab^{-1},\mathbf{f}\,|\,b)a_p^v b_p^{-v}$.

Let us now define the Hecke algebra $\mathbf{H}_{k,w}(S;\mathcal{V})$ (resp. $\mathbf{h}_{k,w}(S;\mathcal{V})$) by the $\mathcal{V}$-subalgebra of $\mathrm{End}_C(\mathbf{M}_{k,w}(S;\mathbf{C}))$ (resp. $\mathrm{End}_\mathbf{C}(\mathbf{S}_{k,w}(S\mathbf{C}))$) generated by $T_0(\mathfrak{n})$'s for $\mathfrak{n}$ outside N, $T(\varpi^m)$'s for $\mathfrak{q}$ dividing N and $T(a,b)$'s for $a, b \in \mathfrak{r}_N^\times$. Put

$$\mathbf{H}_{k,w}(S;A) = \mathbf{H}_{k,w}(S;\mathcal{V}) \otimes_\mathcal{V} A \quad \text{and} \quad \mathbf{h}_{k,w}(S;A) = \mathbf{h}_{k,w}(S;\mathcal{V}) \otimes_\mathcal{V} A$$

for each $\mathcal{V}$-algebra A in $\mathbf{C}$. We also define a topological group $\mathbf{G}$ by

$$\mathbf{G} = \mathbf{G}(N) = Z(N) \times \mathfrak{r}_p^\times \quad \text{for} \quad Z(N) \times F_\mathbf{A}^\times/\overline{F^\times\ U_F(Np^\infty)F_{\infty+}^\times}.$$

Then $(z,a) \in \mathbf{G}$ acts on $\mathbf{M}_{k,w}(S;\mathbf{C})$ via $\langle z,a\rangle = T(a^{-1},1)\langle z\rangle$. Thus the Hecke algebra becomes an algebra over the group algebra $A[\mathbf{G}]$.

THEOREM 2.2. – *Let S be a subgroup of $U_0(N)$ containing $U(N)$.*

(i) *For any finite extension K/K_0 and an $\mathcal{V}$-subalgebra A of K, we have a natural isomorphism : $\mathbf{M}_{k,w}(S;K) \cong \mathbf{M}_{k,w}(S;A) \underset{A}{\otimes} K$.*

(ii) *Let A be an integrally closed domain containing $\mathcal{V}$. Suppose that A is finite flat over either of $\mathcal{V}$ or $\mathbf{Z}_p$. Then $\mathbf{M}_{k,w}(S;A)$ is stable under $T_0(\mathfrak{n})$ for all $\mathfrak{n}$ outside N, $T_0(\varpi^m)$ for each prime element ϖ of $\mathfrak{r}_\mathfrak{q}$ for prime $\mathfrak{q}$ outside N and $T(a,b)$ for all a, $b \in \mathfrak{r}_N{}^\times$.*

(iii) *Let* A *be as in* (ii) *and define a pairing*

$$(\ ,\ ) : \mathbf{m}_{k,w}(S;A) \times \mathbf{H}_{k,w}(S;A) \rightarrow A \quad by \quad (\mathbf{f},h) = \mathbf{a}(1,\mathbf{f}|h) \in A.$$

Then we have the following natural isomorphisms under this pairing :

$$\mathrm{Hom}_A(\mathbf{m}_{k,w}(S;A),A) \cong \mathbf{H}_{k,w}(S;A), \quad \mathrm{Hom}_A(\mathbf{H}_{k,w}(S;A),A) \cong \mathbf{m}_{k,w}(S;A).$$

Proof. — We can prove the second assertion for $T(\mathfrak{n})$ for all $\mathfrak{n}$ outside N and $T(a,a) = \langle a \rangle$ in exactly the same manner as in the proof or [H1], Th. 4.11. As for the assertion for $T_0(\varpi^m)$ and $T(a,b)$ follows from the assertion for $T(a,a)$ by (2.2 a, b, c). The first assertion follows from the third assertion by the argument given in [H1], § 7 in the proof of Theorem 4.10. We now prove the third assertion. First assume that A is a field. For each prime ideal $\mathfrak{q}$, we fix a prime element $\varpi_\mathfrak{q}$. Then each $y \in \hat{\mathfrak{r}} \cap F_{\mathbf{A}_f}^\times$ can be written uniquely that $y = a\prod_\mathfrak{q} \varpi_\mathfrak{q}{}^{e(\mathfrak{q})}u$ with $u \in U_F(N)$, $a \in \mathfrak{r}_{N^\times}$. Write $\mathfrak{n}$ for $(\prod_{\mathfrak{q}\nmid N} \varpi_\mathfrak{q}{}^{e(\mathfrak{q})})\mathfrak{r}$ and define

$$(2.3) \qquad T(y) = T(a,1)T(\mathfrak{n}) \prod_{\mathfrak{q}|N|} T(\varpi_\mathfrak{q}{}^{e(\mathfrak{q})}) \in \mathbf{H}_{k,w}(S;A),$$

$$T_0(y) = T(y)\{y^{-v}\} = T(a,1)T_0(\mathfrak{n}) \prod_{\mathfrak{q}|N} T_0(\varpi_\mathfrak{q}{}^{e(\mathfrak{q})}) \in \mathbf{H}_{k,w}(S;A).$$

Then by (2.2 a, b, c), we know $(\mathbf{f}, T_0(y)) = \mathbf{a}(y,\mathbf{f})$. Thus if $(\mathbf{f},h) = 0$ for all $h \in \mathbf{H}_{k,w}(S;A)$, then $\mathbf{a}(y,\mathbf{f}) = 0$ for all y and hence $\mathbf{f} = 0$. On the other hand, if $(\mathbf{f},h) = 0$ for all $\mathbf{f} \in \mathbf{M}_{k,w}(S;A)$, then

$$0 = (\mathbf{f}|T_0(y),h) = (\mathbf{f}|h,T_0(y)) = \mathbf{a}(y,\mathbf{f}|h) \quad \text{for all } y \text{ and } \mathbf{f}.$$

Thus $\mathbf{f}|h = 0$ for all $\mathbf{f}$ and hence $h = 0$ as an operator. This shows the pairing is non-degenerate at the both side. Since A is a field and $\mathbf{M}_{k,w}(S;A)$ is of finite dimension, the pairing is perfect. The general case where A is no longer a field can be handled in exactly the same way as in the proof of [H1], Th. 5.1.

3. A duality theorem for p-adic Hilbert modular forms.

We now fix a valuation ring $\mathcal{O}$ (in $\bar{\mathbf{Q}}_p$) finite flat over $\mathbf{Z}_p$ containing $t_p(\mathscr{V})$. We also fix an integral ideal N prime to p and take a subgroup S of $U_0(N)$ containing $V_1(N)$ as in [H1], § 2. We put $S(p^\alpha) = S \cap U(p^\alpha)$

and consider the limits

$$\mathbf{M}_{k,w}(S(p^\infty); A) = \varinjlim_{\alpha} \mathbf{M}_{k,w}(S(p^\alpha); A),$$

$$\mathbf{m}_{k,w}(S(p^\infty); A) = \varinjlim_{\alpha} \mathbf{m}_{k,w}(S(p^\alpha); A),$$

$$\mathbf{S}_{k,w}(S(p^\infty); A) = \varinjlim_{\alpha} \mathbf{S}_{k,w}(S(p^\alpha); A)$$

on which the Hecke algebras

$$\mathbf{H}_{k,w}(S(p^\infty); A) = \varprojlim_{\alpha} \mathbf{H}_{k,w}(S(p^\alpha); A),$$

$$\mathbf{h}_{k,w}(S(p^\infty); A) = \varprojlim_{\alpha} \mathbf{h}_{k,w}(S(p^\alpha); A)$$

naturally acts. Here implicitly we think that A is either $\mathcal{O}$, its quotient field K, Φ, $\bar{\mathbf{Q}}$, $\bar{\mathbf{Q}}_p$ or its p-adic completion Ω. For the operator $T(y)$ in § 2; we define a new operator

$$\mathbf{T}(y) = \varprojlim_{\alpha} T(y) y_p^{-v} \quad \text{in} \quad \mathbf{H}_{k,w}(S(p^\infty); A).$$

Using the q-expansion coefficients $\mathbf{a}_p(y,\mathbf{f})$ and $\mathbf{a}_{0,p}(y,\mathbf{f})$ defined in (1.3 b), we introduce a p-adic norm by

$$\|\mathbf{f}\|_p = \mathrm{Sup}_y\,(|\mathbf{a}_p(y,\mathbf{f})|_p, |\mathbf{a}_{0,p}(y,\mathbf{f})|_p) = \mathrm{Sup}_y\,(|\mathbf{a}(y,\mathbf{f})|_p, |\mathbf{a}_0(y,\mathbf{f})|_p),$$

$$|\mathbf{f}|_p = \mathrm{Sup}_y\,(|\mathbf{a}_p(y,\mathbf{f})|_p) = \mathrm{Sup}_y\,(|\mathbf{a}(y,\mathbf{f})|_p.$$

Here we know from (1.3 b) that $|\mathbf{a}_p(y,\mathbf{f})|_p = |\mathbf{a}(y,\mathbf{f})|_p$ and $|\mathbf{a}_{0,p}(y,\mathbf{f})|_p = |\mathbf{a}_0(y,\mathbf{f})|_p$. We denote the completion of $\mathbf{M}_{k,w}$ or $\mathbf{S}_{k,w}$ (resp. $\mathbf{m}_{k,w}$) under the norm $\|\ \|_p$ (resp. $|\ |_p$) by $\bar{\mathbf{M}}_{k,w}$ or $\bar{\mathbf{S}}_{k,w}$ (resp. $\bar{\mathbf{m}}_{k,w}$). For each $\mathbf{f} \in \bar{\mathbf{M}}_{k,w}(S(p^\infty); A)$ and for a p-adically complete ring A (e.g. $A = \mathcal{O}, K$ or Ω), we can regard the function $y \mapsto \mathbf{a}_p(y,\mathbf{f})$ (resp. $\mathbf{a}_{0,p}(y,\mathbf{f})$) as a continuous functions on

$$\mathscr{I} = \hat{\mathfrak{r}} \cap F_{\mathbf{A}_f}^\times / U_F(p^\infty) = \varprojlim_{\alpha} \hat{\mathfrak{r}} \cap F_{\mathbf{A}_f}^\times / U_F(p^\alpha) \quad (\text{resp. } Z = \varprojlim_{\alpha} \mathrm{Cl}_F(p^\alpha)),$$

where on each $\mathscr{I}_\alpha = \hat{\mathfrak{r}} \cap F_{\mathbf{A}_f}^\times / U_F(p^\alpha)$ (resp. $\mathrm{Cl}_F(p^\alpha)$) for finite α, we give the discrete topology. Let Il be the semigroup of all integral ideals of F. Then we see easily that

(3.1 a) $\mathscr{I} \cong \mathfrak{r}_p^\times \times Il$ *as a topological space,* because $\mathscr{I}_\alpha = \mathscr{I}_0 \times \hat{\mathfrak{r}}^\times / U_F(p^\alpha)$ and $\mathscr{I}_0 = Il$ and

$$\hat{\mathfrak{r}}^\times / U_F(p^\alpha) \cong (\mathfrak{r}/p^\alpha \mathfrak{r})^\times .$$

More precisely, we have an exact sequence of topological semigroups

$$(3.1\,b) \qquad 1 \to \mathfrak{r}_p^\times \to \mathscr{I} \to Il \to 1 ,$$

where the first inclusion is induced from the natural inclusion of $\mathfrak{r}_p^\times$ into $F_{\mathbf{A}f}^\times$ and the second projection comes from the natural association of ideals with ideles. Anyway, writing $\mathscr{C}(X;A)$ for the space of continuous functions with values in A on any topological space X, we know that $\mathbf{M}_{k,w}(S(p^\infty);A)$ (resp. $\bar{\mathbf{m}}_{k,w}(S(p^\infty);A)$ and $\bar{\mathbf{S}}_{k,w}(S(p^\infty);A)$) can be embedded into the space $\mathscr{C}(\mathscr{I} \cup Z;A)$ (resp. $\mathscr{C}(\mathscr{I};A)$). We now define a pairing

$$(\ ,\ ) : \mathbf{H}_{k,w}(S(p^\infty);\mathcal{O}) \times \bar{\mathbf{m}}_{k,w}(S(p^\infty);\mathcal{O}) \to \mathcal{O} \quad \text{by} \quad (\mathbf{h},\mathbf{f}) = \mathbf{a}_p(1,\mathbf{f}|\mathbf{h}).$$

For any $\mathcal{O}$-module M, we denote by M^* the $\mathcal{O}$-dual module $\mathrm{Hom}_{\mathcal{O}}(M,\mathcal{O})$. Then we can deduce from Theorem 2.2 the following key duality result in exactly the same manner as in [H3], § 1 :

THEOREM 3.1. — *The above pairing induces isomorphims :*

$$\mathbf{H}_{k,w}(S(p^\infty);\mathcal{O})^* \cong \bar{\mathbf{m}}_{k,w}(S(p^\infty);\mathcal{O}) \text{ and } \bar{\mathbf{m}}_{k,w}(S(p^\infty);\mathcal{O})^* \cong \mathbf{H}_{k,w}(S(p^\infty);\mathcal{O})$$

$$\mathbf{h}_{k,w}(S(p^\infty);\mathcal{O})^* \cong \bar{\mathbf{S}}_{k,w}(S(p^\infty);\mathcal{O}) \text{ and } \bar{\mathbf{S}}_{k,w}(S(p^\infty);\mathcal{O})^* \cong \mathbf{h}_{k,w}(S(p^\infty);\mathcal{O}).$$

As already seen in [H2], Th. 2.3, there exists a canonical algebra isomorphism

$$(3.2) \qquad \mathbf{h}_{k,w}(S(p^\infty);\mathcal{O}) \cong \mathbf{h}_{2t,t}(S(p^\infty);\mathcal{O})$$

for all pair of weight (n,v) with $n \geqslant 0$, $v \geqslant 0$ and $n + 2v \in \mathbf{Z}\cdot t$, which takes $\mathbf{T}(y)$ to $\mathbf{T}(y)$ for all $f \notin \hat{\mathfrak{r}} \cap F_{\mathbf{A}_f}^\times$. This shows that $\bar{\mathbf{S}}_{k,w}(S(p^\infty);\mathcal{O})$ is independent of (k,w) (whenever $n \geqslant 2t$) as a subspace of $\mathscr{C}(\mathscr{I} \cup Z;\mathcal{O})$, which hereafter we denote by $\bar{\mathbf{S}}(S)$ or $\bar{\mathbf{S}}(N)$ if $S = V_1(N)$ for N prime to p. Similarly, we write $\mathbf{h}(S;\mathcal{O})$ (and $\mathbf{h}(N;\mathcal{O})$ if $S = V_1(N)$ for N prime to p) for the Hecke algebra in (3.2). We write $\bar{\mathbf{M}}(S)$ for the completion of $\left\{ \sum_{k,w} \bar{\mathbf{M}}_{k,w}(S(p^\infty);K) \right\} \cap \mathscr{C}(\mathscr{I} \cup Z;\mathcal{O})$ under $\|\ \|_p$ inside $\mathscr{C}(\mathscr{I} \cup Z;\mathcal{O})$, because we do not know whether this space coincides with $\bar{\mathbf{M}}_{k,w}(S(p^\infty);\mathcal{O})$ or not. We simply write $\bar{\mathbf{M}}(N)$ for $\bar{\mathbf{M}}(S)$ when $S = \mathrm{V}_1(N)$. We can extend the operator $\mathbf{T}(a,1)$ for $a \in \mathfrak{r}_p^\times$ on $\mathscr{C}(\mathscr{I} \cup Z; A)$ as follows: Any function $\mathbf{f} \in \mathscr{C}(\mathscr{I} \cup Z; A)$ is a pair of functions (f, f_0) with $f : \mathscr{I} \to A$ and $f_0 : Z \to A$. Then we define

$$f | \mathbf{T}(a,1)(x) = f(ax) \qquad \text{and} \qquad f_0 | \mathbf{T}(a,1)(z) = f_0(az).$$

Then the natural injection $\bar{\mathbf{M}}(S) \to \mathscr{C}(\mathscr{I} \cup Z; \mathcal{O})$ is equivariant under the action of $\mathbf{T}(a,1)$ by (2.2 c). On the other hand, we can extend the character:

$$Il(Np) = \{\mathfrak{a} \in Il \,|\, \mathfrak{a} + Np = \mathfrak{r}\} \ni \mathfrak{n} \mapsto \langle \mathfrak{n} \rangle \in \mathbf{H}_{k,w}(S(p^\infty); \mathcal{O})$$

to a continuous character of $Z(N) = \varprojlim_\alpha \mathrm{Cl}_F(Np^\alpha)$. Write $[n+2v]$ for m if $n + 2v = m \sum_\sigma \sigma$ with $m \in \mathbf{Z}$. Then, $\langle \mathfrak{n} \rangle$ acts on $\mathbf{M}_{k,w}(S(p^\alpha); \mathcal{O})$ via the multiplication of $\mathcal{N}(\mathfrak{n})^{[n+2v]}$ if $\mathfrak{n}$ is trivial in $\mathrm{Cl}_F(Np^\alpha)$. Since $\langle \xi \rangle$ for $\xi \in F^\times$ is the identity operator in $\mathbf{H}_{k,w}(S(p^\infty); \mathcal{O})$, the operator $\langle b \rangle$ for $b \in \mathfrak{r}_p^\times$ with $b \equiv 1 \bmod p^\alpha$ acts on $\mathbf{M}_{k,w}(S(p^\alpha); \mathcal{O})$ via the multiplication of b^{-n-2v}. Namely, for $\mathbf{f} \in \mathbf{M}_{k,w}(S(p^\alpha); \mathcal{O})$, we see

$$\mathbf{a}(y, \mathbf{f} | \langle b \rangle) = \mathbf{a}(y, b^{-n-2v} \mathbf{f} | b) = b^{-n-2v} \mathbf{a}(y, \mathbf{f}).$$

We now define $\mathbf{T}(a,b)$ by $\langle b \rangle \mathbf{T}(ab^{-1}, 1)$. Then we have

$$\mathbf{a}_p(y, \mathbf{f} | \mathbf{T}(a,b)) = \mathbf{a}_p(yab^{-1}, \mathbf{f} | \langle b \rangle) \quad \text{for} \quad a,\ b \in \mathfrak{r}_p^\times.$$

Since $\mathbf{a}(yb, \mathbf{f}) = \mathbf{a}(y, \mathbf{f})$ if $\mathbf{f} \in \mathbf{M}_{k,w}(S(p^\alpha); \mathcal{O})$ and $b \equiv 1 \bmod p^\alpha$, if $a \equiv b \equiv 1 \bmod p^\alpha$,

$$\mathbf{a}_p(y, \mathbf{f} | \mathbf{T}(a,b)) = \mathbf{a}_p(y, \mathbf{f} | \langle b \rangle) a^{-v} b^v = \mathbf{a}_p(y, \mathbf{f})(ab^{-1})^{-v} b^{-n-2v}.$$

It is then easy to prove (by using the above action of $\mathfrak{r}_p^\times$) that

$$\left\{ \sum_{k,w} \bar{\mathbf{M}}_{k,w}(S(p^\infty); K) \right\} \cap \mathscr{C}(\mathscr{I} \cup Z; \mathcal{O}) \cong \left\{ \bigoplus_{k,w} \bar{\mathbf{M}}_{k,w}(S(p^\infty); K) \right\} \cap \mathscr{C}(\mathscr{I} \cup Z; \mathcal{O}).$$

Namely the transition from Fourier expansion to q-expansion is injective on $\bigoplus_{k,w} \bar{\mathbf{M}}_{k,w}(S(p^\infty); K) = \sum_{k,w} \bar{\mathbf{M}}_{k,w}(S(p^\infty); K)$.

We now identify $\mathbf{G}_0 = \varprojlim_\alpha U_0(Np^\alpha)/(V_1(N) \cap U(p^\alpha))\mathfrak{r}^\times$ with a subgroup of $\mathbf{G} = \mathbf{G}(N) = Z(N) \times \mathfrak{r}_p^\times$ by the map:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto (b^{-1}, a_p^{-1} b_p).$$

Thus $(z,a) \in \mathbf{G}$ acts on $\mathbf{S}_{k,w}(U(Np^\alpha);\mathcal{O})$ via the operator $\langle z,a\rangle = T(a^{-1},1)\langle z\rangle$. Defining a new action by $\langle z,a\rangle_{n,v} = \mathcal{N}(z)^{-[n+2v]}\langle z,a\rangle$, we know that this new action factors through the finite quotient $\mathrm{Cl}(Np^\alpha) \times (\mathfrak{r}/p^\alpha\mathfrak{r})^\times$ of $\mathbf{G}$. For each pair of characters $\psi : \mathrm{Cl}_F(Np^\alpha) \to \mathcal{O}^\times$ and $\psi' : (\mathfrak{r}/p^\alpha\mathfrak{r})^\times \to \mathcal{O}^\times$, we define

$$\mathbf{S}_{k,w}(Np^\alpha,\psi',\psi;\mathcal{O}) = \{\mathbf{f} \in \mathbf{S}_{k,w}(U(Np^\alpha);\mathcal{O}) \mid \mathbf{f}|\langle z,a\rangle_{n,v} = \psi(z)\psi'(a)\mathbf{f} \text{ for } (z,a) \in \mathbf{G}(N)\},$$

where we understand $\psi(\mathfrak{n}) = \psi(n)$ for an idele n with $n_{N_p} = n_\infty = 1$ and $\mathfrak{n} = n\mathfrak{r}$. Summing up what we have shown, we get

COROLLARY 3.2. – *In $\mathcal{C}(\mathcal{I};\mathcal{O})$, the closure $\bar{\mathbf{S}}(S)$ of $\mathbf{S}_{k,w}(S(p^\infty);\mathcal{O})$ is independent of the weight (n,v) if $n \geqslant 0$ and $v \geqslant 0$ and the group $\mathbf{G}(N) = Z(N) \times \mathfrak{r}_p^\times$ acts on $\bar{\mathbf{S}}(S)$ continuously such that on $\mathbf{S}_{k,w}(Np^\alpha,\psi',\psi;\mathcal{O})$, (z,a) acts via the multiplication of $\psi(z)\psi'(a)\mathcal{N}(z)^{[n+2v]}a^v$.*

Since $\mathbf{h}(S;\mathcal{O})$ is a compact ring, we can decompose

$$\mathbf{h}(S;\mathcal{O}) = \mathbf{h}^{\mathrm{n.ord}}(S;\mathcal{O}) \oplus \mathbf{h}^{ss}(S;\mathcal{O}) \quad \text{as an algebra direct sum}$$

so that $\mathbf{T}(p)$ is a unit in $\mathbf{h}^{\mathrm{n.ord}}(S;\mathcal{O})$ and is topologically nilpotent in $\mathbf{h}^{ss}(S;\mathcal{O})$ (see [H2]). The idempotent e of the nearly ordinary part $\mathbf{h}^{\mathrm{n.ord}}(S;\mathcal{O})$ in $\mathbf{h}(S;\mathcal{O})$ has a simple expression:

$$e = \lim_{n\to\infty} \mathbf{T}(p)^{n!}$$

after projected down to $\mathbf{h}_{k,w}(S(p^\infty);\mathcal{O})$. Let $\bar{\mathbf{S}}^{\mathrm{n.ord}}(S) = \bar{\mathbf{S}}(S)|e$ (resp. $\bar{\mathbf{S}}^{\mathrm{n.ord}}(N) = \bar{\mathbf{S}}(N)|e$) be the nearly ordinary part of $\bar{\mathbf{S}}(S)$ (resp. $\bar{\mathbf{S}}(N)$). For any ideal $\mathfrak{a}$ of $\mathcal{A} = \mathcal{O}[[\mathbf{G}]]$, we define

$$\bar{\mathbf{S}}^{\mathrm{n.ord}}(N)[\mathfrak{a}] = \{\mathbf{f} \in \bar{\mathbf{S}}^{\mathrm{n.ord}}(N) \mid \mathbf{f}|g = 0 \text{ for all } g \in \mathfrak{a}\}.$$

We also write $\bar{\mathbf{S}}^{\mathrm{n.ord}}(N)[P]$ instead of $\bar{\mathbf{S}}^{\mathrm{n.ord}}(N)[\mathrm{Ker}(P)]$ for any $\mathcal{O}$-algebra holomorphism $P : \mathcal{O}[[\mathbf{G}]] \to \mathcal{O}$.

COROLLARY 3.3. – *Let $P_{n,v,\psi,\psi'}$ be the algebra holomorphism of $\mathcal{O}[[\mathbf{G}]]$ into $\mathcal{O}$ corresponding to the character :*

$$\mathbf{G} \ni (z,a) \mapsto \psi(z)\psi'(a)\mathcal{N}(z)^{[n+2v]}a^v \in \mathcal{O}^\times,$$

where (ψ,ψ') is a character modulo Np^α. Then we have

$$\bar{\mathbf{S}}^{\mathrm{n.ord}}(N)[P_{n,v,\psi,\psi'}] = \mathbf{S}_{k,w}^{\mathrm{n.ord}}(Np^\alpha,\psi',\psi;\mathcal{O})$$

under the identification : $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto (b^{-1}, a_p^{-1}b_p) \in \mathbf{G}$ *on* $\mathfrak{r}_{N_p}^\times \times \mathfrak{r}_p^\times$.

Proof. – By [H2] Th. 2.4, denoting the localization $P = P_{n,v,\psi,\psi'}$ of $\mathbf{h}^{\text{n.ord}}(N;\mathcal{O})$ by $\mathbf{h}^{\text{n.ord}}(\mathrm{N};\mathcal{O})_P$, we have natural isomorphism

$$\mathbf{h}^{\text{n.ord}}(\mathrm{N};\mathcal{O})_P/P\mathbf{h}^{\text{n.ord}}(N;\mathcal{O})_P \cong \mathbf{h}^{\text{n.ord}}_{k,w}(Np^{\alpha},\psi',\psi;K)$$

for the quotient field K, where $\mathbf{h}^{\text{n.ord}}_{k,w}(Np^{\alpha},\psi',\psi;K)$ is the nearly ordinary part of the Hecke algebra over K in $\mathrm{End}_K(\mathbf{S}_{k,w}(Np^{\alpha},\psi',\psi;K))$. Actually, in [H2], Th. 2.4, this fact is proven only when ψ' is the identity character but the argument given there can be applied to the general case. This implies the kernel Ker of the natural map

$$\rho : \mathbf{h}^{\text{n.ord}}(\mathrm{N};\mathcal{O})/P\mathbf{h}^{\text{n.ord}}(N;\mathcal{O}) \to \mathbf{h}^{\text{n.ord}}_{k,w}(Np^{\alpha},\psi',\psi;\mathcal{O})$$

is annihilated by an element X outside P. Since $\mathcal{O}[[\mathbf{G}]]/P \cong \mathcal{O}$, P is a prime ideal of height $\dim(\mathcal{O}[\mathbf{G}]]) - 1$. Thus for the unique maximal ideal $\mathfrak{m}$ of $\mathcal{O}[[\mathbf{G}]]$ containing P, there exists a positive integer m such that $P + X\mathcal{O}[[\mathbf{G}]] \supset \mathfrak{m}^m \ni p^m$. Thus Ker is annihilated by p^m. Since ρ is surjective and the image of ρ is $\mathcal{O}$-free, by dualizing ρ, we have an isomorphism

$$\rho^* : \mathbf{S}^{\text{n.ord}}_{k,w}(U_0(Np^{\alpha}),\psi',\psi;\mathcal{O}) \cong \mathbf{h}^{\text{n.ord}}_{k,w}(Np^{\alpha},\psi',\psi;\mathcal{O})^*$$
$$\cong (\mathbf{h}^{\text{n.ord}}(N;\mathcal{O})/P\mathbf{h}^{\text{n.ord}}(N;\mathcal{O}))^* \cong \bar{\mathbf{S}}^{\text{n.ord}}(N)[P_{n,v,\psi,\psi'}].$$

The last isomorphism follows from [H3], Prop. 7.3.

We now fix a decomposition $\mathbf{G}(N) = \mathbf{W} \times \mathbf{G}_{\text{tor}}(N)$ so that $\mathbf{W}$ is $\mathbf{Z}_p$-torsion free and $\mathbf{G}_{\text{tor}}(N)$ is a finite group. Then $\mathbf{W} \cong \mathbf{Z}_p^r$ for $r = [F:\mathbf{Q}] + 1 + \delta$, where δ is the defect of the Leopoldt conjecture for F (i.e. $0 \leqslant \delta < [F:\mathbf{Q}]$ and $\delta = 0$ if and only if the Leopoldt conjecture holds for F). Thus continuous group algebra $\mathbf{A} = \mathcal{O}[[\mathbf{W}]]$ is isomorphic to the r-variable power series ring over $\mathcal{O}$. Write μ (resp. $Z_{\text{tor}}(N)$) for the torsion part of $\mathfrak{r}_p^{\times}$ (resp. $Z(N)$) and decompose $Z(N) = W \times Z_{\text{tor}}(N)$ and $\mathfrak{r}_p^{\times} = W' \times \mu$ so that $\mathbf{W} = W \times W'$. Then $\mathbf{G}_{\text{tor}}(N) = Z_{\text{tor}}(N) \times \mu$. Let $\mathbf{L}$ be the quotient field of $\mathbf{A}$. We fix an algebraic closure $\bar{\mathbf{L}}$ of $\mathbf{L}$ and the embedding of $\bar{\mathbf{Q}}_p$ into $\bar{\mathbf{L}}$. Let λ: $\mathbf{h}^{\text{n.ord}}(N;\mathcal{O}) \to \bar{\mathbf{L}}$ be an $\mathbf{A}$-linear map. Since $\mathbf{h}^{\text{n.ord}}(N;\mathcal{O})$ is finite over $\mathbf{A}$([H2], Th. 2.4), the image of λ is contained in the integral closure $\mathbf{I}$ of $\mathbf{A}$ in a finite extension $\mathbf{K}$ of $\mathbf{L}$. Let $\mathcal{X}(\mathbf{I})$ be the p-adic space $\mathrm{Hom}_{\mathcal{O}-\text{alg}}(\mathbf{I},\bar{\mathbf{Q}}_p)$ and $\mathcal{A}(\mathbf{I})$ be the subset of $\mathcal{X}(\mathbf{I})$ consisting of $\mathcal{O}$-algebra homomorphisms which coincides with $P_{n,v,\psi,\psi'}$ on $\mathbf{A}$ for some $n \geqslant 0$,

$v \geqslant 0$ and ψ and ψ'. Then, by [H2], Th. 2.4, the composite $\lambda_P = P \circ \lambda$ for $P \in \mathscr{A}(\mathbf{I})$ induces an $\mathcal{O}$-linear map, for a suitable $\alpha > 0$,

$$\lambda_P : \mathbf{h}^{\mathrm{n.ord}}_{k,w}(Np^\alpha; \bar{\mathbf{Q}}_P) \to \bar{\mathbf{Q}}_P \quad \text{for} \quad k = n(P) + 2t \quad \text{and} \quad w = t - v(P)$$

and hence by the duality (Th. 3.1 and Cor. 3.3), we have a unique p-adic form $\mathbf{f}_P \in \mathbf{S}^{\mathrm{n.ord}}_{k,w}(Np^\alpha; \bar{\mathbf{Q}}_p)$ such that $\mathbf{a}_p(y, \mathbf{f}_P) = \lambda_P(\mathbf{T}(y))$ for all integral y. Thus any $\lambda \in \mathrm{Hom}_{\mathbf{A}}(\mathbf{h}^{\mathrm{n.ord}}(N; \mathcal{O}), \mathbf{I})$ gives rise to a family of modular forms parametrized by $\mathscr{A}(\mathbf{I})$. We thus call each $\mathbf{A}$-linear form λ as above an $\mathbf{I}$-adic form. Especially, if λ is an $\mathbf{A}$-algebra homomorphism, each specialization $\mathbf{f}_P$ is a common eigenform and is classical i.e. an element in $\mathbf{S}^{\mathrm{n.ord}}_{k,w}(Np^\alpha, \psi', \psi; \bar{\mathbf{Q}})$ coming from $\mathbf{S}^{\mathrm{n.ord}}_{k,w}(Np^\alpha, \psi', \psi; \mathbf{C})$. Since n, v, ψ and ψ' are determined by P, we write them as $n(P)$, $v(P)$, ψ_P and ψ'_P. When λ is an algebra homomorphism, we can compute ψ_P and ψ'_P explicitly as follows : Let ψ_0 (resp. ψ'_0) be the restriction of λ to $Z_{\mathrm{tor}}(N)$ (resp. μ). Then these maps are characters of $Z_{\mathrm{tor}}(N)$ and μ with values in $\bar{\mathbf{Q}}^\times$. Then

$$\psi_P = \varepsilon_P \psi_0 \omega^{-[n+2v]} \qquad \text{and} \qquad \psi'_P(\zeta) = \varepsilon'_P \psi'_0(\zeta)\, \zeta^{-v(P)} (\zeta \in \mu),$$

where ε_P (resp. ε'_P) is the restriction of $PP^{-1}_{n,v,\mathrm{id},\mathrm{id}}$ to W (resp. W') and ω is the Teichmüller character. We define an equivalence relation « $\approx$ » on the set of all $\mathbf{A}$-algebra homomorphisms: $\bigcup_N \mathrm{Hom}_{\mathbf{A}\text{-alg}}(\mathbf{h}^{\mathrm{n.ord}}(N; \mathcal{O}), \bar{\mathbf{L}})$ by $\lambda \approx \lambda'$ if and only if $\lambda(\mathbf{T}(\mathfrak{q})) = \lambda'(\mathbf{T}(\mathfrak{q}))$ for almost all prime ideals $\mathfrak{q}$ outside Np. Any element λ in equivalence class with minimal level C is called *primitive* and C is called the *conductor* of λ. Then in exactly the same manner as in [H1], Th. 3.6, we can prove

THEOREM 3.4. — *Let $\lambda : \mathbf{h}^{\mathrm{n.ord}}(N; \mathcal{O}) \to \bar{\mathbf{L}}$ be an $\mathbf{A}$-algebra homomorphism. Then the primitive homomorphism equivalent to λ is unique and its conductor C is a divisor of N. If λ itself is primitive, then for all $P \in \mathscr{A}(\mathbf{I})$, the conductor of λ_P in the sense of* [H1], (3.10 *a, b*) *is divisible by N. If the conductor of λ_P is divisible by every prime factor of p, then λ_P itself is primitive.*

4. Adelized Rankin-Selberg convolution.

We explain here how to obtain an integral expression over $G(\mathbf{Q})_+ \backslash G(\mathbf{A})_+ / UF^\times_{\mathbf{A}} C_{\infty+}$ of the standard L-function $\mathscr{D}(s, \mathbf{f}, \mathbf{g})$ in (0.1) for the algebraic group $G \times G$ $(G = \mathrm{Res}_{F/\mathbf{Q}} GL(2))$ for a suitable open compact subgroup U of $G(\mathbf{A}_f)$. This was done in the Hilbert modular case by Shimura [Sh1] in a way near to the original method of Rankin and in a more representation theoretic way by Jacquet [J]. However, to get a formulation optimal to our p-adic interpolation, we give a detailed exposition on this subject. Our approach is more adelic than [Sh1] but not as representation theoretic as [J]. Let B be a linear algebraic subgroup of $G = \mathrm{Res}_{F/\mathbf{Q}}(GL(2)_{/F})$ such that

$$B(A) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in (F \underset{\mathbf{Q}}{\otimes} A)^\times \quad \text{and} \quad b \in (F \underset{\mathbf{Q}}{\otimes} A) \right\}.$$

We put $B(\mathbf{A})_+ = \{ b \in B(\mathbf{A}) \mid \det(b_\infty) \gg 0 \}$ and $B(\mathbf{Q})_+ = B(\mathbf{Q}) \cap B(\mathbf{A})_+$.

Let N be an integral ideal prime to p. We simply write V for $V_1(N)$ and put $V(p^\alpha) = V \cap U(p^\alpha)$. For each pair of weight $(n,v) \in \mathbf{Z}[I]$ with $n + 2v \in \mathbf{Z}t\left(t = \sum_{\sigma \in I} \sigma\right)$, we put $w = t - v$ and $k = n + 2t$ and consider the space $\mathbf{M}_{k,w}(V(p^\alpha); \mathbf{C})$ of modular forms defined in [H1], § 2. Note that the group $\mathbf{G}$ acts on $\mathbf{M}_{k,w}(V(p^\alpha); \mathbf{C})$. In fact, $U_0(Np^r)/(V(p^r))\mathfrak{r}^\times$ is canonically isomorphic to $Z/Z_r \times (\mathfrak{r}/p^r\mathfrak{r})^\times$ via the correspondence:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto (b^{-1}, a_p^{-1} b_p),$$

where Z_r and Z are defined by the following exact sequences:

$$1 \to Z_r \to Z(N) \to \mathrm{Cl}_F(Np^r) \to 1, \quad 1 \to Z \to Z(N) \to \mathrm{Cl}_F(1) \to 1.$$

Especially $Z(N)$ acts via $\langle z \rangle_n$ as in [H1], (3.9) and the action of $(\mathfrak{r}/p^r\mathfrak{r})^\times$ is given by the right translation by $\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ for $a \in \mathfrak{r}_p^\times$. We write this action of $(z,a) \in \mathbf{G}$ on $\mathbf{f} \in \mathbf{M}_{k,w,J}(V(p^\alpha); \mathbf{C})$ as: $\mathbf{f} \mapsto \mathbf{f} | \langle z,a \rangle_{n,v}$. Let A be a subalgebra of either $\bar{\mathbf{Q}}_p$ or $\mathbf{C}$. We then define, for each pair of characters $\psi' : (\mathfrak{r}/p^r\mathfrak{r})^\times \to A^\times$ and $\psi : \mathrm{Cl}_F(Np^\alpha) \to A^\times$,

$$\mathbf{M}_{k,w}(Np^\alpha, \psi', \psi; A) = \{ \mathbf{f} \in \mathbf{M}_{k,w}(V(p^\alpha); A) \mid \mathbf{f} | \langle z,a \rangle_{n,v} = \psi(z)\psi'(a)\mathbf{f} \},$$
$$\mathbf{S}_{k,w}(Np^\alpha, \psi', \psi; A) = \{ \mathbf{f} \in \mathbf{S}_{k,w}(V(p^\alpha); A) \mid \mathbf{f} | \langle z,a \rangle_{n,v} = \psi(z)\psi'(a)\mathbf{f} \}.$$

As seen in § 2, we have the Hecke operators $T(\mathfrak{n})$ for integral ideal prime to p, $T(y)$ for $y \in \mathfrak{r}_p \cap F_p^\times$ and $T(a,b)$ for $(a,b) \in \mathfrak{r}_p^\times \times \mathfrak{r}_p^\times$ acting on $\mathbf{M}_{k,w}(Np^\alpha, \psi', \psi; A)$.

Now let (n,v) be as above and take another pair (ν,υ) of weights. Write $n + 2v = k - 2w = mt$ and $\nu + 2\upsilon = \mu t$ with integers m and μ. Define $k = n + 2t$, $w = t - v$ and $\kappa = \nu + 2t$, $\omega = t - \upsilon$. Write $\mathfrak{m}$ for Np^{α} and let $\mathbf{f} \in \mathbf{S}_{k,w}(\mathfrak{m},\chi',\chi;\mathbf{C})$ and $\mathbf{g} \in \mathbf{S}_{\kappa,\omega}(\mathfrak{m},\psi',\psi;\mathbf{C})$. Define $\mathbf{f}^*(x) = \overline{\mathbf{f}}_\rho(x)$ for $\mathbf{f}_\rho = \mathbf{f}^\rho$ as in (2.1) for complex conjugation ρ. Then

(4.1)

$$\mathbf{f}^*\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y|_{\mathbf{A}} \sum_{0 \ll \xi \in F} \mathbf{a}(\xi yd,\mathbf{f})\{(\xi yd)^{\upsilon}\}(\xi y_\infty)^{-\upsilon}\mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(-\xi x),$$

$$\mathbf{g}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y|_{\mathbf{A}} \sum_{0 \ll \xi \in F} \mathbf{a}(\xi yd,\mathbf{g})\{(\xi yd)^{\upsilon}\}(\xi y_\infty)^{-\upsilon}\mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(\xi x).$$

Write $j(u_\infty, z) = (c_\sigma z_\sigma + d_\sigma)_{\sigma \in I} \in \mathbf{C}^I$ for $u_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty$. Then, these functions satisfy the following automorphic property:

$$\mathbf{f}^*(\gamma x u) = \chi'_*(u)\chi^*(u)\mathbf{f}^*(x)j(u_\infty, z_0)^{-k\rho} \det(u_\infty)^w$$

and

$$\mathbf{g}(\gamma x u) = \psi'_*(u)\psi^*(u)\mathbf{g}(x)j(u_\infty, z_0)^{-\kappa} \det(u_\infty)^{\omega}$$

for $\gamma \in GL_2(F)$ and $u \in U_0(\mathfrak{m}) \cdot C_{\infty+}$, where we define χ^* and χ'_* as follows: The function $\chi^* : G(\mathbf{A}) \to \mathbf{C}$ is given by $\chi^*(x) = \chi_{\mathfrak{m}}(d_{\mathfrak{m}})$ or 0 for $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ according as $x \in B(\mathbf{A})U_0(\mathfrak{m})G_{\infty+}$ or not, where $\chi_{\mathfrak{m}}$ denotes the restriction of χ to $(F_{\mathfrak{m}})^\times = \left(\prod_{\mathfrak{p}|\mathfrak{m}} F_{\mathfrak{p}}\right)^\times$ for $\mathfrak{m} = Np^{\alpha}$. Similarly, the function $\chi'_* : G(\mathbf{A}) \to \mathbf{C}$ is given by $\chi'_*(x) = \chi'(d_p a_p^{-1})$ or 0 for x as above according as $x \in B(\mathbf{A})U_0(\mathfrak{m})G_{\infty+}$ or not.

Then we define the unitarization of $\mathbf{f}$ and $\mathbf{g}$ by

(4.2 a) $$\mathbf{f}^u(x) = |D|^{-(m/2)-1}\mathbf{f}^*(x)j(x_\infty, z_0)^{k\rho}|\det(x)|_{\mathbf{A}}^{m/2}$$

and

$$\mathbf{g}^u(x) = |D|^{-(\mu/2)-1}\mathbf{g}(x)j(x_\infty, z_0)^{\kappa}|\det(x)|_{\mathbf{A}}^{\mu/2}.$$

Then, for any $\gamma \in G(\mathbf{Q})$ and $u \in U_0(\mathfrak{m})C_{\infty+}$,

(4.2 b)

$$\mathbf{f}^u(\gamma x u) = \chi'_*(u)\chi^*(u)\mathbf{f}^u(x)j_\phi(\gamma, x_\infty(z_0))^{k\rho}|\det(u_\infty)|^{k/2} \operatorname{sgn}(\det(u_\infty)^w),$$

$$\mathbf{g}^u(\gamma x u) = \psi'_*(u)\psi^*(u)\mathbf{g}^u(x)j(\gamma, x_\infty(z_0))^{\kappa}|\det(u_\infty)|^{\kappa/2} \operatorname{sgn}(\det(u_\infty)^{\omega}),$$

where $\mathrm{sgn}(x) = x/|x|$ for $x \in \mathbf{R}^\times$. We then have

$$\text{(4.3 a)} \quad \mathbf{f}^u\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{0 \ll \xi \in F} a(\xi yd, \mathbf{f})(\xi y_\infty)^{k/2} \mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(-\xi x),$$

$$\mathbf{g}^u\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{0 \ll \xi \in F} a(\xi yd, \mathbf{g})(\xi y_\infty)^{\kappa/2} \mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(-\xi x),$$

where

$$\text{(4.3 b)} \quad a(y,\mathbf{f}) = \mathcal{N}_{F/\mathbf{Q}}(y\mathfrak{r})^{-(m/2)-1}\{y\mathfrak{r}^v\}\mathbf{a}(y,\mathbf{f})$$

and

$$a(y,\mathbf{g}) = \mathcal{N}_{F/\mathbf{Q}}(y\mathfrak{r})^{-(\mu/2)-1}\{y\mathfrak{r}^\upsilon\}\mathbf{a}(y,\mathbf{g}) \quad \text{(see [H1], (4.1)).}$$

We now write $\Phi(\mathbf{f},\mathbf{g})(x)$ for

$$\text{(4.4 a)} \qquad \mathbf{f}^u(x)\mathbf{g}^u(x)|j(x_\infty, z_0)^{-k-\kappa}|.$$

Then we have for $\gamma \in GL_2(F)$ and $u \in U_0(\mathfrak{m})C_{\infty+}$

$$\text{(4.4 b)} \quad \Phi(\mathbf{f},\mathbf{g})(\gamma x u) = \chi'_*\psi'_*\chi^*\psi^*(u)\Phi(\mathbf{f},\mathbf{g})(x)j(\gamma, x_\infty(z_0))^{\kappa-k}|j(\gamma,x_\infty(z_0))^{k-\kappa}|.$$

We now assume the following condition:

(4.5) *There is a finite order idele character* $\theta : \mathrm{Cl}_F(\mathfrak{m}) \to \mathbf{C}^\times$ *such that* $\chi'\psi' = \theta$ *on* $\mathfrak{r}_\mathfrak{m}^\times$.

By (2.2 c), we have for $u \in \mathfrak{r}_\mathfrak{m}^\times$,

$$a(yu,\mathbf{f}) = a(y,\mathbf{f}|T(u,1)) = \chi'(u)^{-1}a(y,\mathbf{f}).$$

Thus the function $a(y,\mathbf{g})a(y,\mathbf{f})\theta(y)$ is invariant under the multiplication of units in $\mathfrak{r}_\mathfrak{m}^\times$.

We now explain how we normalize the Haar measures. We take the multiplicative Haar measure $d\mu_f^\times$ on $F_{\mathbf{A}_f}^\times$ such that the volume of $\hat{\mathfrak{r}}^\times$ is equal to 1. We take the Lebesgue measure $d\mu_\infty$ on $F_\infty = \mathbf{R}^I$ and define the multiplicative measure $d\mu_\infty^\times(y)$ by $|y_\infty^{-t}|\, d\mu_\infty(y)$. Then we take the product measure $d^\times y = d\mu_f^\times(y)\, d\mu_\infty^\times(y)$ on $F_\mathbf{A}^\times$. As the additive measure on $F_{\mathbf{A}_f}$, we take the measure $d\mu_f$ having volume 1 on $\hat{\mathfrak{r}}$ and define $dx = d\mu_f(x)\, d\mu_\infty(x)$.

We now consider the following integral for the idele character θ in (4.5), which is absolutely convergent if Re (s) is sufficiently large:

$$Z(s,\mathbf{f},\mathbf{g},\theta) = \int_{F_{\mathbf{A}}{}_{+}^{\times}/F_{+}} \int_{F_{\mathbf{A}}/F} \Phi(\mathbf{f},\mathbf{g})\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) dx \theta(y) |y|_{\mathbf{A}}^{s} d^{\times}y$$

$$= D^{(1+2s)/}\theta(d)^{-1} \int_{F_{\mathbf{A}}{}_{+}^{\times}} a(y,\mathbf{f}) a(y,\mathbf{g}) \mathbf{e}_F(2iy_\infty) y_\infty^{(k+\kappa)/2} \theta(y) |y|_{\mathbf{A}}^{s} d^{\times}y,$$

where $D = |d|_{\mathbf{A}}^{-1}$ is the absolute discriminant of $F/\mathbf{Q}$ and to obtain the equality, we have used the classical fact that (e.g. [W] p. 91, Prop.7) $\int_{F_{\mathbf{A}}/F} \mathbf{e}_F((\eta - \xi)x)\,dx = \sqrt{D}$ or 0 according as $\xi = \eta$ or not. Let $C(\theta)$ be the conductor of θ and we use the same letter θ for the ideal character associated with θ; namely, if $\mathfrak{a}$ is prime to $C(\theta)$, then $\theta(\mathfrak{a}) = \theta(\alpha)$ for $\alpha \in F_{\mathbf{A}}^{\times}$ with $\alpha_{C(\theta)} = \alpha_\infty = 1$ and $\mathfrak{a} = \alpha\mathfrak{r}$, and if $\mathfrak{a}$ is not prime to $C(\theta)$, then we agree to put $\theta(\mathfrak{a}) = 0$. Then we have, if Re(s) is sufficiently large,

(4.6) $Z(s,\mathbf{f},\mathbf{g},\theta)$

$$= D^{(1+2s)/2}\theta(d)^{-1}(4\pi)^{-ts-(k+\kappa)/2}\Gamma_F\left(st + \frac{k}{2} + \frac{\kappa}{2}\right) D(s,\mathbf{f},\mathbf{g},\theta),$$

where

$$D(s,\mathbf{f},\mathbf{g},\theta) = \sum_{\mathfrak{r} \supset \mathfrak{a} \neq 0} a(\mathfrak{a},\mathbf{f}) a(\mathfrak{a},\mathbf{g}) \theta(\mathfrak{a}) \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})^{-s}.$$

We shall compute the integral in a little different way: First of all, we note that on $B(\mathbf{A})_+$ the function χ^* and ψ^* are constant with value 1. We write $C_{\infty+}$ for the stabilizer in $G_{\infty+}$ of the point $z_0 = (\sqrt{-1}, \ldots, \sqrt{-1})$ in $\mathscr{Z} = F_\infty + \sqrt{-1}F_{\infty+}^{\times}$ identifying F_∞ with $\mathbf{R}^I$. We also define a function $\eta : G(\mathbf{A}) \to \mathbf{C}$ and θ: $G(\mathbf{A}) \to \mathbf{C}$ by $\eta(x) = |y|_{\mathbf{A}}$ and $\theta(x) = \theta(y)$ if $x = \begin{pmatrix} y & * \\ 0 & 1 \end{pmatrix} au$ or $a \in F_{\mathbf{A}}^{\times}$ and $u \in U(\mathfrak{m})C_{\infty+}$ for ideal $\mathfrak{m} = Np^\alpha$ divisible by $C(\theta)$ and otherwise $\eta(x) = \theta(x) = 0$. Under the assumption (4.5), we know that $\Phi(\mathbf{f},\mathbf{g})(x)\overline{(\chi*\psi*)}(x)\theta(x)$ is invariant under $U_0(\mathfrak{m})C_{\infty+}$ from the right and invariant under $B(\mathbf{Q})_+$ from the left. Thus we can write

$$Z(s,\mathbf{f},\mathbf{g},\theta) = \int_{B(\mathbf{Q})_+\backslash B(\mathbf{A})_+} \Phi(\mathbf{f},\mathbf{g})(x)\overline{(\chi*\psi*)}(x)\theta(x)\eta(x)^{s+1} d\mu_B(x),$$

where $d\mu_B\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y|_{\mathbf{A}}^{-1}\, dx\, d^\times y$ is the (left invariant) Haar measure on $B(\mathbf{A})$. Let U be an open compact subgroup of $G(\mathbf{A}_f)$. Let $d\mu_\infty$ be the Haar measure on the compact group $C_{\infty+}/Z_\infty$ (for the center Z_∞ of $G_{\infty+}$) with volume 1. Then we define a measure μ_U on $B(\mathbf{A})_+ UC_{\infty+}$ such that

$$\int_{B(\mathbf{A})_+ UC_{\infty+}/Z_\infty} \varphi(x)\, d\mu_U(x) = \int_{B(\mathbf{A})_+} \int_{UC_{\infty+}/Z_\infty} \varphi(bu)\, d\mu_0(u)\, d\mu_B(b)$$

for all functions φ on $B(\mathbf{A})_+ UC_{\infty+}$, where $d\mu_0$ is the product measure on $UC_{\infty+}/Z_\infty$ of the Haar measure on the compact group U with volume 1 and the measure $d\mu_\infty$ on $C_{\infty+}/Z_\infty$. This measure induces a measure on $G(\mathbf{Q})_+\backslash G(\mathbf{A})_+/UC_{\infty+}$. In fact, by taking a fundamental domain Φ of $G(\mathbf{Q})_+$ in $B(\mathbf{A})_+ UC_{\infty+}/UC_{\infty+}$, we define

$$\int_{G(\mathbf{Q})_+\backslash G(\mathbf{A})_+/UC_{\infty+}} f(x)\, d\mu_U(x) = \int_\Phi f(x)\, d\mu_U(x).$$

This is possible because of $G(\mathbf{A})_+ = G(\mathbf{Q})_+ \cdot B(\mathbf{A})_+ \cdot U \cdot C_{\infty+}$. The measure we have constructed depends on the choice of U in the following sense: If one takes an open compact subgroup U' of U, then for any right U-invariant function f,

$$\int_{G(\mathbf{Q})_+\backslash G(\mathbf{A})_+/U'C_{\infty+}} f(x)\, d\mu_{U'}(x) = [U{:}U'] \int_{G(\mathbf{Q})_+\backslash G(\mathbf{A}_+/UC_{\infty+}} f(x)\, d\mu_U x),$$

because $d\mu_0$ for U' is $d\mu_0$ for U multiplied the index $[U;U']$. Let $E(U) = F^\times \cap UC_{\infty+}$ and φ be a function on $B(\mathbf{Q})_+\backslash G_+(\mathbf{A})/UC_{\infty+}$ such that φ is supported by $B(\mathbf{A})_+ UC_{\infty+} = B(\mathbf{A}_f)UG_{\infty+}$. Then we have

$$\int_{G(\mathbf{Q})_+\backslash G(\mathbf{A})_+/UC_{\infty+}} \sum_{\gamma \in E(U)B(\mathbf{Q})_+\backslash G(\mathbf{Q})_+} \varphi(\gamma x)\, d\mu_U(x)$$

$$= \int_{E(U)B(\mathbf{Q})_+\backslash B(\mathbf{A})_+ UC_{\infty+}/UC_{\infty+}} \varphi(x)\, d\mu_U(x) = \int_{B(\mathbf{Q})_+\backslash B(\mathbf{A})_+} \varphi(x)\, d\mu_B(x).$$

By applying this argument to $U = U_0(\mathfrak{m})$ and $E(U) = \mathfrak{r}^\times$, we thus have, writing $d\mu_U$ as $d\mu_{\mathfrak{m}}$,

$$(4.7)\quad Z(s,\mathbf{f},\mathbf{g},\theta) = D^{-2-(m+\mu)/2} \int_{X_0} \mathbf{f}^*\mathbf{g}(x)\,|\det(x)|_{\mathbf{A}}^{(m+\mu)/2} \times \sum_i \chi\psi(a_i)\,\mathscr{E}(a_i x)\, j(x_\infty, z_0)^{\kappa-k} |j(x_\infty, z_0)^{k-\kappa}|\, 8 d\mu_{\mathfrak{m}}(x),$$

where $X_0 = X_0(\mathfrak{m}) = G(\mathbf{Q})_+\backslash G(\mathbf{A})_+/U_0(\mathfrak{m})F_{\mathbf{A}}^\times C_{\infty+}, F_{\mathbf{A}}^\times = \bigcup_{i=1}^{h} F^\times a_i \mathfrak{r}^\times F_\infty^\times$ and we used the Eisenstein series $\mathscr{E}(x) = \mathscr{E}_{\kappa-k}(x,\chi^{-1}\psi^{-1},\theta;s+1)$ defined by:

(4.8 a) $\mathscr{E}_k(x,\chi,\theta;s)$
$$= \sum_{\gamma\in\mathfrak{r}^\times B_+(\mathbf{Q})\backslash G(\mathbf{Q})_+} \chi^*(\gamma x)\theta(\gamma x)\eta(\gamma x)^s |j(\gamma,x_\infty(z_0))^{-k}| j(\gamma,x_\infty(z_0))^k .$$

We normalize this Eisenstein series in the following way:

(4.8 b) $E_k^*(u,\chi,\theta;s) = \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{m})\sqrt{D}\sum_{i=1}^{h} \chi^{-1}(a_i)\mathscr{E}_k(a_i u,\chi,\theta;s)$,

(4.8 c) $E_k(u,\chi,\theta;s) = L_{\mathfrak{m}}(2s,\chi^{-1}\theta^2)E_k^*(u,\chi,\theta;s)$.

Moreover we choose a finite idele m such that $m\mathfrak{r} = \mathfrak{m}$ and put $\tau(m) = \begin{pmatrix} 0 & -1 \\ m & 0 \end{pmatrix} = G(\mathbf{A}_f)$. Then we define another Eisenstein series by

(4.8 d) $G_k(u,\chi,\theta;s) = \theta^{-1}(m)\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{m})^{s-1}\chi(\det(u))E_k(u\tau(m),\chi^{-1},\theta;s)$,

whose Fourier coefficients are much easier to compute (see § 6).

We now want to modify the above integral in terms of the newly defined Eisenstein series. In fact, we first define the Petersson inner product for $\mathbf{h}\in\mathbf{M}_{k,w}(\mathfrak{m},\chi',\chi;\mathbf{C})$ and $\mathbf{f}\in\mathbf{S}_{k,w}(\mathfrak{m},\chi',\chi;\mathbf{C})$ by

$$(\mathbf{f},\mathbf{h})_{\mathfrak{m}} = \int_{X_0(\mathfrak{m})} \overline{\mathbf{f}(x)}\mathbf{h}(x)|\det(x)|_A^m d\mu_{\mathfrak{m}}(x).$$

We define $\mathbf{h}|\tau\in\mathbf{M}_{k,w}(\mathfrak{m},\chi'^{-1},\chi^{-1};\mathbf{C})$ for $\tau = \tau(m)$ by

$$\mathbf{h}|\tau(x) = \chi^{-1}(\det(x))\mathbf{h}(x\tau(m)).$$

Then we see easily that $(h,\mathbf{g})_{\mathfrak{m}} = \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{m})^{-m}(h|\tau,\mathbf{g}|\tau)_{\mathfrak{m}}$. We put

$\mathbf{E}_{k-\kappa,w-\omega}(x,\chi^{-1}\psi^{-1},\theta;s)$
$$= |\det(x)|_{\mathbf{A}}^{(\mu-m')/2}E_{\kappa-k,I}(x,\chi^{-1}\psi^{-1},\theta;s)j(x_\infty,z_0)^{\kappa-k}|j(x_\infty,z_0)^{k-\kappa}|,$$

(4.8 e) $\mathbf{G}_{k-\kappa,w-\omega}(x,\chi\psi,\theta;s)$
$$= \theta^{-1}(m)\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{m})^{s-1+(\mu-m)/2}\mathbf{E}_{k-\kappa,w-\omega}(x,\chi^{-1}\psi^{-1},\theta;s)|\tau(m).$$

Then we finally have

$$(4.9)\quad L_{\mathfrak{m}}(2s+2,\chi\psi\theta^2)Z(s,\mathbf{f},\mathbf{g},\theta)$$
$$= \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{m})^{-1}D^{-(5+m+\mu)/2}(\mathbf{f}_\rho,\mathbf{g}\mathbf{E}_{k-\kappa,w-\omega}(x,\chi^{-1}\psi^{-1},\theta;s+1))_{\mathfrak{m}}$$
$$= D^{-(5+m+\mu)/2}\theta(m)\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{m})^{-s-1-(\mu+m)/2}$$
$$\times\ (f_\rho|\tau,(\mathbf{g}|\tau)\mathbf{G}_{k-\kappa,w-\omega}(x,\chi\psi,\theta;s+1))_{\mathfrak{m}}.$$

For our later use, we introduce here the standard L-function for a primitive form $\mathbf{f}: L(s,\mathbf{f}) = \sum_{\mathfrak{a}} a(\mathfrak{a},\mathbf{f})N(\mathfrak{a})^{-s}$, where $a(\mathfrak{a},\mathbf{f})$ is the Fourier coefficient of the unitarization of $\mathbf{f}$. Suppose that $\mathbf{f}$ is a primitive form of conductor N, of weight (k,w) and with central character ψ ant let π be the unitary automorphic representation of $G(\mathbf{A})$ generated by the right translation of $\mathbf{f}^u$. Choose a finite idele n such that $N = n\mathfrak{r}$ and $nn_N^{-1} = 1$ and define $\tau = \tau(n) \in G(\mathbf{A}_f)$ by $\tau = \begin{pmatrix} 0 & -1 \\ n & 0\end{pmatrix}$. Then it is well known that $\mathbf{f}^u|\tau(n) = W(\mathbf{f})(\mathbf{f}^u)^\rho$ for a constant $W(\mathbf{f})$ with absolute value 1. We now want to relate $W(\mathbf{f})$ with the root number of the functional equation of $L(s,\mathbf{f})$. We consider the integral

$$\mathscr{Z}(s,\mathbf{f}) = \int_{F_{\mathbf{A}}^\times/F_+^\times} \mathbf{f}^u\left(\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\right)|y|_{\mathbf{A}}^s d^\times y = D^s G_\infty(s)L(s,\mathbf{f}),$$

where $G_\infty(s) = (2\pi)^{(k/2)+st}\Gamma_F(st+(k/2))\left(t=\sum_\sigma \sigma\right)$. For $\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$, we have $\varepsilon\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\tau = (-1)_f y\begin{pmatrix} ny^{-1} & 0 \\ 0 & 1\end{pmatrix}\varepsilon$. Since $\varepsilon \in C_{\infty+}$, we see that $\mathbf{f}^u\left(\varepsilon\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\tau\right) = \psi_\infty(-1)\psi(y)i^{-k}\mathbf{f}^u\left(\begin{pmatrix} ny^{-1} & 0 \\ 0 & 1\end{pmatrix}\right)$. This shows that

$$W(\mathbf{f})(\mathbf{f}^u)^\rho\left(\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\right) = \psi_\infty(-1)i^{-k}\mathbf{f}^u\left(\begin{pmatrix} ny^{-1} & 0 \\ 0 & 1\end{pmatrix}\right).$$

Therefore we see

$$W(\mathbf{f})\mathscr{Z}(s,\mathbf{f}^\rho) = \psi^\infty(-1)i^{-k}|n|_{\mathbf{A}}^s\mathscr{Z}(-s,\mathbf{f}).$$

This implies that

$$(4.10\text{ a})\quad G_\infty(s)L(s,\mathbf{f}) = i^{-k}W(\mathbf{f})|d^2n|_{\mathbf{A}}^s G_\infty(-s)L(-s,\mathbf{f}^\rho),$$

because $\psi_\infty(-1) = (-1)^k$. Namely the root number of the functional equation is given by $i^{-k}W(\mathbf{f})$. By an easy computation, we have

$$(4.10\text{ b})\qquad \mathbf{f}|\tau = W(\mathbf{f})|n|_{\mathbf{A}}^{-m/2}\mathbf{f}^\rho,$$

where $mt = k - 2w$. Let $\tau = \bigotimes_{\mathfrak{q}} \pi_{\mathfrak{q}}$ be the automorphic representation generated by the right translations of $\mathbf{f}$. As is well known, the constant term of functional equation $i^{-k}W(\mathbf{f})$ can be decomposed into the product of local factors (e.g. [H3], (5.4-5)). We assume that $\pi_{\mathfrak{p}}$ for prime factors $\mathfrak{p}$ of p is either principal or special. For each quasi-character $\xi : F_{\mathfrak{p}}^{\times} \to \mathbf{C}^{\times}$, we define

$$\varepsilon(\xi) = \xi(d\varpi^r)G(\xi^{-1})/|\xi(d\varpi^r)G(\xi^{-1})| \ (G(\xi^{-1}) = \sum_{u \bmod \varpi^r} \xi^{-1}(u)\mathbf{e}_{F_{\mathfrak{p}}}(u/\varpi^r d)),$$

where $\mathbf{e}_{F_{\mathfrak{p}}}(x) = \exp(-2\pi i[\mathrm{Tr}_{F_{\mathfrak{p}}/\mathbf{Q}_p}(x)]_p)$ for the p-fractional part $[y]_p$ of $y \in \mathbf{Q}_p$, ϖ is a prime element if $F_{\mathfrak{p}}$, $\mathfrak{p}^r$ is the conductor of ξ and $dr_{\mathfrak{p}}$ is the different of $F_{\mathfrak{p}}$ for $d \in F_{\mathfrak{p}}$. We understand that $\varepsilon(\xi) = \xi(d)/|\xi(d)|$ if ξ is unramified. Then we can decompose

$$W(\mathbf{f}) = W'(\mathbf{f}) \prod_{\mathfrak{p}|p} W_{\mathfrak{p}}(\mathbf{f}), \tag{4.10 c}$$

where

$$W_{\mathfrak{p}}(\mathbf{f}) = \varepsilon(\xi)\varepsilon(\xi') \text{ if either } \pi_{\mathfrak{p}} = \sigma(\xi,\xi') \text{ with ramified } \xi \text{ or } \pi(\xi,\xi'),$$
$$W_{\mathfrak{p}}(\mathbf{f}) = -\varepsilon(\xi)\varepsilon(\xi')(\xi(\varpi)/|\xi(\varpi)| \text{ if } \pi_{\mathfrak{p}} = \sigma(\xi,\xi') \text{ with unramified } \xi.$$

By abusing a little the language, for any form $\mathbf{f}'$ which spans the same automorphic representation as $\mathbf{f}$, we write $W(\mathbf{f})'$, and $W'(\mathbf{f}')$ for $W(\mathbf{f})$ and $W'(\mathbf{f})$, respectively.

5. Statement of the main result and proof of Theorem I.

We fix two ideals N and J prime to p in $\mathfrak{r}$ and let L be the least common multiple of N and J, i.e. $L = N \cap J$. We consider two primitive $\mathbf{A}$-algebra homomorphisms $\lambda : \mathbf{h}^{\mathrm{n.ord}}(N;\mathcal{O}) \to \mathbf{I}$ and $\varphi : \mathbf{h}^{\mathrm{n.ord}}(J;\mathcal{O}) \to \mathbf{J}$, where $\mathbf{I}$ (resp. $\mathbf{J}$) is the integral closure of $\mathbf{A}$ in a finite extension $\mathbf{K}$ (resp. $\mathbf{M}$) of the quotient fiels $\mathbf{L}$ of $\mathbf{A}$. By extending scalar if necessary, we may assume

$$\mathbf{I} \cap \overline{\mathbf{Q}}_p = \mathbf{J} \cap \overline{\mathbf{Q}}_p = \mathcal{O}. \tag{5.1}$$

We now introduce the congruence module for λ [H6], § 6. Let (ψ,ψ') be the restriction of λ to $\mathbf{G}_{\mathrm{tor}}(N)$(i.e., $\psi : Z_{\mathrm{tor}}(J) \to \mathcal{O}^{\times}$ and $\psi' : \mu \to \mathcal{O}^{\times}$). We can decompose $\mathbf{h}^{\mathrm{n.ord}}(N;\mathcal{O}) \otimes_{\mathcal{O}} K = H(\psi,\psi') \oplus B$ so that $H(\psi,\psi')$ is the maximal quotient of $\mathbf{h}^{\mathrm{n.ord}}(N;\mathcal{O}) \otimes_{\mathcal{O}} K$ on which $\mathbf{G}_{\mathrm{tor}}$ acts via (ψ,ψ'). Let $\mathbf{h}(\psi,\psi')$ be the image of $\mathbf{h}^{\mathrm{n.ord}}(N;\mathcal{O})$ in $H(\psi,\psi')$. Then λ factors through $\mathbf{h}(\psi,\psi')$. Combining $\lambda \otimes \mathrm{id} : \mathbf{h}(\psi,\psi') \otimes_{\mathbf{A}} \mathbf{I} \to \mathbf{I} \otimes_{\mathbf{A}} \mathbf{I}$ with the multiplication : $\mathbf{I} \otimes_{\mathbf{A}} \mathbf{I} \to \mathbf{I}$, we have an $\mathbf{I}$-algebra homomorphism $\hat{\lambda} : \mathbf{h}(\psi,\psi') \otimes_{\mathbf{A}} \mathbf{I} \to \mathbf{I}$. It is known (cf. [H2], Th. 2.4) that $\hat{\lambda}$ induces an $\mathbf{I}$-algebra decomposition :

$$\mathbf{h}(\psi,\psi') \otimes_{\mathbf{A}} \mathbf{K} \cong \mathbf{K} \oplus \mathbf{B}$$

with a complementary direct summand $\mathbf{B}$ so that $\hat{\lambda}$ induces the projection into the first direct summand $\mathbf{K}$. Let $\mathbf{h}(\mathbf{K})$ (resp. $\mathbf{h}(\mathbf{B})$) be the projection of $R = \mathbf{h}(\psi,\psi') \otimes_{\mathbf{A}} \mathbf{I}$ in $\mathbf{K}$ (resp. $\mathbf{B}$). Then, the congruence module is defined by

$$\text{(5.2)} \qquad \mathscr{C}(\lambda;\mathbf{I}) = (\mathbf{h}(\mathbf{K}) \oplus \mathbf{h}(\mathbf{B}))/R \cong \mathbf{h}(\mathbf{K}) \otimes_R \mathbf{h}(\mathbf{B}),$$

which is a torsion cyclic $\mathbf{I}$-module (i.e. $\mathscr{C}(\lambda;\mathbf{I}) \cong \mathbf{I}/\mathfrak{a}$ with a non-trivial ideal $\mathfrak{a}$; see [H6], Lemma 5.2).

Let $\mathbf{f}_P$ (resp. $\mathbf{g}_Q$) be the common eigenform corresponding to $P \in \mathscr{A}(\mathbf{I})$ (resp. $Q \in \mathscr{A}(\mathbf{J})$) belonging to λ (resp. φ) as explained in § 3. To describe these modular forms explicitly, we write Σ for the set of all prime factors of p in $\mathfrak{r}$. For each Σ-tuple $\alpha = (\alpha(\mathfrak{p}))_{\mathfrak{p} \in \Sigma}$ of integers $\alpha(\mathfrak{p})$, we write $\mathfrak{p}^{\alpha} = \prod_{\mathfrak{p} \in \Sigma} \mathfrak{p}^{\alpha(\mathfrak{p})}$. Let (χ,χ') (resp. ψ,ψ') be the restriction of φ (resp. λ) to $\mathbf{G}_{\mathrm{tor}}(J)$ (resp. $\mathbf{G}_{\mathrm{tor}}(N)$) ; i.e., $\chi, \psi : Z_{\mathrm{tor}}(J) \to \mathcal{O}^{\times}$ and $\chi', \psi' : \mu \to \mathcal{O}^{\times}$. Then as seen in § 3, for sufficiently large α,

$$\mathbf{f}_P \in \mathbf{S}^{\mathrm{n.ord}}_{k,w}(N\mathfrak{p}^{\alpha},\psi'_P,\psi_P;\bar{\mathbf{Q}}) \quad \text{and} \quad \mathbf{g}_Q \in \mathbf{S}^{\mathrm{n.ord}}_{\kappa,\omega}(J\mathfrak{p}^{\alpha},\chi'_Q,\chi_S;\bar{\mathbf{Q}})$$

where $\kappa = n(Q) + 2t$, $\omega = t - v(Q)$, $m(Q)t = n(Q) + 2v(Q)$ and $k = n(P) + 2t$, $w = t - v(P)$, $m(P)t = n(P) + 2v(P)$ and

$$\psi_P = \varepsilon_P \psi \omega^{-m(P)} \quad \text{and} \quad \psi'_P(\zeta,w) = \varepsilon'_P(w)\psi'(\zeta)\zeta^{-v(P)} \quad \text{for} \quad \zeta \in \mu,$$
$$\chi_Q = \varepsilon_Q \chi \omega^{-m(Q)} \quad \text{and} \quad \chi'_Q(\zeta,w) = \varepsilon'_Q(w)\chi'(\zeta)\zeta^{-v(Q)} \quad \text{for} \quad \zeta \in \mu.$$

We define the unitarization $\mathbf{f}^u_P \in \mathbf{S}_{k,k/2}(N\mathfrak{p}^{\alpha},\psi'_P,\psi_P;\bar{\mathbf{Q}})$ by

$$\mathbf{f}^u(x) = D^{-(m(P)/2)-1}\,\mathbf{f}(x) j(x_\infty, z_0)^k |\det(x)|_{\mathbf{A}}^{m(P)/2}.$$

Let $\varpi_{\mathfrak{p}}$ be the prime element in $\mathfrak{r}_{\mathfrak{p}}$ for $\mathfrak{p} \in \Sigma$ and let n (resp. j) be a finite idele such that $n\mathfrak{r} = N$ (resp. $j\mathfrak{r} = J$) and $n_{\mathfrak{q}} = 1$ (resp. $j_{\mathfrak{q}} = 1$) outside N (resp. J). Define an idele $n\varpi^{\alpha}$ so that $(n\varpi^{\alpha})_{\mathfrak{q}} = 1$ for $\mathfrak{q}$ outside Np, $(n\varpi^{\alpha})_{\mathfrak{q}} = n_{\mathfrak{q}}$ for prime $\mathfrak{q}$ dividing N and $(n\varpi^{\alpha})_{\mathfrak{p}} = \varpi_{\mathfrak{p}}^{\alpha(\mathfrak{p})}$ for $\mathfrak{p} \in \Sigma$. We then define $\tau(n\varpi^{\alpha})$ by

$$\tau(n\varpi^{\alpha}) = \begin{pmatrix} 0 & -1 \\ n\varpi^{\alpha} & 0 \end{pmatrix} \in G(\mathbf{A}_f).$$

Let ξ be a finite order character of $\mathrm{Cl}(N\mathfrak{p}^{\alpha})$. For each modular form $\mathbf{f}$ in $\mathbf{M}_{k,w}(Np^{\alpha},\xi',\xi;\bar{\mathbf{Q}})$ (i.e. $\mathbf{f}(xz) = \xi(z)|z|_{\mathbf{A}}^{-m}\mathbf{f}(x)$ for $mt = k-2w$), we define

$$\mathbf{f}|\tau(n\varpi^{\alpha})(x) = \xi^{-1}(\det(x))\mathbf{f}(x\tau(n\varpi^{\alpha})) \in \mathbf{M}_{k,w}(U_0(N\mathfrak{p}^{\alpha}),\xi'^{-1},\xi^{-1};\bar{\mathbf{Q}}).$$

As we will see in § 7, for each finite order character θ of $\mathrm{Cl}(\mathfrak{p}^{\delta})$, we can define the twisting operator $\mathbf{f} \mapsto \mathbf{f}|\theta$ by $\mathbf{a}(y,\mathbf{f}|\theta) = \theta(y\mathfrak{r})\mathbf{a}(y,\mathbf{f})$. Here we understand $\theta(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not prime to p. This operator sends $\mathbf{M}_{k,w}(U_0(N\mathfrak{p}^{\alpha}),\xi',\xi;\bar{\mathbf{Q}})$ into $\mathbf{M}_{k,w}(U_0(N\mathfrak{p}^{\alpha+2\delta}),\xi',\xi\theta^2;\bar{\mathbf{Q}})$. We can define a similar but different operator $\mathbf{f} \mapsto \mathbf{f} \otimes \theta$ by $\mathbf{a}(y,\mathbf{f}\otimes\theta) = \theta(y_f)\mathbf{a}(y,\mathbf{f})$, where $\theta(y_f)$ is the value at y_f of the idele character θ. This operator sends $\mathbf{M}_{k,w}(U_0(N\mathfrak{p}^{\alpha}),\xi',\xi;\bar{\mathbf{Q}})$ into $\mathbf{M}_{k,w}(U_0(N\mathfrak{p}^{\alpha+2\delta}),\xi'\theta_p^{-1},\xi\theta^2;\bar{\mathbf{Q}})$. See § 7 for details about these operators.

Let $\mathbf{I} \hat{\otimes}_{\mathcal{O}} \mathbf{J}$ be the profinite completion of $\mathbf{I} \otimes_{\mathcal{O}} \mathbf{J}$ (i.e. the $\mathfrak{m}$-adic completion of $\mathbf{I} \otimes \mathbf{J}$ for the unique maximal ideal $\mathfrak{m}$ of $\mathbf{I}\otimes\mathbf{J}$). Under (5.1), $\mathbf{I} \hat{\otimes}_{\mathcal{O}} \mathbf{J}$ is an integral domain. For each pair of points (P,Q) in $\mathcal{X}(\mathbf{I}) \times \mathcal{X}(\mathbf{J})$, regarding P and Q as $\mathcal{O}$-algebra homomorphisms of $\mathbf{I}$ and $\mathbf{J}$ into $\bar{\mathbf{Q}}_p$, we have an $\mathcal{O}$-algebra homomorphism $P \otimes Q : \mathbf{I} \hat{\otimes}_{\mathcal{O}} \mathbf{J} \mapsto \bar{\mathbf{Q}}_p$. Thus for any $\Phi \in \mathbf{I} \hat{\otimes}_{\mathcal{O}} \mathbf{J}$, we can regard Φ as a p-adic analytic function on $\mathcal{X}(\mathbf{I}) \times \mathcal{X}(\mathbf{J})$ with values in $\bar{\mathbf{Q}}_p$ by $\Phi(P,Q) = P \otimes Q(\Phi)$. Even if $\Phi = \Psi/\Psi'$ is an element in the quotient field of $\mathbf{I} \hat{\otimes}_{\mathcal{O}} \mathbf{J}$, we can think of Φ as a p-adic meromorphic function on $\mathcal{X}(\mathbf{I}) \times \mathcal{X}(\mathbf{J})$ whose value is given by $\Phi(P,Q) = \Psi(P,Q)/\Psi'(P,Q)$ whenever we can choose a denominator Ψ' so that $\Psi'(P,Q) \neq 0$. We first state our result in a crude form and later elaborate it and will eventually reach the expression in Theorem I.

THEOREM 5.1. — *There exists a unique element* $\mathcal{D}$ *in the quotient field of* $\mathbf{I} \hat{\otimes}_{\mathcal{O}} \mathbf{J}$ *satisfying the following interpolation property : Let* (P,Q)

be a pair of arithmetic points in $\mathscr{A}(\mathbf{I}) \times \mathscr{A}(\mathbf{J})$ *satisfying the following two conditions :*

$$\text{(5.3 a)} \qquad t \leqslant n(P)-n(Q), \quad n(Q)-n(P)+2t \leqslant (m(P)-m(Q))t \ \text{ and } \ v(Q) \geqslant v(P),$$

(5.3 b) *There exists a finite order idele class character* θ *unramified outside* p *such that* θ *coincides with* $\chi'_Q\psi'_{P^{-1}}$ *on* $\mathfrak{r}_p^\times$.

Then $\mathscr{D}$ *is finite at* (P,Q), *and we have, for sufficiently large* α,

$$\text{(5.3 c)} \quad (\chi_Q)_\infty(-1)\theta_\infty(-1)\mathscr{D}(P,Q)\theta(d^2 j\varpi^\alpha)^{-1} \times \mathscr{N}_{F/\mathbf{Q}}(N\mathfrak{p}^\alpha)^{m(P)/2}\mathscr{N}_{F/\mathbf{Q}}(J\mathfrak{p}^\alpha)^{-1}$$
$$= C(P,Q)\frac{L_{Lp}(2-m(P)+m(Q),\chi_Q^{-1}\Psi_P)\, D\left(\dfrac{m(Q)-m(P)}{2}, \mathbf{f}_P, \mathbf{g}_\alpha, \theta^{-1}\right)}{((\mathbf{f}_P^u)^\rho \mid \tau(n\varpi^\alpha), \mathbf{f}_P^u)_\alpha},$$

where the subscript Lp *indicates that we have taken out Euler factors at primes dividing* Lp *from the Hecke L-function :* $\mathrm{L}(s,\chi_Q^{-1}\psi_P)$, $(\ ,\ )_\alpha$ *is the Pètersson inner product of level* $N\mathfrak{p}^\alpha$ (defined in § 4), $\mathbf{g}_\alpha = \mathbf{g}_Q|\theta^{-1}|\tau(j\varpi^\alpha)$ *and*

$$C(P,Q) = |D|^{1+m(Q)-(P)}2^{\{-n(P)-n(Q)+2v(P)-2v(Q)-4t\}}\pi^{\{2v(P)-2v(Q)-n(Q)-3t\}}i^{\{n(Q)-n(P)\}} \times \Gamma_F(n(Q)+v(Q)-v(P)+2t)\Gamma_F(v(Q)-v(P)+t).$$

Moreover, for any $H \in \mathbf{I}$ *which annihilates* $\mathscr{C}(\lambda;\mathbf{I})$, $(H \otimes 1)\mathscr{D}$ *is integral* (i.e. $(H \otimes 1)\mathscr{D} \in \mathbf{I} \mathbin{\hat{\otimes}_{\mathcal{O}}} \mathbf{J}$).

This theorem will be proven in § 10. We now deduce from this theorem the following result which is a restatement of Theorem I :

THEOREM 5.2. — *In addition to* (5.3 a), *suppose that*

(5.4) ψ'_P *and* χ'_Q *are both induced by finite order Hecke characters of* $F_A^\times/F^\times$ *unramified outside* p, *for which we use the same symbol, and put, with the notation of* (4.10 c),

$$W(P,Q) = \chi_{Q\infty}\chi'_{Q\infty}\psi_{P\infty}\psi'_{P\infty}(-1)\frac{\mathscr{N}_{F/\mathbf{Q}}(J)^{(m(Q)/2)+1}W'(\mathbf{g}_Q)}{\mathscr{N}_{F/\mathbf{Q}}(N)^{(m(P)/2)}W'(\mathbf{f}_P)} \times \prod_{\mathfrak{p}\in\Sigma}\frac{\xi\xi'(d_\mathfrak{p})|\eta\eta'(d_\mathfrak{p})|G(\xi_\mathfrak{p}^{-1}\psi_\mathfrak{p}'^{-1})G(\xi_\mathfrak{p}'^{-1}\psi_\mathfrak{p}'^{-1})}{\eta\eta'(d_\mathfrak{p})|\xi\xi'(d_\mathfrak{p})|G(\eta_\mathfrak{p}'^{-1}\psi_\mathfrak{p}'^{-1})}.$$

Then we have

$$\mathscr{D}(P,Q) = W(P,Q)C(P,Q)S(P)^{-1}E(P,Q)\frac{\mathscr{D}_p\left(1+\frac{m(Q)-m(P)}{2}, \mathbf{f}_P, \mathbf{g}_Q^\rho\right)}{(\mathbf{f}_P^\circ, \mathbf{f}_P^\circ)_\delta},$$

for all pairs of arithmetic points (P,Q) *satisfying* (5.3 a) *and* (5.4), *where* $\mathbf{f}_P^\circ$ *is the primitive form (in the sense explained below) of conductor* $N\mathfrak{p}^\delta$ *associated with* $\mathbf{f}_P^u \otimes \psi'_P$, *and* $S(P)$ *and* $E(P,Q)$ *are Euler factors at* p *which will be defined in Lemma* 5.3 *below.*

We shall deduce Th. 5.2 from Th. 5.1 in the rest of this section. Namely we compute $\frac{\mathscr{D}(P,Q)}{L_{Lp}(2-m(P)+m(Q),\chi_Q^{-1}\psi_P)}$ which is equal to

$$\text{(5.5)}\quad (\chi_Q)_\infty(-1)\theta_\infty(-1)\theta(d^2j\varpi^\alpha)\mathscr{N}_{F/\mathbf{Q}}(N\mathfrak{p}^\alpha)^{-m(P)/2}\mathscr{N}_{F/\mathbf{Q}}(J\mathfrak{p}^\alpha)$$
$$\times\ C(P,Q)D\left(\frac{m(Q)-m(P)}{2}, \mathbf{f}_P, \mathbf{g}_\alpha, \theta^{-1}\right)\Big/ ((\mathbf{f}_P^\rho)^u|\tau(n\varpi^\alpha), \mathbf{f}_P^u)_\alpha.$$

We put $\theta = \psi_P'^{-1}\chi'_Q$ as a character of $F_{\mathbf{A}}^\times/F^\times$. We reduce our computation to an easier case through several steps: Even if we replace α by $\beta \geqslant \alpha$ (i.e. $\beta(\mathfrak{p})\geqslant\alpha(\mathfrak{p})$ for all $\mathfrak{p}\in\Sigma$), the value (5.5) will not change because of the formula (5.3 c). This can be checked without using the expression (5.3 c) as follows: We write ϖ^α for $n\varpi^\alpha/n$. Then by an easy computation, we can verify

$$\mathbf{f}_P^\rho|\tau(n\varpi^\alpha)|\begin{pmatrix}\varpi^{\alpha-\beta} & 0\\ 0 & 1\end{pmatrix} = |\varpi^{\beta-\alpha}|_{\mathbf{A}}^{m(P)}\mathbf{f}_P^\rho|\tau(n\varpi^\beta).$$

This combined with the well known formula:

$$(\mathbf{f}_P, \mathbf{g}|[UxU'])_\alpha = \psi_P(\det(x))(\mathbf{f}|[U'x^\iota U], \mathbf{g})_\beta \quad \text{(see (7.2 b))}$$

for $U = U_0(N\mathfrak{p}^\alpha)$ and $U' = U_0(N\mathfrak{p}^\beta)$ shows

$$\text{(5.6 a)}\quad (\mathbf{f}_P^\rho|\tau(n\varpi^\beta),\mathbf{f}_P)_\beta = |D|^{m(P)+2}\mathscr{N}_{F/\mathbf{Q}}(Np^\beta)^{m(P)/2}((\mathbf{f}_P^\rho)^u|\tau(n\varpi^\beta),\mathbf{f}_P^u)_\beta$$
$$= (\mathbf{f}_P^\rho|\tau(n\varpi^\alpha),\mathbf{f}_P|T(\varpi^{\beta-\alpha})) = \mathbf{a}_p(\varpi^{\beta-\alpha},\mathbf{f}_P)\varpi^{(\beta-\alpha)v}(\mathbf{f}_P^\rho|\tau(n\varpi^\alpha),\mathbf{f}_P)_\alpha.$$

Meanwhile, we have seen in § 4, writing $\mathbf{g}_\beta$ for $\mathbf{g}_Q|\theta^{-1}|\tau(j\varpi^\beta)$, that

$$D(s,\mathbf{f}_P,\mathbf{g}_\beta,\theta^{-1}) = C(s)\int_{F_{\mathbf{A}_+}^\times} a(y,\mathbf{f})a(y,\mathbf{g}_\beta)\mathbf{e}_F(2iy_\infty)y_\infty^{(k+\kappa)/2}\theta^{-1}(y)|y|_{\mathbf{A}}^{s-1}\,dy$$

with a function $C(s)$ of s independent of β. Since we know

$$\mathbf{g}_Q|\theta^{-1}|\tau(j\varpi^{\alpha})|\begin{pmatrix}\varpi^{\alpha-\beta} & 0\\ 0 & 1\end{pmatrix} = |\varpi^{\beta-\alpha}|_{\mathbf{A}}^{m(Q)}\mathbf{g}_Q|\theta^{-1}|\tau(n\varpi^{\beta})$$

and hence from (4.3 b), $\mathbf{a}(y,\mathbf{g}_\beta) = a(y\varpi^{\alpha-\beta},\mathbf{g}_\alpha)|\varpi^{\alpha-\beta}|_{\mathbf{A}}^{m(Q)/2}$. Similarly, we have

$$a(y\varpi^{\beta-\alpha},\mathbf{f}_P) = \mathbf{a}_p(\varpi^{\beta-\alpha},\mathbf{f}_P)\varpi^{(\beta-\alpha)v}|\varpi^{\beta-\alpha}|_{\mathbf{A}}^{(m(P)/2)+1}a(y,\mathbf{f}_P).$$

Then we see from the above integral expression that

$$(5.6\text{ b})\quad D\left(\frac{m(Q)-m(P)}{2},\mathbf{f}_P,\mathbf{g}_\beta,\theta^{-1}\right)$$
$$= \theta(\varpi^{\alpha-\beta})|\varpi^{\beta-\alpha}|_{\mathbf{A}}\mathbf{a}_p(\varpi^{\beta-\alpha},\mathbf{f}_P)\varpi^{(\beta-\alpha)v}\, D\left(\frac{m(Q)-m(P)}{2},\mathbf{f}_P,\mathbf{g}_\alpha,\theta^{-1}\right).$$

This combined with (5.6 a) and (7.2 c) shows the desired assertion. The above proof of the independence from β of the value :

$$\mathcal{N}_{F/Q}(N\mathfrak{p}^{\beta})^{-m(P)/2}\theta(\varpi^{\alpha})|\varpi^{\alpha}|_{\mathbf{A}}$$
$$\times\ D\left(\frac{m(Q)-m(P)}{2},\mathbf{f}_P,\mathbf{g}_\beta,\theta^{-1}\right)/((\mathbf{f}_P^u)^\rho|\tau(n\varpi^{\beta}),\mathbf{f}_P^u)_\beta$$

is purely formal and by defining the quantity $((\mathbf{f}_P^\rho)^u|\tau(n\varpi^{\beta}),\mathbf{f}_P^u)_\beta$ by (5.6 a) even if $\mathbf{f}_P$ is not of level $N\mathfrak{p}^\beta$, our formula (5.3 c) still holds. Thus we can choose α as we want. Namely we may assume

(5.6 c) $\quad\mathbf{g}_Q\otimes\chi'_Q|\theta^{-1}$ *is of exact level* $J\mathfrak{p}^\alpha$.

The reason for the above assumption is as follows : We know that for any finite order Hecke character ξ,

$$a(y,\mathbf{f}\otimes\xi)a(y,\mathbf{g}_\beta\otimes\xi^{-1}) = a(y,\mathbf{f})a(y,\mathbf{g}_\beta)$$

and hence

$$D\left(\frac{m(Q)-m(P)}{2},\mathbf{f}_P,\mathbf{g}_\alpha,\theta^{-1}\right)=D\left(\frac{m(Q)-m(P)}{2},\mathbf{f}_P\otimes\xi,\mathbf{g}_\alpha\otimes\xi^{-1},\theta^{-1}\right).$$

On the other hand, for $g = \mathbf{g}_Q|\theta^{-1}$, we have

$$(g|\tau(j\varpi^{\alpha}))\otimes\xi^{-1}(x)=\xi^{-1}(\det(\tau(j\varpi^{\alpha}))d^2)(g\otimes\xi)|\tau(j\varpi^{\alpha}).$$

We see easily from definition that $\mathbf{g}_Q|\theta^{-1}\otimes\xi=\mathbf{g}_Q\otimes\xi|\theta^{-1}$. Thus replacing

$\mathbf{g}_Q$ by $\mathbf{g} = \mathbf{g}_Q \otimes \chi'_Q$ (and $\mathbf{f}_P$ by $\mathbf{f}_P \otimes \chi'_Q$), we may assume that $\mathbf{g} \in \mathbf{M}_{\kappa,\omega}(J\mathfrak{p}^{\alpha}; \mathrm{id}, \chi_Q \chi'^2_Q)$. Then (5.5) is equal to

$$(5.7)\quad \mathcal{N}_{F/Q}(N\mathfrak{p}^{\alpha})^{-m(P)/2}(\chi_Q)_{\infty}(-1)\theta_{\infty}(-1)\psi'^{-1}_P(d^2 j\varpi^{\alpha})|j\varpi^{\alpha}|^{-1}_{\mathbf{A}}\, C(P,Q) \times \frac{D\left(\dfrac{m(Q)-m(P)}{2}, \mathbf{f}_P \otimes \chi'_Q, \mathbf{g}|\theta^{-1}|\tau(j\varpi^{\alpha}), \theta^{-1}\right)}{((\mathbf{f}^u_P)^{\rho}|\tau(n\varpi^{\alpha}), \mathbf{f}^u_P)_{\alpha}},$$

where

$$((\mathbf{f}^u_P)^{\rho}|\tau(n\varpi^{\alpha}), \mathbf{f}^u_P)_{\alpha} = \mathbf{a}_p(\varpi^{\alpha-\delta'}, \mathbf{f}_P)\varpi^{(\alpha-\delta')v}|\varpi^{\alpha-\delta'}|^{m(P)/2}_{\mathbf{A}}((\mathbf{f}^u_P)^{\rho}|\tau(n\varpi^{\delta'}), \mathbf{f}^u_P)_{\delta'}$$

for δ' such that $\mathbf{f}_P$ is of level $N\mathfrak{p}^{\delta'}$.

We now want to compute $\mathbf{g}|\theta^{-1}|\tau(j\varpi^{\alpha})$ but begin with a general setting. Let η be a unitary (finite order) character of $F^{\times}_{\mathbf{A}}/F^{\times}$. For any automorphic representation π of $G(\mathbf{A})$ with central character η, we consider its space $V = V_{\pi}$ in $L_2(G(\mathbf{Q})\backslash G(\mathbf{A}), \eta)$, which is the space of square integrable functions f on $G(\mathbf{Q})\backslash G(\mathbf{A})$ with $f(gz) = \eta(z)f(g)$. Suppose that for each infinite place $\sigma \in I$, π_{σ} is the discrete series representation $\sigma(\mu_1, \mu_2)$ with

$$\mu_1(a) = |a|^{(n_{\sigma}+1)/2} \quad \text{and} \quad \mu_2(a) = |a|^{(n_{\sigma}-1)/2}(a/|a|)^{n_{\sigma}},$$

where $n = n(Q)$. We consider a generalization to non-unitary representations: For the modulus character $\alpha(x) = |x|_{\mathbf{A}}$, we also consider the representation space $V(\pi \otimes \alpha^s)$ for $s \in \mathbf{C}$ consisting of functions $f \otimes \alpha^s(x) = \alpha^s(\det(x))f(x)$ for $f \in V(\pi)$, where π is a unitary automorphic representation. The representation $\pi \otimes \alpha^s$ realized on $V(\pi \otimes \alpha^s)$ is also called an automorphic representation. Let π be an automorphic representation in general sense with central character η (which may not be unitary and $\eta = \alpha^{-m}$ up to finite order characters for a weight (k,w) with $k-2w=mt$). Let $\mathfrak{m}$ be an ideal of $\mathfrak{r}$. For any pair (ξ, ξ') of finite order character of $(\mathfrak{r}/\mathfrak{m})^{\times}$, regarding them as characters of $\mathfrak{r}^{\times}_{\mathfrak{m}}$, we consider the subspace $V(\xi,\xi') = V(\pi; \xi, \xi'; \mathfrak{m})$ consisting of functions $f \in V(\pi)$ satisfying the following conditions:

(i) *f is the lowest weight vector of weight k_{σ} at each infinite place* σ (i.e. f corresponds to a holomorphic modular form of weight (k,w));

(ii) $f\left(x\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \xi(a_{\mathfrak{m}})\xi'(d\mathfrak{m})f(x)$ *for all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\mathfrak{m})$.

Since the central character of π is given by η, $V(\xi,\xi')$ is trivial unless $\xi\xi' = \eta$ on $\hat{\mathfrak{r}}^{\times}$. Let $C(\pi)$ be the conductor of π in the sense of [C]. Then $V(\pi;\mathrm{id},\eta;C(\pi))$ is one dimensional and spanned by a unique element $\mathbf{f}$ with $\mathbf{a}(\mathfrak{r},\mathbf{f}) = 1$. This form $\mathbf{f}$ is called the primitive form associated with the representation π. We can formulate the subspaces $V(\xi,\xi')$ locally. Namely, let us write $\pi = \bigotimes_{\mathfrak{q}} \pi_{\mathfrak{q}}$ for local representation $\pi_{\mathfrak{q}}$ at each place $\mathfrak{q}$ and take a representation space $V(\pi_{\mathfrak{q}})$ for $\pi_{\mathfrak{q}}$. Then one can define $V(\pi_{\mathfrak{q}};\xi_{\mathfrak{q}},\xi'_{\mathfrak{q}};\mathfrak{m}_{\mathfrak{q}})$ as a subspace of $V(\pi_{\mathfrak{q}})$ consisting of vectors v satisfying $\pi_{\mathfrak{q}}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)v = \xi(a)\xi'(d)v$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\mathfrak{m})_{\mathfrak{q}}$ when $\mathfrak{q}$ is finite and of lowest weight vectors of weight k_σ when $\mathfrak{q} = \sigma \in I$ is infinite. Then $V(\pi_{\mathfrak{q}};\xi_{\mathfrak{q}},\xi'_{\mathfrak{q}};\mathfrak{m}_{\mathfrak{q}})$ is one dimensional for almost all $\mathfrak{q}$ and, fixing a generator $v_{\mathfrak{q}}$ for such places, we can naturally identify $V(\pi;\xi,\xi';\mathfrak{m}) = \bigotimes_{\mathfrak{q}} V(\pi_{\mathfrak{q}};\xi_{\mathfrak{q}},\xi'_{\mathfrak{q}};\mathfrak{m}_{\mathfrak{q}})$ as modules over the (complex) Hecke algebra with respect to $U_0(\mathfrak{m})$.

Now we specialize the above argument. Let $\pi = \pi(Q) = \bigotimes_{\mathfrak{q}} \pi_{\mathfrak{q}}(Q)(\pi'=\pi'(Q)=\bigotimes_{\mathfrak{q}} \pi'_{\mathfrak{q}}(Q))$ be the irreducible automorphic representation of $GL_2(F_A)$ on the function space $V(\pi)$ (resp. $V(\pi')$) which is generated by all the right translations of $\mathbf{g}_Q$ (resp. $\mathbf{g}'=\mathbf{g}_Q\otimes\chi'_Q|\theta^{-1}$). Let $\mathbf{g}^\circ = \mathbf{g}^\circ_Q$ be the primitive form associated with $V(\pi^u)$. We want to express the spectial value:

$$L_{Lp}(2-m(P)+m(Q),\chi_Q^{-1}\psi_P)D\left(\frac{m(Q)-m(P)}{2},\mathbf{f}_P\otimes\chi'_Q,\mathbf{g}|\theta^{-1}|\tau(j\varpi^\alpha),\theta^{-1}\right)$$

by means of $\mathcal{D}_p(s,\mathbf{f}^\circ_P,(\mathbf{g}^\circ_Q)^\rho)$. By the strong multiplicity one theorem ([C], [M]), π and π' are irreducible and from the construction of the twisting operator given in § 7, we see $\pi' \cong \pi\otimes\psi'$. In fact, we can define an isomorphism of function spaces (but not as $G(\mathbf{A})$-module): $V(\pi) \cong V(\pi')$ by sending $f(x)\in V(\pi)$ to

$$f\otimes\psi'(x) = \psi'(\det(x))f(x)\in V(\pi').$$

Write $C(\psi') = \mathfrak{p}^\varepsilon$ and $C(\pi') = J\mathfrak{p}^\beta$. Then, this map sends $V(\pi;\varphi,\varphi',J\mathfrak{p}^r)$ isomorphically to $V(\pi';\varphi\psi',\varphi'\psi';J\mathfrak{p}^r)$ if $r\geqslant\varepsilon$, and $V(\pi';\mathrm{id},\chi\psi'^2_P;J\mathfrak{p}^\beta)$ is one dimensional and spanned by the primitive form $\mathbf{h}$ associated with $V(\pi')$. Write simply χ' and χ for χ'_Q and χ_Q. By definition, $\mathbf{g} = \mathbf{g}_Q\otimes\chi'_Q$ belongs to $V(\pi;\mathrm{id},\chi\chi'^2;J\mathfrak{p}^\alpha)$ and $\mathbf{g}'$ belongs to $V(\pi';\mathrm{id},\chi\chi'^2\theta^{-2};J\mathfrak{p}^r)$ for some r (see § 7). It is known that $\pi_\mathfrak{p}$ for $\mathfrak{p}\in\Sigma$

is either principal or special and the special representation occurs only when $n_\sigma = 0$ for all the embeddings $\sigma : F \to \bar{\mathbf{Q}} \to \bar{\mathbf{Q}}_p$ which induce the place $\mathfrak{p}$ (see [H1], Lemma 12.2, [H5], Prop. 2.1 and Cor. 2.2). Write $\pi_\mathfrak{p} = \pi(\xi_\mathfrak{p}, \xi'_\mathfrak{p})$ (resp. $\sigma(\xi_\mathfrak{p}, \xi'_\mathfrak{p})$ with $\xi'_\mathfrak{p} = \xi_\mathfrak{p}\alpha_\mathfrak{p}^{-1}$) when $\pi_\mathfrak{p}$ is principal (resp. special) for $\mathfrak{p} \in \Sigma$. Here, we follow the convention explained below [H5], Prop. 2.1 to describe principal series representations and special ones. Writing $C(\xi'_\mathfrak{p}\psi'_\mathfrak{p}) = \mathfrak{p}^{\gamma'(\mathfrak{p})}$ and $C(\xi_\mathfrak{p}\psi'_\mathfrak{p}) = \mathfrak{p}^{\gamma(\mathfrak{p})}$, we now define subsets of Σ as follows :

$$\Sigma_1 = \{\mathfrak{p} \in \Sigma \mid \gamma(\mathfrak{p}) = 0 \quad \text{but} \quad \gamma'(\mathfrak{p}) > 0\}$$
$$\Sigma_2 = \{\mathfrak{p} \in \Sigma \mid \gamma'(\mathfrak{p}) = 0 \quad \text{but} \quad \gamma(\mathfrak{p}) > 0\}$$
$$\Sigma_3 = \{\mathfrak{p} \in \Sigma \mid \gamma(\mathfrak{p}) = \gamma'(\mathfrak{p}) = 0 \quad \text{and} \quad \pi_\mathfrak{p} \text{ is principal}\}$$
$$\Sigma_4 = \{\mathfrak{p} \in \Sigma \mid \gamma(\mathfrak{p}) = \gamma'(\mathfrak{p}) = 0 \quad \text{and} \quad \pi_\mathfrak{p} \text{ is special}\}$$
$$\Xi = \Sigma - \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4 .$$

By [H5], Cor. 2.2, we may assume that $\xi_\mathfrak{p}$ (resp. $\xi'_\mathfrak{p}$) coincides with $\chi'^{-1}_\mathfrak{p}$ on $\mathfrak{r}_\mathfrak{p}^\times$ (resp. $(\chi\chi')_\mathfrak{p}$) and that $\mathbf{g}_Q | T(\varpi_\mathfrak{p}) = \xi_\mathfrak{p}(\varpi_\mathfrak{p})\mathbf{g}_Q$ for the Hecke operator $T(\varpi_\mathfrak{p})$ defined in (2.2 a). Hereafter we suppress the subscript « $\mathfrak{p}$ » (resp. « P » and « Q ») of $\xi_\mathfrak{p}$ (resp. ψ'_P and χ'_Q) if no confusions are likely. Then $\pi'_\mathfrak{p} = \pi(\xi\psi', \xi'\psi')$ or $\sigma(\xi\psi', \xi'\psi')$ according as $\pi_\mathfrak{p} = \pi(\xi, \xi')$ or $\sigma(\xi, \xi')$ because $\theta = \psi'^{-1}\chi'$. We define an operator $[\varpi_\mathfrak{p}]$ (see § 7.B for details) by

$$\mathbf{a}_p(y, \mathbf{f} | [\varpi_\mathfrak{p}]) = \varpi_\mathfrak{p}^{-v}\mathbf{a}_p(y\varpi_\mathfrak{p}^{-1}, \mathbf{f}) (\Leftrightarrow \mathbf{a}(y, \mathbf{f} | [\varpi_\mathfrak{p}]) = \mathbf{a}(y\varpi_\mathfrak{p}^{-1}, \mathbf{f})\{\mathfrak{p}^{-v}\}).$$

Similarly as above, let $\pi(P) = \bigotimes_\mathfrak{q} \pi_\mathfrak{q}(P)$ be the automorphic representation spanned by $\mathbf{f}_P$ and write its $\mathfrak{p}$-component as $\pi(\eta_\mathfrak{p}, \eta'_\mathfrak{p})$ or $\sigma(\eta_\mathfrak{p}, \eta'_\mathfrak{p})$ for $\mathfrak{p} \in \Sigma$ so that $\mathbf{f}_P | T(\varpi_\mathfrak{p}) = \eta(\varpi_\mathfrak{p})\mathbf{f}_P$ (then $\eta_\mathfrak{p}\psi'_\mathfrak{p}$ is unramified at $\mathfrak{p} \in \Sigma$). Writting $C(\eta'_\mathfrak{p}\psi'_\mathfrak{p}) = \mathfrak{p}^{\delta(\mathfrak{p})}$, we define

$$\Sigma_1(P) = \{\mathfrak{p} \in \Sigma \mid \delta(\mathfrak{p}) > 0\}$$
$$\Sigma_2(P) = \{\mathfrak{p} \in \Sigma \mid \delta(\mathfrak{p}) = 0 \quad \text{and} \quad \pi_\mathfrak{p}(P) \text{ is principal}\}$$
$$\Sigma_3(P) = \{\mathfrak{p} \in \Sigma \mid \delta(\mathfrak{p}) = 0 \quad \text{and} \quad \pi_\mathfrak{p}(P) \text{ is special}\}.$$

Then Theorem 5.2 is a direct consequence of Theorem 5.1 and the following result :

LEMMA 5.3. – (i) *We have* $\beta(\mathfrak{p}) = \gamma(\mathfrak{p}) + \gamma'(\mathfrak{p})$ *except when* $\mathfrak{p} \in \Sigma_4$ *and in this exceptional case, we have* $\beta(\mathfrak{p}) = 1$ (where $\mathfrak{p}^{\beta(\mathfrak{p})} = C(\pi'_\mathfrak{p})$).

(ii) *We have* $\alpha(\mathfrak{p}) = \beta(\mathfrak{p})$ *if* $\mathfrak{p} \in \Xi$, $\alpha(\mathfrak{p}) = \beta(\mathfrak{p}) + 1$ *if* $\mathfrak{p} \in \Sigma_1 \cup \Sigma_2 \cup \Sigma_4$ *and* $\alpha(\mathfrak{p}) = 2$ *if* $\mathfrak{p} \in \Sigma_3$.

(iii) *Define operators* $A_{\mathfrak{p}}$ *for each* $\mathfrak{p}\in\Sigma$ *acting on*

$$V(\pi';\mathrm{id},\chi_Q\psi_P'^2;J\mathfrak{p}^\alpha)$$

by

$$A_{\mathfrak{p}}=\mathrm{Id}\quad \text{if}\quad \mathfrak{p}\in\Xi,\quad A_{\mathfrak{p}}=\mathrm{Id}-\xi\psi_P'(\varpi_{\mathfrak{p}})[\varpi_{\mathfrak{p}}]\quad\text{if}\quad \mathfrak{p}\in\Sigma_1\cup\Sigma_4,$$
$$A_{\mathfrak{p}}=\mathrm{Id}-\xi'\psi_P'(\varpi_{\mathfrak{p}})[\varpi_{\mathfrak{p}}]\quad\text{if}\quad\mathfrak{p}\in\Sigma_2,$$

and

$$A_{\mathfrak{p}}=\mathrm{Id}-\psi_P'(\varpi_{\mathfrak{p}})(\xi(\varpi_{\mathfrak{p}})+\xi'(\varpi_{\mathfrak{p}}))[\varpi_{\mathfrak{p}}]+\xi\xi'\psi_P'^2(\varpi_{\mathfrak{p}})[\varpi_{\mathfrak{p}}]^2\quad\text{if}\quad\mathfrak{p}\in\Sigma_3,$$

where we have written simply ξ *for* $\xi_{\mathfrak{p}}$. *Then we have* $\mathbf{g}|\theta^{-1}=\mathbf{h}|A$ *for* $A=\prod_{\mathfrak{p}\in\Sigma}A_{\mathfrak{p}}$.

(iv) *Write* m *for* $m(Q)$ *and define operators* $B_{\mathfrak{p}}$ *for each* $\mathfrak{p}\in\Sigma$ *acting on* $V(\pi'';\mathrm{id},\chi_Q^{-1}\psi_P'^2;J\mathfrak{p}^\alpha)$ *for the unitarization* π'' *of the contragredient of* π' *by*

$$B_{\mathfrak{p}}=\mathrm{Id}\qquad\qquad \text{if}\quad\mathfrak{p}\in\Xi,$$
$$B_{\mathfrak{p}}=\mathrm{Id}-|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{-(m/2)-1}\xi^{-1}\psi_P'^{-1}(\varpi_{\mathfrak{p}})[\varpi_{\mathfrak{p}}]_0\quad\text{if}\quad\mathfrak{p}\in\Sigma_1\cup\Sigma_4,$$
$$B_{\mathfrak{p}}=\mathrm{Id}-|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{-(m/2)-1}\xi'^{-1}\psi_P'^{-1}(\varpi_{\mathfrak{p}})[\varpi_{\mathfrak{p}}]_0\quad\text{if}\quad\mathfrak{p}\in\Sigma_2,$$

and

$$B_{\mathfrak{p}}=\mathrm{Id}-|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{-(m/2)-1}\psi_P'(\varpi_{\mathfrak{p}})^{-1}(\xi^{-1}(\varpi_{\mathfrak{p}})+\xi'^{-1}(\varpi_{\mathfrak{p}}))[\varpi_{\mathfrak{p}}]_0$$
$$+|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{-(m+2)}\xi\xi'\psi_P'^2(\varpi_{\mathfrak{p}}^{-1})[\varpi_{\mathfrak{p}}^2]_0\quad\text{if}\quad\mathfrak{p}\in\Sigma_3.$$

Then we have $\{\mathbf{g}|\theta^{-1}|\tau(j\varpi^\alpha)\}^u=C_0\mathbf{h}^{\rho u}|B$ *for* $B=\prod_{\mathfrak{p}\in\Sigma}B_{\mathfrak{p}}$, *where*

$$C_0=W(\mathbf{h})|j\varpi^\alpha|_{\mathbf{A}}^{-m(Q)/2}\prod_{\mathfrak{p}\in\Sigma_1\cup\Sigma_4}(-\xi\psi_P'(\varpi_{\mathfrak{p}}))|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{(m(Q)/2)+1}$$
$$\times\prod_{\mathfrak{p}\in\Sigma_2}(-\xi'\psi_P'(\varpi_{\mathfrak{p}}))|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{(m(Q)/2)+1}\prod_{\mathfrak{p}\in\Sigma_3}\xi\xi'\psi_P'^2(\varpi_{\mathfrak{p}})|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{m(Q)+2}$$

and $\mathbf{f}|[\varpi_P^r]_0(x)=\mathbf{f}\left(x\begin{pmatrix}\varpi_P^{-r}&0\\0&1\end{pmatrix}\right)$ *for* $\mathbf{f}\in V(\pi'';\mathrm{id},\chi_Q^{-1}\psi_P'^{-2};J\mathfrak{p}^\alpha)$.

(v) *Define an Euler factor* $E_{\mathfrak{p}}(P,Q)$ *at each* $\mathfrak{p}\in\Sigma$ *as follows:*

$$\frac{(1-\xi\eta^{-1}(\varpi))(1-\xi'\eta^{-1}(\varpi))}{(1-\xi^{-1}\eta(\varpi)|\varpi|_{\mathfrak{p}})(1-\xi'^{-1}\eta(\varpi)|\varpi|_{\mathfrak{p}})}\quad\text{if}\quad\mathfrak{p}\in\Sigma_3,$$

$$\xi'\eta^{-1}(\varpi_P^{\gamma'(\mathfrak{p})})\frac{(1-\xi\eta^{-1}(\varpi))}{(1-\xi^{-1}\eta(\varpi)|\varpi|_{\mathfrak{p}})}\quad\text{if}\quad\mathfrak{p}\in\Sigma_1,$$

$$- \xi\eta^{-1}(\varpi^{\mathfrak{p}}) \frac{(1-\xi\eta^{-1}(\varpi)|\varpi|_{\mathfrak{p}}^{-1}}{(1-\xi^{-1}\eta(\varpi)|\varpi|_{\mathfrak{p}}^{2})} \quad \textit{if} \quad \mathfrak{p} \in \Sigma_4,$$

$$\xi\eta^{-1}(\varpi_{\mathfrak{p}}^{\gamma(\mathfrak{p})}) \frac{(1-\xi'\eta^{-1}(\varpi))}{(1-\xi'^{-1}\eta(\varpi)|\varpi|_{\mathfrak{p}})} \quad \textit{if} \quad \mathfrak{p} \in \Sigma_2,$$

$$\xi\eta^{-1}(\varpi_{\mathfrak{p}}^{\gamma(\mathfrak{p})})\, \xi\eta^{-1}(\varpi_{p}^{\gamma'(\mathfrak{p})}) \quad \textit{if} \quad \mathfrak{p} \in \Xi,$$

where we have written ϖ for $\varpi_{\mathfrak{p}}$. *Then we have*

$$L_{Lp}(2-m(P)+m(Q),\ \chi_Q^{-1}\psi_P) \times D\left(\frac{m(Q)-m(P)}{2}, \mathbf{f}_P \otimes \chi'_Q, \mathbf{g}|\theta^{-1}|\tau(j\omega^{\alpha}), \theta^{-1}\right)$$
$$= C_1 E(P,Q)\, \mathcal{D}_p\left(1+\frac{m(Q)-m(P)}{2}, \mathbf{f}_P, \mathbf{g}_Q^{\rho}\right),$$

where $E(P,Q) = \prod_{\mathfrak{p}\in\Sigma} E_{\mathfrak{p}}(P,Q)$ *and*

$$C_1 = W'(\mathbf{g}_Q)\psi'(d^{(p)2}j\omega^{\alpha})|j|_{\mathbf{A}}^{-m(Q)/2} \times \prod_{\mathfrak{p}\in\Sigma} \frac{\xi\xi'\psi'^2(d_{\mathfrak{p}})\eta(\varpi_{\mathfrak{p}}^{\alpha(\mathfrak{p})})|\varpi_{\mathfrak{p}}^{\alpha(\mathfrak{p})}|_{\mathfrak{p}} G(\xi_{\mathfrak{p}}^{-1}\psi_{\mathfrak{p}}'^{-1})\, G(\xi'^{-1}_{\mathfrak{p}}\psi_{\mathfrak{p}}'^{-1})}{|\xi\xi'(d_{\mathfrak{p}})|}$$

(vi) *Write* $\mathbf{f}^{\circ}$ *for the primitive form of the unitarization of* $\pi(P)\otimes\psi'_P$ *and* $C(\pi(P)\otimes\psi'_P) = N\mathfrak{p}^{\delta}$ *and define for* $s = \#\,(\Sigma_3(P))$

$$S(P) = (-1)^s \prod_{\mathfrak{p}\in\Sigma_2(P)} (1-\eta^{-1}\eta'(\varpi_{\mathfrak{p}})|\varpi_{\mathfrak{p}}|_{\mathfrak{p}})(1-\eta^{-1}\eta'(\varpi_{\mathfrak{p}})) \times \prod_{\mathfrak{p}\in\Sigma_1(P)} \eta'\eta^{-1}(\omega_{\mathfrak{p}}^{\delta(\mathfrak{p})})|\varpi_{\mathfrak{p}}^{\delta(\mathfrak{p})}|_{\mathfrak{p}}.$$

Then we have

$$\frac{((\mathbf{f}_P^{\rho})^u|\tau(n\varpi^{\alpha}), \mathbf{f}_P^u)_{\alpha}}{(\mathbf{f}^{\circ},\mathbf{f}^{\circ})_{\delta}} = |\varpi^{\alpha}|_{\mathbf{A}}^{m(P)/2}\eta(\varpi^{\alpha})\psi_{\infty}(-1)W'(\mathbf{f}_P)S(P) \times \prod_{\mathfrak{p}\in\Sigma_1(P)} G(\eta'^{-1}\psi'^{-1}) \prod_{\mathfrak{p}\in\Sigma} \frac{\eta\eta'(d_{\mathfrak{p}})}{|\eta\eta'(d_{\mathfrak{p}})|}.$$

Proof. – The assertions (i) and (ii) follows from (iii) by [H5], Cor. 2.2 and its proof. Thus we prove (iii). What we need to show is that

$$\mathbf{a}(\varpi_{\mathfrak{p}}^n, \mathbf{g}|\theta^{-1}) = \mathbf{a}(\varpi_{\mathfrak{p}}^n, \mathbf{h}|A_{\mathfrak{p}}) \quad \text{for all} \quad n > 0,$$

since the computation and the definition of $A_{\mathfrak{p}}$ is purely local and $\mathbf{a}(y, \mathbf{g}|\theta^{-1})$ and $\mathbf{a}(y, \mathbf{h})$ are transformed in exactly the same way if we replace y by yu for $u \in \mathfrak{r}_{\mathfrak{p}}^{\times}$ (see (2.4)). We only give the explanation of the formula in the most complicated case where $\mathfrak{p} \in \Sigma_3$ since the other cases can be treated similarly. Note that the eigenvalue of $T(\mathfrak{p})$ for $\mathbf{h}$ is given by $\xi\chi'\theta^{-1}(\mathfrak{p}) + \xi'\chi'\theta^{-1}(\mathfrak{p})$, which is equal to

$$\xi\psi'(\varpi) + \xi'\psi'(\varpi) \quad \text{for} \quad \varpi = \varpi_P \quad \text{because of} \quad \theta = \psi_P^{-1}\chi'_Q.$$

This shows that $\mathbf{a}(\varpi^n,\mathbf{h}) = \{\varpi^{-nv}\}\dfrac{\xi\psi'(\varpi)^{n+1} - \xi'\psi'(\varpi)^{n+1}}{\xi\psi'(\varpi) - \xi'\psi'(\varpi)}\cdot$ Then it is easy to check by a simple computation that $\mathbf{a}(\varpi^n,\mathbf{h}|A_{\mathfrak{p}}) = \mathbf{a}(\varpi^n,\mathbf{g}|\theta^{-1})$. The assertion (iv) follows from the fact :

$$\mathbf{h}^u(x) = |D|_{-(m(Q)/2)-1}\mathbf{h}(x)|\det(x)|_{\mathbf{A}}^{m(Q)/2}$$

and the following formulae :

$$\text{(5.8)} \qquad |\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{-(m+1)k}\mathbf{h}|\tau(j\varpi^{\alpha})|[\omega_{\mathfrak{p}}^{k}] = \mathbf{h}|\tau(j\varpi^{\alpha}\varpi_{\mathfrak{p}}^{k}),$$
$$\mathbf{h}|\tau(j\varpi^{\beta}) = W(\mathbf{h})|j\varpi^{\beta}|_{\mathbf{A}}^{-(Q)/2}\mathbf{h}^{\rho}.$$

We now compute each Euler factor $E_{\mathfrak{p}}$ for $\mathfrak{p} \in \Sigma_3$. In fact, if one replaces $|\varpi|_{\mathfrak{p}}^{s}$ by X, we have for

$$a' = \xi\psi'^{-1}(\varpi)|\varpi|_{\mathfrak{p}}^{(m/2)+1}, \qquad b' = \xi'\psi'^{-1}(\varpi)|\varpi|_{\mathfrak{p}}^{(m/2)+1},$$
$$a = \xi(\varpi)|\varpi|_{\mathfrak{p}}^{(m/2)+1}, \quad c = \eta(\varpi)|\varpi|_{\mathfrak{p}}^{(m/2)+1} \quad \text{and} \quad b = \xi(\varpi)|\varpi|_{\mathfrak{p}}^{(m/2)+1},$$

$$E_{\mathfrak{p}}(X) = \sum_{n=0}^{\infty} \chi'\theta^{-1}(\varpi^n)a(\varpi^n,\mathbf{f}_P)a(\varpi^n,\mathbf{h}^{\rho})X^n$$
$$-|\varpi|_{\mathfrak{p}}^{-(m/2)-1}\psi'(\varpi)^{-1}(\xi(\varpi)^{-1}+\xi'(\varpi)^{-1})\sum_{n=0}^{\infty} \chi'\theta^{-1}(\varpi^n)a(\varpi^n,\mathbf{f}_P)a(\varpi^{n-1},\mathbf{h}^{\rho})X^n$$
$$+|\varpi|_{\mathfrak{p}}^{-m-2}\xi\xi'\psi'^2(\varpi^{-1})\sum_{n=0}^{\infty} \chi'\theta^{-1}(\varpi^n)a(\varpi^n,\mathbf{f}_P)a(\varpi^{n-2},\mathbf{h}^{\rho})X^n$$
$$= \sum_{n=0}^{\infty} \psi'(\varpi^n)c^n \frac{a'^{n+1} - b'^{n+1}}{a' - b'} X^n$$
$$\times \{1 - |\varpi|_{\mathfrak{p}}^{-1}(a' + b')c\psi'(\varpi)X + |\varpi|_{\mathfrak{p}}^{-2}a'b'c^2\psi'(\varpi)^2X^2\}$$
$$= abc^2|\omega_{\mathfrak{p}}|_{\mathfrak{p}}^{-2}X^2\frac{(1-a^{-1}c^{-1}|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}X)(1-b^{-1}c^{-1}|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}X)}{(1-acX)(1-bcX)}.$$

Replacing X by $|\varpi|_{\mathfrak{p}}^{(m(Q)-m(P))/2}$ and using the fact : $\xi\xi(\varpi) =$

$|\varpi|_{\mathfrak{p}}^{-m(Q)-1}$ and $\bar{\eta}\eta(\varpi) = |\varpi|_{\mathfrak{p}}^{-m(P)-1}$, we know that

$$\frac{(1-a^{-1}c^{-1}|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}X)(1-b^{-1}c^{-1}|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}X)}{(1-acX)(1-bcX)} = \frac{(1-\xi\eta^{-1}(\varpi))(1-\xi'\eta^{-1}(\varpi))}{(1-\xi^{-1}\eta(\omega)|\varpi|_{\mathfrak{p}})(1-\xi'^{-1}\eta(\varpi)|\omega|_{\mathfrak{p}})}.$$

The contribution from the factor C_0 in (iv) is given by

$$\prod_{\mathfrak{p}\in\Sigma_3} |\varpi^{\alpha(\mathfrak{p})}|_{\mathfrak{p}}^{-m(Q)/2}\,\xi\xi'\psi_P'^2(\varpi_{\mathfrak{p}})|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{m(Q)+2}\,W_{\mathfrak{p}}(\mathbf{h}) = \prod_{\mathfrak{p}\in\Sigma_3} |\varpi^{\alpha(\mathfrak{p})}|_{\mathfrak{p}}^{-m(Q)/2}\frac{\xi\xi'\psi_P'^2(d_{\mathfrak{p}})}{|\xi\xi'(d_{\mathfrak{p}})|_{\mathfrak{p}}}\psi'^2(\varpi_{\mathfrak{p}})\overline{ab}.$$

Then we see that

$$\prod_{\mathfrak{p}\in\Sigma_3} |\varpi^{\alpha(\mathfrak{p})}|_{\mathfrak{p}}^{-m(Q)/2}\frac{\xi\xi'\psi_P'^2(d_{\mathfrak{p}})}{\xi\xi'(d_{\mathfrak{p}})}\psi'^2(\varpi_{\mathfrak{p}})\overline{ab}abc^2|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{2s-2}|_{s=(m(Q)-m(P))/2}$$
$$= \prod_{\mathfrak{p}\in\Sigma_3}\frac{\xi\xi'\psi'^2(d_{\mathfrak{p}})\eta(\varpi_{\mathfrak{p}}^{\alpha(\mathfrak{p})})|\varpi_{\mathfrak{p}}^{\alpha(\mathfrak{p})}|_{\mathfrak{p}}\,G(\xi_{\mathfrak{p}}^{-1}\psi_{\mathfrak{p}}'^{-1})\,G(\xi'^{-1}_{\mathfrak{p}}\psi_{\mathfrak{p}}'^{-1})}{|\xi\xi'(d_{\mathfrak{p}})|},$$

which gives the Σ_3-part of C_1. We can compute similarly the Σ_i-part for $i = 1, 2, 4$ and the Ξ-part. Then, using the well known fact :

$$W'(\mathbf{g}_Q\otimes\psi') = W(\mathbf{g}_Q)\psi'(d^{(p)^2}j) \quad \text{for} \quad d^{(p)} = dd_p^{-1},$$

we conclude the assertion (v) for $\mathfrak{p}\in\Sigma$. All the other Euler factors for $\mathfrak{q}$ outside Σ can be computed similarly and we conclude (v). Now we prove (vi). Let $N\mathfrak{p}^{\alpha'}$ (resp. $N\mathfrak{p}^{\delta'}$) be the exact level of $\mathbf{f}_P$ (resp. $\mathbf{f}_P\otimes\psi'$). First we list several formulae which can be verified from the data we already know :

$$((\mathbf{f}_P^{\rho})^u|\tau(n\varpi^{\alpha'}),\mathbf{f}_P^u)_{\alpha'} = \psi'^{-1}(d^2n\varpi^{\alpha'})((\mathbf{f}_P^{\rho})^u\otimes\psi'^{-1}|\tau(n\varpi^{\alpha'}),\mathbf{f}_P^u\otimes\psi')_{\alpha'},$$

$$((\mathbf{f}_P^{\rho})^u|\tau(n\varpi^{\alpha}),\mathbf{f}_P^u)_{\alpha} = \eta(\varpi)^{\alpha-\alpha'}|\varpi^{\alpha-\alpha'}|_{\mathbf{A}}^{m(P)/2}((\mathbf{f}_P^{\rho})^u|\tau(n\varpi^{\alpha'}),\mathbf{f}_P^u)_{\alpha'} \quad \text{(cf. (5.6 a))},$$

$$((\mathbf{f}_P^{\rho})^u\otimes\psi'^{-1}|\tau(n\varpi^{\alpha'}),\mathbf{f}_P^u\otimes\psi')_{\alpha'} = \eta\psi'(\varpi)^{\alpha'-\delta'}|\varpi^{\alpha'-\delta'}|_{\mathbf{A}}^{m(P)/2}((\mathbf{f}_P^{\rho})^u\otimes\psi'^{-1}|\tau(n\varpi^{\delta'}),\mathbf{f}_P^u\otimes\psi')_{\delta'},$$

$$W(\mathbf{f}_P^{\rho})\,W(\mathbf{f}_P) = \psi_\infty(-1),$$
$$W'(\mathbf{f}_P\otimes\psi') = \psi'(Nd^{(p)^2})W'(\mathbf{f}_P) \quad \text{(see [H3], (5.4 c))},$$

$$W_P(\mathbf{f}_P\otimes\psi')=\prod_{\mathfrak{p}\in\Sigma(P)}\frac{\eta\psi'(d_\mathfrak{p})\eta'\psi'(\varpi_\mathfrak{p}^{\delta(\mathfrak{p})}d_\mathfrak{p})\,G(\eta'^{-1}\psi'^{-1})}{|\eta(d_\mathfrak{p})\eta'(\varpi_\mathfrak{p}^{\delta(\mathfrak{p})}d_\mathfrak{p})\,G(\eta'^{-1}\psi'^{-1})|}\prod_{\mathfrak{p}\in\Sigma_3(P)}\frac{-\eta\psi'(\varpi_\mathfrak{p})}{|\eta(\varpi_\mathfrak{p})|},$$

$$\mathbf{f}_P^u\otimes\psi'=\mathbf{f}^\circ|R \quad \text{for an operator} \quad R=\prod_{\mathfrak{p}\in\Sigma}R_\mathfrak{p} \text{ given by}$$

$$R_\mathfrak{p}=\mathrm{Id} \quad \text{if} \quad \mathfrak{p}\in\Sigma_1(P)\cup\Sigma_3(P)$$

and

$$\mathbf{f}|R_\mathfrak{p}=\mathbf{f}-\psi'\eta'(\varpi_\mathfrak{p})|\varpi_\mathfrak{p}|_\mathfrak{p}^{(m(P)/2)+1}\mathbf{f}|[\varpi_\mathfrak{p}]_0 \quad \text{if} \quad \mathfrak{p}\in\Sigma_2(P).$$

Especially $\delta'(\mathfrak{p})=\delta(\mathfrak{p})+1$ if $\mathfrak{p}\in\Sigma_2(P)$ and otherwise $\delta'(\mathfrak{p})=\delta(\mathfrak{p})$. By the above formulae, we know that

$$\begin{aligned}&((\mathbf{f}_P^\rho)^u|\tau(n\varpi^\alpha),\mathbf{f}_P^u)_\alpha\\ &\qquad=\psi'^{-1}(d^2n\varpi^{\alpha'})\eta(\varpi)^{\alpha-\delta'}|\varpi^{\alpha-\delta'}|_\mathbf{A}^{m(P)/2}((\mathbf{f}^\circ|R)^\rho|\tau(n\varpi^{\delta'}),\mathbf{f}^\circ|R)_{\delta'}.\end{aligned}$$

We now quote some other easy formulae : Let $a=\psi'\eta(\varpi_\mathfrak{p})|\varpi_\mathfrak{p}|_\mathfrak{q}^{m(P)/2}$ and $b=\psi'\eta'(\varpi_\mathfrak{p})|\varpi_\mathfrak{p}|_\mathfrak{p}^{(m(P)/2}$. Then we have, for $\mathfrak{p}\in\Sigma_2(P)$,

$$(5.9)\quad (\mathbf{f}^\circ|[\varpi_\mathfrak{p}]_0,\mathbf{f}^\circ)_{\delta'}=(a+b)(\mathbf{f}^\circ,\mathbf{f}^\circ)_\delta,\ (\mathbf{f}^\circ,\mathbf{f}^\circ|[\varpi_\mathfrak{p}]_0)_{\delta'}=(a^\rho+b^\rho)(\mathbf{f}^\circ,\mathbf{f}^\circ)_\delta,$$

and

$$(\mathbf{f}^\circ,\mathbf{f}^\circ)_{\delta'}=(\mathbf{f}^\circ|[\varpi_\mathfrak{p}]_0,\mathbf{f}^\circ|[\varpi_\mathfrak{p}]_0)_\delta=(1+|\varpi|_{\mathfrak{p})}^{-1})(\mathbf{f}^\circ,\mathbf{f}^\circ)_\delta.$$

Then by (5.8) and (5.9), we get the expression in (vi).

From this lemma combined with (5.7), we conclude the assertion of Theorem 5.2.

6. Fourier expansion of Eisenstein series.

Following the method developed by Shimura in [Sh4] and [Sh5], we shall now give an exposition of the computation of the Fourier expansion of $G_k(x,\chi,\theta;s)$ defined in (4.8 d) for characters χ and θ of $\mathrm{Cl}_F(\mathfrak{m})$. We use the same notation as in § 4. Our purpose is to formulate the result in a manner suitable for our later use. We put

$$(6.1)\quad \varepsilon=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\in G(\mathbf{Q})_+ \quad\text{and}\quad \tau(m)=\begin{pmatrix}0&-1\\m&0\end{pmatrix}\in GL_2(F_{\mathbf{A}_f})$$

for a finite idele m with $m\mathfrak{r}=\mathfrak{m}$. Let D be the discriminant of F. In order to state the Fourier expansion of $G_k(x,\chi,\theta;s)$, we prepare some notations: for $\alpha\in F^\times$, we define a subset of I by $[\alpha]=\{\sigma\in I|\alpha^\sigma>0\}$.

For $k \in \mathbf{Z}[I]$ and a subset J of I, we write $k(J) = \sum_{\sigma \in J} k_\sigma \sigma - \sum_{\sigma \notin J} k_\sigma \sigma$. We define a confluent hypergeometric function $\omega(t;\alpha,\beta)$ for $0 < y \in \mathbf{R}$, $\alpha, \beta \in \mathbf{C}$ by

$$\omega(y;\alpha,\beta) = \Gamma(\beta)^{-1} t^{\beta} \int_0^\infty e^{-yx}(x+1)^{\alpha-1} x^{\beta-1}\, dx,$$

which is absolutely and locally uniformly convergent if $\mathrm{Re}(\beta) > 0$ and is continued as a holomorphic function of α and β to the whole $\mathbf{R}_+ \times \mathbf{C}^2$ (cf. [Sh6], Th. 4.2). We then define out of $\omega(y;\alpha,\beta)$ the Whittaker function $W_J(s,k;y)$ for $s \in \mathbf{C}$, $k \in \mathbf{Z}[I]$ and $y \in F_{\infty+}^\times$ by

$$(6.2) \quad W_J(s,k;y) = (4\pi y)^{-k(J)/2} \mathbf{e}_F(iy) \prod_{\sigma \in J} \omega\left(4\pi y_\sigma; s - \frac{k_\sigma}{2}, s + \frac{k_\sigma}{2}\right) \times \prod_{\sigma \notin J} \omega\left(4\pi y_\sigma; s + \frac{k_\sigma}{2}, s - \frac{k_\sigma}{2}\right),$$

where $\mathbf{e}_F(x) = \exp\left(2\pi i \sum_{\sigma \in I} x_\sigma\right)$. We define functions $\sigma_{m,\chi}$, $\sigma'_{m,\chi}$ for integers m and $c_{s,\chi}$ for complex number s on the group of fractional ideals of F by

$$(6.3) \quad \sigma_{m,\chi}(\mathfrak{a}) = \sum_{\mathfrak{b} \supset \mathfrak{a}} \chi(\mathfrak{b}) \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{b})^m, \quad \sigma'_{m,\chi}(\mathfrak{a}) = \sum_{\mathfrak{b} \supset \mathfrak{a}} \chi(\mathfrak{a}/\mathfrak{b}) \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{b})^m,$$

$$c_{s,\chi}(\mathfrak{a}) = \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})^{s-1} \sum_{\mathfrak{b} \supset \mathfrak{a}} \chi(\mathfrak{b}) \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{b})^{1-2s} \quad \textit{if } \mathfrak{a} \textit{ is integral,}$$

and

$$\sigma_{m,\chi}(\mathfrak{a}) = \sigma'_{m,\chi}(\mathfrak{a}) = c_{s,\chi}(\mathfrak{a}) = 0 \quad \textit{if } \mathfrak{a} \textit{ is not integral.}$$

Here, the value $\chi(\mathfrak{a})$ is defined as follows: When $\mathfrak{a}$ is prime to $\mathfrak{m}$, then we choose $a \in F_f^\times$ such that $a\mathfrak{r} = \mathfrak{a}$ and $a_\mathfrak{m} = 1$ and define $\chi(\mathfrak{a}) = \chi(a)$. When $\mathfrak{a}$ is not prime to $\mathfrak{m}$, we simply put $\chi(\mathfrak{a}) = 0$.

THEOREM 6.1. — *Suppose that* $\mathfrak{m} \neq \mathfrak{r}$. *Then we have*

$$G_k\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, \chi, \theta; s\right) = D^{1-s} i^{\{-k\}} \pi^{[F:\mathbf{Q}]s} \theta^{-1}(y_f)$$
$$\times \left\{ (4\pi)^{(1-s)[F:\mathbf{Q}]} |dy|_{\mathbf{A}}^{1-s} \frac{\Gamma_F((2s-1)t)}{\Gamma_F\left(st - \frac{k}{2}\right)\Gamma_F\left(st + \frac{k}{2}\right)} L_\mathfrak{m}(2s-1, \chi\theta^2) \right.$$
$$\left. + \sum_J \Gamma_F\left(st - \frac{k(J)}{2}\right)^{-1} \sum_{\xi \in F^\times, [\xi] = J} c_{s,\chi\theta^2}(\xi y \mathfrak{d}) W_J(s,k,|\xi| y) \mathbf{e}_F(\xi x) \right\},$$

where $\{k\} = \sum_{\sigma\in I} k_\alpha$ *and* $\mathbf{e}_F : F_{\mathbf{A}}/F \to \mathbf{C}^\times$ *be the standard additive character which coincides with* $\mathbf{e}_F(x_\infty)$ *on* F_∞.

A proof of this theorem will be given in the later half of this section. From the theorem, it is obvious that

$$(6.4\text{ a})\qquad G_k(x,\chi,\theta;s) = \theta^{-1}(\det(x))G_k(x,\chi\theta^2,\mathrm{id};s).$$

Thus we know in particular

$$(6.4\text{ b})\qquad E_k^*(x,\chi^{-1},\theta;s) = \theta(\det(x))\,E_k^*(x,\chi^{-1}\theta^{-2},\mathrm{id};s).$$

Actually we can easily verify this formula from definition. However we will not use this fact in the computation of the Fourier expansion in the theorem.

Next we specialize the Fourier expansion in the theorem to $s = \dfrac{[k]}{2}$ and $1 - \dfrac{[k]}{2}$ when k is parallel and $k = -[k]t$ with $0 < [k]\in\mathbf{Z}$: We quote a formula from [Sh6], (3.17-18) for $0 \leqslant n\in\mathbf{Z}$:

$$(6.5)\quad \omega(z;n+1,\beta) = \sum_{k=0}^{n}\binom{n}{k}\beta(\beta+1)\cdot\cdots\cdot(\beta+k-1)z^{-k} = \sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(\beta+k)}{\Gamma(\beta)}z^{-k},$$

$$\omega(z;\alpha,-n) = \sum_{k=0}^{n}\binom{n}{k}(1-\alpha)(2-\alpha)\cdot\cdots\cdot(k-\alpha)z^{-k} = \sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(k-\alpha+1)}{\Gamma(1-\alpha)}z^{-k}.$$

From this, we get, for $k = -\kappa t$ with $0<\kappa\in\mathbf{Z}$,

$$\Gamma_F\left(s-\frac{k(J)}{2}\right)^{-1} W_J(s,k;|\xi|y_\infty)|_{s=\kappa/2} = \begin{cases}\Gamma(\kappa)^{-[F:\mathbf{Q}]}(4\pi|\xi|y_\infty)^{\kappa t/2}\mathbf{e}_F(i|\xi|y_\infty) & \text{if } J = I\\ 0 \ \text{ otherwise}\end{cases}$$

$$\Gamma_F\left(s-\frac{k(J)}{2}\right)^{-1} W_J(s,k;|\xi|y_\infty)|_{s=1-\kappa/2} = \begin{cases}(4\pi|\xi|y_\infty)^{\kappa t/2}\mathbf{e}_F(i|\xi|y_\infty) & \text{if } J = I\\ 0 \ \text{ otherwise.}\end{cases}$$

From these formulae, we conclude

COROLLARY 6.2. — *Suppose that* $k = -\kappa t$ *for a positive integer* κ. *Suppose either* $F \neq \mathbf{Q}$ *or* $\kappa \neq 2$. *Then we have*

$$G_{-\kappa t}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, \chi, \theta ; \frac{\kappa}{2}\right) = D^{1-\kappa}(2\pi i)^{\kappa[F:\mathbf{Q}]}\Gamma(\kappa)^{-[F:\mathbf{Q}]}\theta^{-1}(y_f)|y|_{\mathbf{A}}^{\kappa/2}$$
$$\times \{\delta_{\kappa 1}2^{-[F:\mathbf{Q}]}L_m(0,\chi\theta^2) + \sum_{\xi \gg 0} \sigma'_{\kappa-1,\chi\theta^2}(\xi y \mathfrak{d})\mathbf{e}_F(i\xi y)\mathbf{e}_F(\xi x)\},$$

$$G_{-\kappa t}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, \chi, \theta ; 1-\frac{\kappa}{2}\right) = (2i)^{\kappa[F:\mathbf{Q}]}\pi^{[F:\mathbf{Q}]}\theta^{-1}(y_f)|y|_{\mathbf{A}}^{\kappa/2}$$
$$\times \{2^{-[F:\mathbf{Q}]}L_m(1-\kappa,\chi\theta^2) + \sum_{\xi \gg 0} \sigma_{\kappa-1,\chi\theta^2}(\xi y \mathfrak{d})\mathbf{e}_F(i\xi y)\mathbf{e}_F(\xi z)\}.$$

We now make explicit the value of Eisenstein series of weight $-\kappa t - 2r$ for $0 \leqslant r \in \mathbf{Z}[I]$ at the integer points $\frac{\kappa}{2}$ and $1 - \frac{\kappa}{2}$ by using Shimura's differential operators. The following relation ([Sh3], (1.16 a, b)) between the defferential operators δ and d in (1.7) is useful for that purpose :

(6.6) $$\delta_k^r = \sum_{0 \leqslant j \leqslant r} \binom{r}{j} \frac{\Gamma_F(k+r)}{\Gamma_F(k+r-j)}(-4\pi y)^{-j}d^{r-j}$$
$$= \sum_{0 \leqslant j \leqslant r} \binom{r}{j} \frac{\Gamma_F(j-k-r+t)}{\Gamma_F(t-k-r)}(4\pi y)^{-j}d^{r-j},$$

where we have written $\binom{r}{j}$ for $\prod_\sigma \binom{r_\sigma}{j_\sigma}$. Then (6.6) combined with (6.5) and Theorem 6.1 yields :

COROLLARY 6.3. — *Suppose that* $k = -\kappa t$ *for a positive integer* κ. *Let* $0 \leqslant r \in \mathbf{Z}[I]$. *Then we have*

$$y_\infty^{\kappa t/2}\Gamma_F(\kappa)\delta_{\kappa t}^r\left\{y_\infty^{-\kappa t/2}G_{-\kappa t}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, \chi, \theta ; \frac{\kappa}{2}\right)\right\}$$
$$= \Gamma_F(\kappa t+r)(-4\pi y_\infty)^{-r}G_{-\kappa t-2r}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, \chi, \theta ; \frac{\kappa}{2}\right),$$

$$y_\infty^{\kappa t/2}\delta_{\kappa t}^r\left\{y_\infty^{-\kappa t/2}G_{-\kappa t}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, \chi, \theta ; 1-\frac{\kappa}{2}\right)\right\}$$
$$= \Gamma_F(r+t)(-4\pi y_\infty)^{-r}G_{-\kappa t-2r}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, \chi, \theta ; 1-\frac{\kappa}{2}\right).$$

Here we let the differential operator $\delta_{\kappa t}^{r}$ act on the function $G_{-\kappa t}$ regarding it as a function on $\mathfrak{Z}$ for a fixed finite part y_f (if one regarded $\delta_{\kappa t}^{r}$ as an operator acting on functions defined on $G(\mathbf{A})$ as in (1.8), then the factor y_∞^{-r} on the right-hand side would have been disappeared).

Before proving the theorem, we list easy lemmas without proof: We first compute the Fourier expansion of $E(x,s) = \mathscr{E}(x\varepsilon_f^{-1},\chi,\theta;s)$ in (4.8 a). We put

$$\alpha(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad x \in F_{\mathbf{A}}.$$

LEMMA 6.4. — $\left(\varepsilon a\alpha(x)\begin{pmatrix} a' & 0 \\ 0 & 1 \end{pmatrix}\varepsilon^{-1}\right)_f \in B(\mathbf{A}_f)U_0(\mathfrak{m})$ *for* $a,\ a' \in F_{\mathbf{A}}^{\times}$ *if and only if the following conditions are satisfied:* (i) $x \in a^{-1}\mathfrak{m}\hat{\mathfrak{r}}$, $a \in a'^{-1}\hat{\mathfrak{r}}$ *and* (ii) $ax\mathfrak{r} + aa'\mathfrak{r} = \mathfrak{r}$.

LEMMA 6.5. — *Under the conditions* (i) *and* (ii) *in Lemma* 6.4, *we have*

$$\chi^*\left(\left(\varepsilon a\alpha(x)\begin{pmatrix} a' & 0 \\ 0 & 1 \end{pmatrix}\varepsilon^{-1}\right)_f\right) = \chi(a_{\mathfrak{m}}a'_{\mathfrak{m}})$$

$$\eta\left(\left(\varepsilon a\alpha(x)\begin{pmatrix} a' & 0 \\ 0 & 1 \end{pmatrix}\varepsilon^{-1}\right)_f\right) = |(a^2a')_f|_{\mathbf{A}},$$

$$\eta((\varepsilon a\alpha(x)w)_\infty) = |a^t|^2 \operatorname{Im}(z)^t |j(\varepsilon a\alpha(x),z)^{-2t}|,$$

$$\theta\left(\left(\varepsilon a\alpha(x)\begin{pmatrix} a' & 0 \\ 0 & 1 \end{pmatrix}\varepsilon^{-1}\right)_f\right) = \theta((a^2a')_f(aa')_{\mathfrak{m}}^{-2}) = \theta(a_f'^{-1})\theta(aa'\mathfrak{r})^2$$
$$= \theta(a'_p)^{-1}\theta(a^2a'\mathfrak{r}) = \theta(a^2a'\mathfrak{r})$$

(the last equality holds only if $a'_{\mathfrak{m}}=1$),

$$\theta((\varepsilon a\alpha(x)w)_\infty) = 1,$$

where $w \in G_+(\mathbf{R})$ *and* $z = w(z_0) \in \mathfrak{Z}$.

We define the Fourier coefficient of $E(x,s)$ at $\xi \in F$ by

$$b(\xi,w,s) = \int_{F_{\mathbf{A}}/F} E(\alpha(x)w,s)\mathbf{e}_F(-\xi x)\,dx, \tag{6.7}$$

where dx is the self dual Haar measure on $F_{\mathbf{A}}$ such that $\int_{F_{\mathbf{A}}/F} dx=1$.

LEMMA 6.6 ([Sh4], § 3). — *If* $b(\xi,u,s) \neq 0$ *for* $u = x\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} w$ *with* $a_{\mathfrak{m}} = a_\infty = 1$ *and* $w \in G_+(\mathbf{R})$, *then* $\xi \in \mathfrak{a}^{-1}\mathfrak{m}^{-1}\mathfrak{d}^{-1}$.

LEMMA 6.7 ([Sh5], Lemma 3). — *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be integral ideals. Then for* $\xi \in \mathfrak{a}^{-1}\mathfrak{m}^{-1}\mathfrak{d}^{-1}$,

$$\sum_{r \in (\mathfrak{b}^{-1}\mathfrak{m}/\mathfrak{a}\mathfrak{m})^\times} \mathbf{e}_F(-\xi r) = \sum_{\mathfrak{c} \supset \mathfrak{a}\mathfrak{b} + \xi\mathfrak{a}\mathfrak{m}\mathfrak{d}} \mu(\mathfrak{a}\mathfrak{b}/\mathfrak{c})\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{c}),$$

where μ *denotes the generalized Möbius function on the ideal group of* F *and* $(\mathfrak{b}^{-1}\mathfrak{m}/\mathfrak{a}\mathfrak{m})^\times = \{x \in (\mathfrak{b}^{-1}\mathfrak{m}/\mathfrak{a}\mathfrak{m}) \mid \mathrm{Ann}(x) = \mathfrak{a}\mathfrak{b}\}$.

Proof of Theorem 6.1. — Now we start with the computation of the Fourier coefficient $b(\xi,x,s)$. Let N be a subgroup of B defined by $N(A) = \left\{\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \middle| \alpha \in F \otimes_{\mathbf{Q}} A\right\}$. The term of $E(x,s)$:

$$\chi^*(\gamma x \varepsilon_f^{-1})\theta(\gamma x \varepsilon_f^{-1})\eta(\gamma x \varepsilon_f^{-1})^s |j(\gamma, x_\infty(z_0))^{-k}|j(\gamma, x_\infty(z_0))^k$$

for $\gamma \in G_+(\mathbf{Q}) = G(\mathbf{Q}) \cap G_{\infty+}$ is non trivial if $\chi^*(\gamma x \varepsilon_f^{-1}) \neq 0$; i.e., $\gamma x \varepsilon_f^{-1} \in B(\mathbf{A})U_0(\mathfrak{m})G_{\infty+}$. Here note that Supp (θ) contains Supp (χ^*). If $x \in B(\mathbf{A})F_{\mathbf{A}}^\times$, then the non-trivially of $\chi^*(\gamma x \varepsilon_f^{-1})$ means that $\gamma x \in B(\mathbf{A})U_0(\mathfrak{m})G(\mathbf{R})\varepsilon_f$ and if we write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $c \neq 0$ and thus $\gamma \in B_+(\mathbf{Q})F^\times \varepsilon N(\mathbf{Q})$ (cf. [Sh4], p. 422). Namely, we know (from (4.8 a))

$$E(x,s) = \sum_{\gamma \in \mathfrak{r}^\times \backslash F^\times N(\mathbf{Q})} \chi^*(\varepsilon\gamma x \varepsilon_f^{-1})\theta(\varepsilon\gamma x \varepsilon_f^{-1})\eta(\varepsilon\gamma x \varepsilon_f^{-1})^s \times |j(\varepsilon\gamma, x_\infty(z_0))^{-k}|j(\varepsilon\gamma, x_\infty(z_0))^k.$$

We take finite ideles a and x with $x_{\mathfrak{m}} = 1$ and $w \in G_{\infty+}$. Then with the notation of Lemma 6.6, we have, writing z for $w(z_0)$,

$$(6.8) \quad b(\xi,w,s) = \int_{F_{\mathbf{A}}/F} \sum_{\gamma \in \mathfrak{r}^\times \backslash F^\times, \xi \in F} \chi^*\theta\left(\varepsilon\gamma x \alpha(\delta+v)\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} w \varepsilon_f^{-1}\right)$$
$$\times \eta\left(\varepsilon\gamma x \alpha(\delta+v)\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} w \varepsilon_f^{-1}\right)^s |j(\varepsilon\gamma\alpha(\delta+v),z)^{-k}|j(\varepsilon\gamma\alpha(\delta+v),z)^k \mathbf{e}_F(-\xi v)\,dv$$
$$= \sum_{\gamma \in (\mathfrak{a}^{-1}\mathfrak{x}^{-1}-\{0\})/\mathfrak{r}^\times} \int_{F_{\mathbf{A}}} \chi^*\theta\left(\varepsilon\gamma x \alpha(v)\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} w \varepsilon_f^{-1}\right)$$
$$\times \eta\left(\varepsilon\gamma x \alpha(v)\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} w \varepsilon_f^{-1}\right)^s |j(\varepsilon\gamma\alpha(v),z)^{-k}|j(\varepsilon\gamma\alpha(v),z)^k \mathbf{e}_F(-\xi v)\,dv,$$

where $\mathfrak{x} = xr$. Here, to get the last identity, we have used Lemma 6.4. We now introduce

$$\text{(6.9 a)} \quad \zeta(y,t;\alpha,\beta) = \int_{-\infty}^{\infty} \exp(-2\pi itx)(x+iy)^{-\alpha}(x-iy)^{-\beta}\,dx$$

for α, $\beta \in \mathbf{C}$, $0 < y \in \mathbf{R}$ and t, $x \in \mathbf{R}$. Note that the self dual measure dx on $F_{\mathbf{A}}$ coincides with the usual Lebesgue measure on F_∞. We know the following formulae from [Sh6], § 3:

(6.9 b)

$$\zeta(y,t;\alpha,\beta) = \begin{cases} i^{\beta-\alpha}(2\pi)^{\alpha}\Gamma(\alpha)^{-1}(2y)^{-\beta}t^{\alpha-1}e^{-2\pi yt}\omega(4\pi yt;\alpha,\beta) & \text{if } t>0 \\ i^{\beta-\alpha}(2\pi)^{\beta}\Gamma(\beta)^{-1}(2y)^{-\alpha}|t|^{\beta-1}e^{-2\pi yt}\omega(4\pi y|t|;\beta,\alpha) & \text{if } t<0 \\ i^{\beta-\alpha}(2\pi)^{\alpha+\beta}\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\Gamma(\alpha+\beta-1)(4\pi y)^{1-\alpha-\beta} & \text{if } t=0. \end{cases}$$

We then put $\zeta(s;y,u) = \prod_{\sigma\in I}\zeta\left(y_\sigma,u_\sigma;s-\frac{k_\sigma}{2},s+\frac{k_\sigma}{2}\right)$ for $u = (u_\sigma)$ and $y = (y_\sigma)$ in $\mathbf{R}^I$. Then by Lemma 6.5, we see

$$\text{(6.10 a)} \quad \int_{F_\infty} \mathbf{e}_F(-\xi v_\infty)\eta((\varepsilon\gamma\alpha(v)w)_\infty)^s|j(\varepsilon\gamma\alpha(v),z)^{-k}|j(\varepsilon\gamma\alpha(v),z)^k\,dv_\infty$$
$$= \operatorname{sgn}(\gamma)^k \operatorname{Im}(z)^{st}\zeta(s;\operatorname{Im}(z),\xi).$$

Then, we have by Lemmas 6.5 and 6.7

$$\text{(6.10 b)} \quad \sum_{\gamma\in\mathfrak{a}^{-1}\mathfrak{x}^{-1}/\mathfrak{r}^\times,\gamma\mathfrak{a}\mathfrak{x}+\mathfrak{m}=\mathfrak{r}} \int_{F_{\mathbf{A}_f}} \chi^*\theta\eta^s\left(\varepsilon\gamma x\alpha(v)\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}w\varepsilon_f^{-1}\right)\mathbf{e}_f(-\xi v_f)\,dv_f$$
$$= D^{-1/2} \sum_{\gamma\in\mathfrak{a}^{-1}\mathfrak{x}^{-1}/\mathfrak{r}^\times,\gamma\mathfrak{a}\mathfrak{x}+\mathfrak{m}=\mathfrak{r}} \theta(a_f^{-1})\theta((\gamma\mathfrak{x}\mathfrak{a})^2)\mathcal{N}(\gamma\mathfrak{x})^{-2s}\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})^{-s}$$
$$\times\ \chi((\gamma a x)_{\mathfrak{m}})\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a}\mathfrak{m})^{-1} \sum_{\mathfrak{c}\supset\gamma\mathfrak{x}\mathfrak{a}+\xi\mathfrak{a}\mathfrak{m}\mathfrak{d}} \mu(\gamma\mathfrak{x}\mathfrak{a}/\mathfrak{c})\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{c}).$$

In fact, by Lemma 6.5, for $\Theta(v) = \chi^*\theta\eta^s\left(\varepsilon\gamma x\alpha(v)\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}w\varepsilon_f^{-1}\right)$, we have

$$\int_{F_{\mathbf{A}_f}} \Theta(v)\mathbf{e}_F(-\xi v_f)\,dv_f$$
$$= |((\gamma x)^2 a)_f|_{\mathbf{A}}^s\theta((\gamma\mathfrak{x})^2\mathfrak{a})\chi((\gamma x a)_{\mathfrak{m}}) \int_{\gamma^{-1}\mathfrak{x}^{-1}\mathfrak{m}\hat{\mathfrak{r}}} \mathbf{e}_F(-\xi v_f)\,dv_f.$$

Here the domain of integration is over $\gamma^{-1}\mathfrak{x}^{-1}\mathfrak{m}\hat{\mathfrak{r}}$ because of Lemma 6.4. Note that $\operatorname{sgn}(\gamma)^k\chi((\gamma ax)_{\mathfrak{m}}) = \chi(ax)\chi^{-1}(\gamma\mathfrak{a}\mathfrak{x})$, where in the right-hand side, we consider χ^{-1} as an ideal character. Then, writing $\mathfrak{n}$ for $\gamma\mathfrak{a}\mathfrak{x}$, by (6.10 a, b), we know from Lemma 6.7 that

$$(6.11)\quad b\left(\xi, x\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}w, s\right) = \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{m})^{-1}D^{-1/2}\chi\theta^{-1}(a_f)\chi(x)\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})^{s-1}\operatorname{Im}(z)^{st} \times \zeta(s;\operatorname{Im}(z),\xi)\sum_{\mathfrak{n}\sim\mathfrak{x}\mathfrak{a}}\theta^2\chi^{-1}(\mathfrak{n})\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{n})^{-2s}\sum_{\mathfrak{b}\supset\mathfrak{n}+\xi\mathfrak{a}\mathfrak{m}\mathfrak{d}}\mu(\mathfrak{n}/\mathfrak{b})\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{b}),$$

where $\mathfrak{n}$ runs over all integral ideals in the same class as $\mathfrak{x}\mathfrak{a}$. We have defined in (4.8 b)

$$E_k^*(u,\chi,\theta;s) = \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{m})\sqrt{D}\sum_{i=1}^{h}(\chi^{-1})(\mathfrak{a}_i)\,\mathscr{E}_k(a_iu,\chi,\theta;s).$$

Then by (6.10 b), the Fourier coefficient of $E_k^*\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}w\varepsilon_f^{-1},\chi,\theta;s\right)$ for $a\in F_{\mathbf{A}}^\times$ with $a_\infty = 1$ at ξ is given by

$$(6.12\text{ a})\quad \chi\theta^{-1}(a_f)\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})^{s-1}\operatorname{Im}(z)^{st}\zeta(s;\operatorname{Im}(z),\xi) \times\sum_{\mathfrak{n}}\chi^{-1}\theta^2(\mathfrak{n})\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{n})^{-2s}\sum_{\mathfrak{b}\supset\mathfrak{n}+\xi\mathfrak{a}\mathfrak{m}\mathfrak{d}}\mu(\mathfrak{n}/\mathfrak{b})\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{b}),$$

where $\mathfrak{n}$ runs over all integral ideals of F and ξ can be zero. Writing $\mathfrak{n} = \mathfrak{c}\mathfrak{b}$ in the above summation and interchanging the two summations, we know that (6.12 a) is equal to

$$(6.12\text{ b})\quad \chi\theta^{-1}(a_f)\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})^{s-1}\operatorname{Im}(z)^{st}\zeta(s;\operatorname{Im}(z),\xi) \times\sum_{\mathfrak{b}\supset\xi\mathfrak{a}\mathfrak{m}\mathfrak{d}}\chi^{-1}\theta^2(\mathfrak{b})\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{b})^{1-2s}\sum_{\mathfrak{c}}\chi^{-1}\theta^2(\mathfrak{c})\mu(\mathfrak{c})\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{c})^{-2s}$$
$$= \chi\theta^{-1}(a_f)\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})^{s-1}\operatorname{Im}(z)^{st}\zeta(s;\operatorname{Im}(z),\xi)\mathrm{L}_{\mathfrak{m}}(2s,\chi^{-1}\theta^2)^{-1} \times\begin{cases}\sum_{\mathfrak{b}\supset\xi\mathfrak{a}\mathfrak{m}\mathfrak{d}}\chi^{-1}\theta^2(\mathfrak{b})\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{b})^{1-2s} & \text{if } \xi\neq 0\\ \mathrm{L}_{\mathfrak{m}}(2s-1,\chi^{-1}\theta^2) & \text{if } \xi = 0.\end{cases}$$

By the formula (6.9 b), we see

$$(6.13)\quad \zeta(s;y_\infty,\xi) = i^{\{k\}}\pi^{[F:\mathbf{Q}]s}\Gamma_F\left(st-\frac{k([\xi])}{2}\right)^{-1}y_\infty^{-st}|\xi|^{st-t}W_{[\xi]}(s,k;|\xi|y_\infty)\quad\text{for}\quad \xi\neq 0,$$

$$\zeta(s;y_\infty,0) = i^{\{k\}}(2\pi)^{\{2st\}}\Gamma_F\left(st-\frac{k}{2}\right)^{-1}\Gamma_F\left(st+\frac{k}{2}\right)^{-1}\Gamma_F((2s-1)t)(4\pi y_\infty)^{(1-2s)t}.$$

We put $\tau(m) = \begin{pmatrix} 0 & -1 \\ m & 0 \end{pmatrix} = -m\begin{pmatrix} m^{-1} & 0 \\ 0 & 1 \end{pmatrix}\varepsilon_f^{-1} \in GL_2(F_{\mathbf{A}_f})$ for a finite idele m with $m\mathfrak{r} = \mathfrak{m}$. Define

$$G_k^*(x,\chi,\theta;s) = \theta^{-1}(m)\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{m})^{s-1}\chi(\det(x))E_k^*(x,\tau(m),\chi^{-1};\theta;s).$$

Thus replacing χ by χ^{-1} in the above argument and keeping in mind the fact that $(-1)^{\{k\}} = \chi_\infty(-1)$ times (6.12 b) gives the coefficient of $G_k^*\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix},\chi,\theta;s\right)$ by the « $-$ » sign in front of $m\begin{pmatrix} m^{-1} & 0 \\ 0 & 1 \end{pmatrix}\varepsilon_f^{-1}$, we conclude that the Fourier coefficient of $G_k^*\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix},\chi,\theta;s\right)$ at $\xi \in F^\times$ is given by

$$(6.14)\quad (-1)^{\{k\}}\theta^{-1}(y_f)|y|_{\mathbf{A}}|y_f|_{\mathbf{A}}^{-s}|y_\infty|^{s-1}\zeta(s;\mathrm{Im}(z),\xi)L_{\mathfrak{m}}(2s,\chi\theta^2)^{-1}$$
$$\times\begin{cases}\sum_{\mathfrak{b}\supset\xi y\mathfrak{d}} \chi\theta^2(\mathfrak{b})\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{b})^{1-2s} & \text{if } \xi \neq 0 \\ L_{\mathfrak{m}}(2s-1,\chi\theta^2) & \text{if } \xi = 0.\end{cases}$$

Then by the definition (4.8 d), Theorem 6.1 is now proven.

7. Operators acting on p-adic Hilbert modular forms.

Here we shall summarize known results on operators acting on p-adic Hilbert modular forms. Throughout this section, N is a given integral ideal prime to p.

A. The action of $\mathbf{G} = Z(N) \times \mathfrak{r}_p^\times$.

For $(\bar{z},a) \in \mathbf{G}$, we let it act as

$$\mathbf{a}_p(y,\mathbf{f}|\langle\bar{z},a\rangle) = \mathbf{a}_p(ya^{-1},\mathbf{f}|\langle z\rangle) \quad\text{and}\quad \mathbf{a}_{0,p}(y,\mathbf{f}|\langle\bar{z},a\rangle) = \mathbf{a}_{0,p}(ya^{-1},\mathbf{f}|\langle z\rangle)$$

by choosing a representive z of $\bar{z}$ in $F_{\mathbf{A}}^\times$ with $z_{N_p} = z_\infty = 1$. Thus on $\mathbf{S}_{k,w}(Np^\alpha,\psi',\psi;\mathcal{O})$, $(\bar{z},a)$ acts via multiplication of $\psi(z)\psi'(a)\mathcal{N}(z)^{[k-2w]}a^v$.

B. The operator $[m]$.

Let M be an integral ideal prime to p and m be an idele with $m\mathfrak{r} = M$ and $m_p = m_\infty = 1$. For $\mathbf{f} \in \bar{\mathbf{M}}(U(MN))$, we define $\mathbf{f}|[m]$ by

$$\mathbf{a}_p(y,\mathbf{f}|[m]) = \mathbf{a}_p(ym^{-1},\mathbf{f}) \quad\text{and}\quad \mathbf{a}_{0,p}(y,\mathbf{f}|[m]) = \mathbf{a}_{0,p}(ym^{-1},\mathbf{f}).$$

This operation corresponds to the action

$$\mathbf{f}|[m](x) = \mathcal{N}_{F/\mathbf{Q}}(M)^{-1}\mathbf{f}|\begin{pmatrix} m^{-1} & 0 \\ 0 & 1 \end{pmatrix}(x) = \mathcal{N}_{F/\mathbf{Q}}(M)^{-1}\mathbf{f}\left(x\begin{pmatrix} m^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

and induces a linear map $[m] : \bar{\mathbf{M}}(U(N)) \to \bar{\mathbf{M}}(U(NM))$. If one restricts the operator $[m]$ to $\bar{\mathbf{M}}(N)$, then the operator $[m]$ only depends on the ideal M but not on the idele m. We can similarly define the operator $[m]$ acting on $\bar{\mathbf{M}}_{k,w}(S(p^{\infty}); K)$ for m with non-trivial m_p by

$$\mathbf{a}_p(y,\mathbf{f}|[m]) = m_p^{-v}\mathbf{a}_p(ym^{-1},\mathbf{f}) \quad \text{and} \quad \mathbf{a}_{0,p}(y,\mathbf{f}|[m]) = m_p^{-v}\mathbf{a}_{0,p}(ym^{-1},\mathbf{f}),$$

which may not preserve integral forms.

C. The action of $x \in SL_2(\hat{\mathfrak{r}})$ *with* $x_p = 1$.

Every element $x \in SL_2(\hat{\mathfrak{r}})$ with $x_p = 1$ acts on $\bar{\mathbf{M}}(U(N))$ (e.g. [H1], Th. 4.9) and sends it into $\bar{\mathbf{M}}(U(N,N))$. This action coincides with the following one on $\bar{\mathbf{M}}_{k,w}(U(N)(p^{\infty}) ; \bar{\mathbf{Q}}) : \mathbf{f}|x(y) = \mathbf{f}(yx)$.

D. The trace operator $Tr_{L/N}$.

Let M and N be the integral ideals prime to p and L be the product MN. Then we put

$$S(N,M) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(N) \mid b \in M\hat{\mathfrak{r}}\right\}$$

and define $Tr_{L/N} : \bar{\mathbf{M}}(S(N,M)) \to \bar{\mathbf{M}}(U(N))$ by $\mathbf{f}|Tr_{L/N} = \sum_{x \in U(N)_L/S(N,M)_L} \mathbf{f}|x$. Note that this operator takes $\mathbf{M}_{k,w}(S(N,M)(p^{\infty}); \bar{\mathbf{Q}})$ into $\mathbf{M}_{k,w}(U(N)(p^{\infty}) ; \bar{\mathbf{Q}})$ and hence naturally extends to an operator as above by C.

E. The twisted trace operator $T_{L/N}$.

Let L be an ideal prime to p and N be a divisor of L. We take finite ideles l and n so that $n\mathfrak{r} = N$ and $l\mathfrak{r} = L$. We define $T_{L/N}$ by

$$\mathbf{f}|T_{L/N}(x) = \mathbf{f}\left|\begin{pmatrix} l/n & 0 \\ 0 & 1 \end{pmatrix}\right| Tr_{L/N}.$$

In fact, the operator $\begin{pmatrix} l/n & 0 \\ 0 & 1 \end{pmatrix} : \mathbf{f}(x) \mapsto \mathbf{f}\left|\begin{pmatrix} l/n & 0 \\ 0 & 1 \end{pmatrix}\right. = \mathbf{f}\left(x\begin{pmatrix} l/n & 0 \\ 0 & 1 \end{pmatrix}\right)$ actually takes $\bar{\mathbf{M}}(S)$ for $S = U(N) \cap U_0(L)$ into $\bar{\mathbf{M}}(S(N,L/N))$, because $\alpha S \alpha^{-1} = S(N,L/N)$ for $\alpha = \begin{pmatrix} l/n & 0 \\ 0 & 1 \end{pmatrix}$. Thus on $\mathbf{M}_{k,w}(U_0(Lp^{\alpha}),\psi',\psi; A)$

for finite order characters ψ of $\mathrm{Cl}_F(Np^\alpha)$ and ψ' of $\mathfrak{r}_p^\times$, the operator $T_{L/N}$ coincides with $[U_0(Np^\alpha)\alpha U_0(Lp^\alpha)]$ defined by

$$\mathbf{f}|[U_0(Np^\alpha)\alpha U_0(Lp^\alpha)](x) = \sum_i \psi(\alpha_i)^{-1}\psi'(\alpha_i)^{-1}\mathbf{f}(x\alpha_i),$$

where α_i is given by the disjoint decomposition:

$$U_0(Np^\alpha)\alpha U_0(Lp^\alpha) = \bigcup_i \alpha_i U_0(Lp^\alpha)$$

and $\psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \psi(d_{N_p})$ and $\psi'\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \psi'(a_p^{-1}d_p)$.

F. The twisting operator for a ray class character χ.

Let M and N be integral ideals prime to p and let $\chi : \mathrm{Cl}_F(Mp^\alpha) \to \mathcal{O}^\times$ be a Hecke character. We then define $\chi : \bar{\mathbf{M}}(N) \to \bar{\mathbf{M}}(NM^2)$ by

$$\mathbf{a}_p(y,\mathbf{f}|\chi) = \chi(y\mathfrak{r})\mathbf{a}_p(y,\mathbf{f}),$$

and

$$\mathbf{a}_{0,p}(y,\mathbf{f}|\chi) = \chi(y\mathfrak{r})\mathbf{a}_{0,p}(y,\mathbf{f}) \quad \text{if} \quad Mp^\alpha = \mathfrak{r} \quad \text{and otherwise } \mathbf{a}_{0,p}(y,\mathbf{f}|\chi) = 0.$$

Here we consider χ to be ideal character such that $\chi(y\mathfrak{r}) = \chi(y)$ if $y_{M_p} \in \mathfrak{r}_{M_p}^\times$, and $\chi(y\mathfrak{r}) = 0$ if $y_{M_p} \notin \mathfrak{r}_{M_p}^\times$. As a special case of this type of operators, we can associate to the trivial character ι_p to $\mathrm{Cl}_F(p)$ the twisting operator $\iota_p : \bar{\mathbf{M}}(N) \to \bar{\mathbf{M}}_{k,w}(N)$, which does not affect $\mathbf{a}_p(y,\mathbf{f})$ if $y\mathfrak{r} + p\mathfrak{r} = \mathfrak{r}$ and simply annihilates $\mathbf{a}_p(y,\mathbf{f})$ if $y\mathfrak{r} + p\mathfrak{r} \neq \mathfrak{r}$. This operator coincides with $1 - T(p) \circ [p]$. For more general character χ, we can show the existence of such operator as follows: We assume that χ is primitive of conductor C. Let c be a finite idele such that $c\mathfrak{r} = C$. Then we define for $\mathbf{f} \in \mathbf{M}_{k,w}(V_1(N)(p^\infty);\bar{\mathbf{Q}})$,

$$\mathbf{f}|R(x) = \chi(\det(x)) \sum_{u \in Y} \chi(u)\mathbf{f}\left(x\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right),$$

where Y is a representative set for $c^{-1}\mathfrak{r}_C^\times$ modulo $\mathfrak{r}_C$. Then we see

$$\mathbf{a}(y,\mathbf{f}|R) = \begin{cases} G(\chi^{-1})\chi(yy_C^{-1})\mathbf{a}(y,\mathbf{f}) & \text{if } \; y_C \in \mathfrak{r}_C^\times, \\ 0 \quad \text{otherwise} \end{cases}$$

where $G(\chi^{-1}) = \sum_{u \in Y} \chi(d^{-1}u)\mathbf{e}_F(d^{-1}u)$. Then $G(\chi^{-1})^{-1}\mathbf{f}|R$ gives the desired form since $\chi(yy_C^{-1}) = \chi(y\mathfrak{r})$ if $y_C \in \mathfrak{r}_C^\times$. By continuity, we can extend the operator χ to $\bar{\mathbf{M}}(N)$. We see easily that if $\mathbf{f} \in \bar{\mathbf{M}}_{k,w}(U_0(Np^\alpha), \psi', \psi; \mathcal{O})$, then $\mathbf{f}|\chi \in \bar{\mathbf{M}}_{k,w}(U_0(NM^2p^\alpha), \psi', \psi\chi^2; \mathcal{O})$.

We now introduce a similar but different operator for each Hecke character $\theta: F_{\mathbf{A}}^\times/F^\times \to \mathbf{C}^\times$ whose conductor is $C(\theta)$ and whose infinity type is given by $\xi \in \mathbf{Z}[I]$, i.e. $\theta(y_\infty) = y_\infty^\xi$. Then automatically ξ is parallel and θ has algebraic value on finite ideles. We then define $\mathbf{f} \otimes \theta$ by $\mathbf{a}(y, \mathbf{f}\otimes\theta) = \theta(y_f)\mathbf{a}(y, \mathbf{f})\{y^\xi\}$ or equivalently, $\mathbf{a}_p(y, \mathbf{f}\otimes\theta) = \theta(y_f)\mathbf{a}_p(y, \mathbf{f})y_p^\xi$. We see easily that $\mathbf{f} \otimes \theta(x) = \theta(d)\theta(\det(x))\mathbf{f}(x)$. Thus

if $\mathbf{f} \in \mathbf{M}_{k,w}(Np^\alpha, \psi', \psi; \mathcal{O})$, *then* $\mathbf{f} \otimes \theta \in \mathbf{M}_{k,w+\xi}(C(\theta)Np^\alpha, \psi', \theta_p^{-1}, \psi\theta^2; \mathcal{O})$.

In the special case when $\theta(y) = |y|_{\mathbf{A}}^{-1}$, we see $\xi = -t$ and

$$\mathbf{a}_p(y, \mathbf{f}\otimes\theta) = \theta(y_f)\mathbf{a}_p(y, \mathbf{f})y_p^{-t} = \mathcal{N}_{F/\mathbf{Q}}(y\mathfrak{r})y_p^{-t}\mathbf{a}_p(y, \mathbf{f}) = \mathcal{N}(y)a_p(y, \mathbf{f}).$$

In general, any Hecke character θ of above type is a product $\theta = \chi|\ |_{\mathbf{A}}^{-j}$ for a finite order character χ and an integer j, and hence $\mathbf{a}_p(y, \mathbf{f}\otimes\theta) = \chi\mathcal{N}^j(y)\mathbf{a}_p(y, \mathbf{f})$. Thus this operator also extends to $\bar{\mathbf{M}}(N)$ by continuity.

G. Differential operator.

For each $\sigma \in I$, there is an operator $d^\sigma: \bar{\mathbf{M}}(N) \to \bar{\mathbf{M}}(N)$ defined by $\mathbf{a}_p(y, \mathbf{f}|d^\sigma) = y_p^\sigma \mathbf{a}_p(y, \mathbf{f})$ and $\mathbf{a}_{0,p}(y, \mathbf{f}|d^\sigma) = 0$. Thus we define for $0 \leqslant r = \sum_\sigma r_\sigma \sigma \in \mathbf{Z}[I]$, $d^r = \prod_\sigma d^{\sigma r_\sigma}$. Then we see

$$\mathbf{a}_p(y, \mathbf{f}|d^r) = y_p^r \mathbf{a}_p(y, \mathbf{f}) \quad \text{and} \quad \mathbf{a}_{0,p}(y, \mathbf{f}|d^r) = 0.$$

In fact, by [K], (2.6.27), there exists a differential operator $\theta(\sigma): V(\mathfrak{c}, \mathcal{O}) \otimes \mathbf{Q}_p \to V(\mathfrak{c}, \mathcal{O}) \otimes \mathbf{Q}_p$ such that

$$\theta(\sigma)\left(\sum_\xi a(\xi)q^\xi\right) = \sum_\xi \xi^\sigma a(\xi)q^\xi,$$

where $V(\mathfrak{c}, \mathcal{O})$ is the space of p-adic modular forms defined in [K], § 1.9. We can show the stability of the subspace $\bar{\mathbf{M}}(N)$ under this operator d^r as follows. Let

$$M_{k,w}(\Gamma(Np^\infty; \mathfrak{a}_i); K) = \lim_{\overrightarrow{\alpha}} M_{k,w}(\Gamma(Np^\alpha; \mathfrak{a}_i); K)$$

and $\bar{M}_{k,w}(\Gamma(Np^\infty; \mathfrak{a}_i); K)$ be the completion inside

$$K[[q]]_i = \left\{a(0) + \sum_{0 \ll \xi \in \mathfrak{a}_i\mathfrak{d}^{-1}} a(\xi)q^\xi \,\middle|\, a(\xi) \in K\right\}$$

of $M_{k,w}(\Gamma(Np^\infty;\mathfrak{a}_i);K)$ under the norm $|f|_p = \mathrm{Sup}_\xi |a(\xi,f)|_p$. Then we can easily show that there is an operator $\theta(\sigma)$ acting on $\bar{M}_{k,w}(\Gamma(Np^\infty;\mathfrak{a}_i);K)$ defined as above. We first choose a_i so that $F_{\mathbf{A}}^\times = \bigcup_{i=1}^{h} F^\times a_i U_F(1) F_{\infty+}^\times$ and then choose $u_j \in \mathfrak{r}_p^\times$ so that

$$F_{\mathbf{A}}^\times = \bigcup_j \bigcup_{i=1}^{h(N)} F^\times a_i u_j V_F(Np^\alpha) F_{\infty+}^\times .$$

Then we can find $f_{\alpha,i,j} \in M_{k,w}(\Gamma(Np^\alpha;\mathfrak{a}_i);K)$ so that $|a(\xi,f_{\alpha,i,j}) - \xi^\sigma a(\xi,\mathbf{f}_i)|_p < p^{-\alpha}$ and $|a(0,f_{\alpha,i,j})|_p < p^{-\alpha}$. Define

$$\mathbf{a}_p(y,\mathbf{f}_\alpha) = a(\xi,f_{\alpha,i,j})(a_i^{-1}u_j^{-1}ud)_p^{-v}(a_i^{-1}u_j^{-1}d)_p^\sigma \mathcal{N}(a_i)^{-1}$$

if $y = \xi a_i^{-1} u_j^{-1} ud$ with $u_p \equiv 1 \bmod p^\alpha$. Then $\mathbf{f}_\alpha$ converges to $\mathbf{f}|d^\sigma$ under the topology in $\bar{\mathbf{M}}_{k,w}(V_1(N)(p^\infty);\mathcal{O})$. Then we know

$$\mathbf{a}_p(y,\mathbf{f}|d^r) = a(\xi,\mathbf{f}_i|\theta^r)(a_i^{-1}ud)^{-v+r}\mathcal{N}(a_i^{-1}) \quad \text{if} \quad y = \xi a_i^{-1}ud .$$

Thus $\mathbf{f}_i|\theta^r$ behaves as if it were a modular form in $M_{k+2r,w+r}(\Gamma(Np^\alpha;\mathfrak{a}_i);K)$. This is reasonable because each element a of the subgroup $\mathfrak{r}_p^\times$ of $\mathbf{G}(N)$ acts on $\mathbf{f}|d^r$ by $\mathbf{a}_p(y,\mathbf{f}|d^r|a) = \mathbf{a}_p(ya,\mathbf{f}|d^r)$ and if $a \equiv 1 \bmod p^\alpha$ for sufficiently large α, then $\mathbf{a}_p(ya,\mathbf{f}|d^r) = \mathbf{a}^{r-v}a_p(y,\mathbf{f}|d^r)$ and hence the p-adic weight of $\mathbf{f}|d^r$ is $(n+2r, v-r)$ (which corresponds to $(k+2r,w+r)$).

H. Relations of operators under Petersson inner product.

We put

$$X_0(\mathfrak{m}) = G(\mathbf{Q})\backslash G(\mathbf{A})/U_0(\mathfrak{m})F_{\mathbf{A}}^\times C_{\infty+} = G(\mathbf{Q})_+\backslash G(\mathbf{A})_+/U_0(\mathfrak{m})F_{\mathbf{A}}^\times C_{\infty+} .$$

Then, we define for $\mathbf{f} \in \mathbf{N}_{k,w,\mu}(\mathfrak{m},\psi',\psi;\mathbf{C})$ and $\mathbf{g} \in \mathbf{N}_{k,w,\mu'}(\mathfrak{m},\psi',\psi;\mathbf{C})$,

$$(7.1) \qquad (\mathbf{f},\mathbf{g})_{\mathfrak{m}} = \int_{X_0(\mathfrak{m})} \bar{\mathbf{f}}(\bar{x})\mathbf{g}(x)|\det(x)|_{\mathbf{A}}^{[k-2w]}\, d\mu_{\mathfrak{m}}(x)$$

whenever the integral is convergent. Here we used the measure $d\mu_{\mathfrak{m}}$ defined in § 4. We first determine the adjoint of the twisted trace operator $T_{L/N}$. Assume $\mathbf{g} \in \mathbf{M}_{k,w}(Lp^\alpha,\psi',\psi;\mathbf{C})$ and $\mathbf{f} \in \mathbf{S}_{k,w}(Np^\alpha,\psi',\psi;\mathbf{C})$ for a divisor N of an ideal L prime to p. Then with the notation in V, we can prove in a standard manner that

$$(7.2\text{ a}) \quad (\mathbf{f},\mathbf{g}|T_{L/N})_{Np^\alpha} = \{(N/L)^v\}\mathcal{N}(L/N)^{1-m}(\mathbf{f}|[L/N],\mathbf{g})_{Lp^\alpha}.$$

Similar computation yields for $\alpha \in G(\mathbf{A}_f)$

$$\text{(7.2 b)} \qquad (\mathbf{f},\mathbf{g}|[U\alpha U'])_{\mathfrak{m}} = \chi(\det(\alpha))(\mathbf{f}|[U'\alpha^{\iota}U],\mathbf{g})_{\mathfrak{m}'},$$

where $\alpha^{\iota}\alpha = \det(\alpha)$ for $U' = U_0(\mathfrak{m}')$. As in § 4, we define the unitarization of $\mathbf{f} \in \mathbf{M}_{k,w}(Lp^{\alpha},\psi',\psi;\mathbf{C})$ by

$$\mathbf{f}^u(x) = D^{-(m/2)-1}\mathbf{f}(x)j(x_\infty,z_0)^k|\det(x)|_{\mathbf{A}}^{m/2} \quad \text{for} \quad m = m(P).$$

Then we see that

$$\text{(7.2 c)} \qquad (\mathbf{f}^u,\mathbf{g}^u)_{\mathfrak{m}} = D^{-m(P)-2}(\mathbf{f},\mathbf{g})_{\mathfrak{m}}.$$

Assuming $k_\sigma > 2m_\sigma$ for all σ, we now define the holomorphic projection map $\mathfrak{h} : \mathbf{N}_{k,w,m}(U;A) \to \mathbf{M}_{k,w}(U;A)$ for any $\mathbf{Q}$-algebra A inside $\mathbf{C}$ by $\mathfrak{h}(\mathbf{f}) = \mathbf{f}_0$ using the expression in Proposition 1.2. The following result has been shown in [Sh1], Lemma 4.11 and [Sh3], Lemma 2.3 :

PROPOSITION 7.1. — *Suppose that* $k_\sigma > 2m_\sigma$ *for all* σ. *If* $\mathbf{f} \in \mathbf{S}_{k,w}(Np^{\alpha},\psi',\psi;A)$, *then* $(\mathbf{f},\mathfrak{h}(\mathbf{g}))_{Np^{\alpha}} = (\mathbf{f},\mathbf{g})_{Np^{\alpha}}$ *for all*

$$\mathbf{g} \in \mathbf{N}_{k,w,m}(Np^{\alpha},\psi',\psi;A).$$

We have a natural multiplication :

$$\mathbf{N}_{k,w,m}(Np^{\alpha},\chi',\chi;A) \times \mathbf{N}_{\kappa,\omega,\mu}(Np^{\alpha},\psi',\psi;A) \to \mathbf{N}_{k+\kappa,w+\omega,m+\mu}(Np^{\alpha},\chi'\psi',\chi\psi;A)$$
$$(\mathbf{f},\mathbf{g}) \mapsto \mathbf{fg}(x) = \mathbf{f}(x)\mathbf{g}(x).$$

The relation of $\mathfrak{h}$ and the differential operators in (1.7) can be given as follows :

PROPOSITION 7.2. — *Suppose that* $k_\sigma > 2m_\sigma$ *and* $\kappa_\sigma > 2\mu_\sigma$ *for all* σ *and* $\mathbf{g} \in \mathbf{N}_{\kappa,\omega,\mu}(U_0(Np^{\alpha}),\chi',\chi;A)$ *and* $\mathbf{f} \in \mathbf{N}_{k,w,m}(U_0(Np^{\alpha}),\psi',\psi;A)$. *Then we have* $\mathfrak{h}(\mathbf{f}\delta_\kappa^r\mathbf{g}) \in \mathbf{S}_{k+\kappa+2r,w+\omega+r}(U_0(Np^{\alpha}),\chi'\psi',\chi\psi;A)$ *if* $r > 0$ *and*

$$\mathfrak{h}(\mathbf{f}\delta_\kappa^r\mathbf{g}) = (-1)^r\mathfrak{h}(\mathbf{g}\delta_k^r\mathbf{f}),$$

where $(-1)^r = (-1)^{\{r\}}$.

Proof. — Note that $\mathbf{f}\delta_\kappa^r\mathbf{g} \in \mathbf{N}_{k+\kappa+2r,w+\omega+r,m+\mu+r}(U_0(Np^{\alpha}),\chi',\chi;A)$ and $k_\sigma + \kappa_\sigma + 2r_\sigma > 2(m_\sigma+\mu_\sigma+r_\sigma)$ for all σ. Thus $\mathfrak{h}$ is well defined. Note that $(\mathbf{f}\delta_\kappa^r\mathbf{g})_i = \mathbf{f}_i\delta_\kappa^r\mathbf{g}_i$. Thus we consider only

$$\mathbf{f}_i\delta_\kappa^r\mathbf{g}_i = \mathfrak{h}(\mathbf{f}_i\delta_\kappa^r\mathbf{g}_i) + \sum_{0<s\leqslant m+\mu+r} \delta_{k+\kappa+2r-2s}^s\mathbf{h}_{s,i}$$

with $\mathbf{h}_s \in \mathbf{M}_{k+\kappa+2r-2s,w+\omega+r-s}(U_0(Np^\alpha),\psi',\psi;A)$ for a fixed i. Define $c:\mathbf{N}_{k,w,m}(\Gamma^i(U_0(Np^\alpha)),\psi;A)\to A[[q]]_i$ by

$$c(f) = a(0,f)(0) + \sum_{0\ll\xi\in\mathfrak{a}_i\mathfrak{d}^{-1}} a(\xi,f)(0)q^\xi,$$

which is the constant term as the polynomial of Y. Note that $d_\sigma((4\pi y)^{-r}q^\xi) = -r_\sigma(4\pi y)^{-r-\sigma}q^\xi + \xi(4\pi y)^{-r}q^\xi$. Thus we know

$$c(d^r(\mathbf{f})) = d^r(c(\mathbf{f})).$$

Then we know from (6.6) that

$$(*)\qquad \mathfrak{h}(\mathbf{f}_i\delta_\kappa^r\mathbf{g}_i) = c(\mathbf{f}_i)d^rc(\mathbf{g}_i) - \sum_{0<s\leqslant m+\mu+r} d^rc(\mathbf{h}_{s,i}).$$

Since $a(0,d^sc(\mathbf{h}))=0$ if $s>0$, we see $a(0,\mathfrak{h}(\mathbf{f}_i\delta_\kappa^s\mathbf{g}_i))=0$. This is true for all translation of $\mathfrak{h}(\mathbf{f}_i\delta_\kappa^r\mathbf{g}_i)|\alpha$ by $\alpha\in G(\mathbf{Q})_+$. Thus $\mathfrak{h}(\mathbf{f}_i\delta_\kappa^r\mathbf{g}_i)$ is a cusp form if $r>0$. Then the result follows from the argument in [H4], Lemma 5.3.

We now define a formal q-expansion

$$c(\mathbf{f}) = \mathcal{N}^{-1}(y)\left\{\mathbf{a}_{0,p}(yd,\mathbf{f})(0) + \sum_{0\ll\xi\in F^\times} \mathbf{a}_p(\xi yd,\mathbf{f})(0)q^\xi\right\}$$

for each $\mathbf{f}\in\mathbf{N}_{k,w,m}(U_0(Np^\alpha),\psi',\psi;A)$. By (*) we have

$$a(\xi,\mathfrak{h}(\mathbf{f}_i\delta_\kappa^r\mathbf{g}_i)) = a(\xi,c(\mathbf{f}_i)d^rc(\mathbf{g}_i)) - \sum_{0<s\leqslant m+\mu+r}\xi^s a(\xi,\mathbf{h}_{s,i}).$$

Thus if $y=\xi a_i^{-1}du$, by (1.3 b), we see

$$\begin{aligned}\mathcal{N}(a_i)\mathbf{a}_p(y,\mathfrak{h}(\mathbf{f}\delta_\kappa^r\mathbf{g}))\\ &= \mathcal{N}(a_i)\mathbf{a}_p(y,c(\mathbf{f}\delta_\kappa^r\mathbf{g})) - \sum_{0<s\leqslant m+\mu+r}\mathcal{N}(a_i)y_p^s\mathbf{a}_p(y,\mathbf{h}_{s,i}).\end{aligned}$$

This shows that

$$(7.3)\quad \mathfrak{h}(\mathbf{f}\delta_\kappa^r\mathbf{g}) = c(\mathbf{f}\delta_\kappa^r\mathbf{g}) - \sum_{0<s\leqslant m+\mu+r}\mathbf{h}_s|d^s \quad \textit{inside}\quad \mathscr{C}(\mathscr{I}\cup Z;A).$$

Since $\mathbf{a}_p(y,\mathbf{f}|T_0(\varpi^m)) = \mathbf{a}_p(y\varpi^m,\mathbf{f})\{\varpi^{-mv}\}\varpi_p^{mv}$ if $\mathbf{f}\in\mathbf{M}_{k,w}(S(p^\infty);\mathcal{O})$, we know that for the quotient field K of $\mathcal{O}$,

$$|\mathbf{a}_p(y,\mathbf{h}_s|d^s|T_0(p^m))|_p = |\mathbf{a}_p(yp^m,\mathbf{h}_s|d^s)|_p = |p^{ms}y_p^s\mathbf{a}(y,\mathbf{h}_s)|_p \leqslant |p^{ms}|_p\|\mathbf{h}_s\|_p$$

and thus $\mathbf{h}_s|d^s|e = 0$ for the idempotent e defined in § 3. Namely we have :

Proposition 7.3. – *Suppose that* $k_\sigma > 2m_\sigma$ *and* $\kappa_\sigma > 2\mu_\sigma$ *for all* σ *and* $\mathbf{g} \in \mathbf{N}_{\kappa,\omega,\mu}(Np^\alpha,\chi',\chi;\bar{\mathbf{Q}})$ *and* $\mathbf{f} \in \mathbf{N}_{k,w,m}(Np^\alpha,\psi',\psi;\bar{\mathbf{Q}})$. *Then* $c(\mathbf{f})$ *exists in* $\bar{\mathbf{M}}_{k,w}(Np^\infty,\psi',\psi;\Omega)$ *and we have*

$$e(\mathfrak{h}(\mathbf{f}\delta_\kappa^r\mathbf{g})) = e(c(\mathbf{f}\delta_\kappa^r\mathbf{g})) = e(c(\mathbf{f})d^r c(\mathbf{g})).$$

The existence of $c(\mathbf{f})$ in the space of p-adic modular form follows from (7.3) and G by replacing $\delta_\kappa^r\mathbf{g}$ by the constant 1. The other assertion follows from the above argument.

Now we can prove the p-adic version of Proposition 7.2 :

Proposition 7.4. – *In* $\mathbf{M}_{k,w}(U_0(Np^\alpha),\psi',\psi;K)$, *we have*

$$e((\mathbf{f}|\chi)\mathbf{g}) = \chi_\infty(-1)e((\mathbf{g}|\chi)\mathbf{f}) \quad \textit{and} \quad e((\mathbf{f}|d^r)\mathbf{g}) = (-1)^r e((\mathbf{g}|d^r)\mathbf{f}).$$

Proof. – As already seen, $e((\mathbf{f}|d^\sigma)\mathbf{g}) + e((\mathbf{g}|d^\sigma)\mathbf{f}) = e((\mathbf{gf})|d^\sigma) = 0$. From this, the second assertion follows. Note that, by choosing sufficiently large n so that $p^n \equiv 1 \bmod N$, we see $yp_p^n = \xi p^n a_i^{-1} dup'^{-n}$ and $up'^{-n} \in U_F(N)$ if $y = \xi a_i^{-1}ud$ with $u \in U_F(N)$ where $p'p_p = p$ in $F_\mathbf{A}$. Thus

$$\mathbf{a}_p(y,\mathbf{f}|T_0(p^n)) = \mathbf{a}_p(yp_p^n,\mathbf{f})(\{p^v\}p^v)^n = \mathbf{a}(\xi p^n,\mathbf{f}_i)(a_i^{-1}ud)_p^{-v}\mathcal{N}^{-1}(a_i).$$

Then the first assertion follows from the fact that $(\mathbf{fg})_i = \mathbf{f}_i\mathbf{g}_i$ and

$$\begin{aligned} a(\xi p^n,\mathbf{f}_i(\mathbf{g}_i|\chi)) &= \sum_{\alpha+\beta=\xi p^n} \chi(\beta\mathfrak{a}_i^{-1}\mathfrak{d})a(\alpha,\mathbf{f}_i)a(\beta,\mathbf{g}_i) \\ &= \sum_{\alpha+\beta=\xi p^n} \chi(-\alpha\mathfrak{a}_i^{-1}\mathfrak{d})a(\alpha,\mathbf{f}_i)a(\beta,\mathbf{g}_i) = \chi_\infty(-1)a(\xi p^n,\mathbf{g}_i(\mathbf{f}_i|\chi)). \end{aligned}$$

8. Eisenstein measure.

In this section, we interpret the result of § 6 into p-adic setting. Namely fixing an ideal L of $\mathfrak{r}$ prime to p, we give a definition of the Eisenstein measures on the p-adic group $\bar{\mathbf{G}}(L) = \bar{Z}(L) \times \mathfrak{r}_p^\times$, where

$$\bar{Z}(L) = \bar{Z}(L)/F_\infty^\times = F_\mathbf{A}^\times/\overline{F^\times U_F(Lp^\infty)F_\infty^\times}.$$

Although our construction is covered by Katz [K], (4.2), our view point is more adelic. We use the same notation introduced in § 3. We write

$U^* = \bar{\mathbf{M}}(L)$ (thus we write U for the $\mathcal{O}$-dual of U^*). Then we can state the following theorem-definition:

THEOREM 8.1. – *We have a unique $\mathcal{O}$-linear map $E: \mathscr{C}(\mathbf{G}(L);\mathcal{O}) \to U^*$ such that*

$$\text{(8.1 a)} \quad \mathcal{N}(y)^{-1}\mathbf{a}_p\left(y, \int_{\mathbf{G}(L)} \phi(w^{-1},z)w^{-t}\mathcal{N}(z)\,dE\right) = \begin{cases} \sum_{\mathfrak{a} \supset y\mathfrak{r}, \mathfrak{a}+Lp=\mathfrak{r}} \phi(y_p,[\mathfrak{a}]), & \text{if } y\mathfrak{r} + Lp = \mathfrak{r} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbf{a}_{0,p}\left(y, \int_{\mathbf{G}(L)} w^{-t-r}\phi(z)\mathcal{N}(z)\,dE\right) = 0$$

for any continuous function $\phi: Z(L) \times \mathfrak{r}_p^\times \to \mathcal{O}$ such that $\phi(w,z)\mathcal{N}(z)$ factors through $\bar{\mathbf{G}}(L)$, where w (resp. z) *denotes the variable on $\mathfrak{r}_p^\times$* (resp. $Z(L)$). *If the Leopoldt conjecture holds for F and p, then E has values in $\bar{\mathbf{S}}(L)$.*

Proof. – We first show the existence of E with values in U^*. Consider a quadruple (θ,χ,κ,r) consisting of a finite order Hecke character θ of p-ower conductor, an integer $\kappa > 0$, a finite order character χ of $Z(L)$ with $\chi_\infty(x_\infty) = x_\infty^{\kappa t}/|x_\infty^{\kappa t}|$ and a weight $0 \leqslant r \in \mathbf{Z}[I]$. By (8.1 a), we see, if $y_{L_p} \in \mathfrak{r}_{Lp^\times}$,

$$\mathbf{a}_p\left(y, \int_{\bar{\mathbf{G}}(L)} \theta(w^{-1})w^{-r}\chi(z)\mathcal{N}^\kappa(z)\,dE\right) = \mathcal{N}(y)\theta(y_p)y_p^r \sum_{\mathfrak{a} \supset y\mathfrak{r}, \mathfrak{a}+Lp=\mathfrak{r}} \chi(\mathfrak{a})\mathcal{N}(\mathfrak{a})^{\kappa-1}.$$

Thus, it is sufficient to prove for all quadruples (θ,χ,κ,r), we have $\mathbf{E} = \mathbf{E}(\theta,\chi,\kappa,r) \in U^*$ such that

$$\text{(8.1 b)} \quad \mathcal{N}(y)^{-1}\mathbf{a}_p(y,\mathbf{E}) = \begin{cases} \theta(y_p)y_p^r \sum_{\mathfrak{a} \supset y\mathfrak{r}, \mathfrak{a}+Lp=\mathfrak{r}} \chi(\mathfrak{a})\,\mathcal{N}(\mathfrak{a})^{\kappa-1} & \text{if } y\mathfrak{r} + Lp = \mathfrak{r} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbf{a}_{0,p}(y,\mathbf{E}) = 0,$$

because functions of the form $(a,z) \mapsto \theta(a)^{-1}a^{-r}\chi(z)\mathcal{N}^{\kappa}(z)$ are dense in $\mathscr{C}(\mathbf{G}(L);\Omega)$ for the p-adic completion Ω of $\bar{\mathbf{Q}}_p$. We see easily from Corollary 6.2 and § 7.F-G that $\mathbf{E}(\theta,\chi,\kappa,0)$ is a constant multiple of

$$|y|_{\mathbf{A}}^{-(\kappa/2)} G_{-\kappa t}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, \chi\theta^2, \theta^{-1}; 1-\frac{\kappa}{2}\right)\Big|\theta^{-1}.$$

Here we use the twisting operator ι_p in place of θ when θ is trivial. This shows

$$\mathbf{E}(\theta,\chi,\kappa,0) \in \mathbf{M}_{\kappa t,t}(V_1(L)(p^{\infty}); \bar{\mathbf{Q}})$$

and

$$\mathbf{E}(\theta,\chi,\kappa,r) = d^r\mathbf{E}(\theta,\chi,\kappa,0) \in U^*,$$

and we know the existence of E having values in U^*. If the Leopoldt conjecture is true for F and p, we can find an Eisenstein series G_n of $\mathbf{M}_{mt,t}(U_0(p),\mathbf{Q})$ with the following property [Col]:

(8.2) $G_n \in \mathbf{M}_{rt,t}(U_0(p),\mathbf{Q})$ *for each positive integer* n *such that* (i) $r = p^{n-1}(p-1)$, (ii) $G_n \equiv 1 \bmod p^{n-\varepsilon}\mathbf{Z}_p$ *for* $0 \leqslant \varepsilon \in \mathbf{Z}$ *independent of* n *and* (iii) $\mathbf{a}_{0,p}(y, G_n|\gamma) = 0$ *if* $\gamma \in G(\mathbf{A}_f) - B(\mathbf{A}_f)F_{\mathbf{A}}^{\times}U_0(p)$.

It is easy to show that $G_n|[p^r]$ satisfies the above condition for $U_0(p^r)$ instead of $U_0(p)$. Then we see that $G_n\mathbf{E} \in \bar{\mathbf{S}}(L)$ and $\mathbf{E} = \lim_{n\to\infty} G_n\mathbf{E} \in \bar{\mathbf{S}}(L)$. This shows that E has values in $\bar{\mathbf{S}}(L)$. Thus, in fact, the p-adic cuspidality of E follows if one assumes the existence of sequence of modular forms $\{G_n\}$ as in (8.2) without assuming the Leopoldt conjecture.

For a finite order character $\varepsilon: Z(L) \to \Omega$ with $\varepsilon\mathcal{N}^j (j>0)$ factoring through $\bar{Z}(L)$, we see

(8.3 a) $$\mathbf{a}_p\left(y, \int_{\bar{\mathbf{G}}(L)} \theta(w^{-1})w^{-r}\varepsilon(z)\,\mathcal{N}^j(z)\,dE\right) = \theta(y_p)y_p^r\mathcal{N}(y) \sum_{\mathfrak{a}\supset y\mathfrak{r},\,\mathfrak{a}+Lp=\mathfrak{r}} \varepsilon(\mathfrak{a})\,\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})^{j-1} \quad \text{if} \quad y\mathfrak{r} + Lp = \mathfrak{r},$$

(8.3 b) $$\mathbf{a}_p\left(y, \int_{\bar{\mathbf{G}}(L)} \theta(w^{-1})w^{-jt+t-r}\varepsilon(z)\,\mathcal{N}^{2-j}(z)\,dE\right) = \mathcal{N}(y)^{2-j}y_p^r\theta(y_p) \sum_{\mathfrak{a}\supset y\mathfrak{r},\,\mathfrak{a}+Lp=\mathfrak{r}} \varepsilon(y\mathfrak{r}/\mathfrak{a})\,\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})^{j-1} \quad \text{if} \quad y\mathfrak{r} + Lp = \mathfrak{r},$$

because $\mathcal{N}(y_p) = \mathcal{N}_{F/\mathbf{Q}}(y\mathfrak{r})^{-1}\mathcal{N}(y)$. Thus the measure E has values in $\bar{\mathbf{M}}_{\kappa t,t}(V_1(L)(p^\infty);K)$ for any $\kappa > 0$. Then modifying the action of $\mathbf{G}$ on $\bar{\mathbf{M}}_{\kappa t,t}(V_1(L)(p^\infty);K)$ so that $\mathbf{f}|_t\langle z,a\rangle = a^t\mathbf{f}|\langle z,a\rangle$, we see that

$$(8.4)\quad \int_{\bar{\mathbf{G}}(L)} \phi(w,z)\, dE(z)|_t\langle z',a\rangle = \int_{\bar{\mathbf{G}}(L)} \phi(aw,z'z)\, \mathcal{N}(z')^2\, dE(z).$$

9. Convoluted measure.

In this section, we define a convolution of the Eisenstein measure E and an $\mathcal{O}[[\mathbf{G}]]$-linear map $\varphi : M^* \to \bar{\mathbf{S}}(L)$ for an $\mathcal{O}[[\mathbf{G}]]$-module M. Let L be an ideal of $\mathfrak{r}$ prime to p. As in § 3, we decompose $\mathbf{G}(L) = \mathbf{W} \times \mathbf{G}_{\mathrm{tor}}(L)$ for the torsion-free part $\mathbf{W}$ and a finite group $\mathbf{G}_{\mathrm{tor}}(L)$. We write $\mathbf{A}$ for the continuous group algebra $\mathcal{O}[[\mathbf{W}]]$. We begin with

LEMMA 9.1. — *Any $\mathbf{A}$-submodule M of an $\mathbf{A}$-free module of finite rank satisfies the following conditions :* (i) $M \cong \varprojlim_\alpha M_\alpha$ *as $\mathbf{A}$-module ;* (ii) M_α *is an $\mathbf{A}$-module and is $\mathcal{O}$-free of finite rank ; and* (iii) *The transition maps : $M_\beta \to M_\alpha$ are all surjective. Moreover if an $\mathbf{A}$-module M satisfies* (i), (ii) *and* (iii), *we have*

$$M^* = \mathrm{Hom}_{\mathcal{O}}(M,\mathcal{O}) \cong \varprojlim_\alpha \left(\left(\varinjlim_\beta M_\beta^*\right)\otimes_{\mathcal{O}}\mathcal{O}/p^\alpha\mathcal{O}\right),$$

$$(M^*)^* \cong M \quad\text{and}\quad \mathrm{Hom}_{\mathbf{A}}(M,\mathbf{A}^*) \cong M^* \quad\text{and}\quad \mathrm{Hom}_{\mathbf{A}}(M^*,\mathbf{A}^*) \cong M.$$

Proof. — Let $E \cong \mathbf{A}^r$ be a $\mathbf{A}$-free module of finite rank and suppose that M is a $\mathbf{A}$-submodule of E. Writing $\mathbf{W}^\alpha = \mathbf{W}/\mathbf{W}_p^\alpha$, we see $\mathbf{A} = \varprojlim_\alpha \mathcal{O}[\mathbf{W}^\alpha]$, and hence E satisfies the conditions (i)-(iii) for $E_\alpha = \mathcal{O}[\mathbf{W}^\alpha]^r$. Since $\mathbf{A}$ is noetherian, M is a closed submodule of E and hence M is compact. In the category of pro-objects made of compact abelian groups, the functor of projective limit is exact. Thus we know $M = \varprojlim_\alpha M_\alpha$ for the image M_α of M in E_α. Then it is easy to verify the conditions (i)-(iii) for M_α. The second part of the lemma follows from the proof of [H3], Proposition 7.1. Especially, the pairing

$\langle\ ,\ \rangle : M \times M^* \to \mathbf{A}^*$ is given by

$$\langle m,m^*\rangle(a) = m^*(am) \quad \text{for} \quad m \in M,\ m^* \in M^* \quad \text{and} \quad a \in \mathbf{A}.$$

We now recall the space U^* introduced in Theorem 8.1, which is the closure of the space $\left\{\sum_{k,w} \bar{\mathbf{M}}_{k,w}(V_1(L)(p^\infty);L)\right\} \cap \mathscr{C}(\mathscr{I}\cup Z;\mathcal{O})$ in $\mathscr{C}(\mathscr{I}\cup Z;\mathcal{O})$ under the norm $\|\ \|_p$ in § 3 (thus U is the $\mathcal{O}$-dual of U^*). Note that U satisfies the condition (i), (ii) and (iii) of Lemma 9.1. In fact, we can take U_α to be the $\mathcal{O}$-dual of $\sum_{0<k<\alpha t,\, t\geqslant w\geqslant -\alpha t} \bar{\mathbf{M}}_{k,w}(V_1(L)(p^\alpha);K) \cap \mathscr{C}(\mathscr{I}\cup Z;\mathcal{O})$. Then we see $U = \varprojlim_\alpha U_\alpha$. On the other hand, the multiplication induces a product $m' : U^* \times \mathbf{S} \to |D|\mathbf{S}$, where $\mathbf{S} = \bar{\mathbf{S}}(L)$ (see (1.4)). Thus we define $m : U^* \times \mathbf{S} \to \mathbf{S}$ by $m(\mathbf{f},\mathbf{g}) = |D|^{-1}m'(\mathbf{f},\mathbf{g})$. Then by definition, we see

$$m(\mathbf{f}|_t\langle z,a\rangle,\mathbf{g}|\langle z,a\rangle) = m(\mathbf{f},\mathbf{g})|\langle z,a\rangle$$

where the action $\mathbf{f} \mapsto \mathbf{f}|_t\langle z,a\rangle$ is given above (8.4) and, as seen in (8.4), the Eisenstein measure $E : \mathscr{C}(\mathbf{G}(L);\mathcal{O}) \to U^*$ becomes an $\mathcal{O}[[\mathbf{G}(L)]]$-linear map under the action : $\mathbf{f} \mapsto \mathbf{f}|_t\langle z,a\rangle$ on U^* and the action :

$$\phi \mapsto \phi|_2(z,a)(w,z') = \phi(aw,z'z)\mathcal{N}(z')^2 \quad \text{on} \quad \mathscr{C}(\mathbf{G}(L);\mathcal{O}).$$

Let J be a divisor of L and consider a compact $\mathcal{O}[[\mathbf{G}(J)]]$-module M satisfying the conditions (i)-(iii) of Lemma 9.1. We consider M as a $\mathcal{O}[[\mathbf{G}(L)]]$-module via a natural projection of $Z(L)$ to $Z(J)$. Let $\varphi : M^* \to \mathbf{S}$ be an $\mathcal{O}[[\mathbf{G}]]$-linear map for $\mathbf{G} = Z(L) \times \mathfrak{r}_p^\times$. Define

$$\hat{\varphi} : M^* \mathbin{\hat{\otimes}_{\mathcal{O}}} U^* \to \mathbf{S} \qquad \text{by} \qquad \hat{\varphi} = m \circ (\varphi \otimes \mathrm{id}),$$

where $M^* \mathbin{\hat{\otimes}_{\mathcal{O}}} U^*$ is a p-adic completion $\varprojlim (M^* \otimes U^*)/p^j(M^* \otimes U^*)$.

On the other hand, $M \mathbin{\hat{\otimes}_{\mathcal{O}}} U$ denotes the profinite completion $\varprojlim_{i\ j} M_i \otimes U_j$. We say that a function $\varphi : M \to U^*$ is continuous if it is continuous under the p-adic topology on U^* and under the topology of the profinite group M. Thus if ϕ is $\mathcal{O}$-linear, then ϕ is continuous if and only if there exists $i > 0$ for any $j > 0$ such that $\phi \bmod p^j : M/p^jM \to U^*/p^jU^*$ factors through M_i/p^jM_i, where $M = \varprojlim M_i$ as in Lemma 9.1. Then, since $U = \varprojlim U_k$ for $\mathcal{O}$-free modules U_k of finite rank satisfying the condition of Lemma 9.1, the image of

$\phi \bmod p^j$ is actually contained in $U_k^*/p^jU_k^*$ for some k. We denote by $\mathrm{Hom}_c(M, U^*)$ the space of all continuous $\mathcal{O}$-linear maps. Then we have

$$\mathrm{Hom}_c(M,U^*) = \varprojlim_j \varinjlim_i \varinjlim_k \mathrm{Hom}_{\mathcal{O}}(M_i/p^jM_i,\ U_k^*/p^jU_k^*)$$
$$= \varprojlim_j \varinjlim_{i\,k} \mathrm{Hom}_{\mathcal{O}}(M_i/p^jM_i \otimes_{\mathcal{O}} U_k, \mathcal{O}/p^j\mathcal{O}) = (M\hat{\otimes}_{\mathcal{O}}U)^*,$$

where $M\hat{\otimes}_{\mathcal{O}}U$ is the profinite completion of $M \otimes_{\mathcal{O}} U$.

LEMMA 9.2. – *Let M and U be* **A**-*modules satisfying the conditions* (i) (ii) *and* (iii) *of Lemma* 9.1. *Then, we have*

$$M^* \hat{\otimes}_{\mathcal{O}} U^* \simeq \mathrm{Hom}_c(M,U^*) \simeq (M \hat{\otimes}_{\mathcal{O}} U)^*,$$

where the identification is given by $\phi \otimes u^* \mapsto (\phi\otimes u^*)(m) = \phi(m)u^*$.

Proof. – We have a natural map: $M^* \otimes_{\mathcal{O}} U^* \to \mathrm{Hom}_c(M,U^*)$ given by $\phi \otimes u \mapsto (m \to \phi(m)u)$. By definition, $M \hat{\otimes}_{\mathcal{O}} U = \varprojlim_{i\,j} M_i \otimes U_j$ satisfies the assumption of Lemma 9.1 and thus

$$(M\hat{\otimes}_{\mathcal{O}}U)^* \cong \varprojlim_m \left(\varprojlim_{i\,j} M_i^* \otimes U_j^*\right) \otimes_{\mathcal{O}} \mathcal{O}/p^m\mathcal{O} = M^*\hat{\otimes}_{\mathcal{O}}U^*,$$

which proves the assertion.

For each continuous function $\Phi \in \mathscr{C}(M \times \mathbf{G}; \mathcal{O})$, we define an action of $\mathbf{G} = \mathbf{G}(L)$ by $(\Phi|(z,a))(m,z',a') = \Phi((z^{-1},a^{-1})m, zz', aa')$. We now define $E_* : \mathscr{C}(M \times \mathbf{G}; \mathcal{O}) \to \mathscr{C}(M, U^*)$ by

$$E_*(\Phi)(m) = \int_{\mathbf{G}(L)} (\Phi|(z,a))(m,1)\, dE(z,a).$$

Then E_* induces on $M^* \hat{\otimes}_{\mathcal{O}} \mathscr{C}(\mathbf{G};\mathcal{O})(\cong \mathrm{Hom}_c(M, \mathscr{C}(\mathbf{G};\mathcal{O}))$ which can be regarded as subspace of $\mathscr{C}(M \times \mathbf{G}; \mathcal{O}))$ a morphism into $\mathrm{Hom}_c(M,U^*)(\cong M^*, \hat{\otimes}_{\mathcal{O}} U^*)$. Then we have

$$E_*(\Phi)(m) = \int_{\mathbf{G}(L)} \Phi((z^{-1},a^{-1})m)(z,a)\, dE(z,a) \quad \text{for} \quad \Phi \in \mathrm{Hom}_c(M, \mathscr{C}(\mathbf{G};\mathcal{O})).$$

We now define the convoluted measure $E * \varphi : M^* \hat{\otimes}_{\mathcal{O}} \mathscr{C}(\mathbf{G};\mathcal{O}) \to \mathbf{S}$ by

$$E * \varphi(\Phi) = \hat{\varphi}(E_*(\Phi)).$$

By our construction, if we let $(z,a) \in Z(L) \times \mathfrak{r}_p^\times$ act on Φ via $\Phi\|(z,a)(m)(x,b) = \Phi(m)(zx,ab)\mathcal{N}(z)^2$ regarding Φ as an element of $\mathrm{Hom}_c(M,\mathscr{C}(\mathbf{G};\mathcal{O}))$, then

(9.1) $E * \varphi$ *is a morphism of* $\mathcal{O}[[\mathbf{G}]]$*-modules*.

In fact, under the diagonal action of $\mathbf{G}$ on $M^* \hat{\otimes}_{\mathcal{O}} U^*$, $\hat{\varphi}$ is $\mathcal{O}[[\mathbf{G}]]$-linear. Thus we see

$$(9.2)\quad (E_*(\Phi)|(z,a))(m) = \int_{\mathbf{G}(L)} (\Phi|(x,b))((z,a)m,1)\,dE(x,b)|_t(z,a)$$
$$= \int_{\mathbf{G}(L)} (\Phi|(zx,ab))((z,a)m,1)\,dE(x,b)$$
$$= \int_{\mathbf{G}(L)} \Phi(x^{-1},b^{-1})m,(zx,ab))\mathcal{N}(z)^2\,dE$$
$$= \int_{\mathbf{G}(L)} (\Phi\|(z,a)|(x,b))(m,1)\,dE(x,b).$$

This shows the compatibility of the action with $E * \varphi$.

We now fix another divisor N of L. We can naturally identify the maximal torsion free quotient of $\mathbf{G}(N)$ with $\mathbf{W}$ via the natural surjection : $\mathbf{G}(L) \to \mathbf{G}(N)$. As in § 3, we fix decompositions : $\mathbf{G}(N) = \mathbf{W} \times \mathbf{G}_{\mathrm{tor}}(N)$, $Z(N) = W \times Z_{\mathrm{tor}}(N)$ and $\mathfrak{r}_p^\times = W' \times \mu$. Then we may assume that $\mathbf{W} = W \times W'$ and $\mathbf{G}_{\mathrm{tor}}(N) = Z_{\mathrm{tor}}(N) \times \mu$. Let $\mathbf{L}$ be the quotient field of $\mathbf{A} = \mathcal{O}[[\mathbf{W}]]$. We fix a finite extension $\mathbf{K}$ of $\mathbf{L}$ inside the fixed algebraic closure $\bar{\mathbf{L}}$ and denote by $\mathbf{I}$ the integral closure of $\mathbf{A}$ in $\mathbf{K}$. Then there exists an $\mathbf{A}$-free submodule M of $\mathbf{K}$ of rank $[\mathbf{K} : \mathbf{L}]$ containing $\mathbf{I}$ (cf. [B2], V.1.6, Lemma 3). Thus $\mathbf{I}$ satisfies the conditions of Lemma 9.1. We now take a primitive $\mathbf{A}$-algebra homomorphism $\lambda : \mathbf{h}^{\mathrm{n.ord}}(N;\mathcal{O}) \to \mathbf{K}$, which factors through $\mathbf{h}(\psi,\psi')$ defined in § 5. Then λ has values in $\mathbf{I}$. By composing the multiplication map : $\mathbf{I} \otimes_{\mathbf{A}} \mathbf{I} \to \mathbf{I}$ with $\lambda \otimes \mathrm{id}$, we can naturally extend λ to an $\mathbf{I}$-algebra homomorphism of $\mathbf{h}(\psi,\psi') \otimes_{\mathbf{A}} \mathbf{I}$ into $\mathbf{I}$, which we again denote by λ. We now consider the $\mathbf{I}$-algebra decomposition $\mathbf{h}(\psi,\psi') \otimes_{\mathbf{A}} \mathbf{K} = \mathbf{K} \oplus \mathbf{B}$ induced by λ such that the first projection to $\mathbf{K}$ coincides with λ, and we let 1_λ denote the indempotent of the factor $\mathbf{K}$. Let pr be the second projection into $\mathbf{B}$ and put $\mathscr{C}(\lambda) = (\mathbf{I} \oplus pr(\mathbf{h}(\psi,\psi') \otimes_{\mathbf{A}} \mathbf{I}))/\mathbf{h}(\psi,\psi') \otimes_{\mathbf{A}} \mathbf{I}$. Then by [H2], Th. 2.4, $\mathscr{C}(\lambda)$ is a torsion $\mathbf{I}$-module. We take $0 \neq H \in \mathbf{I}$ which annihilates $\mathscr{C}(\lambda)$. Then $H1_\lambda \in \mathbf{h}(\psi,\psi') \otimes_{\mathbf{A}} \mathbf{I}$. We now decompose $\mathbf{S}^{\mathrm{n.ord}}(N;K) = \mathbf{S}(\psi,\psi';K) \oplus X$ so that $\mathbf{G}_{\mathrm{tor}}(N)$ acts on $\mathbf{S}(\psi,\psi';K)$ via the character (ψ,ψ') and on X

via other characters. We write $\mathbf{S}(\psi,\psi')$ for the intersection of $\mathbf{S}^{\mathrm{n.ord}}(N;\mathcal{O})$ with $\mathbf{S}(\psi,\psi';K)$. We also write $\hat{\mathbf{M}} = \mathrm{Hom}_{\mathbf{A}}(\mathbf{M},\mathbf{A})$ for any $\mathbf{A}$-module $\mathbf{M}$ and we define a pairing $\langle\, ,\rangle$ between $\mathbf{h}(\psi,\psi')\otimes_{\mathbf{A}}\mathbf{I}$ and $\mathbf{S}(\psi,\psi')\otimes_{\mathbf{A}}\hat{\mathbf{I}}$ as follows :

$$\text{(9.3 a)} \qquad \langle \mathbf{h}\otimes\mathbf{i},\mathbf{f}\otimes\hat{\mathbf{j}}\rangle = (\mathbf{h}\hat{\mathbf{j}}(\mathbf{i}),\mathbf{f}) = (\mathbf{h},\mathbf{f}|\hat{\mathbf{j}}(\mathbf{i})),$$

where $(\, ,)$ is the pairing in Theorem 3.1. We then define

$$\text{(9.3 b)} \qquad l_\lambda : \mathbf{S}(\psi,\psi')\otimes_{\mathbf{A}}\hat{\mathbf{I}} \to \mathcal{O} \quad \text{by} \quad l_\lambda(\mathbf{f}) = \langle H1_\lambda,\mathbf{f}\rangle .$$

For $P\in\mathcal{X}(\mathbf{I})$, we have an $\mathcal{O}$-algebra homomorphism

$$\lambda_P = P\circ\lambda : \mathbf{h}(\psi,\psi')\otimes_{\mathbf{A}}\mathbf{I} \to \bar{\mathbf{Q}}_p .$$

Suppose $P\in\mathcal{A}(\mathbf{I})$. Then, we have characters ψ_P and ψ'_P associated to P as introduced in § 5. For a suitable Σ-tuple $\alpha = (\alpha(\mathfrak{p}))_{\mathfrak{p}\in\Sigma}$ and for $k = n(P) + 2t$ and $w = t - v(P)$, as seen in § 3, we find $\mathbf{f}_P \in \mathbf{S}^{\mathrm{n.ord}}_{k,w}(N\mathfrak{p}^\alpha,\psi'_P,\psi_P;\bar{\mathbf{Q}})$ such that

$$\text{(9.4)} \quad \mathbf{a}_p(y,\mathbf{f}_P) = \lambda_P(T(y))y_p^{-v} = P(\lambda(\mathbf{T}(y))) \quad \textit{or equivalently,}$$
$$\mathbf{a}(y,\mathbf{f}_P) = \lambda_P(T(y))\{y^{-v}\} = P(\lambda(\mathbf{T}(y)))y_p^{v}\{y^{-v}\} \quad \textit{for } y\in\hat{\mathfrak{r}}\cap F_{\mathbf{A}_f}^\times .$$

For complex conjugation ρ, we write $\mathbf{f}_P^\rho$ for the cusp form in $\mathbf{S}^{\mathrm{n.ord}}_{k,w}(N\mathfrak{p}^\alpha,\psi_P'^{-1},\psi_P^{-1};\bar{\mathbf{Q}})$ defined by $\mathbf{a}(y,\mathbf{f}_P^\rho) = \mathbf{a}(y,\mathbf{f}_P)^\rho$ (see (2.1 a)).

LEMMA 9.3. – *Let $\mathbf{f}_P$ be as above for $P\in\mathcal{A}(\mathbf{I})$. Then we have*

$$l_\lambda(\mathbf{g}) = H(P)(\mathbf{f}_P^\rho|\tau(n\varpi^\alpha),\mathbf{g}|e)_\alpha/(\mathbf{f}_P^\rho|\tau(n\varpi^\alpha),\mathbf{f}_P)_\alpha$$

for all $\mathbf{g}\in\mathbf{S}_{k,w}(N\mathfrak{p}^\alpha,\psi'_P,\psi_P;\bar{\mathbf{Q}})$, where $(\, ,)_\alpha = (\, ,)_{N\mathfrak{p}^\alpha}$.

Proof. – By definition, we have $\mathbf{f}|\tau(x) = \chi^{-1}(\det(x))\mathbf{f}(x\tau(\mathfrak{m}))$ for $\mathbf{f}\in\mathbf{S}_{k,w}(\mathfrak{m},\chi',\chi;\mathbf{C})$. We write U for $U_0(\mathfrak{m})$. Then it is easy to see from this formula that

$$\chi^{-1}(\det(\alpha))(\mathbf{f}^\rho|\tau)|[U\alpha U] = (\mathbf{f}^\rho|[U\alpha^\iota U])|\tau,$$

where $\alpha = \begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}$ with a prime element ϖ of $\mathfrak{r}_\mathfrak{q}$ for a prime $\mathfrak{q}$. We now suppose that $\mathfrak{m} = N\mathfrak{p}^\alpha$, $\chi = \psi_P$ and $\chi' = \psi'_P$. Since

$$\mathbf{h}_{k,w}(\mathfrak{m},\chi',\chi;\mathbf{C}) = \mathbf{h}_{k,w}(\mathfrak{m},\chi',\chi;\mathfrak{r}_\Phi)\otimes_{\mathfrak{r}_\Phi}\mathbf{C}$$

and

$$\mathbf{h}_{k,w}(\mathfrak{m},\chi',\chi;\mathcal{O}) = \mathbf{h}_{k,w}(\mathfrak{m},\chi',\chi;\mathfrak{r}_\Phi)\otimes_{\mathfrak{r}_\Phi}\mathcal{O},$$

we can extend $\lambda_P: \mathbf{h}_{k,w}(\mathfrak{m},\chi',\chi;\mathfrak{r}_\Phi) \to \mathbf{C}$ linearly to unique algebra homomorphism of $\mathbf{h}_{k,w}(\mathfrak{m},\chi',\chi;\mathbf{C})$ into $\mathbf{C}$, which we will denote again by λ_P. We first show that the linear form $l: \mathbf{f} \in \mathbf{S}_{k,w}(\mathfrak{m},\chi',\chi;\mathbf{C}) \to \mathbf{C}$ defined by $l(\mathbf{g}) = (\mathbf{f}_P^\rho|\tau(N\mathfrak{p}^\alpha),\mathbf{g}|e)_\alpha/(\mathbf{f}_P^\rho|\tau(N\mathfrak{p}^\alpha),\mathbf{f}_P)_\alpha$ is the unique one satisfying $l(\mathbf{g}|h) = \lambda_P(h)l(\mathbf{g})$ for all $h \in \mathbf{h}_{k,w}(\mathfrak{m},\chi',\chi;\mathbf{C})$ and $l(\mathbf{f}_P) = 1$. The uniqueness is obvious by the duality theorem (Theorem 2.2) and the fact : $l(\mathbf{f}_P) = 1$ which follows directly from the definition. By (7.2 b), we have

$$(\mathbf{f}_P^\rho|\tau,\mathbf{g}|[U\alpha U]) = \lambda_P([U\alpha U])(\mathbf{f}_P^\rho|\tau,\mathbf{g}).$$

This shows the desired property of l. Now, by definition, l_λ satisfies the same property, i.e. $l_\lambda(\mathbf{g}|h) = \lambda_P(h)l_\lambda(\mathbf{g})$ for operators $h \in \mathbf{h}_{k,w}(\mathfrak{m},\chi',\chi;\bar{\mathbf{Q}}_p)$. Thus l_λ is a constant multiple of l on $\mathbf{S}_{k,w}(N\mathfrak{p}^\alpha,\psi'_P,\psi_P;\bar{\mathbf{Q}})$. Since $l_\lambda(\mathbf{f}_P) = H(P)$, we know that $l_\lambda = H(P)l$ on $\mathbf{S}_{k,w}(N\mathfrak{p}^\alpha,\psi'_P,\psi_P;\bar{\mathbf{Q}})$. This finishes the proof of Lemma 9.3. Note that the expression on the left-hand side does not depend on the choice of $\alpha = (\alpha(\mathfrak{p}))_{\mathfrak{p}\in\Sigma}$.

Now we take an $\mathbf{A}$-free submodule $\mathbf{X}$ in $\mathbf{K}$ containing $\mathbf{I}$. Then $\mathbf{X}/\mathbf{I}$ is a torsion $\mathbf{A}$-module. We use the symbol : "*" to indicate $\mathcal{O}$-dual module while we use the symbol : "^" to indicate $\mathbf{A}$-dual module. We write $\psi: Z_{\mathrm{tor}}(N) \to \mathcal{O}^\times$ and $\psi': \mu \to \mathcal{O}^\times$ for the characters which gives the restriction of λ to $\mathbf{G}_{\mathrm{tor}}(N)$. Define a subspace of $\mathscr{C}(\mathbf{G};\mathcal{O})$ by

$$\mathscr{C}(\mathbf{G};\mathcal{O})[\psi,\psi'] = \{f \in \mathscr{C}(\mathbf{G};\mathcal{O}) \mid f(\zeta x) = \psi\psi'(\zeta)f(x) \text{ for all } \zeta \in \mathbf{G}_{\mathrm{tor}}\}.$$

We know, by restricting functions on $\mathbf{G}(L)$ to $\mathbf{W}$,

$$\mathscr{C}(\mathbf{G};\mathcal{O})[\psi,\psi'] \cong \mathscr{C}(\mathbf{W};\mathcal{O}) \quad \text{as } \mathbf{A}\text{-module}.$$

Similarly, we define

$$\bar{\mathbf{S}}(L;\psi,\psi') = \{f \in \bar{\mathbf{S}}(L) \mid f|\langle\zeta,\zeta'\rangle = \psi(\zeta)\psi'(\zeta')f \text{ for all } (\zeta,\zeta') \in \mathbf{G}_{\mathrm{tor}}\}.$$

Then $\mathbf{S}(\psi,\psi') = \bar{\mathbf{S}}(N;\psi,\psi') \cap \mathbf{S}^{\mathrm{n.ord}}(N;\mathcal{O}) = e(\bar{\mathbf{S}}(N;\psi,\psi'))$. On the other hand, we see

$$\mathscr{C}(\mathbf{W};\mathcal{O}) \otimes_{\mathbf{A}} \mathrm{Hom}_{\mathbf{A}}(\mathbf{X},\mathbf{A}) \cong \mathrm{Hom}_{\mathbf{A}}(\mathbf{A},\mathbf{A}^*) \otimes_{\mathbf{A}} \mathrm{Hom}_{\mathbf{A}}(\mathbf{X},\mathbf{A})$$
$$\cong \mathrm{Hom}_{\mathbf{A}}(\mathbf{A}\otimes_{\mathbf{A}}\mathbf{X},\mathbf{A}\otimes_{\mathbf{A}}\mathbf{A}^*) \cong \mathrm{Hom}_{\mathbf{A}}(\mathbf{X},\mathbf{A}^*) \cong \mathbf{X}^*.$$

Here, to get the second isomorphism, we have used the $\mathbf{A}$-freeness of $\mathbf{X}$. Thus we know $M^* \hat{\otimes}_{\mathcal{O}}(\mathscr{C}(\mathbf{G};\mathcal{O})[\psi,\psi'])\otimes_{\mathbf{A}}\hat{\mathbf{X}} = M^* \hat{\otimes}_{\mathcal{O}}\mathbf{X}^*$. We then define

$$\Psi : M^* \hat{\otimes}_{\mathcal{O}}(\mathscr{C}(\mathbf{G};\mathcal{O})[\psi,\psi']) \xrightarrow{E*\varphi} \bar{\mathbf{S}}(L;\psi,\psi') \xrightarrow{T_{L/N}} \bar{\mathbf{S}}(N;\psi,\psi') \xrightarrow{e} \bar{\mathbf{S}}(\psi,\psi')$$

and

$$E *_{\lambda} \varphi : M^* \hat{\otimes}_{\mathcal{O}} \mathbf{X}^* \to \mathcal{O} \quad \text{by} \quad l_{\lambda} \circ (\Psi \otimes \mathrm{id}).$$

Here we naturally consider $\hat{\mathbf{X}}$ as a submodule of $\hat{\mathbf{I}}$ and hence $l_{\lambda} : \mathbf{S}(\psi,\psi')\otimes_{\mathbf{A}}\hat{\mathbf{I}} \to \mathcal{O}$ is well defined on $\mathbf{S}(\psi,\psi')\otimes_{\mathbf{A}}\hat{\mathbf{X}}$. We shall regard $E *_{\lambda} \varphi$ naturally as an element of the profinite completion $M \hat{\otimes}_{\mathcal{O}} \mathbf{X}$. We now show that $E *_{\lambda} \varphi$ is independent of the auxiliary choice of $\mathbf{X}$. Let first assume that $\mathbf{X}' \supset \mathbf{X} \supset \mathbf{I}$ and write $E *_{\lambda} \varphi(\mathbf{X})$ for $E *_{\lambda} \varphi$ obtained from $\mathbf{X}$. Then naturally, we have $\hat{\mathbf{X}} \supset \hat{\mathbf{X}}'$ and a natural map : $\mathbf{X}'^* \to \mathbf{X}^*$ by duality. Thus we have a commutative diagram :

$$\begin{array}{ccccccc} E*_{\lambda}\varphi(\mathbf{X}') : M^* \hat{\otimes}_{\mathcal{O}}(\mathscr{C}(\mathbf{G};\mathcal{O})[\psi,\psi'])\otimes_{\mathbf{A}}\hat{\mathbf{X}}' & \cong & M^* \hat{\otimes}_{\mathcal{O}}\mathbf{X}'^* & \to & \mathbf{S}(\psi,\psi')\otimes_{\mathbf{A}}\hat{\mathbf{X}}' & \to & \mathcal{O} \\ \downarrow & & \downarrow & & \downarrow & & \\ E*_{\lambda}\varphi(\mathbf{X}) : M^* \hat{\otimes}_{\mathcal{O}}(\mathscr{C}(\mathbf{G};\mathcal{O})[\psi,\psi'])\otimes_{\mathbf{A}}\hat{\mathbf{X}} & \cong & M^* \hat{\otimes}_{\mathcal{O}}\mathbf{X}^* & \to & \mathbf{S}(\psi,\psi')\otimes_{\mathbf{A}}\hat{\mathbf{X}} & \to & \mathcal{O}. \end{array}$$

This implies the image of $E *_{\lambda} \varphi(\mathbf{X})$ in $M \hat{\otimes}_{\mathcal{O}} \mathbf{X}'$ is given by $E *_{\lambda} \varphi(\mathbf{X}')$. Thus $E *_{\lambda} \varphi$ is uniquely determined in $\varinjlim_{\mathbf{X}} M \hat{\otimes}_{\mathcal{O}} \mathbf{X}$ and is contained in $X = \bigcap_{\mathbf{X}} M \hat{\otimes}_{\mathcal{O}} \mathbf{X}$, where $\mathbf{X}$ runs all $\mathbf{A}$-free submodules containing $\mathbf{I}$.

LEMMA 9.4. – $E *_{\lambda} \varphi \in M \hat{\otimes}_{\mathcal{O}} \mathbf{I}$.

Proof. – We only need to show that

(*) $\mathbf{I} = \bigcap_{\mathbf{X}} \mathbf{X}$ *for finitely many* $\mathbf{A}$*-free lattices* $\mathbf{X}$.

Pick one $\mathbf{A}$-free lattice $\mathbf{X} \supset \mathbf{I}$. Then $\mathrm{Ass} = \mathrm{Ass}_{\mathbf{A}}(\mathbf{X}/\mathbf{I})$ consists of finitely many height one primes $P_1, \dots, P_r$. Let X_i denote the localization at P_i for any $\mathbf{A}$-module X. Then by the approximation theorem of Krull domains [B2] VII.1.5, we can find $\alpha_i \in \mathrm{Aut}_{\mathbf{L}}(\mathbf{K})$ such that $\alpha_i\mathbf{X}_i = \mathbf{I}_i$ for each i and $\alpha_i\mathbf{X} \supset \mathbf{I}$. Then $\mathbf{I} = \bigcap_i \alpha_i\mathbf{X} \cap \mathbf{X}$, which shows the assertion.

10. Proof of Theorem 5.1.

We take a primitive homomorphism $\varphi : \mathbf{h}^{\text{n.ord}}(\mathbf{J};\mathcal{O}) \to \mathbf{J}$ for the integral closure $\mathbf{J}$ of $\mathbf{A}$ in a finite extension $\mathbf{F}$ of $\mathbf{L}$. By the duality (Th. 3.1), we have $\varphi^* : \mathbf{J}^* \to \mathbf{S}^{\text{n.ord}}(J;\mathcal{O})$. Let $L = N \cap J$. We take $\mathbf{J}^*$ as M^* and $[L/J] \circ \varphi^*$ as φ in the previous section. We therefore write φ' for $[L/J] \circ \varphi^*$. Let (χ,χ') be the restriction of φ to $\mathbf{G}_{\text{tor}}(J)$; i.e., $\chi : Z_{\text{tor}}(J) \to \mathcal{O}^\times$ and $\chi' : \mu \to \mathcal{O}^\times$. Then $(E*_\lambda\varphi'^*)$ gives an element of $(\mathbf{J}^* \hat{\otimes}_\mathcal{O} \mathbf{X}^*)^* \cong \mathbf{J} \hat{\otimes}_\mathcal{O} \mathbf{X}$. We define an element $\mathcal{D} = \lambda * \varphi$ in the quotient field of $\mathbf{J} \hat{\otimes}_\mathcal{O} \mathbf{I}$ by $(E*_\lambda\varphi'^*)/1 \otimes H$. We now want to evaluate $\mathcal{D}$ at an arithmetic point $(P,Q) \in \mathcal{A}(\mathbf{I}) \times \mathcal{A}(\mathbf{J})$. Let $A \neq 0$ be an element of $\mathbf{I}$ which annihilate $\mathbf{X}/\mathbf{I}$ and write Φ_A for $(1 \otimes A)(E*_\lambda\varphi'^*)$ in $\mathbf{J} \hat{\otimes}_\mathcal{O} \mathbf{I}$. Since the localization $\mathbf{I}_P = \mathbf{I} \otimes_\mathbf{A} \mathbf{A}_P$ of $\mathbf{I}$ at P is $\mathbf{A}_P$-free for $P \in \mathcal{A}(\mathbf{I})$ ([H2], Th. 2.4, Cor. 2.5), we can always choose an $\mathbf{A}$-free module $\mathbf{X} \supset \mathbf{I}$ so that $A(P) \neq 0$. We can also choose H so that $H(P) \neq 0$ and H kills the congruence module $\mathcal{C}(\lambda;\mathbf{I})$. We first compute $\Psi = T_{T/L} \circ e \circ (E*\varphi)$. By extending scalars if necessary, we may assume that $P : \mathbf{I} \to \bar{\mathbf{Q}}_p$ and $Q : \mathbf{J} \to \bar{\mathbf{Q}}_p$ have values in $\mathcal{O}$. Then especially $Q \in \mathbf{J}^*$ and thus $Q \otimes \eta \in M^* \hat{\otimes}_\mathcal{O} \mathcal{C}(\mathbf{G};\mathcal{O})$ for any character η of $\mathbf{G}(L)$ with values in $\mathcal{O}$. We first compute $\Psi(Q \otimes \eta_P) \in \mathbf{S}(\psi,\psi')$ for the character $\eta_P : \mathbf{G}(L) \to \mathcal{O}^\times$ given by $\eta_P(z) = \psi_P(z)\psi'_P(a)\mathcal{N}(z)^{m(P)}a^{v(P)}$, where $v(P) \in \mathbf{Z}[I]$ and $m(P)t = n(P) + 2v(P) > 0$. Let

$$\mathbf{g} = \mathbf{g}_Q \in \mathbf{S}^{\text{n.ord}}_{\kappa,\omega}(J\mathfrak{p}^\alpha, \chi'_Q, \chi_Q; \mathcal{O})$$

be the cusp form corresponding to Q, where $m(Q) = [n(Q)+2v(Q)]$, $\kappa = n(Q) + 2t$, $\omega = t - v(Q)$ and $\alpha = (\alpha(\mathfrak{p}))_{\mathfrak{p} \in \Sigma}$ is a suitable Σ-tuple. Then regarding $Q \otimes \eta_P \in M^* \hat{\otimes}_\mathcal{O}(\mathcal{C}(\mathbf{G};\mathcal{O})[\psi,\psi'])$, we have by definition (see also (1.4))

$$(10.1)\quad |D|\Psi(Q \otimes \eta_P)$$
$$= T_{L/N} \circ e\,(\mathbf{g}_Q | [L/J] \cdot (\mathbf{E}(\chi'_Q \psi'^{-1}_P, \chi_Q^{-1}\psi_P, m(P) - m(Q), v(Q) - v(P))))$$

where $\mathbf{E}(\theta,\chi,\kappa,r)$ is as in (8.1 b). The discriminant $|D|$ appears in front of Ψ because we have divided the multiplication map m' by $|D|$.

Hereafter we assume that

(10.2) $\chi'_Q \psi'^{-1}_P$ *is a restriction to* $\mathfrak{r}_p^\times$ *of a character* θ *of* $\mathrm{Cl}(\mathfrak{p}^\alpha)$.

We now compute the value of Φ_A. We start from the algebra homomorphism $P : \mathbf{I} \to \mathcal{O}$ given by the point $P \in \mathcal{A}(\mathbf{I})$ as above. Suppose that $A(P) \neq 0$ and consider $\Phi_A(Q\otimes P) = A(P)(E*_\lambda\varphi)(Q\otimes P)$. Since $\mathbf{X}$ is $\mathbf{A}$-free of finite rank, we see from Lemma 9.1

$$\mathbf{S}(\psi,\psi') \otimes_{\mathbf{A}} \hat{\mathbf{X}} \cong \mathrm{Hom}_{\mathbf{A}}(\mathbf{h}(\psi,\psi'),\mathbf{A}^*) \otimes_{\mathbf{A}} \hat{\mathbf{X}}$$

$$\cong \mathrm{Hom}_{\mathbf{A}}(\mathbf{h}(\psi,\psi') \otimes_{\mathbf{A}} \mathbf{X},\mathbf{A}^*) \cong \mathbf{S}(\psi,\psi') \hat{\otimes}_{\mathcal{O}} \mathbf{X}^*.$$

Let $\mathbf{I}(\mathbf{X}) = \{i \in \mathbf{I} | \mathbf{X} \supset i\mathbf{X}\}$ (i.e. the order of $\mathbf{X}$), which is a subring of $\mathbf{I}$. Let $\mathbf{P} = P \cap \mathbf{I}(\mathbf{X})$. Note that $\Psi \otimes \mathrm{id}(Q\otimes P) \in \mathbf{S}(\psi,\psi') \hat{\otimes}_{\mathcal{O}} \mathbf{X}^*[\mathbf{P}]$ and $\mathbf{S}(\psi,\psi') \hat{\otimes}_{\mathcal{O}} \mathbf{X}^*[\mathbf{P}] \cong (\mathbf{h}(\psi,\psi')\hat{\otimes}_{\mathcal{O}}\mathbf{X}/\mathbf{PX})^*$ (cf. [H3], Prop. 7.3). Tensoring $I(\mathbf{X})/\mathbf{P}$ to the exact sequence : $0 \to \mathbf{I} \to \mathbf{X} \to \mathbf{X}/\mathbf{I} \to 0$, we see that the natural morphism : $\mathbf{I}/\mathbf{PI} \to \mathbf{X}/\mathbf{PX}$ is an isomorphism up to p-torsion, i.e., having finite kernel and cokernel. We also have $\mathbf{I}/\mathbf{PI} \cong \mathbf{I}/P\mathbf{I} \cong \mathcal{O}$ up to p-torsion. Thus the torsion-free part $\mathbf{X}/\mathbf{PX}^\circ$ of $\mathbf{X}/\mathbf{PX}$ contains naturally $\mathbf{I}/P\mathbf{I}(\cong \mathcal{O})$ and $A(P)$ annihilates $(\mathbf{X}/\mathbf{PX}^\circ)/(\mathbf{I}/P\mathbf{I})$. Thus we see

$$(\mathbf{h}(\psi,\psi')\hat{\otimes}_{\mathcal{O}} \mathbf{X}/\mathbf{PX})^* \cong \mathbf{S}^{\mathrm{n.ord}}(N\mathfrak{p}^\alpha,\psi'_P,\psi_P;\mathbf{X}/\mathbf{PX}^\circ)$$

which is a subspace of $\mathbf{S}^{\mathrm{n.ord}}(N\mathfrak{p}^\alpha,\psi'_P,\psi_P;K)$. Thus we see

$$|D|A(P)\Psi \otimes \mathrm{id}(Q\otimes P)$$
$$= A(P)T_{L/N}\circ e\{(\mathbf{g}_Q|[L/J]\cdot(\mathbf{E}(\theta_p,\chi_Q^{-1}\psi_P,m(P)-m(Q),v(Q)-v(P))\}$$
$$\in \mathbf{S}^{\mathrm{n.ord}}(U_0(\mathfrak{p}^\alpha),\varepsilon_P\psi_c\omega^{-m(P)};\mathcal{O}).$$

Write $K_0 = \bar{\mathbf{Q}} \cap K$. Then by Lemma 9.3, on $\mathbf{S}^{\mathrm{n.ord}}(Np^\alpha,\psi'_P,\psi_P;K_0)$, we have

$$l_\lambda(\mathbf{h}) = H(P)(\mathbf{f}_P^\rho|\tau(N\mathfrak{p}^\alpha),\mathbf{h})_\alpha/(\mathbf{f}_P^\rho|\tau(N\mathfrak{p}^\alpha),\mathbf{f}_P)_\alpha,$$

where $(\ ,\ )_\alpha = (\ ,\ )_\mathfrak{m}$ for $\mathfrak{m} = N\mathfrak{p}^\alpha$. Now we have

$$|D|\Phi_A(Q\otimes P) = A(P)H(P)(\mathbf{f}_P^\rho|\tau(N\mathfrak{p}^\alpha),\mathbf{h})_\alpha/(\mathbf{f}_P^\rho|\tau(N\mathfrak{p}^\alpha),\mathbf{f}_P)_\alpha$$

for

$$\mathbf{h} = T_{L/N}\circ e(\mathbf{g}_Q|[L/J]\int_{\bar{G}(L)} \theta_p^{-1}(a)a^{v(P)-v(Q)}\chi_Q^{-1}\psi_P(z)\mathcal{N}(z)^{m(P)-(Q)}dE(z,a)).$$

Recall that $\mathcal{D} = \lambda * \varphi = \Phi_A/AH$. We now compute the value $\mathcal{D}(P,Q)$ for arithmetic points P and Q. As in the theorem, we now suppose that

(10.3 a) $\qquad n(P) - n(Q) \geqslant t,$

(10.3 b) $\qquad n(Q)-n(P) + 2t \leqslant (m(P)-m(Q))t \quad$ and $\quad v(Q) \geqslant v(P).$

Then, (10.3 b) is equivalent to

$$\text{(10.3 c)}\quad n(Q) - n(P) + 2t \leqslant (m(P)-m(Q))t \leqslant n(P) - n(Q).$$

We divide our argument into the following two cases:

Case I: $m(P) - m(Q) \geqslant 1$ and Case II: $m(P) - m(Q) \leqslant 1$.

We put $j = m(P) - m(Q)$ in case I and $j = 2 - m(P) + m(Q)$ in Case II. Replacing k by $n(P) + 2t$, κ by $n(Q) + 2t$, w by $t - v(P)$ and ω by $t - v(Q)$, we can apply the result obtained in § 4:

Case I: $(j=m(P) - m(Q)\geqslant 1, v(Q)\geqslant v(P)\,; t \leqslant jt \leqslant n(P)-n(Q))$.

We see

$$|D|A(P)\Psi \otimes \mathrm{id}(Q\otimes P) = A(P)T_{L/N}\circ e\{\mathbf{g}_Q|[L/J]d^{v(Q)-v(P)}\,\mathbf{E}(\theta_p, \chi_Q^{-1}\psi_P, j, 0)\}.$$

We now claim that, for the Eisenstein series $\mathbf{G}_{jt,0}$ introduced in (4.8 e),

$$\text{(10.4 a)}\quad \mathbf{G}_{jt,0}\left(x,\chi_Q^{-1}\psi_P\theta^2,\theta^{-1};1-\frac{j}{2}\right)\Big|\theta^{-1} = c|D|^{-1}\theta^{-1}(d)\,\mathbf{E}(\theta_P,\chi_Q^{-1}\psi_P,j,0)$$

with $c = (2i)^{j[F:\mathbf{Q}]}\pi^{[F:\mathbf{Q}]}$. Writing $\mathbf{E}$ for $\mathbf{E}(\theta_p,\chi_Q^{-1}\psi_P,j,0)$, we see from the definition that

$$\mathbf{a}_p(y,\mathbf{E}) = \mathcal{N}(y)\theta(y_p)\sigma_{j-1,\chi}(y\mathfrak{r}) \quad \text{for} \quad \chi = \chi_Q^{-1}\psi_P \quad \text{and} \quad y_p \in \mathfrak{r}_p^\times$$
$$\text{and otherwise} \quad \mathbf{a}_p(y,\mathbf{E}) = 0.$$

Since $\mathbf{a}_p(ya,\mathbf{E}) = a^{-t}\theta(a)\mathbf{a}_p(y,\mathbf{E})$ for $a \in \mathfrak{r}_p^\times$ (see (8.4) or (2.2*c*)), the subgroup $\mathfrak{r}_p^\times$ of $\mathbf{G}(L)$ acts on $\mathbf{E}$ via the character $\mathfrak{r}_p^\times \ni a \mapsto a^{-t}$ (up to finite order character θ) and hence we can compute formally the Fourier expansion coefficient $\mathbf{a}(y,\mathbf{E})$ out of the q-expansion coefficient $\mathbf{a}_p(y,\mathbf{E})$. The tip for this transition is given in (1.3*b*): $\mathbf{a}(y,\mathbf{f}) = \mathbf{a}_p(y,\mathbf{f})\{y^{-t}\}(y_p)^t$. Then

$$\mathbf{a}(y,\mathbf{E}) = \{y^{-t}\}(y_P)^t\mathcal{N}(y)\theta(y_P)\sigma_{j-1,\chi}(y\mathfrak{r}) = |y|_{\mathbf{A}}^{-1}\theta(y_P)\sigma_{j-1,\chi}(y\mathfrak{r})y_\infty^t\{y^{-t}\},$$

since $\mathbf{a}_p(y,\mathbf{E}) = 0$ unless $y_p \in \mathfrak{r}_p^\times$. Now we compute the Fourier coefficients of $\mathbf{G} = \mathbf{G}_{jt,0}\left(x,\chi_Q^{-1}\psi_P\theta^2,\theta^{-1};1-\frac{j}{2}\right)$. Noting

$$\mathbf{G} = |y|_{\mathbf{A}}^{-j/2}G_{-jt}\left(x,\chi_Q^{-1}\psi_P\theta^2,\theta^{-1};1-\frac{j}{2}\right),$$

we see from Corollary 6.2, for $c = (2i)^{j[F:\mathbf{Q}]}\pi^{[F:\mathbf{Q}]}$,

$$\mathbf{a}(y,\mathbf{G}) = c|D|^{-1}\theta(d)^{-1}\theta(y\mathfrak{r})\mathbf{a}(y,\mathbf{E}) \quad \text{if} \quad y_p \in \mathfrak{r}_p^\times .$$

This shows (10.4 a).

Write $\mathfrak{m} = L\mathfrak{p}^\alpha$ and $m = l\varpi^\alpha$ with a finite idele l such that $lr = L$ and $ll_L^{-1} = 1$, and recall that the p-adic L-function $\mathscr{D}$ is defined by Φ_A/AH. Then, we see from Lemma 9.3

$$\begin{aligned}
&(\mathbf{f}_P^\rho|\tau(N\mathfrak{p}^\alpha),\mathbf{f}_P)_\alpha\mathscr{D}(P,Q)\\
&= c^{-1}\theta(d)(\mathbf{f}_P^\rho|\tau(N\mathfrak{p}^\alpha), T_{L/N}\circ e\{(\mathbf{g}_Q|[L/J])\cdot(d^{v(Q)-v(P)}\mathbf{G}|\theta^{-1})\})_\alpha\\
&= c^{-1}\theta(d)(\mathbf{f}_P^\rho|\tau(\mathfrak{m}), e\{(\mathbf{g}_Q|[L/J])\cdot(d^{v(Q)-v(P)}\mathbf{G}|\theta^{-1})\})_\mathfrak{m} \text{ (by (7.2 a))}\\
&= c^{-1}\theta(d)\theta_\infty(-1)(\mathbf{f}_P^\rho|\tau(\mathfrak{m}), e\{(\mathbf{g}_Q|[L/J]|\theta^{-1})\\
&\qquad\qquad\cdot(d^{v(Q)-v(P)}\mathbf{G})\})_\mathfrak{m} \text{ (Prop. (7.4)}\\
&= c^{-1}\theta(d)\theta(J/L)\theta_\infty(-1)(\mathbf{f}_P^\rho|\tau(\mathfrak{m}), e\{(\mathbf{g}_Q|\theta^{-1}|[L/J])\cdot(d^{v(Q)-v(P)}\mathbf{G})\})_\mathfrak{m}\\
&= c^{-1}\theta(d)\theta(J/L)(\theta^{-1}\chi_Q)_\infty(-1)\mathcal{N}_{F/\mathbf{Q}}(J/L)\mathcal{N}_{F/\mathbf{Q}}(L\mathfrak{p}^\alpha)^{-m(Q)}\\
&\qquad\qquad\times(\mathbf{f}_P^\rho|\tau(\mathfrak{m}), e\{(\mathbf{g}_Q|\theta^{-1}|\tau(J\mathfrak{p}^\alpha)|\tau(\mathfrak{m}))\cdot(d^{v(Q)-v(P)}\mathbf{G})\})_\mathfrak{m}\\
&= c^{-1}\theta(d)\theta(J/L)(\theta^{-1}\chi_Q)_\infty(-1)\mathcal{N}_{F/\mathbf{Q}}(J/L)\mathcal{N}_{F/\mathbf{Q}}(L\mathfrak{p}^\alpha)^{-m(Q)}\\
&\qquad\qquad\times(\mathbf{f}_P^\rho|\tau(\mathfrak{m}), \mathfrak{h}\{(\mathbf{g}_Q|\theta^{-1}|\tau(J\mathfrak{p}^\alpha)|\tau(\mathfrak{m}))\cdot(\delta_{jt}^{v(Q)-v(P)}\mathbf{G})\})_\mathfrak{m} \text{ (Prop. 7.3)}.
\end{aligned}$$

Here we used the fact : $\mathbf{h} = \mathbf{g}_Q|\theta^{-1} \in \mathbf{S}_{k(Q),w(Q)}(J\mathfrak{p}^\alpha, \chi'_Q, \chi_Q\theta^{-2};\mathbf{C})$ and

$$\mathbf{h}|\tau(J\mathfrak{p}^\alpha)|\tau(\mathfrak{m})(x) = (\chi_Q\theta^{-2})_\infty(-1)\mathcal{N}_{F/\mathbf{Q}}(L\mathfrak{p}^\alpha)^{m(Q)}\mathcal{N}_{F/\mathbf{Q}}(L/J)\mathbf{h}|[L/J].$$

By (4.9), we know that, for $\mathbf{g} = \mathbf{g}_Q|\theta^{-1}|\tau(J\mathfrak{p}^\alpha)$ and $r = v(Q) - v(P) \geqslant 0$

$$\begin{aligned}
&|D|^{(5+m(P)+m(Q))/2}L_N(2-j,\chi_Q^{-1}\psi_P)Z\left(-\frac{j}{2},\mathbf{f}_P,\mathbf{g},\theta^{-1}\right)\\
&\qquad = \theta^{-1}(m)\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{m})^{-1-m(Q)}\\
&\qquad\times\left(\mathbf{f}_P^\rho|\tau(\mathfrak{m}),\mathfrak{h}\left\{(\mathbf{g}|\tau(\mathfrak{m}))\mathbf{G}_{jt+2r,r}\left(x,\chi_Q^{-1}\psi_P\theta^2,\theta^{-1};1-\frac{j}{2}\right)\right\}\right)_\mathfrak{m}.
\end{aligned}$$

Moreover we see from corollary 6.3 that

$$\begin{aligned}
&\delta_{jt}^r\left\{\mathbf{G}_{jt,0}\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix},\chi,\theta;1-\frac{j}{2}\right)\right\}\\
&\qquad = \Gamma_F(r+t)(-4\pi)^{-\{r\}}\mathbf{G}_{jt+2r,r}\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix},\chi,\theta;1-\frac{j}{2}\right).
\end{aligned}$$

Here it might be worth making the following remark: By (1.8) and the definition of the correspondence of adelic automorphic forms and classical Hilbert modular forms, $(\delta^r_{\kappa t}\mathbf{G})_i = y_\infty^{-r}\,\delta^r_{\kappa t}\mathbf{G}_i$. Hence in the above formula, we have $(-4\pi)^{-\{r\}}$ instead of $(-4\pi y_\infty)^{-r}$. Thus $(\mathbf{f}^\rho_P|\tau(N\mathfrak{p}^\alpha),\mathbf{f}_P)_\alpha\mathscr{D}(P,Q)$ is equal to

(10.4 b)

$$c^{-1}\theta(J/L)(\theta^{-1}\chi_Q)_\infty(-1)\mathscr{N}_{F/\mathbf{Q}}(J\mathfrak{p}^\alpha)\Gamma_F(v(Q)-v(P)+t)(-4\pi)^{-\{v(Q)-v(P)\}}$$
$$\times\ \theta(dm)D^{(5+m(P)+m(Q))/2}L_{L_p}(2-j,\chi_Q^{-1}\psi_P)Z\left(-\frac{j}{2},\mathbf{f}_P,\mathbf{g},\theta^{-1}\right)$$

for $\mathbf{g} = \mathbf{g}_Q|\theta^{-1}|\tau(J\mathfrak{p}^\alpha)$. Now by (4.6), we have, with the notation there,

$$Z(s,\mathbf{f},\mathbf{g},\theta) = D^{(1+2s)/2}\theta(d)^{-1}(4\pi)^{-ts-(k+\kappa)/2}\Gamma_F\left(S+\frac{k}{2}+\frac{\kappa}{2}\right)D(s,\mathbf{f},\mathbf{g},\theta)$$

and hence

$$Z\left(-\frac{j}{2},\mathbf{f}_P,\mathbf{g},\theta^{-1}\right) = \theta(d)D^{(1-m(P)+m(Q))/2}(4\pi)^{v(P)-n(Q)-v(Q)-2t}$$
$$\times\ \Gamma_F(n(Q)+v(Q)-v(P)+2t)D\left(\frac{m(Q)-m(P)}{2},\mathbf{f}_P,\mathbf{g},\theta^{-1}\right).$$

This combined with (7.2 c), $(\mathbf{f}^u_P)^\rho|\tau(N\mathfrak{p}^\alpha) = |n\varpi^\alpha|_{\mathbf{A}}^{m(P)/2}(\mathbf{f}^\rho_P|\tau(N\mathfrak{p}^\alpha))^u$, $(\mathbf{f}^\rho_P|\tau(N\mathfrak{p}^\alpha),\mathbf{f}_P)_\alpha = D^{m(P)+2}((\mathbf{f}^\rho_P|\tau(N\mathfrak{p}^\alpha))^u,\mathbf{f}^u_P)_\alpha$ and $\theta(L/J) = \theta(m)/\theta(j\varpi^\alpha)$ shows

(10.4 c) $$\theta(d^2)^{-1}\theta\chi_{Q\infty}(-1)|n\varpi^\alpha|_{\mathbf{A}}^{-m(P)/2}((\mathbf{f}^u_P)^\rho|\tau(N\mathfrak{p}^\alpha),\mathbf{f}^u_P)_\alpha\mathscr{D}(P,Q)$$
$$= \theta(j\varpi^\alpha)\mathscr{N}_{F/\mathbf{Q}}(J\mathfrak{p}^\alpha)C(P,Q)$$
$$\times\ \mathrm{L}_{Lp}(2-\mathrm{m(P)}+m(Q),\chi_Q^{-1}\psi_P)D\left(\frac{m(Q)-m(P)}{2},\mathbf{f}_P,\mathbf{g}_\alpha,\theta^{-1}\right),$$

where $\mathbf{g}_\alpha = \mathbf{g}_Q|\theta^{-1}|\tau(J\mathfrak{p}^\alpha)$ and $C(P,Q)$ is as in the theorem.

Case II: $(1\leqslant j=2-m(P)+m(Q),\ v(Q)\geqslant v(P);\ t\leqslant jt\leqslant n(P)-n(Q))$. From (10.3 b), we have

(10.5) $$r = n(P)-n(Q)+v(P)-v(Q)-t\geqslant 0.$$

Writing $\mathbf{E}$ for $\mathbf{E}(\theta_p,\chi,2-j,j-1)(\chi=\chi_Q^{-1}\psi_P)$, we have similarly to Case I:

$$\mathbf{a}_p(y,\mathbf{E}) = \mathscr{N}(y)^{2-j}\theta(y_p)\sigma'_{j-1,\chi}(y\mathfrak{r})\quad\text{for}\quad\chi=\chi_Q^{-1}\psi_P\quad\text{and}\quad y_p\in\mathfrak{r}_p^\times$$
$$\text{and otherwise}\quad\mathbf{a}_p(y,\mathbf{E}) = 0.$$

Moreover, the subgroup $\mathfrak{r}_p^\times$ acts on $\mathbf{E}$ via the character $\mathfrak{r}_p^\times \ni a \mapsto a^{(j-2)t}$ up to finite order character θ, and hence we have from Corollary 6.2

$$\mathbf{a}(y,\mathbf{G}') = D^{-j}\theta(d)^{-1}c'\theta(y\mathfrak{r})\mathbf{a}(y,\mathbf{E}) \quad \text{if} \quad y_p \in \mathfrak{r}_p^\times$$

for $\mathbf{G}' = |y|_{\mathbf{A}}^{-j/2}G_{-jt}\left(x,\chi_Q^{-1}\psi_P\theta^2,\theta^{-1};\frac{j}{2}\right)$ and $c' = (2\pi i)^{j[F:\mathbf{Q}]}\Gamma_F(jt)^{-1}$. Since $\mathbf{G}_{jt,0}\left(x,\chi_Q^{-1}\psi_P\theta^2,\theta^{-1};\frac{j}{2}\right) = |y|_{\mathbf{A}}^{(j-2)/2}G_{-jt}\left(x,\chi_Q^{-1}\psi_P\theta^2,\theta^{-1};\frac{j}{2}\right)$, keeping the fact : $\mathbf{f}\otimes\theta(x) = \theta(d)\theta(\det(x))\mathbf{f}(x)$ in mind, we know

$$\text{(10.6 a)} \quad \left\{\mathbf{G}_{jt,0}\left(x,\chi_Q^{-1}\psi_P\theta^2,\theta^{-1};\frac{j}{2}\right)\otimes|\det(x)|_{\mathbf{A}}^{1-j}\right\}|\theta^{-1} = c'D^{-1}\theta^{-1}(d)\mathbf{E}(\theta_p,\chi_Q^{-1}\psi_P,(2-j)t,(j-1)t).$$

Then, we see from the definition in § 7.G that

$$\left\{d^r(\mathbf{G}_{jt,0}\left(x,\chi_Q^{-1}\psi_P\theta^2,\theta^{-1};\frac{j}{2}\right)\otimes|\det(x)|_{\mathbf{A}}^{1-j})\right\}|\theta^{-1} = c'|D|^{-1}\theta^{-1}(d)\mathbf{E}(\theta_p,\chi_Q^{-1}\psi_P,(2-j)t,(j-1)t+r).$$

Especially, if we take r as in (10.5), we will have $(j-1)t + r = v(Q) - v(P)$.

Now, writing $\mathbf{G}$ for $\mathbf{G}_{jt,0}\left(x,\chi_Q^{-1}\psi_P\theta^2,\theta^{-1};\frac{j}{2}\right)$ and $\mathfrak{m} = L\mathfrak{p}^\alpha$, by a similar computation as in Case I, we see from Lemma 9.3

$$\begin{aligned}&(\mathbf{f}_P^\rho|\tau(N\mathfrak{p}^\alpha),\mathbf{f}_P)_\alpha\mathscr{D}(P,Q)\\ &\quad = c'^{-1}\theta(d)\theta(J/L)(\chi_Q\theta^{-1})_\infty(-1)\mathcal{N}_{F/\mathbf{Q}}(L\mathfrak{p}^\alpha)^{-1-2m(Q)+m(P)}\\ &\quad \times \mathcal{N}_{F/\mathbf{Q}}(J/L)(\mathbf{f}_P^\rho|\tau(\mathfrak{m}),\mathfrak{h}\{(\mathbf{g}_Q|\theta^{-1}|\tau(J\mathfrak{p}^\alpha)\otimes|\det(x)|_{\mathbf{A}}^{1-j}|\tau(\mathfrak{m}))\cdot(\delta_{jt}^r\mathbf{G})\})_{\mathfrak{m}}.\end{aligned}$$

The only difference in computation form Case I is that we used the following fact to conclude the above identity :

$$f\otimes\theta|\tau(\mathfrak{m}) = \bar{\theta}(d)^{-1}\theta(d\mathfrak{m})((f|\tau(\mathfrak{m})\otimes\bar{\theta}).$$

Again by Corollary 6.3, we have

$$\Gamma_F(jt)\delta_{jt}^r\left\{\mathbf{G}_{jt,0}\left(\begin{pmatrix}y & x\\ 0 & 1\end{pmatrix},\chi,\theta;\frac{j}{2}\right)\right\} = \Gamma_F(jt+r)(-4\pi)^{-r}\mathbf{G}_{jt+2r,r}\left(\begin{pmatrix}y & x\\ 0 & 1\end{pmatrix},\chi,\theta;\frac{j}{2}\right).$$

Now we want to apply (4.9). Note that the weight of $\mathbf{g}_Q|\theta^{-1}|\tau(J\mathfrak{p}^\alpha)\otimes|\det(x)|_{\mathbf{A}}^{1-j}$ is given by $(n(Q), v(Q)+(j-1)t)$ (see § 7.F) and hence μ in (4.9) is equal to $m(Q)+2j-2$ and m in (4.9) is given by $m(P)$. Then we see

$$\begin{aligned}(10.6\text{ b})\quad & (\mathbf{f}_P^\rho|\tau(N\mathfrak{p}^\alpha),\mathbf{f}_P)_\alpha\mathscr{D}(P,Q)\\ &= c'^{-1}\Gamma_F(jt)^{-1}D^{(5+m(P)+m(Q))/2}\theta(dm)\theta(J/L)(\chi_Q\theta^{-1})_\infty(-1)\mathcal{N}_{F/\mathbf{Q}}(J\mathfrak{p}^\alpha)\\ &\times\Gamma_F(t-v(P)+v(Q))(-4\pi)^{-r}L_m(j,\chi_Q^{-1}\psi_P)Z\left(\frac{j}{2}-1,\mathbf{f}_P,\mathbf{g},\theta^{-1}\right),\end{aligned}$$

where $\mathbf{g}=\mathbf{g}_Q|\theta^{-1}|\tau(J\mathfrak{p}^\alpha)$. In fact, what we get in the right-hand side is $Z\left(\frac{j}{2}-1,\mathbf{f}_P,\mathbf{g}\otimes|\ |_{\mathbf{A}}^{1-j},\theta^{-1}\right)$, but it is equal to $Z\left(\frac{j}{2}-1,\mathbf{f}_P,\mathbf{g},\theta^{-1}\right)$ because this function only depends in the unitarizations of $\mathbf{f}_P$ and $\mathbf{g}$ by definition and the unitarization of $\mathbf{g}$ and $\mathbf{g}\otimes|\ |_{\mathbf{A}}^{1-j}$ are the same. Similarly to (10.4*c*), we conclude

$$\begin{aligned}(10.6\text{ c})\quad & \theta(d^2)^{-1}(\theta\chi_Q)_\infty(-1)|n\varpi^\alpha|_{\mathbf{A}}^{-m(P)/2}((\mathbf{f}_P^u)^\rho|\tau(N\mathfrak{p}^\alpha),\mathbf{f}_P^u)_\alpha\mathscr{D}(P,Q)\\ &= \theta(j\varpi^\alpha)\mathcal{N}_{F/\mathbf{Q}}(J\mathfrak{p}^\alpha)C(P,Q)\\ &\times L_{Lp}(2-m(P)+m(Q),\chi_Q^{-1}\psi_P)D\left(\frac{m(Q)-m(P)}{2},\mathbf{f}_P,\mathbf{g}_\alpha,\theta^{-1}\right)\end{aligned}$$

from

$$\begin{aligned}Z\left(\frac{j}{2}-1,\mathbf{f}_P,\mathbf{g},\theta^{-1}\right) &= \theta(d)|D|^{(1-m(P)+m(Q))/2}(4\pi)^{v(P)-n(Q)-v(Q)-2t}\\ &\times\Gamma_F(v(Q)-v(P)+n(Q)+2t)D\left(\frac{m(Q)-m(P)}{2},\mathbf{f}_P,\mathbf{g},\theta^{-1}\right).\end{aligned}$$

Thus we have

$$\mathscr{D}=\Phi_A/AH=A(E*_\lambda\varphi'^*)/AH=(E*_\lambda\varphi'^*)/H$$

is the desired p-adic L-function. The last assertion of the theorem is then clear from the above formula and Lemma 9.4.

BIBLIOGRAPHY

[B1] N. Bourbaki, Algèbre, Hermann, Paris, 1970.

[B2] N. Bourbaki, Algèbre commutative, Hermann, Paris, 1961.

[BR] D. Blasius and J. D. Rogawski, Galois representations for Hilbert modular forms, Bull. A.M.S., (1) 21 (1989), 65-69.

[C] W. CASSELMAN, On some results of Atkin and Lehner, Math. Ann., 201 (1973), 301-314.

[Co] J. COATES, Motivic p-adic L-functions, Proceedings of the Durham Symposium, July, 1989, LMS Lecture Note Series, 153 (1991), 141-172.

[Col] P. COLMEZ, Résidu en $s = 1$ des fonctions zêta p-adiques, Inventiones Math., 91 (1988), 371-389.

[D] P. DELIGNE, Valeurs de fonctions L et périodes d'intégrales, Proc. Symp. Pure Math., 33 (1979), part. 2, 313-346.

[H1] H. HIDA, On p-adic Hecke algebras for GL_2 over totally real fields, Ann. of Math., 128 (1988), 295-384.

[H2] H. HIDA, On nearly ordinary Hecke algebras for $GL(2)$ over totally real fields, Adv. Studies in Pure Math., 17 (1989), 139-169.

[H3] H. HIDA, A p-adic measure attached to the zeta functions associated with two elliptic modular forms II, Ann. l'Institut Fourier, 38-3 (1988), 1-83.

[H4] H. HIDA, A p-adic measure attached to the zeta functions associated with two elliptic modular forms I, Inventiones Math., 79 (1985), 159-195.

[H5] H. HIDA, Nearly ordinary Hecke algebras and Galois representations of several variables, JAMI Inaugural Conference Proceedings, 1988 May, Baltimore, Supplement of Amer. J. Math., (1990), 115-134.

[H6] H. HIDA, Modules of congruence of Hecke algebras and L-functions associated with cusp forms, Amer. J. Math., 110 (1988), 323-382.

[H7] H. HIDA, Le produit de Petersson et de Rankin p-adique, Sém. Théorie des Nombres, 1988-1989, 87-102.

[H8] H. HIDA, Algebraicity theorems for standard L-functions of $GL(2)$ and $GL(2) \times GL(2)$, preprint.

[HT1] H. HIDA and J. TILOUINE, Katz p-adic L-functions, congruence modules and deformation of Galois representations, Proceedings of the Durham Symposium, July, 1989, LMS Lecture Note Series, 153 (1991), 271-293.

[HT2] H. HIDA and J. TILOUINE, Anti-cyclotomic Katz p-adic L-functions and congruence modules, preprint.

[HT3] H. HIDA and J. TILOUINE, On the anticyclotomic main conjecture for CM fields, preprint.

[J] H. JACQUET, Automorphic forms on $GL(2)$, II, Lecture notes in Math., 278 (1972), Springer.

[K] N. M. KATZ, p-adic L-functions for CM fields, Inventiones Math., 49 (1978), 199-297.

[M] T. MIYAKE, On automorphic forms on GL_2 and Hecke operators, Ann. of Math., 94 (1971), 174-189.

[P1] A. A. PANCHISHKIN, Convolutions of Hilbert modular forms and their non-Archimedean analogues, Mat. Sbornik, 64 (1988), 574-587 (Russian ; English translation : Math. USSR Sbornik, 64 (1989), 571-584).

[P2] A. A. PANCHISHKIN, Convolutions of Hilbert modular forms, motives, and p-adic L-functions, preprint, Max-Planck-Institut für Mathematik.

[Sh1] G. SHIMURA, The special values of the zeta functions associated with Hilbert modular forms, Duke Math. J., 45 (1978), 637-679.

[Sh2] G. SHIMURA, On some arithmetic properties of modular forms of one and several variables, Ann. of Math., 102 (1975), 491-515.

[Sh3] G. SHIMURA, On certain zeta functions attached to two Hilbert modular forms : I. The case of Hecke characters, Ann. of Math., 114 (1981), 127-164.

[Sh4] G. SHIMURA, On Eisenstein series, Duke Math., J., 50 (1983), 417-476.

[Sh5] G. SHIMURA, On the holomorphy of certain Dirichlet series, Proc. London Math. Soc., 31 (1975), 79-98.

[Sh6] G. SHIMURA, Confluent hypergeometric functions on tube domains, Math. Ann., 260 (1982), 269-302.

[Sh7] G. SHIMURA, Algebraic relations between critical values of zeta functions and inner products, Amer. J. Math., 104 (1983), 253-285.

[Sh8] G. SHIMURA, On the critical values of certain Dirichlet series and the periods of automorphic forms, Inventiones Math., 94 (1988), 245-305.

[T] R. TAYLOR, On Galois representations associated to Hilbert modular forms, Inventiones Math., 98 (1989), 265-280.

[W] A. WEIL, Basic number theory, Springer, 1974.

Manuscrit reçu le 24 avril 1990,
révisé le 21 mai 1991.

Haruzo HIDA,
Department of Mathematics
University of California, Los Angeles
Los Angeles, Ca 90024 (U.S.A.).