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## DISTRIBUTION FUNCTION INEQUALITIES FOR THE DENSITY OF THE AREA INTEGRAL

by R. BAÑUELOS (\*) & Ch.N. MOORE (\*\*)

### 0. Introduction.

Let  $X_t$  be a continuous martingale starting at zero and define  $X^* = \sup_{t>0} |X_t|$  and  $S(X) = \langle X \rangle^{\frac{1}{2}}$ , where  $\langle X \rangle$  is the quadratic variation process at time  $\infty$ . The Burkholder-Gundy inequalities state that for  $0 < p < \infty$ ,

$$c_p \|X^*\|_p \leq \|S(X)\|_p \leq C_p \|X^*\|_p,$$

where  $c_p$  and  $C_p$  are constants depending only on  $p$ . In [4], M. Barlow and M. Yor proved that the maximal local time of  $X_t$  also has  $L^p$  norm equivalent to the  $L^p$  norm of  $S(X)$ . More precisely, let  $L(t; a)$  be the local time at  $a$  and let  $L^* = \sup\{L(\infty; a) : a \in \mathbf{R}\}$ . The occupation of time formula [16] gives

$$(0.1) \quad \int_0^t f(X_s) d\langle X \rangle_s = \int_{\mathbf{R}} f(a) L(t; a) da$$

for all Borel functions  $f$  in  $\mathbf{R}$ . If we take  $f \equiv 1$ , it follows from this that

$$S(X) = \left( \int_{\mathbf{R}} L(\infty; a) da \right)^{\frac{1}{2}} = \left( \int_{-X^*}^{X^*} L(\infty; a) da \right)^{\frac{1}{2}}$$

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and therefore

$$(0.2) \quad S(X) \leq \sqrt{2}(X^*)^{\frac{1}{2}}(L^*)^{\frac{1}{2}}.$$

This, the Cauchy-Schwarz inequality and the Burkholder-Gundy inequalities above give that  $\|S(X)\|_p \leq C_p \|L^*\|_p$  for all  $0 < p < \infty$ . Thus, it is natural to ask if the opposite inequality holds. The result of Barlow and Yor [4] is precisely this. Let  $0 < p < \infty$ . Then

$$(0.3) \quad \|L^*\|_p \leq C_p \|S(X)\|_p.$$

The original proof of (0.3) given in [4] made use of the Ray-Knight theorem on the Markov structure of the Brownian local time. In [5], the same authors gave a different proof based on Tanaka's formula and a real variable lemma of Garsia, Rodemich and Rumsey. However, more recently, R. Bass [6], and independently, B. Davis [12], have shown that the good- $\lambda$  inequalities between  $X^*$  and  $S(X)$  used by Burkholder and Gundy in their proof of their inequalities continue to hold for  $L^*$  and  $S(X)$ .

In the setting of harmonic functions, the nontangential maximal function is the analogue of  $X^*$  and the Lusin area function is the analogue of  $S(X)$ . In [14], R. Gundy introduced a new operator on harmonic functions which he called the maximal density of the area integral; it is a harmonic function analogue of  $L^*$ . The purpose of this paper is to prove good- $\lambda$  inequalities between the maximal density of the area integral and the nontangential maximal and Lusin area functions. Besides answering the question posed in Gundy [15], page 9, our good- $\lambda$  inequalities and the methods of Burkholder and Gundy [9] can be used to give a different proof of the recent results of J. Brossard [7] on the local properties of the maximal density. We also prove a Kesten type law of the iterated logarithm for harmonic functions. Our Theorems 1 and 3 below are for Lipschitz domains. However, all our results are new even in the case of  $\mathbb{R}_+^2$ .

Let  $u$  be a harmonic function in the upper half space  $\mathbb{R}_+^{n+1} = \{(x, y) \in \mathbb{R}^{n+1} : y > 0\}$ . For  $\alpha > 0, x \in \mathbb{R}^n$  and  $y \geq 0$ , we set  $\Gamma_\alpha(x, y) = \{(s, t) : |s - x| < \alpha(t - y)\}$  which is a cone having a vertical axis and vertex  $(x, y)$ . When  $y = 0$  we simply write  $\Gamma_\alpha(x)$ . We define the nontangential maximal function and the Lusin area function of  $u$  by

$$(0.4) \quad \begin{aligned} N_\alpha u(x, y) &= \sup\{|u(s, t)| : (s, t) \in \Gamma_\alpha(x, y)\}, \\ A_\alpha u(x, y) &= \left( \int_{\Gamma_\alpha(x, y)} (t - y)^{1-n} |\nabla u(s, t)|^2 ds dt \right)^{\frac{1}{2}}. \end{aligned}$$

respectively. When  $y = 0$  we will simply write  $N_\alpha u(x)$  and  $A_\alpha u(x)$ . As is well known,  $N_\alpha u$  and  $A_\alpha u$  have equivalent  $L^p$  norms for all  $0 < p < \infty$ .

Let  $a \in \mathbb{R}$  and note that since  $u$  is harmonic on  $\mathbb{R}_+^{n+1}$ , the function  $(u - a)^+$  is subharmonic and its distributional Laplacian,  $\Delta(u - a)^+$ , is a positive measure in  $\mathbb{R}_+^{n+1}$ . We then define (as in Gundy [15])

$$D_\alpha u((x, y); a) = \int_{\Gamma_\alpha(x, y)} (t - y)^{1-n} \Delta(u(s, t) - a)^+(ds dt)$$

and

$$(0.5) \quad D_\alpha u(x, y) = \sup\{D_\alpha u((x, y); a) : a \in \mathbb{R}\}$$

and refer to  $D_\alpha u(x, y)$  as the maximal density at  $(x, y)$ . Again, when  $y = 0$ , we write these as  $D_\alpha u(x; a)$  and  $D_\alpha u(x)$  respectively.

In [16], Gundy and Silverstein proved a change of variables formula for  $D_\alpha u((x, y); a)$  similar to (0.1) :

$$(0.6) \quad \int \psi(s, t) f(u(s, t)) |\nabla u(s, t)|^2 ds dt = \int \psi(s, t) f(r) \Delta(u(s, t) - r)^+(ds dt) dr$$

whenever  $\psi, f$  are Borel functions on  $\mathbb{R}_+^{n+1}$  and  $\mathbb{R}$ . With  $\psi(s, t) = (t - y)^{1-n} \chi_{\Gamma_\alpha(x, y)}(s, t)$  and  $f \equiv 1$  in (0.6), and the fact that  $\Delta(u - a)^+ = 0$  on  $\Gamma_\alpha(x, y)$  whenever  $|a| > N_\alpha u(x, y)$ , we obtain

$$(0.7) \quad A_\alpha^2 u(x, y) = \int_{-N_\alpha u(x, y)}^{N_\alpha u(x, y)} D_\alpha u((x, y); a) da$$

and it is for this reason that  $D_\alpha u((x, y); a)$  and  $D_\alpha u(x, y)$  are called the density of the area integral and the maximal density. Formula (0.7) immediately gives an inequality similar to (0.2) and, as in the martingale case, it follows that  $\|A_\alpha u(x)\|_p \leq C_p \|D_\alpha u(x)\|_p$  for all  $0 < p < \infty$ . In [14], Gundy showed that if  $n = 1$  then for  $0 < p < \infty$  we also have

$$(0.8) \quad \|D_\alpha u(x)\|_p \leq C_p \|A_\alpha u(x)\|_p$$

which provides a harmonic function analogue of (0.3). In Gundy's proof he shows that  $D_\alpha u(x) \leq C_\alpha E^x(L^*(u))$ , where  $E^x$  represents expectation with respect to Brownian motion conditioned to exit the upper half-space at  $x$ ,  $C_\alpha$  is a constant depending on  $\alpha$ , and  $L^*(u)$  is the maximal local time of the martingale  $u(B_t)$ , where  $B_t$  is Brownian motion in  $\mathbb{R}_+^{n+1}$ . Then for  $1 \leq p < \infty$  the inequality (0.8) follows from this and the  $L^p$  equivalence of the Brownian maximal function and the nontangential maximal function. The case  $0 < p < 1$  is obtained from this and what Gundy calls a "good-enough  $\lambda$ " inequality. In [16], Gundy and Silverstein give a proof of (0.8) for all dimensions using real variable techniques. They adopt the second

proof of Barlow and Yor of (0.3) to the harmonic function setting and obtain “good enough  $\lambda$ ” inequalities for  $D_\alpha u$  and  $N_\alpha u$ . These lead to  $L^p$  inequalities such as (0.8) but are weaker than the results we will obtain; for example, they are not sufficient to obtain the local results of Brossard [7] or Corollary 2 and certainly not sufficient to prove Corollary 1. Our first result is :

**THEOREM 1.** — *Let  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$  be a Lipschitz function with Lipschitz constant  $M$ . Let  $D = \{(x, y) : x \in \mathbb{R}^n, y > \Phi(x)\}$  be the Lipschitz domain above the graph of  $\Phi$ . Suppose  $u$  is harmonic in  $D$  and  $0 < \alpha < \beta < 1/M$ . There exist constants  $K_1, K_2, C_1, C_2, C_3$ , and  $C_4$  all depending only on  $\alpha, \beta, n$ , and  $M$ , such that if  $\lambda > 0$  and  $0 < \varepsilon < 1$ , then*

$$(a) \quad \begin{aligned} & |\{x \in \mathbb{R}^n : N_\alpha u(x, \Phi(x)) > K_1 \lambda, D_\beta u(x, \Phi(x)) \leq \varepsilon \lambda\}| \\ & \leq C_1 \exp\left(\frac{-C_2}{\varepsilon}\right) |\{x \in \mathbb{R}^n : N_\alpha u(x, \Phi(x)) > \lambda\}| \end{aligned}$$

and

$$(b) \quad \begin{aligned} & |\{x \in \mathbb{R}^n : A_\alpha u(x, \Phi(x)) > K_2 \lambda, D_\beta u(x, \Phi(x)) \leq \varepsilon \lambda\}| \\ & \leq C_3 \exp\left(\frac{-C_4}{\varepsilon^2}\right) |\{x \in \mathbb{R}^n : A_\alpha u(x, \Phi(x)) > \lambda\}|. \end{aligned}$$

One reason for proving subgaussian inequalities on Lipschitz domains such as (b), is that they immediately imply (using the Lipschitz domains as stopping times) laws of the iterated logarithm for harmonic functions in  $\mathbb{R}_+^{n+1}$ . The following corollary of Theorem 1(b) gives two equivalent laws of the iterated logarithm; an analogue of the lower half of Kesten’s [19] *LIL* for local time and an analogue of the upper half of Kolmogorov’s classical *LIL* for independent random variables. Of course, for Brownian motion this is the trivial part of Kesten’s Theorem. However, for harmonic functions even this part is nontrivial. To see how this *LIL* is related to other *LIL*’s for harmonic functions, see [1].

**COROLLARY 1.** — *Let  $0 < \alpha < \beta$  and suppose  $u$  is a harmonic function in  $\mathbb{R}_+^{n+1}$  with the property that there exists a point  $(x_0, y_0) \in \mathbb{R}_+^{n+1}$  and an  $\alpha' > \alpha$  such that  $A_{\alpha'} u(x_0, y_0) < \infty$ . Then*

$$(a) \quad \liminf_{y \rightarrow 0} \left( \frac{\log \log A_\alpha u(x, y)}{A_\alpha^2 u(x, y)} \right)^{\frac{1}{2}} D_\beta u(x, y) \geq C > 0 \text{ for almost every } x \text{ with } A_\alpha u(x, 0) = \infty.$$

- (b)  $\limsup_{y \rightarrow 0} \frac{A_\alpha u(x, y)}{\sqrt{D_\beta^2 u(x, y) \log \log D_\beta u(x, y)}} \leq C^{-1} < \infty$  for almost every  $x$   
 with  $A_\alpha u(x, 0) = \infty$ ,  
 with  $C$  depending only on  $\alpha, \beta$ , and  $n$ .

Our second theorem is similar to Theorem 1 (a) but with the roles of  $Nu$  and  $Du$  reversed. However, we have not been able to obtain this result for Lipschitz domains. For this theorem it will be necessary to work with a slightly smoother version of  $Du$ . We fix a function  $\psi$  which has the following properties :  $\psi > 0$  on  $\mathbf{R}^n$ ,  $\text{supp} \psi \subseteq B(0, \alpha)$  (here  $\alpha > 0$  is fixed),  $\int_{\mathbf{R}^n} \psi dx = 1$ , and  $\psi$  is radial. We define  $\psi_t(x) = \frac{1}{t^n} \psi(x/t)$  and set

$$(0.9) \quad D_\alpha u(x; a) = \int_{\Gamma_\alpha(x)} t \psi_t(x-s) \Delta(u(s, t) - a)^+ ds dt,$$

$$D_\alpha u(x) = \sup\{D_\alpha u(x; a) : a \in \mathbf{R}\}.$$

In practice, the versions (0.9) and (0.5) behave similarly, since given  $\alpha > 0$  and any  $\gamma < \alpha < \gamma'$  we can always find a  $C^\infty$  function  $\psi$  with  $\psi \equiv 1$  on  $B(0, \gamma)$ ,  $\text{supp} \psi \subseteq B(0, \alpha)$ , so that  $D_\alpha u(x; a)$  defined by (0.9) using this  $\psi$  is bounded below by  $D_\gamma u(x; a)$  defined using (0.5) and dominated above by  $D_{\gamma'} u(x; a)$  defined using (0.5).

**THEOREM 2.** — *Let  $u$  be a harmonic function on  $\mathbf{R}_+^{n+1}$  and let  $0 < \alpha < \beta$  and define  $D_\alpha u(x)$  using (0.9). Then there exists constants  $K_3, C_5, C_6$ , depending only on  $\alpha, \beta, n$  and  $\psi$  and such that if  $\lambda > 0$  and  $0 < \epsilon < 1$ ,*

$$|\{x \in \mathbf{R}^n : D_\alpha u(x) > K_3 \lambda, N_\beta u(x) < \epsilon \lambda\}| \leq C_5 \exp\left(\frac{-C_6}{\epsilon}\right) |\{x \in \mathbf{R}^n : D_\alpha u(x) > \lambda\}|.$$

If we replace  $N_\beta(u)$  by  $A_\beta(u)$  we are able to prove this result in Lipschitz domains. More precisely we have

**THEOREM 3.** — *Let  $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}$  be a Lipschitz function with Lipschitz constant  $M$ . Let  $D = \{(x, y) : x \in \mathbf{R}^n, y > \Phi(x)\}$  be the Lipschitz domain above the graph of  $\Phi$ . Suppose  $u$  is harmonic in  $D$  and  $0 < \alpha < \beta < 1/M$ . There are constants  $K_4, C_7$  and  $C_8$ , depending only on  $\alpha, \beta, n$ , and  $M$  such that if  $\lambda > 0$  and  $0 < \epsilon < 1$ , then*

$$|\{x \in \mathbf{R}^n : D_\alpha u(x, \Phi(x)) > K_4 \lambda, A_\beta u(x, \Phi(x)) \leq \epsilon \lambda\}| \leq C_7 \exp\left(\frac{-C_8}{\epsilon}\right) |\{x \in \mathbf{R}^n : D_\alpha u(x, \Phi(x)) > \lambda\}|.$$

The proof of this result is a combination of probabilistic and analytical methods using the Barlow–Yor result together with estimates on Green functions and harmonic measure in Lipschitz domains. If  $D = \mathbb{R}_+^{n+1}$  we can give a purely analytical proof of this result but with  $\epsilon$  replaced by  $\epsilon^{2/3}$  which is not as sharp. Such a proof is presented in §5.

The following corollary answers a question of Gundy [15], p. 9; it follows from Theorems 1, 2, and 3 by well known methods (see Burkholder and Gundy [9]).

**COROLLARY 2.** — *Let  $u$  be harmonic in  $\mathbb{R}_+^{n+1}$ . Let  $\Phi$  be a non-decreasing function with  $\Phi(0) = 0$  and such that for some constant  $C_0$ ,  $\Phi(2\lambda) < C_0\Phi(\lambda)$  for all  $\lambda > 0$ . Then*

$$C_1 \int_{\mathbb{R}^n} \Phi(D_\alpha u(x)) dx \leq \int_{\mathbb{R}^n} \Phi(A_\alpha u(x)) dx \leq C_2 \int_{\mathbb{R}^n} \Phi(D_\alpha u(x)) dx$$

where  $C_1$  and  $C_2$  depend only on  $\alpha, n$ , and  $C_0$ .

A word about the proofs. In the case of the area function one has a free local  $L^2$ -estimate which comes essentially from Green's Theorem. The global  $L^2$ -estimate for the area integral also comes free from Green's Theorem or the Fourier transform. In the case of the  $D$ -functional there are no  $L^p$ -estimates, local or global, which are as easy as in the area integral case. This makes the proofs of the above results much more difficult in comparison. To obtain a local  $L^2$ -estimate needed for Theorem 2 we use the theory of vector valued singular integrals together with the Garsia–Rodemich–Rumsey lemma and for Theorem 3 we use the Barlow–Yor result. It is interesting to note that in the martingale case, one does have a free  $L^2$ -estimate which comes from the scaling properties of the local time.

The paper is organized as follows. In §1 we prove two lemmas which are needed for the proof of Theorem 1 and Corollary 1 in §2. In §3, we prove Theorem 2. In §4, we prove Theorem 3 and in §5 we present an alternative analytic approach to Theorem 3 for the upper half space. In §6, we make some comments as to the sharpness of our results.

Throughout the paper, the notation  $C, C_1, C_2, C_{\alpha, \beta, \gamma, n} \dots$  will be used to denote constants depending only on  $\alpha, \beta, \gamma$  and  $n$  and whose value may not be the same from line to line. For all our results below in Lipschitz domains the apertures of the cones,  $\alpha, \beta, \gamma$ , etc., are always assumed to be smaller than the inverse of the Lipschitz constants of the domains even if this is not mentioned.

**1. Two Lemmas.**

Our first lemma is a  $D$ -functional analogue of an estimate for  $Au$  and  $Nu$  found in [20], p. 207. Our second lemma is also a  $D$ -functional analogue of a lemma for  $Au$  found in [2]. Unfortunately the proofs of these lemmas are somewhat longer and more technical than their corresponding results for  $Au$ .

LEMMA 1. — *Suppose  $\beta > \gamma$ . There exists an absolute constant  $C = C(\beta, \gamma, n)$  such that if  $u$  is harmonic on  $\mathbb{R}_+^{n+1}$  and if  $(s, t) \in \Gamma_\gamma(x)$ , then  $t|\nabla u(s, t)| \leq CD_\beta u(x)$ .*

In the proof of Theorem 1 and its corollary we will need to compare  $D_\beta u(x, y_1)$  and  $D_\beta u(x, y_2)$  for  $y_2 > y_1$ . If  $n = 1$  it is clear that  $D_\beta u(x, y_1) \geq D_\beta u(x, y_2)$ , but for  $n \geq 2$  this is no longer clear. The next lemma allows us to make the necessary comparisons.

LEMMA 2. — *There exists a constant  $L$  depending only on  $\beta$  and  $n$  such that if  $y_1 < y_0$  then  $D_\beta u(x, y_0) \leq LD_\beta u(x, y_1)$  for any  $x \in \mathbb{R}^n$ .*

*Proof of Lemma 1.* — Fix  $(s, t) = z_0 \in \Gamma_\gamma(x)$ . We may assume that  $u(z_0) = 0$ , otherwise we consider the function  $u - u(z_0)$ . We first note that there exists an  $r_0$  such that  $B(z_0, 4r_0) \subseteq \Gamma_\beta(x)$  with  $r_0 \sim Ct$ , where  $C$  depends only on  $\beta$  and  $\gamma$ . For  $i = 1, 2, 3, 4$  we set  $B_i = B(z_0, ir_0)$  and set  $M_i = \sup\{|u(z, y)| : (z, y) \in B_i\}$ . By the subharmonicity of  $|\nabla u|$  and the change of variables formula (0.6) we have :

$$\begin{aligned} t^2|\nabla u(z_0)|^2 &\leq C \int_{B_2} |\nabla u(z, y)|^2 y^{1-n} dz dy \\ &= C \int_{-M_2}^{M_2} \int_{B_2} \Delta(u(z, y) - a)^+ y^{1-n} dz dy da \\ &\leq CM_2 D_\beta u(x). \end{aligned}$$

Thus we have :

$$(1.1) \quad t|\nabla u(z_0)| \leq C\sqrt{M_2} \sqrt{D_\beta u(x)}.$$

Using similar reasoning we can conclude that if  $(z, y) \in B_2$  then  $y|\nabla u(z, y)| \leq C\sqrt{M_3} \sqrt{D_\beta u(x)}$ . Since  $u(z_0) = 0$ , and for  $(z, y) \in B_2$  we have  $y \sim C \cdot 2r_0$ , we then have the estimate

$$(1.2) \quad M_2 \leq C\sqrt{M_3} \sqrt{D_\beta u(x)}.$$



Now consider  $B_4$  and apply Green's theorem to  $|u(w) - a|$ ,  $a \in \mathbb{R}$  and  $G(w, z_0) = \frac{1}{|w - z_0|^{n-1}} - \frac{1}{(4r_0)^{n-1}}$ . (Technically, we must approximate  $|u(w) - a|$  by smooth functions of  $u(w) - a$ , and then take limits. See [15] for such applications of Green's theorem.) We then obtain :

$$\int_{B_4} \Delta |u(w) - a| G(w, z_0) dw = \frac{C_n}{r_0^n} \int_{\partial B_4} |u - a| d\sigma - C|a|.$$

Therefore, since  $\Delta |u(w) - a| = 2\Delta(u(w) - a)^+$ , we have :

$$(1.3) \quad \frac{1}{r_0^n} \int_{\partial B_4} |u| d\sigma \leq C|a| + C \int_{B_4} \Delta(u(w) - a)^+ G(w, z_0) dw.$$

We need to analyze the integral on the right hand side of (1.3). For  $w \in \mathbb{R}_+^{n+1}$ , write  $w = (\tilde{w}, w')$ ,  $\tilde{w} \in \mathbb{R}^n$ ,  $w' \geq 0$ . Then,

$$\begin{aligned} & \int_{B_4} \Delta(u(w) - a)^+ G(w, z_0) dw \\ & \leq \int_{B_1} \Delta(u(w) - a)^+ G(w, z_0) dw + \int_{B_4 \setminus B_1} \Delta(u(w) - a)^+ \frac{C}{r_0^{n-1}} dw \\ & \leq \int_{B_1} \Delta(u(w) - a)^+ G(w, z_0) dw + C \int_{B_4 \setminus B_1} \Delta(u(w) - a)^+ (w')^{1-n} dw \\ & \leq \int_{B_1} \Delta(u(w) - a)^+ G(w, z_0) dw + CD_\beta u(x). \end{aligned}$$

Combine (1.3) with this last inequality to obtain

$$(1.4) \quad \frac{1}{r_0^n} \int_{\partial B_4} |u| d\sigma \leq C|a| + \int_{B_1} \Delta(u(w) - a)^+ G(w, z_0) dw + CD_\beta u(x)$$

Choosing  $a = M_2$  yields :

$$(1.5) \quad \frac{1}{r_0^n} \int_{B_4} |u| d\sigma \leq C(M_2 + D_\beta u(x)).$$

However, elementary estimates on the Poisson kernel for  $B_4$  show that for  $z \in B_3$ ,  $|u(z)| \leq \frac{C}{r_0^n} \int_{\partial B_4} |u| d\sigma$ . Therefore, using this and (1.5) we conclude

$$(1.6) \quad M_3 \leq C(M_2 + D_\beta u(x)).$$

If we substitute (1.6) into (1.2) we obtain

$$M_2 \leq C \sqrt{M_2 + D_\beta u(x)} \sqrt{D_\beta u(x)}$$

so that  $M_2 \leq CD_\beta u(x)$ . Then by (1.1),  $t|\nabla u(z_0)| \leq CD_\beta u(x)$ , which completes the proof of Lemma 1.

*Proof of Lemma 2.* — Without loss of generality, we may assume that  $y_1 = 0$  and  $x = 0$ . For convenience we write  $\Gamma_\beta(0, y_0) = \Gamma(y_0)$ . We note that there exists a constant  $C_\beta$  depending only on  $\beta$  such that the ball  $B((0, y_0), 2C_\beta y_0) \subseteq \Gamma_{\frac{\beta}{2}}(0, 0)$ . Set  $B_i = B((0, y_0), iC_\beta y_0)$  for  $i = 1, 2$ . If  $(x, t) \in \Gamma(y_0)B_1$ , then  $t - y_0 \approx t$  and thus, if  $a \in \mathbb{R}$ ,

$$(1.7) \quad \int_{\Gamma(y_0) \setminus B_1} \Delta(u(s, t) - a)^+ (t - y_0)^{1-n} ds dt \leq C \int_{\Gamma(y_0) \setminus B_1} \Delta(u(s, t) - a)^+ t^{1-n} ds dt \leq CD_\beta u(0).$$

Let  $G(w, z)$  be the Green's function for  $B_2$ . Let  $w = (s, t) \in \Gamma(y_0) \cap B_1$ . Then we have the estimates :

$$(t - y_0)^{1-n} \approx |w - (0, y_0)|^{1-n} \approx G(w, (0, y_0)).$$

Therefore, using this and Green's theorem, we have :

$$(1.8) \quad \int_{\Gamma(y_0) \cap B_1} \Delta(u(s, t) - a)^+ (t - y_0)^{1-n} ds dt \leq C \int_{B_2} \Delta(u(w) - a)^+ G(w, (0, y_0)) dw = \frac{C}{y_0^n} \int_{\partial B_2} \left( (u(w) - a)^+ - (u(0, y_0) - a)^+ \right) d\sigma(w).$$

However,  $B_2 \subseteq \Gamma_{\frac{\beta}{2}}(0, 0)$  and so by Lemma 1,  $t|\nabla u(s, t)| \leq CD_\beta u(0)$  for all  $(s, t) \in B_2$ . Then (1.8) implies :

$$(1.9) \quad \int_{\Gamma(y_0) \cap B_1} \Delta(u(s, t) - a)^+ (t - y_0)^{1-n} ds dt \leq CD_\beta u(0).$$

The lemma follows from (1.7) and (1.9).

## 2. The proof of Theorem 1 and Corollary 1.

With lemmas 1 and 2 proved, the proof of Theorem 1 follows the same strategy used in [2] for the corresponding result for  $N_\alpha u$  and  $A_\alpha u$ . We first construct a "sawtooth" region over  $\{D_\beta u(x, \Phi(x)) < \varepsilon\lambda\}$  and then estimate the nontangential maximal function or the area function on the boundary of this region. Since the proofs of part (a) and part (b) are essentially the same, we will only do part (b). The following proposition will allow us to make the necessary estimates on the boundary of this sawtooth region.

PROPOSITION 1. *Suppose  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$  is a Lipschitz function and set  $W = \{(x,t) : x \in \mathbb{R}^n; t > \psi(x)\}$ . Suppose that  $\rho > \alpha > 0, D_\rho u(x, \psi(x)) \leq 1$  for every  $x \in \mathbb{R}^n$ , and that there exists a  $z_0 \in \mathbb{R}^n$  for which  $A_\alpha u(z_0, \psi(z_0)) < \infty$ . Then  $\|A_\alpha^2 u(x, \psi(x))\|_{BMO} \leq C$  where  $C$  is a constant which depends on  $\rho, \alpha, n$  and the Lipschitz constant of  $\psi$ .*

*Proof.* — Fix a cube  $Q \subseteq \mathbb{R}^n$ , let  $Q' = \{(x, \psi(x)) : x \in Q\}$  denote the graph of  $Q$ , and let  $x_0$  denote the center of  $Q$ . We now borrow a construction from [2] page 646. Pick  $\gamma'$  and  $\gamma$  so that  $\rho > \gamma' > \gamma > \alpha$  and set  $\widetilde{W} = \bigcup_{x \in Q} \Gamma_\rho(x, \psi(x))$ . Then trivially  $\widetilde{W} \subseteq W$ . There exists an  $R > 0$  with the following properties :

- (i) Set  $P^* = (x_0, \psi(x_0) + R\ell(Q))$ , where  $\ell(Q)$  denotes the side length of  $Q$ . Then for every  $P \in Q', \{tP + (1-t)P^* : 0 \leq t < 1\} \subseteq \widetilde{W}$ .
- (ii) For every  $P \in Q'$  the cone  $\Gamma_\gamma^h(P)$  with aperture  $\gamma$ , vertical axis  $\overline{PP^*}$ , vertex at  $P$ , and height  $h = |P - P^*|$  is completely contained in  $\widetilde{W}$ .
- (iii) For every  $P = (x, \psi(x)) \in Q'$ , the cone  $\Gamma_\alpha^h(P)$  with vertex at  $P$ , vertical axis  $\{(x, \psi(x) + s) : s > 0\}$ , and height  $h = \psi(x_0) + R\ell(Q) - \psi(x)$  is completely contained in the cone  $\Gamma_\gamma^h(P)$  given by ii).

We assume that  $R$  is the smallest such constant for which i), ii) and iii) holds for all cubes  $Q \subseteq \mathbb{R}^n$ . Then the constant  $R$  depends only on  $\rho$  and  $\alpha$  and the Lipschitz constant of  $\psi$ .

We now set  $\Omega = \bigcup_{P \in Q'} \Gamma_{\gamma'}(P) \cap \{(x, y) : y < \psi(x_0) + 2R\ell(Q)\}$ .

The domain  $\Omega$  is Lipschitz and starlike with respect to the point  $P^*$ . Furthermore,  $\Omega$  has the property that there exists an  $\varepsilon_0 > 0$  ( $\varepsilon_0$  depends only on  $\rho$  and the Lipschitz constant of  $\psi$ ) such that for every point  $P \in \partial\Omega$  there exists an  $\varepsilon$  with  $\gamma \geq \varepsilon \geq \varepsilon_0$  so that the cone  $\Gamma_\varepsilon^h(P)$  with vertex at  $P$ , height  $|P - P^*|$ , aperture  $\varepsilon$  and vertical axis  $\overline{PP^*}$  is completely contained in  $\Omega$ . In fact, for  $P \in Q' \subseteq \partial\Omega$ , we may take  $\Gamma_\varepsilon^h(P)$  to be the cone given by ii) above.

For  $P \in Q'$  set  $\Gamma_1(P) = \Gamma_\alpha^h(P)$  where the latter is the cone given by iii) above and set  $\Gamma_2(P) = \Gamma_\alpha(P) \setminus \Gamma_1(P)$ . For  $j = 1, 2$ , and  $x \in Q$ , we set

$$A_j u(x, \psi(x)) = \left( \int_{\Gamma_j(x, \psi(x))} |\nabla u(s, t)|^2 (t - \psi(x))^{1-n} ds dt \right)^{\frac{1}{2}}$$

so that  $A_\alpha^2 u(x, \psi(x)) = A_1^2 u(x, \psi(x)) + A_2^2 u(x, \psi(x))$ . We need estimates for  $A_1 u$  and  $A_2 u$ . We first estimate  $A_2 u$ .

LEMMA . — Under the hypothesis of the proposition we have

$$|A_2^2(x, \psi(x)) - A_2^2(x_0, \psi(x_0))| \leq C$$

for every  $x \in Q$ . Here  $C$  is a constant which depends only on  $\alpha, \rho$ , and the Lipschitz constant of  $\psi$ .

The proof of this lemma is essentially the same as Lemma 6 of [2]. All that is necessary is an estimate on  $(t + \psi(s))|\nabla u(s, t + \psi(s))|, s \in \mathbb{R}^n, t > 0$  and this is provided by Lemma 1. Note that this estimate and the fact that  $A_\alpha^2 u(z_0, \psi(z_0)) < \infty$  for some  $z_0$  implies that  $A_2 u(x, \psi(x)) < \infty$  for all  $x \in Q$ . We now estimate  $A_1 u$ .

LEMMA 4. — Under the hypothesis of the theorem,

$$\frac{1}{|Q|} \int_Q A_1^2 u(x, \psi(x)) dx \leq C$$

where  $C$  is a constant which depends only on  $\rho$  and the Lipschitz constant of  $\psi$ .

*Proof.* — We may assume that  $u(P^*) = 0$  since both  $Du$  and  $Au$  remain unchanged if we add a constant to  $u$ . Recall that for every  $P \in \partial\Omega$  we have a cone  $\Gamma_\epsilon^h(P)$  with vertical axis  $\overline{PP^*}$  and such that for  $P \in Q', \Gamma_1(P) \subseteq \Gamma_\gamma(P) = \Gamma_\epsilon^h(P)$ . For  $P \in \partial\Omega$  we set

$$Du(P; a) = \int_{\Gamma_\epsilon^h(P)} \Delta(u(z, y) - a)^+ d((z, y), P)^{1-n} dz dy,$$

$$Du(P) = \sup\{Du(P, a) : a \in \mathbb{R}\},$$

$$Au(P) = \left( \int_{\Gamma_\epsilon^h(P)} |\nabla u(z, y)|^2 d((z, y), P)^{1-n} dz dy \right)^{\frac{1}{2}}$$

and

$$Nu(P) = \sup\{|u(z, y)| : (z, y) \in \Gamma_\epsilon^h(P)\}.$$

Since  $D_\rho u(x, \psi(x)) \leq 1$  for all  $x \in \mathbb{R}^n$  then a slight variation of the proof of Lemma 2 shows that  $Du(P) \leq C$  for all  $P \in \partial\Omega$ . Then by (0.7),  $A^2 u(P) \leq C Nu(P)$  for all  $P \in \partial\Omega$ . Therefore,

$$\begin{aligned} \frac{1}{\sigma(\partial\Omega)} \int_{\partial\Omega} A^2 u(P) d\sigma(P) &\leq C \frac{1}{\sigma(\partial\Omega)} \int_{\partial\Omega} Nu(P) d\sigma(P) \\ &\leq C \left( \frac{1}{\sigma(\partial\Omega)} \int_{\partial\Omega} Nu(P)^2 d\sigma(P) \right)^{\frac{1}{2}} \\ &\leq C \left( \frac{1}{\sigma(\partial\Omega)} \int_{\partial\Omega} Au(P)^2 d\sigma(P) \right)^{\frac{1}{2}} \end{aligned}$$

where the last inequality follows from Dahlberg [11]. The above inequality, combined with the fact that  $\Gamma_1(P) \subseteq \Gamma_\epsilon^h(P)$  for  $P \in Q'$  and the fact that  $\psi$  is Lipschitz allow us to conclude the Lemma.

We can now complete the proof of the proposition. Set  $a_Q = A_2^2(x_0, \psi(x_0))$ . Then

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |A_\alpha^2 u(x, \psi(x)) - a_Q| dx \\ & \leq \frac{1}{|Q|} \int_Q A_1^2 u(x, \psi(x)) dx + \frac{1}{|Q|} \int_Q |A_2^2 u(x, \psi(x)) - a_Q| dx \\ & \leq C \end{aligned}$$

by Lemmas 3 and 4. The proposition follows from this.

We are now ready to complete the proof of theorem 1b). Let  $E = \{x : D_\beta u(x, \Phi(x)) < \epsilon \lambda\}$  and set  $\eta = \frac{\alpha + \beta}{2}$  and  $W = \bigcup_{x \in E} \Gamma_\eta(x, \Phi(x))$ . Then  $W$  is a subdomain of  $D$  and  $\partial W$  is the graph of a Lipschitz function, call it  $\psi(x)$ . Thus, for  $x \in E, \Phi(x) = \psi(x)$ . Now set  $\rho = \frac{2\alpha + \beta}{3}$ . By a slight variation of the proof of Lemma 2 it follows that  $D_\rho u(x, \psi(x)) \leq L\epsilon \lambda$  for every  $x \in \mathbb{R}^n$ . Then by the proposition,  $\|A_\alpha^2 u(x, \psi(x))\|_{BMO} \leq C(L\epsilon \lambda)^2$ . (Note we may assume  $A_\alpha u(z_0, \psi(z_0)) < \infty$  for some  $z_0$ , for if not, then the result is trivial). By Lemma 4 of [2], we then have the distributional inequality,

$$\begin{aligned} & |\{x \in \mathbb{R}^n : A_\alpha^2 u(x, \psi(x)) > 2\tilde{\lambda}\}| \\ & \leq C \exp\left(\frac{c}{-c\tilde{\lambda}}(L\epsilon \lambda)^2\right) |\{z \in \mathbb{R}^n : A_\alpha^2 u(x, \psi(x)) > \tilde{\lambda}\}| \end{aligned}$$

for every  $\tilde{\lambda} > 0$ . Here  $C, c$  are constants which depend only on  $\alpha, \beta, n$ . Substitute  $\tilde{\lambda} = \lambda^2$  into this inequality to obtain :

$$\begin{aligned} & |\{x \in \mathbb{R}^n : A_\alpha u(x, \psi(x)) > \sqrt{2} \lambda\}| \\ & \leq C \exp\left(\frac{-c}{\epsilon^2}\right) |\{x \in \mathbb{R}^n : A_\alpha u(x, \psi(x)) > \lambda\}| \end{aligned}$$

for all  $\lambda > 0$ . It is shown in [2], Lemma 11 that  $A_\alpha u(x, \psi(x)) \leq L_1 A_\alpha u(x, \Phi(x))$  where  $L_1$  is a constant depending only on  $\alpha$  and  $n$ . Now

set  $K = \sqrt{2}L_1$ . Then for every  $\lambda > 0$ ,

$$\begin{aligned} & |\{x \in \mathbb{R}^n : A_\alpha u(x, \Phi(x)) > K\lambda, D_\beta u(x, \Phi(x)) < \varepsilon\lambda\}| \\ & = |\{x \in E : A_\alpha u(x, \Phi(x)) > K\lambda\}| \\ & = |\{x \in E : A_\alpha u(x, \psi(x)) > K\lambda\}| \\ & \leq C \exp\left(\frac{-c}{\varepsilon^2}\right) |\{x \in \mathbb{R}^n : A_\alpha u(x, \psi(x)) > \frac{K}{\sqrt{2}}\lambda\}| \\ & \leq C \exp\left(\frac{-c}{\varepsilon^2}\right) |\{x \in \mathbb{R}^n : A_\alpha u(x, \Phi(x)) > \frac{K}{\sqrt{2}L_1}\lambda\}| \\ & = C \exp\left(\frac{-c}{\varepsilon^2}\right) |\{x \in \mathbb{R}^n : A_\alpha u(x, \Phi(x)) > \lambda\}| \end{aligned}$$

which completes the proof of Theorem 1b).

We now prove Corollary 1. The proof of a) is similar to the proof of Theorem 2 in [3], however, there are some added complications in this setting. Part b) follows from a) by noting that if a) holds at a point  $x$ , then

$$\limsup_{y \rightarrow 0} \frac{A_\alpha u(x, y)}{\sqrt{D_\beta^2 u(x, y) \log \log A_\alpha u(x, y)}} \leq \frac{1}{C}.$$

But if also  $A_\alpha u(x, y) \rightarrow \infty$  as  $y \rightarrow 0$ , then this implies that

$$\limsup_{y \rightarrow 0} \left( \frac{\log A_\alpha u(x, y)}{\log D_\beta u(x, y)} \right) \leq 1$$

and thus, that

$$\limsup_{y \rightarrow 0} \left( \frac{\log \log A_\alpha u(x, y)}{\log \log D_\beta u(x, y)} \right) \leq 1$$

and b) follows.

*Proof of Corollary 1a).* — Fix  $M$ ; it suffices to consider those  $x$  with  $|x| \leq M$ . We first note that the hypothesis of the corollary imply that  $A_\alpha u(x, y) < \infty$  for all  $x \in \mathbb{R}^n$  and  $y > 0$ . Then we may assume that  $D_\beta u(x_1, y_1) < \infty$  for some  $(x_1, y_1)$  with  $|x_1| \leq M$ , otherwise the result is trivial. Then for  $k = 1, 2, \dots$  and  $|x| \leq M$ , we may define

$\rho_k(x) = \inf\{y : D_\beta u(x, y) \leq \left(\frac{2^k}{0 \log k}\right)^{\frac{1}{2}}\}$ , where  $c_0$  is a constant to be chosen later. Let  $\gamma = \frac{\alpha + \beta}{2}$  and set  $W_k = \bigcup_{\{x: |x| \leq M\}} \Gamma_\gamma(x, \rho_k(x))$ . Then

$W_k$  is a Lipschitz domain, say  $\partial W_k$  is the graph of  $\eta_k(x)$ . Note that for every  $x$  with  $|x| \leq M$  we have  $\eta_k(x) \leq \rho_k(x)$ . Then we have :

a) Set  $\gamma' = \frac{2\alpha + \beta}{3}$  so that  $\alpha < \gamma' < \gamma < \beta$ . Then  $D_{\gamma'}u(x, \eta_k(x)) \leq L \left( \frac{2^k}{c_0 \log k} \right)^{\frac{1}{2}}$  for all  $x \in \mathbb{R}^n$ .

Proof : Let  $\varepsilon > 0$ . Then  $(x, \eta_k(x) + \varepsilon) \in W_k$  so there exists  $\tilde{x}$  with  $|\tilde{x}| \leq M$  such that  $(x, \eta_k(x) + \varepsilon) \in \Gamma_\gamma(\tilde{x}, \rho_k(\tilde{x})) \subseteq \Gamma_\beta(\tilde{x}, \rho_k(\tilde{x}))$ . Then by a slight variation of Lemma 2, we conclude that  $D_{\gamma'}u(x, \eta_k(x) + \varepsilon) \leq L \left( \frac{2^k}{c_0 \log k} \right)^{\frac{1}{2}}$ . Fatou's lemma then implies a).

b) If  $x \in \mathbb{R}^n$ , then  $\eta_{k+1}(x) \leq \eta_k(x)$ .

Proof : Let  $(x, t) \in W_k$ . Then  $(s, t) \in \Gamma_\gamma(\tilde{x}, \rho_k(\tilde{x}))$  for some  $\tilde{x}$  with  $|\tilde{x}| \leq M$ . But since  $\rho_{k+1}(\tilde{x}) \leq \rho_k(\tilde{x})$  then  $(s, t) \in \Gamma_\gamma(\tilde{x}, \rho_{k+1}(\tilde{x}))$ , hence  $(s, t) \in W_{k+1}$ . Thus,  $W_k \subseteq W_{k+1}$  and b) follows.

c) If  $|x| \leq M$  and  $y < \eta_k(x)$  then  $D_\beta u(x, y) > \left( \frac{2^k}{c_0 \log k} \right)^{\frac{1}{2}}$ .

Proof : This follows from the fact that  $\eta_k(x) \leq \rho_k(x)$  and the definition of  $\rho_k(x)$ .

Since  $W_k$  was defined using cones of aperture  $\gamma < \beta$ , then there exists an  $\widetilde{M}$  independent of  $k$  such that if  $|x| > \widetilde{M}$  then  $(x, \eta_k(x)) \in \Gamma_{\gamma''}(x_1, y_1)$  where  $\gamma'' = \frac{\gamma + \beta}{2}$  and  $(x_1, y_1)$  is the point at which we have assumed  $D_\beta(x_1, y_1) < \infty$ . Consider an  $x$  with  $|x| > \widetilde{M}$ . Then since  $(x, \eta_k(x)) \in \Gamma_{\gamma''}(x_1, y_1) \subseteq \Gamma_\beta(x_1, y_1)$ , Lemma 1 implies that  $(t - y_1)|\nabla u(s, t - y_1)| \leq CD_\beta u(x_1, y_1)$  for all  $(s, t) \in \Gamma_\alpha((x, \eta_k(x)))$ . Also, the hypothesis of the corollary implies that if we take  $\alpha'' = \frac{\alpha + \alpha'}{2}$ , then  $A_{\alpha''}u(x_1, y_1) < \infty$ . We may assume  $\alpha'' < \gamma$ . Then we can conclude that

$$A_\alpha u(x, \eta_k(x)) \leq CA_{\alpha''}u(x_1, y_1) + CD_\beta u(x_1, y_1) = C_1,$$

by splitting the integral defining  $A_\alpha u(x, \eta_k(x))$  into an integral over a top part of  $\Gamma_\alpha(x, \eta_k(x))$  and estimating this by  $A_{\alpha''}u(x_1, y_1)$  and an integral over the remainder of  $\Gamma_\alpha(x, \eta_k(x))$  and estimating this by the gradient estimate above. (The proof is similar to that of Lemma 2 and so we omit it). Hence, if we take  $k_0$  large enough,  $|\{x \in \mathbb{R}^n : A_\alpha u(x, \eta_k(x)) > \sqrt{2^k}\}| \leq C$  for every  $k \geq k_0$ , where  $C$  is a finite constant. Let  $k \geq k_0$ , and apply

Theorem 1b) to  $D_{\gamma'}u$  and  $A_{\alpha}u$  to obtain:

$$\left| \left\{ x \in \mathbb{R}^n : A_{\alpha}u(x, \eta_k(x)) > K_1\sqrt{2^k}, D_{\gamma'}u(x, \eta_k(x)) < L \left( \frac{2^k}{c_0 \log k} \right)^{\frac{1}{2}} \right\} \right| \leq CC_3 \exp \left( -C_4 \frac{c_0 \log k}{L^2} \right) = \frac{C}{k^2}$$

if we take  $c_0 = \frac{2L^2}{C_4}$ . Since for all  $x \in \mathbb{R}^n$ , we have  $D_{\gamma'}u(x, \eta_k(x)) < L \left( \frac{2^k}{c_0 \log k} \right)^{\frac{1}{2}}$  by a), then it follows that

$$\sum_{k=k_0}^{\infty} |\{x \in \mathbb{R}^n : |x| \leq M \text{ and } A_{\alpha}u(x, \eta_k(x)) > K_1\sqrt{2^k}\}| < \infty.$$

The Borel–Cantelli lemma implies that for almost all  $x$  with  $|x| \leq M$  we have  $A_{\alpha}u(x, \eta_k(x)) < K_1\sqrt{2^k}$  eventually. Pick such an  $x$  for which  $A_{\alpha}u(x, 0) = \infty$ , say  $A_{\alpha}u(x, \eta_k(x)) < K_1\sqrt{2^k}$  for all  $k \geq N$ . Then  $\eta_k(x) > 0$  for every  $k$  and since  $\eta_k(x)$  decreases by b) then either  $\eta_k(x) \searrow t_0 > 0$  or  $\eta_k(x) \searrow 0$ . In the first case,  $D_{\beta}u(x, y) = \infty$  for all  $y < t_0$  by c) and the result holds trivially for this  $x$ . In the second case, let  $y < \eta_N(x)$ , say  $\eta_{k+1}(x) \leq y < \eta_k(x)$  for some  $k \geq N$ . Then by c), Lemma 11 of [2] and the fact that the function  $f(r) = \left( \frac{r}{c_0 \log \log r} \right)^{\frac{1}{2}}$  is increasing we obtain :

$$\begin{aligned} D_{\beta}u(x, y) &> \left( \frac{2^k}{c_0 \log k} \right)^{\frac{1}{2}} > C \left( \frac{2^{k+1}}{c_0 \log(k+1)} \right)^{\frac{1}{2}} \\ &\geq C \left( \frac{A_{\alpha}^2 u(x, \eta_{k+1}(x))}{\log \log A_{\alpha} u(x, \eta_{k+1}(x))} \right)^{\frac{1}{2}} \\ &\geq C \left( \frac{A_{\alpha}^2 u(x, y)}{\log \log A_{\alpha} u(x, y)} \right)^{\frac{1}{2}} \end{aligned}$$

which gives the result.

In the case when the Lipschitz function  $\Phi$  of Theorem 1 is identically zero, that is, when  $D = \mathbb{R}_+^{n+1}$ , it is possible to give a quick proof based on the inequality  $A_{\alpha}^2 u(x) \leq 2N_{\alpha}u(x)D_{\alpha}u(x)$  and a result from [3]. We will do this since this is very short. Let us recall the following Theorem from [3].

**THEOREM A.** — *Let  $\beta > \gamma$ . There exists constants  $\tilde{K}, C_1, C_2$  depending only on  $\beta, \gamma$ , and  $n$  such that if  $u$  is harmonic on  $\mathbb{R}_+^{n+1}$  and  $0 < \varepsilon < 1$ , then*



$$\begin{aligned} & \left| \left\{ x \in \mathbf{R}^n : N_\gamma u(x) > \tilde{K}\lambda, A_\beta u(x) \leq \varepsilon\lambda \right\} \right| \\ & \leq C_1 \exp\left(\frac{-C_2}{\varepsilon^2}\right) |\{x \in \mathbf{R}^n : N_\gamma u(x) > \lambda\}|. \end{aligned}$$

Now set  $\gamma = \frac{\alpha}{2}$  where  $\alpha$  is as in the statement of Theorem 1. By Lemma 1,  $N_\gamma u(x) \geq N_\alpha u(x) - \tilde{C}D_\beta u(x)$  where  $\tilde{C}$  is a constant depending only on  $\alpha, \beta$ , and  $n$ . We then set  $K = 2\tilde{K} + \tilde{C}$  so that for  $j = 1, 2, \dots$  and  $0 < \varepsilon < 1$  we have  $2^{j-1}K - \tilde{C}\varepsilon > 2^j\tilde{K}$ . Then we have

$$\begin{aligned} & |\{N_\alpha u > K\lambda, D_\beta u \leq \varepsilon\lambda\}| \\ & = \sum_{j=1}^{\infty} |\{2^j K\lambda \geq N_\alpha u > 2^{j-1} K\lambda, D_\beta u < \varepsilon\lambda\}| \\ & = \sum_{j=1}^{\infty} \left| \left\{ 2^j K\lambda \geq N_\alpha u > 2^{j-1} K\lambda, D_\beta u < \varepsilon\lambda, A_\alpha u < \sqrt{2^{j+1}K\varepsilon}\lambda \right\} \right| \\ & \leq \sum_{j=1}^{\infty} \left| \left\{ N_\gamma u > (2^{j-1}K - \tilde{C}\varepsilon)\lambda, A_\alpha u < \sqrt{2^{j+1}K\varepsilon}\lambda \right\} \right| \\ & \leq \sum_{j=1}^{\infty} \left| \left\{ N_\gamma u > \tilde{K}(2^j\lambda), A_\alpha u < \sqrt{\frac{\varepsilon K}{2^{j-1}}} 2^j\lambda \right\} \right| \\ & \leq C_1 \sum_{j=1}^{\infty} \exp\left(\frac{-C_2 2^{j-1}}{\varepsilon K}\right) |\{N_\gamma u > 2^j\lambda\}| \\ & \leq C_1 \left( \sum_{j=1}^{\infty} \exp\left(\frac{-C_2 2^{j-1}}{\varepsilon}\right) \right) |\{N_\alpha u > \lambda\}| \\ & \leq C \exp\left(\frac{-C_3}{\varepsilon}\right) |\{N_\alpha u > \lambda\}| \end{aligned}$$

and this gives Theorem 1(a) when  $D = \mathbf{R}_+^{n+1}$ .

The reason why this proof does not work in Lipschitz domains is that we do not know Theorem A in a Lipschitz domain other than when the Lipschitz domain is in  $\mathbf{R}^2$ .

### 3. The proof of Theorem 2.

The proof of Theorem 2 involves using roughly the same strategy as was used in the proof of Theorem 1. We will consider a sawtooth region

$W$  on which  $u$  is bounded and will define a version  $D^*u(x)$  of  $D_\alpha u(x)$  by using the definition (0.9) but restricting the integration to  $\Gamma_\alpha(x) \cap W$ . We then show that  $D^*u \in BMO$ . To do this, we will consider a typical cube  $Q$  and break the integral defining  $D^*u$  into a “top” part and a “bottom” part.

The difficult part to control will be the “bottom” part; in the course of the proof we shall apply Green’s theorem several times. The following lemma allows us to control the boundary terms which arise from these and it is more convenient to state and prove it first.

LEMMA 5. — Suppose  $\alpha < \gamma, E \subseteq \mathbf{R}^n$  and  $W = \bigcup_{x \in E} \Gamma_\gamma(x)$ .

Suppose also that  $\rho$  is a function supported on  $B(0, \alpha)$ . Let  $h > 0$  and set  $\Gamma(x) = \{(s, t) : |x - s| \leq \alpha t, t \leq h\} \cap W$ . Then there exists a constant  $C$  depending on  $\alpha, \gamma$  and  $n$  such that for every  $x_0 \in \mathbf{R}^n$

$$\left| \int_{\partial\Gamma(x_0)} t^{-n} \rho\left(\frac{x_0 - s}{t}\right) d\sigma(s, t) \right| \leq C \|\rho\|_\infty.$$

Here  $\sigma$  denotes surface measure on  $\partial\Gamma(x_0)$ .

*Proof.* — Clearly,  $\partial\Gamma(x_0) \subseteq \{(s, h) : |x_0 - s| \leq \alpha h\} \cup (\Gamma_\alpha(x_0) \cap \partial W) \cup \{(s, t) : |x_0 - s| = \alpha t\}$ . Since  $\rho$  is supported on  $B(0, \alpha)$ , the integral of  $\rho_t(x_0 - s)$  vanishes on the third set. The first set has measure  $Ch^n$  and so the integral of  $\rho_t(x_0 - s)$  over this set is then bounded by  $C\|\rho\|_\infty$ . To control the integral over the second set we first claim :

$$(3.1) \quad \sigma(\Gamma_\alpha(x_0) \cap \partial W) \leq C(\inf\{t : (s, t) \in \Gamma_\alpha(x_0) \cap \partial W\})^n.$$

To see this, we note that  $\partial W$  is the graph of a Lipschitz function with Lipschitz constant at most  $\frac{1}{\gamma}$ . Set  $t_0 = \inf\{t : (s, t) \in \Gamma_\alpha(x_0) \cap \partial W\}$ . Then

there exists an  $s_0 \in \mathbf{R}^n$  such that  $(s_0, t_0) \in \partial W$  and  $(s_0, t_0) \in \overline{\Gamma_\alpha(x_0)}$ . Then, in particular,  $|x_0 - s_0| \leq \alpha t_0$ . Let  $(s, t) \in \Gamma_\alpha(x_0) \cap \partial W$ , then  $|s - x_0| < \alpha t < \alpha(t - t_0) + \alpha t_0 \leq \frac{\alpha}{\gamma}|s - s_0| + \alpha t_0 < \frac{\alpha}{\gamma}|s - x_0| + \frac{\alpha}{\gamma}|s_0 - x_0| + \alpha t_0 < \frac{\alpha}{\gamma}|s - x_0| +$

$\left(\frac{\alpha^2}{\gamma} + \alpha\right) t_0$ . Since  $\frac{\alpha}{\gamma} < 1$ , we conclude that  $|s - x_0| \leq C_{\alpha, \gamma} t_0$ . Therefore,

we have shown that if  $(s, t) \in \partial W \cap \Gamma_\alpha(x_0)$ , then  $s \in B(x_0, C_{\alpha, \gamma} t_0)$ . Since  $\partial W$  is Lipschitz this gives  $\sigma(\partial W \cap \Gamma_\alpha(x_0)) \leq C|B(x_0, C_{\alpha, \gamma} t_0)| \leq Ct_0^n$ , which is (3.1). Then we have :

$$\begin{aligned} & \left| \int_{\Gamma_\alpha(x_0) \cap \partial W} t^{-n} \rho\left(\frac{x_0 - s}{t}\right) d\sigma(s, t) \right| \\ & \leq \|\rho\|_\infty \sigma(\Gamma_\alpha(x_0) \cap W) (\inf\{t : (s, t) \in \Gamma_\alpha(x_0) \cap W\})^{-n} \leq C \|\rho\|_\infty \end{aligned}$$

which completes the proof of Lemma 5.

We now begin the proof of Theorem 2. We consider  $\varepsilon > 0, \lambda > 0$  fixed for the rest of this section. We set  $\gamma = \frac{\alpha + \beta}{2}$  and let  $E = \{x : N_\beta u(x) < \varepsilon\lambda\}$  and define  $W = \bigcup_{x \in E} \Gamma_\gamma(x)$ . Then we have :

$$(3.2) \quad \begin{aligned} |u| &\leq \varepsilon\lambda \text{ on } \overline{W} \\ t|\nabla u(s, t)| &\leq C\varepsilon\lambda \text{ for } (s, t) \in \overline{W}. \end{aligned}$$

The first of these statements is obvious, and the second follows from the Lemma on page 207 of [20].

For  $x \in \mathbb{R}^n$  and  $a \in \mathbb{R}$  we now set

$$D^*u(x; a) = \int_{\Gamma_\alpha(x) \cap W} t\psi_t(x - s)\Delta(u(s, t) - a)^+ ds dt$$

and

$$D^*u(x) = \sup\{D^*u(x; a); a \in \mathbb{R}\}.$$

Before proceeding further, it will be convenient to state and prove a lemma which will allow us to estimate the contribution to  $D^*u(x)$  from a "top" part of the region of integration  $\Gamma_\alpha(x) \cap W$ . This lemma is analogous to Lemma 3 in which a similar result held for  $Au(x)$ .

LEMMA 6. — *Let  $k > 0$  and suppose  $\Omega \subseteq W$  is a subdomain of  $W$ . Define  $D_\Omega^T u(x; a) = \int_{\Omega \cap \{t \geq k\}} t\psi_t(x - s)\Delta(u(s, t) - a)^+ ds dt$  and  $D_\Omega^T u(x) = \sup\{D_\Omega^T u(x; a) : a \in \mathbb{R}\}$ . There exists a constant  $C = C(\alpha, \beta, n)$  such that if  $x, y \in \mathbb{R}^n$  then*

- i)  $|D_\Omega^T u(x; a) - D_\Omega^T u(y; a)| \leq C\varepsilon\lambda \frac{|x - y|}{k}$
- ii)  $|D_\Omega^T u(x) - D_\Omega^T u(y)| \leq C\varepsilon\lambda \frac{|x - y|}{k}$

*Proof.* — Fix  $x, y \in \mathbb{R}^n$  and set  $\mathcal{R} = (\Gamma_\alpha(x) \cup \Gamma_\alpha(y)) \cap \Omega \cap \{t \geq k\}$ . Then

$$\begin{aligned} |D_\Omega^T u(x; a) - D_\Omega^T u(y; a)| &= \left| \int_{\mathcal{R}} t(\psi_t(x - s) - \psi_t(y - s))\Delta(u(s, t) - a)^+ ds dt \right| \\ &\leq C|x - y| \int_{\mathcal{R}} t^{-n} \Delta(u(s, t) - a)^+ ds dt. \end{aligned}$$

Now set  $\gamma' = \frac{\beta + \gamma}{2}$  and form  $\widetilde{W} = \cup_{z \in E} \Gamma_{\gamma'}(z)$ . Then  $\widetilde{W} \supseteq W \supseteq \Omega$  and again by the Lemma on page 207 of [20],  $t|\nabla u(s, t)| \leq C\varepsilon\lambda$  for  $(s, t) \in \widetilde{W}$ .

We note that there exists a  $C = C(\alpha, \beta, n)$  such that if  $(v, w) \in \mathbb{R}_+^{n+1}$  and  $\text{dist}((v, w), W) < Cw$  then  $(v, w) \in \widetilde{W}$ .

We may assume  $C < 1/2$ . Therefore, we can find a  $C^\infty$  function  $\phi(s, t)$  such that  $0 \leq \phi(s, t) \leq 1$  for all  $(s, t) \in \mathbb{R}_+^{n+1}$ ,  $\phi(s, t) \equiv 1$  on  $\mathcal{R}$ ,  $\text{supp}\phi(s, t) = \{(v, w) : \text{dist}((v, w), \mathcal{R}) < Cw\} \subseteq \widetilde{W}$  and  $|\nabla\phi(s, t)| \leq \frac{C}{t}$ . Note that for  $t < (1 - C)k$ ,

$$|\{s : (s, t) \in \text{supp}\phi\}| = 0$$

and for  $t \geq (1 - C)k$ ,

$$|\{s : (s, t) \in \text{supp}\phi\}| \leq |\{s : \text{dist}((s, t), \Gamma_\alpha(x)) < Ct\}| + |\{s : \text{dist}((s, t), \Gamma_\alpha(y)) < Ct\}| \leq Ct^n.$$

Therefore,

$$\begin{aligned} \int_{\mathcal{R}} t^{-n} \Delta(u(s, t) - a)^+ ds dt &\leq \int_{\text{supp}\phi} t^{-n} \phi(s, t) \Delta(u(s, t) - a)^+ ds dt \\ &\leq C \int_{\text{supp}\phi} [t^{-n} |\nabla\phi(s, t)| + t^{-n-1} \phi(s, t)] |\nabla u(s, t)| ds dt \\ &\leq C\varepsilon\lambda \int_{\text{supp}\phi} t^{-n-2} ds dt \\ &= C\varepsilon\lambda \int_{(1-C)k}^\infty \int_{\text{supp}\phi} t^{-n-2} ds dt \leq C\varepsilon\lambda \int_{(1-C)k}^\infty \frac{dt}{t^2} = \frac{C\varepsilon\lambda}{k}. \end{aligned}$$

This combined with the previous computation gives i); ii) follows by taking supremums.

We now state a proposition, and show how to deduce Theorem 2 from it.

PROPOSITION 2. — *With  $D^*u$  as above we have*

$$\|D^*u\|_{BMO} \leq C\varepsilon\lambda \text{ where } C = C(\alpha, \beta, n).$$

To see how Theorem 2 follows from Proposition 2 we first note that by Lemma 4 of [2],  $\|D^*u\|_{BMO} \leq C\varepsilon\lambda$  implies that for every  $\eta > 0$ ,

$$|\{x \in \mathbb{R}^n : D^*u(x) > 2\eta\}| \leq C_1 \exp\left(\frac{-C_2\eta}{\varepsilon\lambda}\right) |\{x \in \mathbb{R}^n : D^*u(x) > \eta\}|$$

where  $C_1, C_2$  depend only on  $\alpha, n$  and the constant  $C$  of Proposition 2.

Then,

$$\begin{aligned}
 & |\{x \in \mathbf{R}^n : D_\alpha u(x) > 2\lambda, N_\beta u(x) < \varepsilon\lambda\}| \\
 & = |\{x \in E : D_\alpha u(x) > 2\lambda\}| = |\{x \in E : D^*u(x) > 2\lambda\}| \\
 & \leq |\{x \in \mathbf{R}^n : D^*u(x) > 2\lambda\}| \\
 & \leq C_1 \exp\left(\frac{-C_2}{\varepsilon}\right) |\{x \in \mathbf{R}^n : D^*u(x) > \lambda\}| \\
 & \leq C_1 \exp\left(\frac{-C_2}{\varepsilon}\right) |\{x \in \mathbf{R}^n : D_\alpha u(x) > \lambda\}|.
 \end{aligned}$$

Thus, it remains to prove the proposition; the remainder of this section is devoted to this.

Fix a cube  $Q \subseteq \mathbf{R}^n$ ; we will show that there exists a constant  $a_Q$  such that

$$\frac{1}{|Q|} \int_Q |D^*u(x) - a_Q| dx \leq C\varepsilon\lambda$$

with  $C$  independent of  $Q$ . Set  $h = \ell(Q)$  and  $\Gamma_\alpha^h(x) = \{(s, t) : |x - s| < \alpha t, t < h\}$ , and put  $\Gamma_1(x) = \Gamma_\alpha^h(x) \cap W$ ,  $\Gamma_2(x) = (\Gamma_\alpha(x) - \Gamma_1(x)) \cap W$ . For  $j = 1, 2$  we set

$$D_j u(x; a) = \int_{\Gamma_j(x)} t \psi_t(x - s) \Delta(u(s, t) - a)^+ ds dt$$

and

$$D_j u(x) = \sup\{D_j u(x; a) : a \in \mathbf{R}\}.$$

Thus, we have :

$$\begin{aligned}
 (3.3) \quad & \text{(i)} \quad D_j(x; a) = 0 \text{ if } |a| \geq \varepsilon\lambda \\
 & \text{(ii)} \quad D^*u(x; a) = D_1 u(x; a) + D_2 u(x; a) \\
 & \text{(iii)} \quad D^*u(x) \leq D_1 u(x) + D_2 u(x).
 \end{aligned}$$

We now need a local estimate for  $D_1 u(x; a)$ ; this is provided by Lemma 7.

LEMMA 7. — Let  $1 \leq p < \infty$ . Let  $\Omega = \bigcup_{x \in Q} \Gamma_1(x)$ . Then if  $|a| \leq \varepsilon\lambda$ ,

$$\int_{\mathbf{R}^n} \left( \int_{\mathbf{R}_+^{n+1}} t \psi_t(s_0 - s) \chi_\Omega(s, t) \Delta(u(s, t) - a)^+ ds dt \right)^p ds_0 \leq C(\varepsilon\lambda)^p |Q|.$$

Here  $C = C(p, \alpha, \beta, n)$ .

*Proof.* — Let  $\tilde{Q}$  be the cube in  $\mathbf{R}^n$  which is concentric with  $Q$  and with side length  $(1 + 4\alpha)\ell(Q)$ . Then  $|\tilde{Q}| \leq C|Q|$  and if  $(s, t) \in \Omega, s_0 \notin \tilde{Q}$  then  $\psi_t(s_0 - s) = 0$ . Let  $J$  be any subcube of  $\tilde{Q}$  and form

$$V = \left( \bigcup_{z \in J} \Gamma_1(z) \right) \cap \Omega \cap \{(s, t) : t < \ell(J)\}.$$

Then by Green's theorem,

$$(3.4) \quad \int_J \left| \int_{\mathbf{R}_+^{n+1} \cap \{(s,t):t<\ell(J)\}} t\psi_t(s_0 - s)\chi_\Omega(s, t)\Delta(u(s, t) - a)^+ ds dt \right| ds_0 \\ \leq \int_V t\Delta(u(s, t) - a)^+ ds dt \leq C \int_{\partial V} t \left| \frac{\partial}{\partial n}(u - a)^+ \right| + (u - a)^+ \left| \frac{\partial t}{\partial n} \right| d\sigma \\ \leq C(\varepsilon\lambda)|J|$$

where we have used (3.2) and the fact that  $\sigma(\partial V) \leq C|J|$ . (Technically, to apply Green's theorem we first need to smooth  $\partial V$  and the function  $(u - a)^+$  and then pass to the limit. The details of this are as in [14] or [15]). Also, if we let  $s_1$  denote the center of  $J$ , and set

$$D_\Omega^T u(s_0; a) = \int_{\mathbf{R}_+^{n+1} \cap \{t \geq \ell(J)\}} t\psi_t(s_0 - s)\chi_\Omega(s, t)\Delta(u(s, t) - a)^+ ds dt$$

for  $s_0 \in J$ , then Lemma 6 i) implies that

$$(3.5) \quad |D_\Omega^T u(s_0; a) - D_\Omega^T u(s_1; a)| \leq C\varepsilon\lambda \text{ for every } s_0 \in J.$$

Combining (3.4) and (3.5) with the triangle inequality yields :

$$\frac{1}{|J|} \int_J \left| \int_{\mathbf{R}_+^{n+1}} t\psi_t(s_0 - s)\chi_\Omega(s, t)\Delta(u(s, t) - a)^+ ds dt - D_\Omega^T u(s_1; a) \right| ds_0 \\ \leq C\varepsilon\lambda.$$

Therefore,  $\int_{\mathbf{R}_+^{n+1}} t\psi_t(s_0 - s)\chi_\Omega(s, t)\Delta(u(s, t) - a)^+ ds dt$  is in  $BMO$  on  $\tilde{Q}$  with  $BMO$  norm less than  $C\varepsilon\lambda$ . Since (3.4) holds for  $J = \tilde{Q}$ , and  $|\tilde{Q}| \leq C|Q|$ , the conclusion of the Lemma follows.

We now need to turn this local estimate into an estimate for  $D_1 u(x)$ . We first need to create a slightly different version of  $D_1 u(x)$  which approximates  $D_1 u(x)$  but is easier to estimate. Let  $x \in \mathbf{R}^n$  and  $|a| \leq \varepsilon\lambda$  be fixed. Then,

$$D_1 u(x; a) = \int_{\Gamma_1(x)} t\psi_t(x - s)\Delta(u(s, t) - a)^+ ds dt \\ = - \int_{\Gamma_1(x)} \nabla(t\psi_t(x - s)) \cdot \nabla(u(s, t) - a)^+ ds dt \\ + \int_{\partial\Gamma_1(x)} t\psi_t(x - s) \frac{\partial}{\partial n}(u(s, t) - a)^+ d\sigma \\ = I + II.$$

However, by (3.2) and Lemma 5,  $|II| \leq C\varepsilon\lambda$ . Also,

$$\begin{aligned} I &= - \int_{\Gamma_1(x)} \nabla_s t \psi_t(x-s) \cdot \nabla_s (u(s,t) - a)^+ ds dt \\ &\quad - \int_{\Gamma_1(x)} t \frac{\partial}{\partial t} \psi_t(x-s) \frac{\partial}{\partial t} (u(s,t) - a)^+ ds dt \\ &\quad - \int_{\Gamma_1(x)} \psi_t(x-s) \frac{\partial}{\partial t} (u(s,t) - a)^+ ds dt = I_a + I_b + I_c. \end{aligned}$$

Now set  $k_i(s) = s_i \psi(s)$  for  $i = 1, \dots, n$ . Then elementary computations show that

$$\frac{\partial}{\partial t} \psi_t(x-s) = \sum_{i=1}^n \frac{\partial}{\partial s_i} [(k_i)_t(x-s)].$$

Then by the divergence theorem we have :

$$\begin{aligned} I_c &= - \int_{\Gamma_1(x)} \psi_t(x-s) \frac{\partial}{\partial t} (u(s,t) - a)^+ ds dt \\ &= - \int_{\Gamma_1(x)} \sum_{i=1}^n (k_i)_t(x-s) \frac{\partial}{\partial s_i} (u(s,t) - a)^+ ds dt \\ &\quad + \int_{\partial\Gamma_1(x)} [(u(s,t) - a)^+][(k_1)_t(x-s), \dots, (k_n)_t(x-s), -\psi_t(x-s)] \cdot n d\sigma(s,t) \\ &= I_{c_1} + I_{c_2}. \end{aligned}$$

By (3.2), the fact that  $|a| \leq \varepsilon\lambda$ , and Lemma 5, we conclude that  $|I_{c_2}| \leq C\varepsilon\lambda$ . Now we set

$$\begin{aligned} \tilde{D}_1 u(x; a) &= I_a + I_b + I_{c_1} \\ &= \int_{\Gamma_1(x)} -\nabla_s t \psi_t(x-s) \cdot \nabla_s (u(s,t) - a)^+ - t \frac{\partial}{\partial t} \psi_t(x-s) \frac{\partial}{\partial t} (u(s,t) - a)^+ \\ &\quad - \sum_{i=1}^n (k_i)_t(x-s) \frac{\partial}{\partial s_i} (u(s,t) - a)^+ ds dt. \end{aligned}$$

The above computations show that for  $|a| \leq \varepsilon\lambda$ ,

$$(3.6) \quad |D_1 u(x; a) - \tilde{D}_1 u(x; a)| \leq C\varepsilon\lambda.$$

For  $i = 1, \dots, n$ , let  $\psi_i$  denote the partial derivative of  $\psi$  with respect to the  $i^{\text{th}}$  coordinate. Then elementary computations with the chain rule show that if we set

$$\Phi(x) = (\psi_1(x) - k_1(x), \dots, \psi_n(x) - k_n(x), n\psi(x) + \sum_{i=1}^n \psi_i(x)x_i)$$

then we may write  $\tilde{D}_1 u(x; a)$  more succinctly as :

$$\tilde{D}_1 u(x; a) = \int_{\Gamma_1(x)} \Phi_t(x - s) \cdot \nabla(u(s, t) - a)^+ ds dt.$$

Also, we note that since  $\psi$  is radially symmetric, then each of the coordinate functions of  $\Phi(x)$  has mean value 0 on  $\mathbb{R}^n$ .

We now prove a smoothness lemma for  $\tilde{D}_1 u(x; a)$  which is similar to Lemma 2 in [15] or Lemma 3 in [16].

LEMMA 8. — For  $2 \leq p < \infty, |a| \leq \varepsilon\lambda, |b| \leq \varepsilon\lambda,$

$$\int_Q |\tilde{D}_1 u(x; a) - \tilde{D}_1 u(x; b)|^p dx \leq C|Q| |a - b|^{\frac{p}{2}} (\varepsilon\lambda)^{\frac{p}{2}}.$$

*Proof.* — Let  $\varphi$  be a function supported on  $Q$  with  $\|\varphi\|_q = 1$  where  $\frac{1}{p} + \frac{1}{q} = 1$ . Then, setting  $\Omega = \bigcup_{x \in Q} \Gamma_1(x)$ , we have

$$\begin{aligned} & \int_Q (\tilde{D}_1 u(x; a) - \tilde{D}_1 u(x; b)) \varphi(x) dx \\ &= \int_Q \int_{\Gamma_1(x)} \Phi_t(x - s) \cdot \nabla((u(s, t) - a)^+ - (u(s, t) - b)^+) ds dt \varphi(x) dx \\ &= \int_Q \int_{\mathbb{R}_+^{n+1}} \chi_\Omega(s, t) [\Phi_t(x - s) \cdot \nabla((u(s, t) - a)^+ - (u(s, t) - b)^+)] ds dt \varphi(x) dx \\ &= \int_{\mathbb{R}_+^{n+1}} [\Phi_t * \varphi(s) \cdot \nabla((u(s, t) - a)^+ - (u(s, t) - b)^+)] \chi_\Omega(s, t) ds dt \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}_+^{n+1}} \psi_t(s_0 - s) [\Phi_t * \varphi(s) \cdot \nabla((u(s, t) - a)^+ - (u(s, t) - b)^+)] \chi_\Omega(s, t) ds dt ds_0 \\ &\leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_+^{n+1}} \psi_t(s_0 - s) |\Phi_t * \varphi(s)|^2 \frac{ds dt}{t} \right)^{\frac{1}{2}} \\ &\quad \cdot \left( \int_{\mathbb{R}_+^{n+1}} t \psi_t(s_0 - s) \chi_\Omega(s, t) |\nabla((u(s, t) - a)^+ - (u(s, t) - b)^+)|^2 ds dt \right)^{\frac{1}{2}} ds_0 \\ &\leq \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_+^{n+1}} \psi_t(s_0 - s) |\Phi_t * \varphi(s)|^2 \frac{ds dt}{t} \right)^{\frac{q}{2}} ds_0 \right)^{\frac{1}{q}} \\ &\quad \cdot \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_+^{n+1}} t \psi_t(s_0 - s) \chi_\Omega(s, t) |\nabla((u(s, t) - a)^+ - (u(s, t) - b)^+)|^2 ds dt \right)^{\frac{p}{2}} ds_0 \right)^{\frac{1}{p}} \\ &= I \cdot II. \end{aligned}$$



$|I| \leq C_q \|\varphi\|_q \leq C_q$  by well-known results in Littlewood–Paley theory. Also, by (0.6),

$$\begin{aligned} \int_{\mathbf{R}_+^{n+1}} t\psi_t(s_0 - s)\chi_\Omega(s, t) |\nabla((u(s, t) - a)^+ - (u(s, t) - b)^+)|^2 ds dt \\ = \int_a^b \int_{\mathbf{R}_+^{n+1}} t\psi_t(s_0 - s)\chi_\Omega(s, t)\Delta(u(s, t) - c)^+ ds dt dc. \end{aligned}$$

Then by Jensen’s inequality and Lemma 7,

$$\begin{aligned} II &\leq \left( \int_{\mathbf{R}^n} |a - b|^{\frac{1}{2} - \frac{1}{p}} \int_a^b \left( \int_{\mathbf{R}_+^{n+1}} t\psi_t(s_0 - s)\chi_\Omega(s, t)\Delta(u(s, t) - c)^+ ds dt \right)^{\frac{p}{2}} dc ds_0 \right)^{\frac{1}{p}} \\ &= |a - b|^{\frac{1}{2} - \frac{1}{p}} \left( \int_a^b \int_{\mathbf{R}^n} \left( \int_{\mathbf{R}_+^{n+1}} t\psi_t(s_0 - s)\chi_\Omega(s, t)\Delta(u(s, t) - c)^+ ds dt \right)^{\frac{p}{2}} ds_0 dc \right)^{\frac{1}{p}} \\ &\leq |a - b|^{\frac{1}{2} - \frac{1}{p}} |a - b|^{\frac{1}{p}} C(\varepsilon\lambda)^{\frac{1}{2}} |Q|^{\frac{1}{p}} \\ &= C|a - b|^{\frac{1}{2}} (\varepsilon\lambda)^{\frac{1}{2}} |Q|^{\frac{1}{p}} \end{aligned}$$

and this gives Lemma 8.

For  $x \in Q$ , we now set

$$B(x) = \left( \int_{-\varepsilon\lambda}^{\varepsilon\lambda} \int_{-\varepsilon\lambda}^{\varepsilon\lambda} \left| \frac{\tilde{D}_1 u(x; a) - \tilde{D}_1 u(x; b)}{|a - b|^{\frac{1}{2}}} \right|^p da db \right)^{\frac{1}{p}} \text{ for } 2 < p < \infty.$$

Then Lemma 8 and Fubini’s theorem imply that

$$(3.7) \quad \frac{1}{|Q|} \int_Q B^p(x) dx \leq C(\varepsilon\lambda)^{\frac{p}{2+2p}}.$$

Lemma 8 and Kolmogorov’s continuity theorem imply that for almost every  $x \in Q$ ,  $\tilde{D}_1 u(x; a)$  is a continuous function of  $a$  on the interval  $[-\varepsilon\lambda, \varepsilon\lambda]$ . Therefore, for such an  $x$ , we can apply a lemma of Garsia, Rodemich and Rumsey exactly as in [15] or [16] and combine this with the fact that  $\tilde{D}_1 u(x; a) = 0$  for  $a \geq \varepsilon\lambda$  to conclude that  $|\tilde{D}_1 u(x; a)| \leq C_p B(x) (\varepsilon\lambda)^{\frac{1}{2} - \frac{2}{p}}$  whenever  $|a| \leq \varepsilon\lambda$  and  $p > 4$ . But then by (3.6),  $|D_1 u(x)| \leq C_p B(x) (\varepsilon\lambda)^{\frac{1}{2} - \frac{2}{p}} + C\varepsilon\lambda$ . Integrating this last inequality and using (3.7) gives  $\frac{1}{|Q|} \int_Q |D_1 u(x)|^p dx \leq C_p (\varepsilon\lambda)^p$  for  $p > 4$ . Fix  $p = 5$ , say, then by Jensen’s inequality

$$(3.8) \quad \frac{1}{|Q|} \int_Q D_1 u(x) dx \leq C\varepsilon\lambda.$$

where  $C = C(\alpha, \beta, n)$ .

Finally, we can now finish the proof of Proposition 2. Let  $x_0$  be the center of  $Q$ . Then

$$\begin{aligned} \frac{1}{|Q|} \int_Q |D^*u(x) - D_2u(x_0)|dx &\leq \frac{1}{|Q|} \int_Q |D^*u(x) - D_2u(x)|dx \\ &+ \frac{1}{|Q|} \int_Q |D_2u(x) - D_2u(x_0)|dx \\ &\leq \frac{1}{|Q|} \int_Q D_1u(x)dx + C\varepsilon\lambda \\ &\leq C\varepsilon\lambda \end{aligned}$$

where we have obtained the second to the last inequality by using (3.3) iii) and Lemma 6 ii) and the last inequality by using (3.8).

#### 4. The proof of Theorem 3.

We start with a proposition which is similar to Proposition 1.

**PROPOSITION 3.** — *Suppose  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$  is a Lipschitz function and set  $W = \{(x, t) : x \in \mathbb{R}^n, t > \psi(x)\}$ . Suppose  $\rho > \alpha > 0, A_\rho u(x, \psi(x)) \leq 1$  for every  $x \in \mathbb{R}^n$ , and that there exists a point  $z_0 \in \mathbb{R}^n$  such that  $D_\alpha u(z_0, \psi(z_0)) < \infty$ . Then  $\|D_\alpha u(x, \psi(x))\|_{BMO} \leq C$  where  $C$  is a constant depending only on  $\alpha, \rho, n$  and the Lipschitz constant of  $\psi$ .*

Assume the Proposition. Let  $E = \{x : A_\beta u(x, \Phi(x)) < \varepsilon\lambda\}$  and let  $\rho, W$  and  $\psi$  be as in the proof of Theorem 1b. Then by Lemma 11 in [2],  $A_\rho u(x, \psi(x)) \leq L_1\varepsilon\lambda$  for every  $x \in \mathbb{R}^n$ . Thus, by Proposition 3,  $\|D_\alpha u(x, \psi(x))\|_{BMO} \leq C(L_1\varepsilon\lambda)$ . Therefore by Lemma 4 in [2],

$$\begin{aligned} &|\{x \in \mathbb{R}^n : D_\alpha u(x, \psi(x)) > 2\tilde{\lambda}\}| \\ &\leq C_1 \exp\left(\frac{-C_2\tilde{\lambda}}{L_1\varepsilon\lambda}\right) |\{x \in \mathbb{R}^n : D_\alpha u(x, \psi(x)) > \tilde{\lambda}\}|, \end{aligned}$$

for all  $\tilde{\lambda} > 0$ . By Lemma 2,  $D_\alpha u(x, \psi(x)) \leq LD_\alpha u(x, \Phi(x))$ . Now take  $K_5 = 2L$  and proceed as in the proof of Theorem 1b to obtain the conclusion of Theorem 3.

For the proof of the proposition, fix a cube  $Q \subseteq \mathbb{R}^n$  with center at  $x_0$ . Construct  $Q', \Omega, \Gamma_1$  and  $\Gamma_2$  exactly as in the proof of Proposition 1 and

define for  $j = 1, 2$ ,

$$D_j u((x, \psi(x)); a) = \int_{\Gamma_j(x, \psi(x))} (t - \psi(x))^{1-n} \Delta(u(s, t) - a)^+ (ds dt)$$

and

$$D_j u(x, \psi(x)) = \sup_{a \in \mathbf{R}} D_j u((x, \psi(x)); a).$$

We have

LEMMA 9. — *Under the hypothesis of Proposition 2,*

$$|D_2 u(x, \psi(x)) - D_2 u(x_0, \psi(x_0))| \leq C$$

for every  $x \in Q$ . The constant  $C$  depends only on  $\rho$  and the Lipschitz constant of  $\psi$ .

The proof of this lemma is essentially the same as the proof of Lemma 6 and we leave it to the reader. (All that is really needed in the proof of Lemma 6 is the estimate on  $t|\nabla u|$  and in the case of Lemma 9 we also have this by the lemma on page 207 of [20]).

Now let  $\omega_{P^*}$  be the harmonic measure of  $\Omega$  at the point  $P^*$ , (recall that  $\Omega$  is starlike with respect to  $P^*$ ).

LEMMA 10. — *There exist a constant  $C$  depending on  $n$  and the Lipschitz constant for  $\Omega$ , (hence only on  $n, \alpha$ , and  $\rho$ ), such that*

$$\int_{Q'} (D_1 u(P))^2 d\omega_{P^*}(P) \leq C.$$

Letting  $P_0 = (x_0, \psi(x_0))$  and  $P = (x, \psi(x))$ , Lemmas 9 and 10 give that

$$\begin{aligned} \int_{Q'} |D_\alpha u(P) - D_2 u(P_0)|^2 d\omega_{P^*}(P) &\leq 2 \int_{Q'} |D_\alpha u(P) - D_2 u(P)|^2 d\omega_{P^*}(P) \\ &\quad + 2 \int_{Q'} |D_2 u(P) - D_2 u(P_0)|^2 d\omega_{P^*}(P) \\ &\leq C. \end{aligned}$$

By an estimate of Hunt and Wheeden, (see Jerison and Kenig [18]), and the way our Lipschitz domain  $\Omega$  was constructed,  $\omega_{P^*}(Q') \geq C$ , for some constant  $C$ . It follows from this, Chebychev's inequality and the  $A_\infty$  property of  $\omega_{P^*}$  with respect to surface measure ([18]), that there exists a constant  $b > 0$  depending on the Lipschitz constant of  $\Omega$  such that

$$|\{x \in Q : |D_\alpha u(x, \psi(x)) - D_2 u(x_0, \psi(x_0))| > \lambda\}| \leq \frac{C}{\lambda^b} |Q|.$$

Choosing  $\lambda_0$  such that  $\frac{C}{\lambda_0^b} = \frac{1}{4}$ , it follows from Strömberg [21] that

$$\|D_\alpha u(x, \psi(x))\|_{BMO} \leq C\lambda_0$$

and the Proposition is proved.

We now turn to the proof of Lemma 10. Let  $G(z, w), z, w \in \Omega$ , be the Green's function for  $\Omega$  and let  $K(z, P), P \in \partial\Omega$ , be its Poisson kernel. We recall three facts about Lipschitz domains; the first is trivial and the other two are due to Dahlberg, a proof can be found in [18]. These state that there exist two positive constants  $r_0$  and  $C_0$  such that

(1) If  $r < r_0$  and  $P \in \partial\Omega$ , then there exists a point  $A_r(P) \in \Omega$  such that  $C_0^{-1}r < |A_r(P) - P| < C_0r$  and  $C_0^{-1}r < \text{dist}(A_r(P), \partial\Omega)$ ;

(2) if  $r < r_0, P \in \partial\Omega$  and  $z \in \Omega \setminus B(A_r(P), \frac{1}{2}C_0^{-1}r)$ , where  $B(A_r(P), \frac{1}{2}C_0^{-1}r)$  is the ball in  $\mathbb{R}_+^{n+1}$  with  $A_r(P)$  and radius  $\frac{1}{2}C_0^{-1}r$ , then

$$C_0^{-1} \leq \frac{G(z, A_r(P))r^{n-1}}{\omega_z(\Delta(P, r))} \leq C_0$$

where  $\Delta(P, r) = B(P, r) \cap \partial\Omega$ ; the surface ball centered at  $P$  and radius  $r$ ;

(3) for all  $z \in \Omega \setminus B(P, C_0r)$ ,

$$K(z, P) \approx K(A_r(P), P)\omega_z(\Delta(P, r)).$$

Now let  $B_t$  be Brownian motion starting at  $P^* \in \Omega$  and let  $\tau_\Omega$  be its exit time from  $\Omega$ . Since  $u$  is harmonic in  $\Omega, u(B_{t \wedge \tau_\Omega})$  is a martingale. Let  $Lu(t; a)$  denote its local time. It follows from (0.1) and the Itô formula that for all Borel functions  $f$  in  $\mathbb{R}$ ,

$$(4.1) \quad \int_0^{\tau_\Omega} f(u(B_s))|\nabla u(B_s)|^2 ds = \int_{\mathbb{R}} f(a)Lu(\tau_\Omega; a)da.$$

Let  $E_{P^*}$  be the expectation for Brownian motion starting at  $P^*$  and let  $E_{P^*}^P, P \in \partial\Omega$ , be the expectation for this motion conditioned to exit  $\Omega$  at  $P$ ; the Doob  $h$ -process corresponding to the harmonic function  $h(z) = K(z, P)$ . Let  $G_P(z, \omega)$  be the Green's function for the conditional process. Taking  $E_{P^*}^P$  of both sides of (4.1) and using Fubini's theorem we get

$$(4.2) \quad \int_{\Omega} G_P(P^*, z)f(u(z))|\nabla u(z)|^2 dz = \int_{\mathbb{R}} f(r)E_{P^*}^P(Lu(\tau_\Omega; r))dr.$$

Applying (0.6) to the left hand side of (4.2) we get

$$\begin{aligned} \int_{\mathbf{R}} \int_{\Omega} G_P(P^*, z) f(r) \Delta(u-r)^+(dz) dr \\ = \int_{\mathbf{R}} f(r) E_{P^*}^P(Lu(\tau_{\Omega}; r)) dr. \end{aligned}$$

It follows from this formula, (we refer the reader to Brossard [7] for the proof), that

$$(4.3) \quad E_{P^*}^P(Lu(\tau_{\Omega}; a)) = \frac{1}{K(P^*, P)} \int_{\Omega} G(P^*, z) K(z, P) \Delta(u-a)^+(dz).$$

Next, let  $\Gamma_{\varepsilon}^h(P), P \in \partial\Omega$ , be as in the proof of Proposition 1. Define

$$\begin{aligned} Du(P; a) &= \int_{\Gamma_{\varepsilon}^h(P)} d(z, P)^{1-n} \Delta(u(z) - a)^+(dz), \\ Du(P) &= \sup_{a \in \mathbf{R}} (Du(P; a)), \end{aligned}$$

and

$$Au(P) = \left( \int_{\Gamma_{\varepsilon}^h(P)} d(z, P)^{1-n} |\nabla u(z)|^2 dz \right)^{\frac{1}{2}}.$$

It follows from facts (2), (3) above and (4.3) that there exists a constant  $C$  depending only on the Lipschitz constant of  $\Omega$  such that

$$(4.4) \quad Du(P) \leq C \sup_{a \in \mathbf{R}} E_{P^*}^P(Lu(\tau_{\Omega}; a)) \leq C E_{P^*}^P(L^*u(\tau_{\Omega}))$$

where  $L^*u(\tau_{\Omega})$  is the maximal local time of  $u(B_t), t < \tau_{\Omega}$ . Integrating gives

$$\begin{aligned} \int_{\partial\Omega} (Du(P))^2 d\omega_{P^*}(P) &\leq C \int_{\partial\Omega} (E_{P^*}^P(L^*u(\tau_{\Omega}))^2) d\omega_{P^*}(P) \\ (4.5) \quad &= C E_{P^*}((E_{P^*}^P(L^*u(\tau_{\Omega})))^2) \\ &\leq C E_{P^*} \cdot E_{P^*}^P(L^*u(\tau_{\Omega}))^2 \\ &= C E_{P^*}(L^*u(\tau_{\Omega}))^2 \\ &\leq C E_{P^*}|u(B_{\tau_{\Omega}})|^2 \end{aligned}$$

where for the last inequality we have used the Barlow-Yor [4] result applied to the martingale  $u(B_t)$ ; (we are assuming, as we may, that  $u(P^*) = 0$ ). Rewriting the right hand side of (4.5) in terms of harmonic measure and using the result of Dahlberg [10] we find that

$$\begin{aligned} \int_{\partial\Omega} (Du(P))^2 d\omega_{P^*}(P) &\leq C \int_{\partial\Omega} |u(P)|^2 d\omega_{P^*}(P) \\ (4.6) \quad &\leq C \int_{\partial\Omega} (Au(P))^2 d\omega_{P^*}(P). \end{aligned}$$

Since  $A_\rho u(x, \psi(x)) \leq 1$  for all  $x \in \partial\Omega$  by a slight variation of the proof of Lemma 11 of [2] we have  $Au(P) \leq C$  for all  $P \in \partial\Omega$ . Thus

$$\int_{\partial\Omega} (Du(P))^2 d\omega_{P^*}(P) \leq C.$$

Since  $\Gamma_1(P) \subseteq \Gamma_\epsilon^h(P)$  for  $P \in Q'$ , we have that

$$\int_{Q'} (D_1u(P))^2 d\omega_{P^*}(P) \leq \int_{\partial\Omega} (Du(P))^2 d\omega_{P^*}(P) \leq C$$

and Lemma 10 is proved.

**5. An alternative proof of Theorem 3 for the upper half space.**

It is highly desirable, in order to get a better understanding of the analytic structure of the  $D$ -functional and as Gundy and Silverstein [16] put it, to better understand “its possible status in the catalogue of artifacts under the label Littlewood–Paley, singular integral theory,” to have a “classical” proof of the good- $\lambda$  inequalities. For this reason we provide an analytical proof of Theorem 3. However, we have not been able to provide such proof in the Lipschitz domains setting and, in addition we only obtain the result with  $\exp(C/\epsilon^{2/3})$  on the right hand side. More precisely, we have

**THEOREM 3'.** — *Let  $u$  be a harmonic function on  $\mathbf{R}_+^{n+1}$  and let  $0 < \alpha < \beta$  and define  $D_\alpha u(x)$  using (0.9). There are constants  $K_5, C_9$  and  $C_{10}$  depending only on  $\alpha, \beta, n$  and  $\psi$  and such that if  $\lambda > 0$  and  $0 < \epsilon < 1$ ,*

$$\begin{aligned} & |\{x \in \mathbf{R}^n : D_\alpha u(x) > K_s \lambda, A_\beta u(x) < \epsilon \lambda\}| \\ & \leq C_9 \exp\left(\frac{C_{10}}{\epsilon^{2/3}}\right) |\{x \in \mathbf{R}^n : D_\alpha u(x) > \lambda\}|. \end{aligned}$$

The proof of Theorem 3' is short; we merely have to combine what was done in section 3 with a lemma from [3]. An elementary argument shows that  $\{x : D_\alpha u(x) > \lambda\}$  is open and so we let  $\{Q_j\}$  be a Whitney decomposition of this set. Then it suffices to show :

$$(5.1) \quad |\{x \in Q_j : D_\alpha u(x) > K\lambda, A_\beta u(x) < \epsilon\lambda\}| \leq C_1 \exp\left(\frac{-C_2}{\epsilon^{2/3}}\right) |Q_j|$$

for every  $Q_j$ .

We now fix such a  $Q_j$ , call it  $Q$ , and let  $x_0$  be the center of  $Q$  and set  $h = \ell(Q)$ . Let  $\rho > 0$  be a constant to be fixed momentarily, and for

$\mu > 0$  set  $\Gamma_\mu^{\rho h}(x) = \{(s, t) : |x - s| < \mu t, t < \rho h\}$ . Now set  $\gamma = \frac{\alpha + \beta}{2}$  and define  $D_\alpha^{\rho h}u(x)$  and  $N_\gamma^{\rho h}u(x)$  by taking the same definitions as before but by using the cones  $\Gamma_\alpha^{\rho h}(x)$  and  $\Gamma_\gamma^{\rho h}(x)$  respectively.

Since  $Q$  is a cube in a Whitney decomposition of  $\{D_\alpha u(x) > \lambda\}$ , there exists a constant  $C(n)$  such that if  $\tilde{Q}$  represents the cube in  $\mathbb{R}^n$  having center  $x_0$ , sides parallel to those of  $Q$  and with  $\ell(\tilde{Q}) = C(n)\ell(Q)$ , then  $\tilde{Q} \cap \{D_\alpha u(x) \leq \lambda\} \neq \emptyset$ . Now note that since  $\alpha < \gamma$ , then we may choose  $\rho$  large enough (and depending only on  $\alpha, \beta, n$ ) such that if  $x, y \in \tilde{Q}$ , then  $\Gamma_\alpha(x) \setminus \Gamma_\alpha^{\rho h}(x) \subseteq \Gamma_\gamma(y)$ . Let  $y_0$  be a point of  $Q$  for which  $A_\beta u(y_0) < \varepsilon \lambda$ . (We may assume  $y_0$  exists, else (5.1) is trivial for  $Q$ ). Set  $\gamma' = \frac{\beta + \gamma}{2}$ , then by the Lemma on page 207 of [20] we have

$$(5.2) \quad t|\nabla u(s, t)| \leq C\varepsilon\lambda$$

for all  $(s, t) \in \Gamma_{\gamma'}(y_0)$ .

Now define  $D_\alpha^T u(x)$  as in (0.9) but with the integration taken only over  $\Gamma_\alpha(x) \setminus \Gamma_\alpha^{\rho h}(x)$ . Pick  $z_0 \in \tilde{Q} \cap \{D_\alpha u(x) \leq \lambda\}$ . Then  $D_\alpha^T u(z_0) \leq \lambda$  and we claim that then  $|D_\alpha^T u(x)| \leq \lambda + C\varepsilon\lambda$  for every  $x \in Q$ . To see this, we recall that  $\Gamma_\alpha(x) \setminus \Gamma_\alpha^{\rho h}(x) \subseteq \Gamma_\gamma(y_0) \subseteq \Gamma_{\gamma'}(y_0)$  for all  $x \in \tilde{Q}$  and reasoning as in Lemma 6, we conclude that  $|D_\alpha^T u(x) - D_\alpha^T u(z_0)| \leq C\varepsilon\lambda$  for all  $x \in \tilde{Q}$ . (All that is required in the proof of Lemma 6 is an estimate such as (5.2)). Since  $D_\alpha u(x) \leq D_\alpha^T u(x) + D_\alpha^{\rho h}u(x)$  then by setting  $L = K - (1 + C\varepsilon)$ , (5.1) follows if we show :

$$(5.3) \quad |\{x \in Q : D_\alpha^{\rho h}u(x) > L\lambda, A_\beta u(x) < \varepsilon\lambda\}| \leq C_1 \exp\left(\frac{-C_2}{\varepsilon^{\frac{2}{3}}}\right) |Q|.$$

Since both  $D_\alpha^{\rho h}u(x)$  and  $A_\beta u(x)$  remain unchanged when we add a constant to  $u$  we may assume that  $u(x_0, Ch) = 0$  where  $C$  will be determined momentarily. We break apart the set on the left hand side of (5.3) using  $N_\gamma^{\rho h}u(x)$  to obtain :

$$\begin{aligned} & |\{x \in Q : D_\alpha^{\rho h}u(x) > L\lambda, A_\beta u(x) < \varepsilon\lambda\}| \\ & \leq |\{x \in Q : D_\alpha^{\rho h}u(x) > L\lambda, A_\beta u(x) < \varepsilon\lambda, N_\gamma^{\rho h}u(x) \geq L\varepsilon^{\frac{2}{3}}\lambda\}| \\ & \quad + |\{x \in Q : D_\alpha^{\rho h}u(x) > L\lambda, A_\beta u(x) < \varepsilon\lambda, N_\gamma^{\rho h}u(x) < L\varepsilon^{\frac{2}{3}}\lambda\}| \\ & \leq |\{x \in Q : N_\gamma^{\rho h}u(x) \geq L\varepsilon^{\frac{2}{3}}\lambda, A_\beta u(x) < \varepsilon\lambda\}| \\ & \quad + |\{x \in Q : D_\alpha^{\rho h}u(x) > L\lambda, N_\gamma^{\rho h}u(x) < L\varepsilon^{\frac{2}{3}}\lambda\}| \\ & = I + II. \end{aligned}$$

To estimate  $I$ , we use Lemma 4.1 of [3]. This lemma states that there exists constants  $\tilde{L}$  and  $C$  depending only on  $\gamma, \beta, n$  such that if  $\tilde{h} = Ch$  and  $u(x_0, \tilde{h}) = 0$  then for every  $\lambda > 0, 0 < \varepsilon < 1$ , we have

$$(5.4) \quad |\{x \in Q : N_{\gamma}^{\tilde{h}}u(x) > \tilde{L}\lambda, A_{\beta}u(x) < \varepsilon\lambda\}| \leq C_1 \exp\left(\frac{-C_2}{\varepsilon^2}\right) |Q|.$$

We now assume that  $u(x_0, \tilde{h}) = 0$ . Note that if  $\rho h > \tilde{h}$  then for an  $x$  with  $A_{\beta}u(x) < \varepsilon\lambda$  we have  $N_{\gamma}^{\rho h}u(x) \leq N_{\gamma}^{\tilde{h}}u(x) + C\varepsilon\lambda$  by the estimate (5.2). If  $\rho h < \tilde{h}$  then this estimate is still true. At any rate, by choosing  $L$  large enough we have :

$$\begin{aligned} & \{x \in Q : N_{\gamma}^{\rho h}u(x) \geq L\lambda, A_{\beta}u(x) < \varepsilon\lambda\} \\ & \subseteq \{x \in Q : N_{\gamma}^{\tilde{h}}u(x) > \tilde{L}\lambda, A_{\beta}u(x) < \varepsilon\lambda\}. \end{aligned}$$

Then by this fact, and (5.4) with  $\lambda$  replaced by  $\varepsilon^{\frac{2}{3}}\lambda$  and  $\varepsilon$  replaced by  $\varepsilon^{\frac{1}{3}}$  we obtain  $I \leq C_1 \exp\left(\frac{-C_2}{\varepsilon^{\frac{2}{3}}}\right) |Q|$ . To control  $II$ , we proceed as before.

We form  $E = \{x \in Q : N_{\gamma}^{\rho h}u(x) < L\varepsilon^{\frac{2}{3}}\lambda\}$ , set  $\gamma'' = \frac{\gamma + \alpha}{2}$  and form  $W = \bigcup_{x \in E} \Gamma_{\gamma''}^{\rho h}(x)$ . There exists a cube  $Q' \subseteq Q$  such that  $\ell(Q') \leq C\ell(Q)$

and if  $x \notin Q'$ , then  $\Gamma_{\gamma}^{\rho h}(x) \cap W = \emptyset$ . For  $x \in Q'$  we then define  $D^*u(x)$  as before but this time by restricting the integration to  $W \cap \Gamma_{\alpha}^{\rho h}(x)$ . Then as in section 3, we can show  $\|D^*u(x)\|_{BMO(Q')} \leq CL\varepsilon^{\frac{2}{3}}\lambda$ . Also, we then have the analogue of (3.8) :

$$\frac{1}{|Q'|} \int_{Q'} D^*u(x) dx \leq CL\varepsilon^{\frac{2}{3}}\lambda.$$

These facts and the John-Nirenberg theorem imply that

$$|\{x \in Q : D^*u(x) > \eta\}| \leq C_1 \exp\left(\frac{-C_2\eta}{L\varepsilon^{\frac{2}{3}}\lambda}\right) |Q'|$$

for every  $\eta > 0$ . Since  $D^*u(x) = D_{\alpha}^{\rho h}u(x)$  when  $N_{\gamma}^{\rho h}u(x) < L\varepsilon^{\frac{2}{3}}\lambda$ , we have :

$$II \leq |\{x \in Q : D^*u(x) > L\lambda\}| \leq C_1 \exp\left(\frac{-C_2}{\varepsilon^{\frac{2}{3}}}\right) |Q|,$$

which proves (5.3) and completes the proof.

### 6. Concluding Remarks.

As mentioned in the introduction, the good- $\lambda$  inequalities for the maximal local time were proved by Bass [6] and independently by Davis



[12]. Even though these authors did not obtain sharp estimates, these proofs can easily be adapted to give the following sharp good- $\lambda$  inequalities between  $L^*$ ,  $X^*$  and  $S(X)$  :

$$(6.1) \quad P\{X^* > 2\lambda, L^* \leq \varepsilon\lambda\} \leq C_1 \exp\left(\frac{-C_2}{\varepsilon}\right) P\{X^* > \lambda\},$$

$$(6.2) \quad P\{S(X) > 2\lambda, L^* \leq \varepsilon\lambda\} \leq C_1 \exp\left(\frac{-C_2}{\varepsilon^2}\right) P\{S(X) > \lambda\},$$

$$(6.3) \quad P\{L^* > 2\lambda, X^* \leq \varepsilon\lambda\} \leq C_1 \exp\left(\frac{-C_2}{\varepsilon}\right) P\{L^* > \lambda\},$$

and

$$(6.4) \quad P\{L^* > 2\lambda, S(X) \leq \varepsilon\lambda\} \leq C_1 \exp\left(\frac{-C_2}{\varepsilon^2}\right) P\{L^* > \lambda\}.$$

In the setting of harmonic functions, using (0.7), the inequality  $\|A_\alpha u\|_p \leq C_p \|N_\alpha u\|_p$  with  $C_p = O(\sqrt{p})$  as  $p \rightarrow \infty$  proved in [2], and integrating out the good- $\lambda$  inequality in Theorem 2(a), it follows that Theorem 2(a) is also sharp in terms of the decay in  $\varepsilon$  as  $\varepsilon \rightarrow 0$ . It seems by analogy with the martingale case that Theorem 1 is also sharp in this respect but we have not been able to prove this. In the case of the area function in place of the  $D$ -functional, the sharpness of such results is proved by explicitly computing the area integral of a lacunary series. It seems to be very nontrivial to compute the  $D$ -functional explicitly for any function. With regards to Theorem 3, we believe that the sharp estimate should be  $\exp\left(\frac{-C}{\varepsilon^2}\right)$  as in the martingale case. However, before one can prove this, one will have to prove the following more basic *conjecture* :

Suppose  $u$  is a harmonic function in  $\mathbb{R}_+^{n+1}$  with the property that  $A_\beta u(x) \leq 1$  for almost every  $x \in \mathbb{R}^n$ . Let  $0 < \alpha < \beta$  and suppose there exist an  $x_0 \in \mathbb{R}^n$  such that  $D_\alpha u(x_0) < \infty$ . Then  $D_\alpha u$  has a subgaussian estimate on every cube. That is, given a cube  $Q \subset \mathbb{R}^n$  there exist a constant  $C_Q$  such that for all  $\lambda > 0$ ,

$$(6.5) \quad |\{x \in Q : |D_\alpha u(x) - C_Q| > \lambda\}| \leq C_1 \exp(-C_2 \lambda^2) |Q|$$

where  $C_1$  and  $C_2$  depend only on  $\alpha, \beta$ , and  $n$ .

If we replace  $D_\alpha u$  by  $N_\alpha u$ , (6.5) follows from a result of Chang, Wilson and Wolff [10], (see [2] for full details). Also we believe, (although we have

not written all the details down), that if we replace  $D_\alpha u(x)$  by  $D_\alpha u(x; a)$  we can prove (6.5) and with  $C_1$  and  $C_2$  independent of  $a$ .

In [19], Kesten proves two LIL's for local time :

$$(6.6) \quad \limsup_{t \rightarrow \infty} \frac{L_t^*}{\sqrt{2S_t^2(X) \log \log S_t^2(X)}} = 1$$

a.s. on  $\{S(X) = \infty\}$  and

$$(6.7) \quad \limsup_{t \rightarrow \infty} \left( \frac{\log \log S_t^2(X)}{S_t^2(X)} \right)^{\frac{1}{2}} L_t^* = \gamma$$

a.s. on  $\{S(X) = \infty\}$ , where  $\gamma$  is a constant such that  $q_0/2 \leq \gamma \leq q_0^2/\sqrt{2}$ , and where  $q_0$  is the smallest positive zero of the Bessel function  $J_0(x)$ .

The Kesten LIL (6.7) is perhaps the deepest and most difficult of all the LIL's for Brownian motion. We believe that the corresponding result for harmonic functions is a very challenging and interesting problem. Corollary 1(a) gives an analogue for the lower bound. The upper bound is open. Also, we have been unable to obtain an analogue of either half of (6.6). The upper half analogue of (6.6) would follow if we could prove Theorem 3 with  $\exp(-c/\varepsilon^2)$ . This, however, requires proving (6.5) not only in flat space but in Lipschitz domains. Finally we mention that since its discovery by P. Lévy, the local time has been of fundamental importance in the study and applications of Brownian motion. We believe that when the  $D$ -functional is as well understood as the local time, it will play a correspondingly useful role in the study of harmonic functions. J. Brossard and L. Chevalier [8] have already made interesting progress in this direction with their new characterization of  $L \log L$ .

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