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ON MEROMORPHIC EQUIVALENCE OF LINEAR DIFFERENCE OPERATORS

par Gertrude K. IMMINK

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0. Introduction.

We consider linear difference operators of the type

$$\Delta_A y(z) = y(z+1) - A(z)y(z) \ ,$$

where $A \in G\ell(n; \mathbb{C}\{z^{-1}\}[z])$, $n \in \mathbb{N}$.

Two difference operators Δ_A and Δ_B will be called formally equivalent, if there exists a matrix $F \in G\ell(n; \mathbb{C}[z^{-1}][z])$ such that $F(z+1)^{-1}A(z)F(z) = B(z)$. The difference operators Δ_A and Δ_B will be called meromorphically equivalent if there exists a matrix $F \in G\ell(n; \mathbb{C}\{z^{-1}\}[z])$ with this property.

Meromorphically equivalent difference operators are also formally equivalent. This paper is concerned with the meromorphic classification of difference operators belonging to the same formal equivalence class. The meromorphic equivalence classes will be characterized by a system of "meromorphic invariants". It is shown that this system is both complete and free. In other words, if two difference operators have the same set of invariants they must be equivalent. Moreover, every possible set of invariants corresponds to some equivalence class.

Key-words : Linear difference operator – Formal equivalence – Meromorphic equivalence – Meromorphic invariants – Inverse problem – Riemann-Hilbert problem.
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Neither the results nor the methods of this paper are entirely new. We have merely rearranged and extended some results already established by Birkhoff (*cf.* [1], [2]).

In §1 we define a system of meromorphic invariants. Its completeness is proved by an argument familiar from the theory of differential equations (*cf.* [7], [8], [11]). The main difficulty is to solve the inverse problem, *i.e.* to establish the existence of a difference operator having a given set of invariants. This is done in §2. The problem is reduced to a Riemann-Hilbert boundary value problem (also called Riemann problem by some and Hilbert problem by others) on two intersecting contours (*cf.* also [4]). Here we have resorted to well-known existence theorems (*cf.* [9], [13]), rather than adopting Birkhoff's constructive but elaborate method. In both sections we pay special attention to the important subclass of difference operators with rational coefficients.

We do not go into the difficult problem of the actual evaluation of the meromorphic invariants. An attempt in that direction was undertaken in [6]. For a very profound study of the analytic invariants of various local objects we refer the reader to the work of J. Ecalle (*cf.* [3]).

1. A complete system of meromorphic invariants.

We use the following notations

$$K = \mathbb{C}\{z^{-1}\}[z]\,, \qquad \widehat{K} = \mathbb{C}[z^{-1}][z]\,,$$
$$K_p = \mathbb{C}\{z^{-1/p}\}[z^{1/p}]\,, \qquad \widehat{K}_p = \mathbb{C}[z^{-1/p}][z^{1/p}]\,, \quad p \in \mathbb{N}\,.$$

Let $A \in G\ell(n;K)$. It is known that there exists a positive integer p and a matrix $F \in G\ell(n;\widehat{K}_p)$ such that the transformation

$$A \longrightarrow A^F \equiv F(z+1)^{-1}A(z)F(z)$$

changes A into a matrix function $\overset{c}{A}$ of the form

$$\overset{c}{A}(z) = \exp\{Q(z+1)-Q(z)\}\big(1+\frac{1}{z}\big)^G$$

where $G = \operatorname{diag}\{G_1,\dots,G_m\}$, $Q = \operatorname{diag}\{q_1 I_{n_1},\dots,q_m I_{n_m})$ with $m \in \mathbb{N}$ and, for all $i \in \{1,\dots,m\}$, $G_i = \gamma_i I_{n_i} + N_i$, $\gamma_i \in \mathbb{C}$, $0 \le \operatorname{Re}\gamma_i < \frac{1}{p}$, N_i is a nilpotent $n_i \times n_i$ matrix, $q_i(z) = d_i z\log z + \sum_{h=1}^{p} \mu_{i,h} z^{h/p}$, $d_i \in \frac{1}{p}\mathbb{Z}$, $\mu_{i,h} \in \mathbb{C}$, $0 \le \operatorname{Im}\mu_{i,p} < 2\pi$.

We shall call $\overset{c}{A}$ a canonical form of A. It is uniquely determined by A up to permutations of the diagonal blocks (*cf.* [10]).

We shall write

$$d_i - d_j = d_{ij},\ q_i - q_j = q_{ij},\ \gamma_i - \gamma_j = \gamma_{ij},\ \mu_{i,h} - \mu_{j,h} = \mu_{ij,h}$$

for all $i,j \in \{1,\dots,m\}$, $h \in \{1,\dots,p\}$.

LEMMA 1.1. — *Let $A \in \mathrm{G}\ell(n;K)$, $F_1, F_2 \in \mathrm{G}\ell(n;\widehat{K}_p)$ with $p \in \mathbb{N}$, and suppose that*

$$A^{F_1} = A^{F_2} = \overset{c}{A} \ .$$

Then there exists a constant invertible $n \times n$ matrix C such that

$$[\overset{c}{A}, C] = 0 \quad \textit{and} \quad F_2 = F_1 C \ .$$

Proof. — Let $F = F_1^{-1} F_2$. Then we have

$$F(z+1) = \overset{c}{A}(z) F(z) \overset{c}{A}(z)^{-1} \ .$$

Hence the block F_{ij} in the partition of F induced by $\overset{c}{A}$, must satisfy the equation

$$Y(z+1) = \exp\{q_{ij}(z+1) - q_{ij}(z)\}(1+1/z)^{\gamma_{ij}}(1+1/z)^{N_i} Y(z)(1+1/z)^{-N_j} \ , \quad i,j \in \{1,\dots,m\} \ .$$

This equation has no nonvanishing solutions $\in \mathrm{Hom}(\widehat{K}_p^{n_j}, \widehat{K}_p^{n_i})$ unless $q_{ij} \equiv 0$, $\gamma_{ij} = 0$. In the latter case the only solutions $\in \mathrm{Hom}(\widehat{K}_p^{n_j}, \widehat{K}_p^{n_i})$ are the constant $n_i \times n_j$ matrices C_{ij} with the property that

$$N_i C_{ij} = C_{ij} N_j \ .$$

Thus F is a constant matrix with the property that

$$\overset{c}{A}_{ii} F_{ij} = F_{ij} \overset{c}{A}_{jj} \quad \text{for all} \quad i,j \in \{j,\dots,m\}$$

and hence $[\overset{c}{A}, F] = 0$. Obviously, $\det F \neq 0$.

DEFINITION. — (i) *A quadrant is a region Γ of $\mathbb{C}$ of the following form :*

$$\Gamma = \{z \in \mathbb{C} : k\frac{\pi}{2} < \arg(z - z_0) < (k+1)\frac{\pi}{2} \ , \ |z| > R_0\}$$

where $z_0 \in \mathbb{C}$, $k \in \mathbb{Z}$ and R_0 is a positive number.

(ii) *Let* Γ *be a region of* $\mathbb{C}$ *and* R *a positive number. By* Γ^* *and* $\Gamma(R)$ *we shall denote the regions*

$$\Gamma^* = \{z \in \mathbb{C} : \bar{z} \in \Gamma\}$$

and

$$\Gamma(R) = \{z \in \Gamma : |z| > R\} .$$

DEFINITION. — *Let* Γ *be an unbounded region of* $\mathbb{C}$, φ *an analytic function on* Γ *and* f *a formal series of the form* $f = \sum_{n \geq n_0} f_n z^{-n/p}$, $n_0 \in \mathbb{Z}$. *We shall say that* φ *is represented asymptotically by* f *(or admits the asymptotic expansion* f*) as* $z \to \infty$ *in* Γ, *if*

$$\sup_{z \in \Gamma} \left| (\varphi(z) - \sum_{n=n_0}^{N-1} f_n z^{-n/p}) z^{N/p} \right| < \infty \quad \text{for all } N > n_0 .$$

In that case we write

$$\varphi(z) \sim f \ , \ z \to \infty \text{ in } \Gamma$$

and put $f = \widehat{\varphi}$.

DEFINITION. — *A matrix function* Φ *will be called non singular on a set* S *if* $\det \Phi(z) \neq 0$ *for all* $z \in S$.

The following theorem is a slightly improved version of theorem 3.2 in [5].

THEOREM 1.2. — *Let* $A \in G\ell(n; K)$ *and let* $\overset{c}{A}$ *be a canonical form of* A. *Let* Γ *be a quadrant. There exists a positive number* R *and a matrix function* Φ *with the following properties :*

(i) Φ *is non singular and analytic in* $\Gamma(R) \cup \Gamma(R)^*$

(ii) Φ *is represented asymptotically by a matrix* $F \in G\ell(n; \widehat{K}_p)$ $(p \in \mathbb{N})$, *as* $z \to \infty$ *in* $\Gamma(R)$

(iii) $A^\Phi = \overset{c}{A}$.

Proof. — We shall consider the case that Γ (and hence Γ^*) is contained in some right half plane $\operatorname{Re} z > a$, $a \in \mathbb{R}$. The proof for a left half plane is analogous. Without loss of generality we may assume that Γ contains a strip S_0 of the form $S_0 = \{z \in \mathbb{C} : \operatorname{Re} z > b, \ |\operatorname{Im} z| < c\}$ where

$b > 0$, $c > 0$. Let R be a large positive number and let $\Gamma_+ = \Gamma(R)$ if $\operatorname{Im} z$ is bounded from below on Γ, and $\Gamma_+ = \Gamma(R)^*$ otherwise. Let $\Gamma_- = \Gamma_+^*$. According to theorem 3.2 in [5], if R is sufficiently large, there exist matrix functions Φ^+ and Φ^- with the following properties :

(i) $\Phi^\pm$ is analytic in $\Gamma_\pm$

(ii) $\Phi^\pm$ admits an asymptotic expansion $F^\pm \in G\ell(n; \widehat{K}_p)$ as $z \to \infty$ in $\Gamma_\pm$

(iii) $A^{\Phi_\pm} \overset{c}{=} A$.

The second property implies that

$$\det \Phi^\pm(z) \neq 0 \text{ for all } z \in \Gamma_\pm \tag{1.3}$$

provided R has been chosen sufficiently large. In view of lemma 1.1 we may assume that $F^+ = F^- = F$ (if this is not the case, Φ^- can be replaced by $\Phi^- C$, where $C = (F^-)^{-1}F^+$). Let

$$Y^\pm(z) = \Phi^\pm(z)\exp\{Q(z)\}z^G \tag{1.4}$$

(we take $\arg z \in (-\pi, \pi)$) and

$$P = (Y^+)^{-1}Y^- \ ; \tag{1.5}$$

P is periodic matrix function of period 1, analytic in $S \equiv \Gamma_+ \cap \Gamma_-$. Furthermore, in the partition induced by $\overset{c}{A}$

$$P_{ij}(z) = \exp\{-q_{ij}(z)\}(z^{-G}(\Phi^+)^{-1}\Phi^- z^G)_{ij}$$

for all $i, j \in \{1, \dots, m\}$. Due to the fact that $(\Phi^+)^{-1}\Phi^- \sim I$ as $z \to \infty$ in S, $\lim_{z\to\infty} P_{ij}(z) = \delta_{ij}$ and hence $P_{ij} \equiv \delta_{ij}$, unless $\exp q_{ij}(z) \to 0$ as $z \to \infty$ in S. For convenience we shall assume that the blocks of $\overset{c}{A}$ are ordered in such a way that for all sufficiently large positive numbers z

$$\operatorname{Re} q'_{ij}(z) \geq 0 \ \text{ if } \ i \geq j \ . \tag{1.6}$$

This implies that, for $i \geq j$, $\operatorname{Re}(q_{ij})$ is bounded from below on the positive real axis and hence $P_{ij} \equiv \delta_{ij}$. Following Birkhoff and Trjitzinsky (*cf.* [2]) we are going to look for periodic matrix functions P^+ and P^-, upper-block-triangular in the partition induced by $\overset{c}{A}$, analytic in Γ_+ and Γ_- respectively, such that

$$P = P^+P^- \ . \tag{1.7}$$

We choose $P^+_{ii} = P^-_{ii} = I_{n_i}$ for all $i \in \{1, \dots, m\}$. The remaining blocks can be determined recursively from the relations

$$P_{ij} = P^+_{ij} + P^-_{ij} + \sum_{i<h<j} P^+_{ih}P^-_{hj} \ , \quad i < j \tag{1.8}$$

by means of induction on $j-i$. Obviously, the factorization (1.7) is not unique. We may impose the additional condition that, for $i<j$, P_{ij}^{+} and P_{ij}^{-} have the form

$$(1.9)\qquad P_{ij}^{+}(z)=\sum_{n\geq n_{ij}}(P_n^{+})_{ij}e^{2n\pi iz}\ ,\quad P_{ij}^{-}(z)=\sum_{n<n_{ij}}(P_n^{-})_{ij}e^{2n\pi iz}$$

where the numbers n_{ij} are arbitrary integers. Now let

$$(1.10)\qquad \Phi(z)=Y^{+}(z)P^{+}(z)\exp\{-Q(z)\}z^{-G}\ ,\quad z\in\Gamma_{+}\ .$$

One easily verifies that $A^{\Phi}\overset{c}{=}A$. Moreover, due to (1.5) and (1.7), Φ may be continued analytically to Γ_{-} and we have

$$(1.11)\qquad \Phi(z)=Y^{-}(z)P^{-}(z)^{-1}\exp\{-Q(z)\}z^{-G}\ ,\quad z\in\Gamma_{-}\ .$$

Noting that $\det P^{\pm}=1$ and using (1.3), (1.4), (1.10) and (1.11), we conclude that $\det\Phi(z)\neq 0$ for all $z\in\Gamma_{+}\cup\Gamma_{-}$.

Now let us consider the asymptotic behaviour of Φ as $z\to\infty$ in $\Gamma_{+}\cup\Gamma_{-}$. For all, $i,j\in\{1,\dots,m\}$ we have

$$(1.12)\quad \Phi_{ij}(z)-\Phi_{ij}^{\pm}(z)=\sum_{h<j}\Phi_{ih}^{\pm}(z)\exp\{q_{hj}(z)\}z^{G_h}P_{hj}^{\pm}(z)z^{-G_j}\ ,\quad z\in\Gamma_{\pm}\ .$$

First, suppose that, for some $h<j$, $\operatorname{Re}q'_{hj}(z)=0$ for all sufficiently large positive values of z. Due to (1.6), the same is true of $\operatorname{Re}q'_{hk}$ for all $k\in\{h+1,\dots,j\}$. Consequently, $P_{hk}\equiv 0$ for all $k\in\{h+1,\dots,j\}$. With (1.8) and (1.9) it follows that $P_{hk}^{+}\equiv P_{hk}^{-}\equiv 0$ for all $k\in\{h+1,\dots,j\}$. In particular, $P_{hj}^{\pm}\equiv 0$. Thus the only non-vanishing terms in the right-hand side of (1.12) are the ones for which $\operatorname{Re}q'_{hj}(z)<0$ for sufficiently large positive z. This implies that either $d_{hj}<0$, or else $d_{hj}=0$ and there is a number $h_0\in\{1,\dots,p\}$ such that $\operatorname{Re}\mu_{hj,k}=0$ for all $k<h_0$ whereas $\operatorname{Re}\mu_{hj,h_0}<0$. In both cases there exists a number $\delta\in(0,1)$ such that

$$\exp(q_{hj}(z))=O(\exp\{-z^{\delta}\})\ ,\quad z\to\infty \text{ in } S\ .$$

As $\Phi^{\pm}(z)\sim F\in G\ell(n;\widehat{K}_p)$ as $z\to\infty$ in S, it follows that there exists a $\delta\in(0,1)$ such that

$$(1.13)\qquad \Phi_{ij}(z)-\Phi_{ij}^{\pm}(z)=O(\exp\{-z^{\delta}\})\ ,\quad z\to\infty \text{ in } S\ .$$

If $\Gamma(R)=\Gamma_{+}$ we choose the integers n_{ij} in (1.9) in such a way that

$$2n_{ij}\pi+d_{ij}\frac{\pi}{2}+\operatorname{Im}\mu_{ij,p}>0\ ,\quad \text{for } i<j\ .$$

Then it is easily seen that all terms in the right-hand side of (1.12) decrease exponentially as $z\to\infty$ in Γ_{+}, provided $\arg z>\varepsilon$ for some positive number ε. Combining this with (1.13) and applying a well-known theorem of Phragmén-Lindelöf (*cf.* [12], p. 177) we conclude that, for all $i,j \in \{1,\dots,m\}$,

$$\Phi_{ij}(z) - \Phi^+_{ij}(z) = O(\exp\{-z^\delta\}) \ , \quad z \to \infty \text{ in } \Gamma_+$$

and, consequently, $\Phi^+(z) \sim F$ as $z \to \infty$ in Γ_+. If, on the other hand, $\Gamma(R) = \Gamma_-$ we choose the integers n_{ij} in such a way that

$$2(n_{ij}-1)\pi - d_{ij}\frac{\pi}{2} + \operatorname{Im}\mu_{ij,p} < 0 \ \text{ for } \ i<j \ .$$

By means of an argument similar to the one used above we find that $\Phi^-(z) \sim F$ as $z \to \infty$ in Γ_- in that case.

The next theorem follows immediately from theorem 1.2 and lemma 1.1.

THEOREM 1.14. — *Let $A, B \in G\ell(n;K)$ and suppose there exists a matrix $F \in G\ell(n;\widehat{K}_p)$ such that $A^F = B$. Let Γ be a quadrant. There exists a positive number R and a matrix function Φ with the following properties :*

(i) *Φ is non singular and analytic in $\Gamma(R) \cup \Gamma(R)^*$*

(ii) *$\Phi(z) \sim F(z)$ as $z \to \infty$ in $\Gamma(R)$*

(iii) *$A^\Phi = B$.*

Proof. — Let $\overset{c}{A}$ be a canonical form of A. Obviously, $\overset{c}{A}$ is a canonical form of B as well. According to theorem 1.2 there exists a positive number R and matrix functions Φ_1 and Φ_2 with the following properties :

(i) Φ_1 and Φ_2 are non singular and analytic in $\Gamma(R) \cup \Gamma(R)^*$

(ii) $\Phi_j(z) \sim \widehat{\Phi}_j \in G\ell(n;\widehat{K}_p)$ as $z \to \infty$ in $\Gamma(R)$, $j = 1,2$

(iii) $A^{\Phi_1} = B^{\Phi_2} = \overset{c}{A}$.

Consequently, $A^{\widehat{\Phi}_1} = B^{\widehat{\Phi}_2} = A^{F\widehat{\Phi}_2} = \overset{c}{A}$. By lemma 1.1 this implies that $F\widehat{\Phi}_2 = \widehat{\Phi}_1 C$, where C is a constant invertible matrix which commutes with $\overset{c}{A}$. Hence it follows that

$$A^{\Phi_1 C \Phi_2^{-1}} = \overset{c}{A}{}^{\Phi_2^{-1}} = B \ .$$

One easily verifies that the matrix function $\Phi \equiv \Phi_1 C \Phi_2^{-1}$ has the required properties.

DEFINITION. — *Let* Φ *be a meromorphic matrix function in* $\mathbb{C}$. *By* P_Φ *we shall denote the set*

$$P_\Phi = \{z \in \mathbb{C} : \Phi \text{ has a pole in } z\} .$$

COROLLARY of theorem 1.14. — *If, in addition to the assumptions made in theorem 1.14, A and B are matrices of rational functions, then any matrix* Φ *with the properties* (i)–(iii) *can be continued analytically to a meromorphic matrix function. Moreover*

$$P_\Phi \subset P_A \cup P_{B^{-1}} + \mathbb{N} , \quad P_{\Phi^{-1}} \subset P_{A^{-1}} \cup P_B + \mathbb{N} \tag{1.15}$$

if Γ *is contained in a left half plane, whereas*

$$P_\Phi \subset P_{A^{-1}} \cup P_B - \mathbb{N}_0 , \quad P_{\Phi^{-1}} \subset P_A \cup P_{B^{-1}} - \mathbb{N}_0 \tag{1.16}$$

if Γ *is contained in a right half plane.*

Proof. — It Γ is contained in a left half plane the matrix function Φ with the properties (i)–(iii) mentioned in theorem 1.14 may be continued analytically to the right by means of the relation

$$\Phi(z+1) = A(z)\Phi(z)B(z)^{-1} . \tag{1.17}$$

If, on the other hand, Γ is contained in a right half plane, Φ may be continued to the left by means of

$$\Phi(z) = A(z)^{-1}\Phi(z+1)B(z) . \tag{1.18}$$

(1.15) and (1.16) follow immediately from (1.17) and (1.18), respectively.

THEOREM 1.19. — *Let* $A, B, M \in \mathrm{G}\ell(n; K)$ *and suppose that* Δ_A *and* Δ_B *are formally equivalent to* Δ_M. *Let* Γ_j, $j = 1, \ldots, 4$, *be quadrants such that* $\Gamma_{j+1} = e^{i\frac{\pi}{2}}\Gamma_j$ *for* $j = 1, 2, 3$, *and* $\Gamma_1, \ldots, \Gamma_4$ *cover a neighbourhood of* ∞. Δ_A *and* Δ_B *are meromorphically equivalent if and only if there exists a positive number R and matrix functions* Φ_j *and* Ψ_j, $j = 1, \ldots, 4$, *with the following properties :*

(i) Φ_j *and* Ψ_j *are non singular and analytic in* $\Gamma_j(R)$

(ii) Φ_j *and* Ψ_j *admit asymptotic expansions* $\widehat{\Phi}_j$ *and* $\widehat{\Psi}_j \in \mathrm{G}\ell(n; \widehat{K})$ *as* $z \to \infty$ *in* $\Gamma_j(R)$ *and both* $\widehat{\Phi}_j$ *and* $\widehat{\Psi}_j$ *are independent of* $j \in \{1, \ldots, 4\}$

(iii) $A^{\Phi_j} = B^{\Psi_j} = M$ *for all* $j \in \{1, \ldots, 4\}$

(iv) $\Phi_j^{-1}\Phi_{j+1} = \Psi_j^{-1}\Psi_{j+1}$ *for* $j \in \{1, 2, 3\}$ *and* $\Phi_4^{-1}\Phi_1 = \Psi_4^{-1}\Psi_1$.

Proof. — Suppose that Δ_A and Δ_B are meromorphically equivalent. Then there exists a matrix function $F \in G\ell(n;K)$ such that $A^F = B$. According to theorem 1.14 there exists a positive number R and matrix functions Φ_j, $j = 1,\dots,4$, with the properties (i) – (iii) mentioned above. Let $\Psi_j = F^{-1}\Phi_j$, $j \in \{1,\dots,4\}$. One easily verifies that conditions (i) – (iv) of theorem 1.19 are satisfied.

Conversely, suppose that (i) – (iv) hold. (iv) implies that

$$\Phi_j(z)\Psi_j^{-1}(z) = \Phi_{j+1}(z)\Psi_{j+1}^{-1}(z)\ ,\quad z \in \Gamma_j(R) \cap \Gamma_{j+1}(R),\ j = 1,2,3$$

and

$$\Phi_4(z)\Psi_4^{-1}(z) = \Phi_1(z)\Psi_1^{-1}(z)\ ,\quad z \in \Gamma_4(R) \cap \Gamma_1(R)\ .$$

Hence the matrix function $F \equiv \Phi_1\Psi_1^{-1}$ may be continued analytically to $\bigcup_{j=1}^{4} \Gamma_j(R)$, *i.e.* to a reduced neighbourhood of ∞. Moreover, F admits an asymptotic expansion $\widehat{F} = \widehat{\Phi}_j\widehat{\Psi}_j^{-1} \in G\ell(n;\widehat{K})$ as $z \to \infty$ in $\bigcup_{j=1}^{4} \Gamma_j(R)$. Therefore, $F \in G\ell(n;K)$. Obviously,

$$A^F = A^{\Phi_1\Psi_1^{-1}} = M^{\Psi_1^{-1}} = B\ .$$

The connection matrices $T_j \equiv \Phi_j^{-1}\Phi_{j+1}$, $j = 1,2,3$, and $T_4 = \Phi_4^{-1}\Phi_1$ are uniquely determined by M and A up to transformations of the following type

$$\begin{aligned} T_j &\longrightarrow S_j^{-1}T_jS_{j+1}\ ,\quad j = 1,2,3 \\ T_4 &\longrightarrow S_4^{-1}T_4S_1 \end{aligned} \tag{1.20}$$

where S_j is a non singular and analytic matrix function in $\Gamma_j(R)$, admitting an asymptotic expansion $\widehat{S}_j \in G\ell(n;\widehat{K})$ as $z \to \infty$ in $\Gamma_j(R)$, independent of j, with the additional property that $M^{S_j} = M$ for all $j \in \{1,\dots,4\}$. The set of connection matrices $\{T_1,\dots,T_4\}$ modulo transformations of the form (1.20) constitutes a complete system of meromorphic invariants of the difference operator Δ_A.

It is easily seen that theorem 1.19 remains valid if (i) is replaced by

(i)' Φ_j and Ψ_j are non singular and analytic in $\Gamma_j(R) \cup \Gamma_j(R)^*$.

This implies that, in the case that A, B and M have rational coefficients, the matrix functions Φ_j and Ψ_j may be continued analytically to meromorphic functions in $\mathbb{C}$ (*cf.* the corollary of theorem 1.14). Thus the meromorphic equivalence classes of matrices of rational functions which are formally equivalent to a matrix of rational functions, can be characterized by a set of meromorphic connection matrices $\{T_1, \dots, T_4\}$ with the property that $T_1T_2T_3T_4 = I$, modulo transformations of the type (1.20), where S_j is meromorphic in $\mathbb{C}$ for all $j \in \{1, \dots, 4\}$.

2. The inverse problem.

DEFINITION. — *Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $z_0 \in \widehat{\mathbb{C}}$ and let C be a simple closed contour in $\widehat{\mathbb{C}}\backslash\{z_0\}$. C is the positively oriented boundary of a domain $D^+ \subset \widehat{\mathbb{C}}\backslash\{z_0\}$ and the negatively oriented boundary of a domain $D^- \subset \widehat{\mathbb{C}}\backslash\{z_0\}$. We shall call D^+ the interior and D^- the exterior of C.*

We shall call Φ a sectionally holomorphic function in $\widehat{\mathbb{C}}\backslash\{z_0\}$, relative to C, if

(i) *Φ is holomorphic in $D^+ \cup D^-$, and*

(ii) *for any $t \in C$, Φ approaches a definite limiting value $\Phi^+(t)$ or $\Phi^-(t)$ as $z \to t$ along any path in D^+ or D^-, respectively.*

A matrix function Φ will be called non singular and sectionally holomorphic in $\widehat{\mathbb{C}}\backslash\{z_0\}$, relative to C, if in addition to (i) *and* (ii) *above, Φ is non singular in $D^+ \cup D^-$ and both Φ^+ and Φ^- are non singular on C.*

We begin by stating a well-known result (*cf.* [9], [13]).

THEOREM 2.1. — *Let $z_0 \in \widehat{C}$ and let C be a simple, smooth, closed contour in $\widehat{\mathbb{C}}\backslash\{z_0\}$. Let A be a non singular, Hölder continuous matrix function on C.*

There exists a matrix function Φ, non singular and sectionally holomorphic in $\widehat{\mathbb{C}}\backslash\{z_0\}$, relative to C, with the following properties :

(i) *$\Phi^+(z) = \Phi^-(z)A(z)$ for all $z \in C$*

(ii) *Φ has at most a pole in z_0.*

Moreover, Φ^+ and Φ^- are Hölder continuous on C.

Remark. — The usual version of this theorem applies to the case that $z_0 = \infty$. However, the general situation can be easily reduced to that case by means of a linear fractional transformation φ of the form $\varphi(z) = a\dfrac{z - z_1}{z - z_0}$, with $a \neq 0$, $z_1 \neq z_0$.

Theorem 2.1 will enable us to solve the inverse problem mentioned in the introduction. We shall take $z_0 = 0$ and put $\widehat{\mathbb{C}}\backslash\{0\} = \widehat{C}*$. Throughout this section $\{\Gamma_j,\ j = 1,\dots,4\}$ will denote a set of quadrants such that $\Gamma_{j+1} = e^{i\frac{\pi}{2}}\Gamma_j$ for $j = 1,2,3$, and $\Gamma_1,\dots,\Gamma_4$ cover a neighbourhood of ∞. We define : $S_j = \Gamma_j \cap \Gamma_{j+1}$ for $j = 1,2,3$ and $S_4 = \Gamma_4 \cap \Gamma_1$. Furthermore, we shall assume that

$$\sup_{\zeta,z\in S_j} \mathrm{Re}(\zeta - z) > 1 \quad \text{for all} \quad j \in \{1,\dots,4\} .$$

Let $j \in \{1,\dots,4\}$, $R > 0$, $\varepsilon > 0$. By $\Gamma_{j,\varepsilon}(R)$ we shall denote the quadrant

$$\Gamma_{j,\varepsilon}(R) = \{z \in \Gamma_j(R) \ :\ |z - \zeta| > \varepsilon \quad \text{for all} \quad \zeta \in \delta\Gamma_j(R)\} .$$

THEOREM 2.2. — *Suppose we are given a matrix function $M \in G\ell(n;K)$ and matrix functions T_j, $j = 1,\dots,4$, with the following properties :*

(i) *T_j is analytic in S_j*

(ii) *$T_j \sim I$ as $z \to \infty$ in S_j*

(iii) *$M^{T_j} = M$.*

There exists a positive number R, a matrix function $A \in G\ell(n;K)$ and matrix functions Φ_j, $j = 1,\dots,4$, with the following properties :

(1) *Φ_j is analytic in $\Gamma_j(R)$ and admits an asymptotic expansion $F \in G\ell(n;\widehat{K})$, independent of j, as $z \to \infty$ in $\Gamma_{j,\varepsilon}(R)$, for every $\varepsilon > 0$*

(2) *$A^{\Phi_j} = M$*

(3) *$\Phi_j^{-1}\Phi_{j+1} = T_j$ for $j = 1,2,3$ and $\Phi_4^{-1}\Phi_1 = T_4$.*

We shall prove theorem 2.2 in two steps. Let C_1 be a smooth contour in $\Gamma_2 \cup \Gamma_3$ consisting of a half line L_1 in S_1, a half line L_2 in S_3 and an arc connecting the starting points of L_1 and L_2. Since $T_j \sim I$ as $z \to \infty$ in S_j, there exists a positive number R_0 such that

$$\det T_j(z) \neq 0 \quad \text{for all} \quad z \in S_j(R_0),\ j \in \{1,\dots,4\} . \tag{2.3}$$

Let T be a matrix function on $C_1 \cup S_1(R_0) \cup S_3(R_0)$ with the following properties :

(i) T is Hölder continuous on C_1

(ii) $\det T \neq 0$ on C_1

(iii) $T(z) = T_1(z)$ for $z \in S_1(R_0)$, $T(z) = T_3(z)^{-1}$ for $z \in S_3(R_0)$.

PROPOSITION 2.4. — *There exists a positive number R, a matrix function Φ^+, non singular and holomorphic in $\Gamma_2(R) \cup \Gamma_3(R)$ and a matrix function Φ^-, non singular and holomorphic in $\Gamma_4(R) \cup \Gamma_1(R)$ such that*

(i) $\Phi^+(z) = \Phi^-(z)T(z)$ *for all* $z \in S_1(R) \cup S_3(R)$

(ii) Φ^+ *and* Φ^- *admit the same asymptotic expansion* $\widehat{\Phi} \in G\ell(n;\mathbb{C}[z^{-1}])$ *as* $z \to \infty$ *in* $\Gamma_{2,\varepsilon}(R) \cup \Gamma_{3,\varepsilon}(R)$ *and* $\Gamma_{4,\varepsilon}(R) \cup \Gamma_{1,\varepsilon}(R)$, *respectively, for any* $\varepsilon > 0$.

Proof. — Let D^+ and D^- denote the interior and exterior of C_1, respectively. According to theorem 2.1 there exists a matrix function Φ, non singular and sectionally holomorphic in $\widehat{\mathbb{C}}*$, relative to C_1, such that

$$\Phi^+(z) = \Phi^-(z)T(z) \quad \text{for all } z \in C_1 \tag{2.5}$$

and Φ has at most a pole in 0. By Cauchy's theorem,

$$\Phi(z) = \frac{z}{2\pi i}\int_{C_1} \frac{\Phi^+(t)}{t(t-z)}dt \ , \ z \in D^+$$

and

$$\int_{C_1} \frac{\Phi^-(t)}{t(t-z)}dt = \int_{C_0} \frac{\Phi(t)}{t(t-z)}dt \ , \ z \in D^+ \ ,$$

where C_0 is a simple, closed contour in D^- enclosing 0. With (2.5) it follows that

$$\Phi(z) = \frac{z}{2\pi i}\int_{C_1} \frac{\Phi^-(t)(T(t)-I)}{t(t-z)}dt + \frac{z}{2\pi i}\int_{C_0} \frac{\Phi(t)}{t(t-z)}dt \ , \ z \in D^+ \ . \tag{2.6}$$

Similarly, we have

$$\Phi(z) = \frac{z}{2\pi i}\int_{C_1} \frac{\Phi^+(t)(I-T(t)^{-1})}{t(t-z)}dt + \frac{z}{2\pi i}\int_{C_0(z)} \frac{\Phi(t)}{t(t-z)}dt, \ z \in D^- \tag{2.7}$$

where $C_0(z)$ is a simple closed contour in D^-, enclosing 0, but not z. In view of (2.3) and the third property of T, both T and T^{-1} are holomorphic in $S_1(R_0) \cup S_3(R_0)$. Hence we deduce, by deforming the contour C_1 in (2.6) and (2.7), that Φ^+ and Φ^- may be continued analytically to $D^+ \cup S_1(R_0) \cup S_3(R_0)$ and $D^- \cup S_1(R_0) \cup S_3(R_0)$, respectively. Consequently, (2.5) holds for all $z \in S_1(R_0) \cup S_3(R_0)$. As Φ is non singular in $\widehat{\mathbb{C}}*$ and T is non singular in $S_1(R_0) \cup S_3(R_0)$, due to (2.5), the analytic continuations of Φ^+ and Φ^- are non singular in $S_1(R_0) \cup S_3(R_0)$.

Next we consider the asymptotic behaviour of Φ^+ and Φ^- as $z \to \infty$. Note that the second integral in the right-hand side of (2.6) and (2.7) is holomorphic at ∞. Furthermore, $T(z) \sim I$ as $z \to \infty$ in $S_1(R_0) \cup S_3(R_0)$, hence

$$\sup_{t \in S_1(R_0) \cup S_3(R_0) \cup C_1} |(T(t) - I)t^n| < \infty \quad \text{for all } n \in \mathbb{N}\,.$$

Putting

$$-\frac{1}{2\pi i} \int_{C_1} \Phi^-(t)(T(t) - I)t^{n-1} dt = F_n$$

we have, for all $z \in D^+$,

$$\frac{z}{2\pi i} \int_{C_1} \frac{\Phi^-(t)(T(t)-I)}{t(t-z)} dt - \sum_{n=0}^{N} F_n z^{-n} = \frac{1}{2\pi i} z^{-N} \int_{C_1} \frac{\Phi^-(t)(T(t)-I)}{t-z} t^N dt.$$

The right-hand side of this identity is $O(z^{-N})$ as $z \to \infty$, uniformly on $D_\varepsilon^+ \equiv \{z \in D^+ : |z-t| > \varepsilon \text{ for all } t \in C_1\}$ for any $\varepsilon > 0$. Hence, it follows that Φ admits an asymptotic expansion $\widehat{\Phi}^+$ as $z \to \infty$ in D_ε^+ for any $\varepsilon > 0$. Moreover, $\lim_{z \to \infty\ ,\ z \in D^+} \Phi(z) = \Phi^+(\infty)$ and $\det \Phi^+(\infty) \neq 0$. This implies that $\widehat{\Phi}^+ \in G\ell(n; \mathbb{C}[z^{-1}])$. By varying the contour C_1 we find that $\Phi^+(z) \sim \widehat{\Phi}^+(z)$ as $z \to \infty$ in $\Gamma_{2,\varepsilon}(R) \cup \Gamma_{3,\varepsilon}(R)$ for any $\varepsilon > 0$ and a sufficiently large number R. In a similar manner one proves that Φ^- admits an asymptotic expansion $\widehat{\Phi}^-$ as $z \to \infty$ in $\Gamma_{4,\varepsilon}(R) \cup \Gamma_{1,\varepsilon}(R)$ for any $\varepsilon > 0$ and a sufficiently large number R. By (2.5),

$$\widehat{\Phi}^+ = \widehat{\Phi}^- \widehat{T} = \widehat{\Phi}^-\,.$$

This completes the proof of proposition 2.4.

Proof of theorem 2.2. — Now let R_1 be some sufficiently large positive number and let C_2 be a smooth contour in $\Gamma_3(R_1) \cup \Gamma_4(R_1)$ consisting of a half line L_3 in $S_2(R_1)$, a half line L_4 in $S_4(R_1)$ and an arc connecting the starting points of L_3 and L_4. Let S be a matrix function on $C_2 \cup S_2(R_1) \cup S_4(R_1)$ with the following properties :

(i) S is Hölder continuous on C_2

(ii) $\det S \neq 0$ on C_2

(iii) $S(z) = \Phi^+(z)T_2(z)\Phi^+(z)^{-1}$ for $z \in S_2(R_1)$,
$S(z) = \Phi^-(z)T_4^{-1}(z)\Phi^-(z)^{-1}$ for $z \in S_4(R_1)$,

where Φ^+ and Φ^- are matrix functions with the properties mentioned in proposition 2.4.

Obviously, $S(z) \sim I$ as $z \to \infty$ in $S_2(R_1) \cup S_4(R_1)$. Thus we may apply proposition 2.4, with $\Gamma_2 \cup \Gamma_3$ replaced by $\Gamma_3(R_1) \cup \Gamma_4(R_1)$, T by S, etc. Hence there exists a positive number R, a matrix function Ψ^+, non singular and holomorphic in $\Gamma_3(R) \cup \Gamma_4(R)$ and a matrix function Ψ^-, non singular and holomorphic in $\Gamma_1(R) \cup \Gamma_2(R)$, such that

(i) $\Psi^+(z) = \Psi^-(z)S(z)$ for all $z \in S_2(R) \cup S_4(R)$

(ii) Ψ^+ and Ψ^- admit the same asymptotic expansion $\hat{\Psi} \in G\ell(n; \mathbb{C}[z^{-1}])$, as $z \to \infty$ in $\Gamma_{3,\varepsilon}(R) \cup \Gamma_{4,\varepsilon}(R)$ and $\Gamma_{1,\varepsilon}(R) \cup \Gamma_{2,\varepsilon}(R)$, respectively, for any $\varepsilon > 0$.

We now define the matrix functions Φ_j, $j = 1, \ldots, 4$ as follows :

$$\begin{aligned}
\Phi_1(z) &= \Psi^-(z)\Phi^-(z), \quad z \in \Gamma_1(R)\\
\Phi_2(z) &= \Psi^-(z)\Phi^+(z), \quad z \in \Gamma_2(R)\\
\Phi_3(z) &= \Psi^+(z)\Phi^+(z), \quad z \in \Gamma_3(R)\\
\Phi_4(z) &= \Psi^+(z)\Phi^-(z), \quad z \in \Gamma_4(R)\ ,
\end{aligned}$$

Φ_j is non singular and holomorphic in $\Gamma_j(R)$ and represented asymptotically by $F \equiv \hat{\Psi}\hat{\Phi}$ as $z \to \infty$ in $\Gamma_{j,\varepsilon}$, for every $\varepsilon > 0$, $j \in \{j, \ldots, 4\}$. Furthermore, we have

$$\begin{aligned}
\Phi_1(z)^{-1}\Phi_2(z) &= \Phi^-(z)^{-1}\Phi^+(z) = T_1(z), && z \in S_1(R)\\
\Phi_2(z)^{-1}\Phi_3(z) &= \Phi^+(z)^{-1}S(z)\Phi^+(z) = T_2(z), && z \in S_2(R)\\
\Phi_3(z)^{-1}\Phi_4(z) &= \Phi^+(z)^{-1}\Phi^-(z) = T_3(z), && s \in S_3(R)\\
\Phi_4(z)^{-1}\Phi_1(z) &= \Phi^-(z)^{-1}S(z)^{-1}\Phi^-(z) = T_4(z), && z \in S_4(R)\ .
\end{aligned}$$

Hence it follows that

$$M^{\Phi_j^{-1}}(z) = M^{T_j\Phi_{j+1}^{-1}}(z) = M^{\Phi_{j+1}^{-1}}(z), \quad z \in S_j(R) \cap S_j(R) - 1,\ j \in \{1, 2, 3\}$$

and

$$M^{\Phi_4^{-1}}(z) = M^{T_4\Phi_1^{-1}}(z) = M^{\Phi_1^{-1}}(z), \quad z \in S_4(R) \cap S_4(R) - 1\ .$$

Consequently, the matrix function A defined by

$$A(z) = M^{\Phi_1^{-1}}(z), \quad z \in \Gamma_1(R) \cap \Gamma_1(R) - 1$$

may be continued analytically to a reduced neighbourhood of ∞. Moreover,

$$A(z) \sim F(z+1)M(z)F(z)^{-1} \quad \text{as} \quad z \to \infty \quad \text{in} \quad \bigcup_{j=1}^{4} \Gamma_{j,\varepsilon}(R) \cap \Gamma_{j,\varepsilon}(R) - 1$$

for every $\varepsilon > 0$, and this implies that $A \in G\ell(n; K)$.

Remark. — An alternative proof of theorem 2.2 can be given by adapting an argument used by J. Martinet and J.P. Ramis in [14], which makes essential use of the theorem of Newlander-Nirenberg.

Finally, we consider the particular case that M is a matrix of rational functions and the matrix functions T_j are meromorphic in $\mathbb{C}$ for each $j \in \{1,\dots,4\}$.

DEFINITION. — *If Φ is a meromorphic matrix function in $\mathbb{C}$ with the property that* $\det \Phi \not\equiv 0$, *then* $\sum(\Phi)$ *will denote the set of all singularities of* Φ, *i.e.*

$$\sum(\Phi) = P_\Phi \cup P_{\Phi^{-1}} .$$

THEOREM 2.8. — *In addition to the assumptions made in theorem 2.2, suppose that M is a matrix of rational functions, that T_j is meromorphic in $\mathbb{C}$ for each $j \in \{1,\dots,4\}$ and that*

$$T_1T_2T_3T_4 = I .$$

Then there exists a matrix A of rational functions and meromorphic matrix functions Φ_j, $j \in \{1,\dots,4\}$, with the properties mentioned in theorem 2.2. Moreover

$$\sum(\Phi_j) \subset \bigcup_{i=1}^{4} \sum(T_i), \quad j \in \{1,\dots,4\} ,$$

and

$$\sum(A) \subset \bigcup_{i=1}^{4} \sum(T_j) \cup \sum(T_j) - 1 \cup \sum(M) .$$

Proof. — From property (3) in theorem 2.2 we deduce that the matrix function Φ_1 can be continued analytically to a meromorphic function in some reduced neighbourhood U of ∞. Moreover, the singular points of Φ_1 in U will form a subset of $\bigcup_{j=1}^{4} \sum(T_j)$. Using an idea of Birkhoff (*cf.* [1]), we shall remove the singularities of Φ_1 outside U by means of a simple transformation.

Let C be a simple closed contour in $\mathbb{C}$ with interior D^+ and exterior D^-, such that Φ_1 is non singular and analytic on C and $\mathbb{C}\backslash U \subset D^+$. According to theorem 2.1 (with $z_0 = \infty$) there exists a matrix function X, non singular and sectionally holomorphic in $\mathbb{C}$, relative to C, with the property that

$$X^+(z) = X^-(z)\Phi_1(z) \quad \text{for all} \quad z \in C . \tag{2.9}$$

Moreover, X has at most a pole at ∞. Consequently, at ∞ X admits a Laurent series representation $\widehat{X} \in G\ell(n;K)$.

For each $j \in \{1,\dots,4\}$ let $\widetilde{\Phi}_j$ be defined by

$$\widetilde{\Phi}_j(z) = X(z)\Phi_j(z), \quad z \in D^- \cap \Gamma_j(R) .$$

Thus, for a sufficiently large number R_1, $\widetilde{\Phi}_j$ is a non singular and analytic function in $\Gamma_j(R_1)$, admitting the asymptotic expansion $\widehat{X}F \in \mathrm{G}\ell(n;K)$ as $z \to \infty$ in $\Gamma_{j,\varepsilon}(R_1)$, for every $\varepsilon > 0$, $j \in \{1,\dots,4\}$. Moreover, $\widetilde{\Phi}_1$ is meromorphic in D^-. Due to (2.9), it may be continued analytically to D^+. Thus it becomes a meromorphic function in $\mathbb{C}$ and the same is true of $\widetilde{\Phi}_1^{-1}$. Furthermore, we have

$$\widetilde{\Phi}_j^{-1}\widetilde{\Phi}_{j+1} = \Phi_j^{-1}\Phi_{j+1} = T_j \quad \text{in} \quad S_j(R_1),\ j = 1,2,3,$$

and

$$\widetilde{\Phi}_4^{-1}\widetilde{\Phi}_1 = \Phi_4^{-1}\Phi_1 = T_4 \quad \text{in} \quad S_4(R_1) .$$

Hence it follows that all $\widetilde{\Phi}_j$ may be continued to meromorphic functions in $\mathbb{C}$ with the property that

$$\sum(\widetilde{\Phi}_j) \subset \bigcup_{i=1}^{4} \sum(T_i) , \quad j \in \{1,\dots,4\} . \tag{2.10}$$

Now let A be defined by

$$A(z) = M^{\widetilde{\Phi}_1^{-1}}(z), \quad z \in \mathbb{C} .$$

One easily verifies that A has the properties mentioned in theorem 2.2 (with respect to $\widetilde{\Phi}_j$ instead of Φ_j). In particular, $A \in \mathrm{G}\ell(n;K)$. At the same time, A is a meromorphic matrix function in $\mathbb{C}$. Hence its entries must be rational functions. Moreover,

$$\sum(A) \subset \sum(\widetilde{\Phi}_1) \cup \sum(\widetilde{\Phi}_1) - 1 \cup \sum(M) .$$

With (2.10) the last statement of the theorem follows.

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