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ON RIEMANNIAN FOLIATIONS WITH MINIMAL LEAVES

by Jesús A. ALVAREZ LÓPEZ

Introduction.

Let M be a smooth manifold which carries a smooth foliation $\mathcal{F}$ of dimension p and codimension q. Then we have the associated spectral sequence (E_i, d_i) (defined for example in [22]). It carries the vector space topology induced by the C^∞−topology on the de Rham complex. We further have the cohomologies $H(\overline{O}_1)$ and $\mathcal{E}_2 = H(\mathcal{E}_1)$ of the differential spaces $\overline{O}_1$ (the closure of the trivial subspace in E_1) and $\mathcal{E}_1 = E_1/\overline{O}_1$, respectively.

$\mathcal{F}$ is said to be taut if there is some Riemannian metric on M for which all the leaves are minimal submanifolds. This property depends only on the transverse structure of $\mathcal{F}$ [11], and there are several papers studying its relation with cohomological properties of $\mathcal{F}$ [11], [12], [14], [15], [19], [21], [24], involving cohomology spaces which always can be considered as parts of E_2, $\mathcal{E}_2$, or $H(\overline{O}_1)$. All the results of this type are based on Rummler's mean curvature formula [21], which implies the following criterion of Rummler-Sullivan (the definition of positiveness along the leaves is given in Section 2).

THEOREM [21], [24]. — *An oriented smooth foliation is taut if and only if some element in $E_2^{0,p}$ can be defined by a p−form which is positive along the leaves.*

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$E_2^{\cdot,p}$ is isomorphic to the $\mathcal{F}$−relative cohomology used in [21], [24] (see for example [16], [23]).

When $\mathcal{F}$ is Riemannian [20], the associated spectral sequence verifies some properties of finiteness and duality [1], [2], [3], [7], [8], [13], [14], [15], [22], [23]. Thus, in this case, the cohomological study of the minimality of the leaves has special interest. We have the following conjecture (see e.g. [5]).

TAUTNESS CONJECTURE. — *An oriented Riemannian foliation on a compact connected oriented manifold is taut if and only if* $E_2^{q,0} \neq 0$.

This conjecture makes sense by the criterion of Rummler-Sullivan, and because some duality relation may be expected between $E_2^{q,0}$ and $E_2^{0,p}$. By now it has only been given partial solutions : by A. Haefliger if the structural Lie algebra is compact or nilpotent [11], [12], by Kamber-Tondeur if the mean curvature is a basic form [14], [15], by Molino-Sergiescu for Riemannian flows [19], and by Y. Carrière if the leaves have polynomial growth [6].

The aim of this paper is to establish some more steps towards resolving the Tautness Conjecture using results proved in [1], [2]. Firstly, under the condition $E_2^{q,0} \neq 0$, it is proved (Section 2) that the Rummler-Sullivan criterion is verified in a weaker sense, namely replacing $E_2^{0,p}$ by $\mathcal{E}_2^{0,p}$. Secondly, some arguments are made on the canonical long exact sequence relating $H(\overline{O}_1)$, E_2, and $\mathcal{E}_2$, obtaining that the Tautness Conjecture is true if and only if the connecting homomorphism from $\mathcal{E}_2^{0,p}$ to $H^{1,p}(\overline{O}_1)$ is zero (Section 3). This is a generalization of the results in [17] for Lie foliations with dense leaves. Then, when $q \leq 2$ it is proved that $E_2 \cong \mathcal{E}_2$ canonically (Section 4), obtaining a proof of the Tautness Conjecture in this case.

Finally, I want to express my deep gratitude to F. Kamber, X. Masa, and Ph. Tondeur for helpful comments.

1. Spectral sequence of Rieamannian foliations.

1.1. Let $\mathcal{F}$ be a smooth foliation of dimension p and codimension q on a smooth manifold M. In this paper all the manifolds will be assumed to be connected. Let $T\mathcal{F} \subset TM$ be the subbundle of vectors tangent to $\mathcal{F}$, and let $\mathcal{X}(\mathcal{F}) = \Gamma T\mathcal{F}$.

The de Rham differential algebra $(A(M), d)$ of M is filtered by differential ideals. A differential form of degree r is of filtration degree $\geq k$ if it vanishes whenever evaluated on $r - k + 1$ vector fields in $\mathcal{X}(\mathcal{F})$. This filtration defines the spectral sequence $(E_i(\mathcal{F}), d_i)$ (or simply (E_i, d_i)) which converges to the de Rham cohomology of M.

For each vector subbundle $Q \subset TM$, complementary of $T\mathcal{F}$, we have the bigradation of $A(M)$ given by

$$A^{u,v}(M) \equiv \Gamma(\Lambda^v T^*\mathcal{F} \otimes \Lambda^u Q^*)$$

for integers u, v [1], [2]. The de Rham differential operator can be decomposed as a sum of bihomogeneous operators $d_{0,1}$, $d_{1,0}$, and $d_{2,-1}$, where the double subindices denote the corresponding bidegrees, obtaining the following canonical identities of bigraded differential algebras [1], [2].

$$(1.1) \qquad (E_0, d_0) \equiv (A(M), d_{0,1}) \;, \qquad (E_1, d_1) \equiv (H(A(M), d_{0,1}), d_{1,0*}).$$

$(A(M), d)$ is a topological differential algebra with the C^∞–topology, and each (E_i, d_i) is a topological differential algebra with the induced topology and the identities (1.1) are also topological. E_1 in general is not Hausdorff [11] obtaining two new bigraded topological differential algebras : the closure $\overline{O}_1$ of the trivial subspace of E_1, and $\mathcal{E}_1 = E_1/\overline{O}_1$, (or more explicitly $\overline{O}_1(\mathcal{F})$ and $\mathcal{E}_1(\mathcal{F})$). Let $\mathcal{E}_2 = H(\mathcal{E}_1)$. Clearly we have $E_1^{\cdot,0} = \mathcal{E}_1^{\cdot,0}$ and $E_2^{\cdot,0} = \mathcal{E}_2^{\cdot,0}$, and we also obtain the associated long exact sequence in cohomology

$$(1.2) \qquad \cdots \longrightarrow H^{u,\cdot}(\overline{O}_1) \longrightarrow E_2^{u,\cdot} \longrightarrow \mathcal{E}_2^{u,\cdot} \longrightarrow H^{u+1,\cdot}(\overline{O}_1) \longrightarrow \cdots \;.$$

If $\alpha \in A(M)$ defines an element in E_i or $\mathcal{E}_i$ $(i = 1, 2)$ it will be represented by $[\alpha]_i$ or $\overline{[\alpha]}_i$ respectively.

If $\mathcal{F}$ is Riemannian and M compact, then $E_2, \mathcal{E}_2$, and $H(\overline{O}_1)$ are of finite dimension [1], [2], [22], [23] (see also [7], [8], [13]). In particular, $E_2^{q,0}$ is of dimension zero or one [7], [8]. Further if M is also oriented, we have the isomorphisms [2] (see also [13])

$$(1.3) \qquad \mathcal{E}_2^{u,v} \cong \mathcal{E}_2^{q-u,p-v} \;, \qquad H^{u,v}(\overline{O}_1) \cong H^{q-u-1,p-v+1}(\overline{O}_1) \;,$$

induced by the de Rham duality map, implying

$$(1.4) \qquad H^{q,\cdot}(\overline{O}_1) = 0 \;, \qquad E_2^{q,\cdot} \cong \mathcal{E}_2^{q,\cdot} \;.$$

1.2. P. Molino in [18] describes the structure of Riemannian foliations, using transversally parallelizable (TP) foliations and Lie foliations.

Firstly, for a transversally oriented Riemannian foliation on a compact manifold M he considers the principal $SO(q)-$bundle of oriented orthonormal transverse frames, $\pi : \widehat{M} \to M$, with the transverse Levi-Civita connection, and he proves that the canonical horizontal lifting $\widehat{\mathcal{F}}$ or $\mathcal{F}$ is TP. In this situation we have the isomorphisms

$$E_2^{0,\cdot}(\widehat{\mathcal{F}}) \cong E_2^{0,\cdot}(\mathcal{F}) \,, \qquad \mathcal{E}_2^{0,\cdot}(\widehat{\mathcal{F}}) \cong \mathcal{E}_2^{0,\cdot}(\mathcal{F}) \,, \tag{1.5}$$

which are induced by averaging along the fibers of π. The first isomorphism of (1.5) is proved in [1] using continuous operators, the second one can be proved with similar arguments. Then, by (1.3), (1.4), and (1.5) we obtain

$$\mathcal{E}_2^{\hat{q},v}(\widehat{\mathcal{F}}) \cong E_2^{\hat{q},v}(\widehat{\mathcal{F}}) \cong E_2^{q,v}(\mathcal{F}) \cong \mathcal{E}_2^{q,v}(\mathcal{F}) \tag{1.6}$$

for any integer v, where $\hat{q} = \operatorname{codim}(\widehat{\mathcal{F}}) = q + q(q-1)/2$.

1.3. P. Molino further proves in [18] that, for a TP foliation $\mathcal{F}$ on a compact manifold M, the closures of the leaves are the fibers of a fiber-bundle $\pi_b : M \to W$, such that the restriction of $\mathcal{F}$ to each fiber is a Lie foliation with dense leaves. The foliation $\overline{\mathcal{F}}$ defined by the fibers of π_b is called the basic foliation. Moreover, the local trivializations of π_b can be taken compatible with $\mathcal{F}$. This means that there exists a Lie foliation $\mathcal{F}_0$ with dense leaves on the standard fiber M_0 of π_b with the following property. The diffeormorphisms $h : \pi_b^{-1}(U) \to U \times M_0$ of triviality of π_b, over small enough open subsets $U \subset W$, can be choosen so that $\mathcal{F}_{|\pi_b^{-1}(U)}$ corresponds to the foliation $U \times \mathcal{F}_0$ with leaves $\{y\} \times L$ (for points $y \in U$ and leaves L of $\mathcal{F}_0$). Let $q_0 = \dim(\mathcal{F}_0)$ and $q_1 = \dim(W)$. The codimension of $\mathcal{F}$ is $q = q_0 + q_1$.

In this case we have the bigraded presheaves $\mathcal{O}_i$, $\mathcal{P}_i$, and $\mathcal{Q}_i$ on W $(i = 1, 2)$ given by

$$\mathcal{O}_1(U) = \overline{O}_1(\mathcal{F}_U) \,, \qquad \mathcal{O}_2(U) = H(\overline{O}_1(\mathcal{F}_U)) \,,$$
$$\mathcal{P}_i(U) = E_i(\mathcal{F}_U) \,, \qquad \mathcal{Q}_i(U) = \mathcal{E}_i(\mathcal{F}_U) \,,$$

where $\mathcal{F}_U = \mathcal{F}_{|\pi_b^{-1}(U)}$, with the canonical restrictions.

Let $\mathcal{C}_i$ be any one of the presheaves $\mathcal{O}_i$, $\mathcal{P}_i$, or $\mathcal{Q}_i$, and let d_1 denote the corresponding differential on each $\mathcal{C}_1(U)$. Then, for a fixed suitable open covering $\mathcal{U} = \{U_m\}$ of W we have the graded differential Čech spaces $(\check{C}(\mathcal{U}, \mathcal{C}_i), \delta)$, and the operator D given by $D = \delta + (-1)^k d_1$ on $\check{C}^k(\mathcal{U}, \mathcal{C}_1)$, turning $(\check{C}(\mathcal{U}, \mathcal{C}_1), D)$ into a bigraded differential space.

With a slight sharpening of the arguments of Proposition 8.5 of [4] it can be proved that

$$0 \longrightarrow \mathcal{C}_1(W) \xrightarrow{r_1} \check{C}^0(\mathcal{U}, \mathcal{C}_1) \xrightarrow{\delta} \check{C}^1(\mathcal{U}, \mathcal{C}_1) \xrightarrow{\delta} \cdots \tag{1.7}$$

is an exact sequence, where r_1 is given by the restrictions. Thus we have (Proposition 8.8 of [4])

$$r_{1*} : \mathcal{C}_2(W) \xrightarrow{\cong} H(\check{C}(\mathcal{U},\mathcal{C}_1), D). \tag{1.8}$$

We also have a spectral sequence $(\check{E}_i, \check{d}_i)$ converging to $H(\check{C}(\mathcal{U},\mathcal{C}_1), D)$ (Theorem 14.14 of [4]) such that

$$\check{E}_1^{k,t} = \check{C}(\mathcal{U},\mathcal{C}_2^t) \ , \qquad \check{E}_2^{k,t} = H^k(\check{C}^{\cdot}(\mathcal{U},\mathcal{C}_2^t),\delta) \tag{1.9}$$

where t denotes the total degree of $\mathcal{C}_2$. Then, since the bidegree of d_1 is $(1,0)$, from (1.8) and (1.9) we have

$$\mathcal{C}_2^{0,v}(W) \cong H^0(\check{C}^{\cdot}(\mathcal{U},\mathcal{C}_2^{0,v}),\delta), \tag{1.10}$$

for each integer v. Therefore the homomorphism

$$r_2 : \mathcal{C}_2^{0,\cdot}(W) \to \check{C}^0(\mathcal{U},\mathcal{C}_2^{0,\cdot}), \tag{1.11}$$

defined by the restrictions, is injective.

1.4. For a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ with dense leaves on a compact manifold, $E_1^{\cdot,0}$ can be identified with $\Lambda^{\cdot}\mathfrak{g}^*$, so $E_2^{\cdot,0} \equiv H^{\cdot}(\mathfrak{g})$ [17]. Moreover, if $\mathfrak{g}$ is unimodular and $\mathcal{F}$ oriented, then $\mathcal{E}_1^{\cdot,p}$ also can be identified with $\Lambda^{\cdot}\mathfrak{g}^*$, obtaining $\mathcal{E}_2^{\cdot,p} \equiv H^{\cdot}(\mathfrak{g})$ [17] (this is a consequence of sections 2.1 and 3.1 of [11]). Hence, in this last case, any p-form on M defines an element in $\mathcal{E}_2^{0,p}$.

1.5. Now let $\mathcal{F}$ be any Riemannian foliation on a compact manifold M. In [18] the structural Lie algebra $\mathfrak{g}$ of $\mathcal{F}$ is defined as the Lie algebra given by the Lie foliation with dense leaves corresponding to $\mathcal{F}$ by the above structure theorems of P. Molino. It is an intrinsic invariant of $\mathcal{F}$, and we have the following result.

PROPOSITION 1. — *If $E_2^{q,0} \neq 0$, then $\mathfrak{g}$ is unimodular.*

Proof. — Using standard arguments, by passing to the 2-fold covering of transverse orientations we can assume that $\mathcal{F}$ is transversally oriented. Then, by (1.6) we can also suppose that $\mathcal{F}$ is TP. In this case, by (1.3) and the injectivity of (1.11) we have $\mathcal{E}_2^{0,p}(\mathcal{F}_{U_m}) \neq 0$ for some U_m. On the other hand we have $\mathcal{E}_2^{0,p}(\mathcal{F}_{U_m}) = \mathcal{E}_2^{0,p}(\mathcal{F}_0)$ because U_m is contractible. Hence the result follows by (1.3) and the properties mentioned in 1.4. □

2. Condition of Rummler-Sullivan for $\mathcal{E}_2$.

A differential form on a manifold M is said to be positive along the leaves of an oriented foliation $\mathcal{F}$ on M, if its restriction to the leaves is a volume form defining the orientation of $\mathcal{F}$. The condition of Rummler-Sullivan for $\mathcal{E}_2$ can be stated as follows.

PROPOSITION 2. — *Let $\mathcal{F}$ be an oriented Riemannian foliation on a compact manifold M, and assume $E_2^{q,0} \neq 0$. Then there exists a p-form positive along the leaves defining an element in $\mathcal{E}_2^{0,p}$.*

Using standard arguments, by passing to the 2-fold covering of orientations of M we can suppose that M is oriented in the following proof. Therefore $\mathcal{F}$ is also transversally oriented.

Integration along the fibers of $\pi : \widehat{M} \to M$, after exterior multiplication with the invariant volume form along the fibers, assigns p-forms on M positive along the leaves of $\mathcal{F}$ to p-forms on $\widehat{M}$ positive along the leaves of $\widehat{\mathcal{F}}$. Then (1.5) and (1.6) imply that we can suppose that $\mathcal{F}$ is TP. Let then $\mathcal{F}$ be an oriented TP foliation on a compact manifold M. For a fixed bundle-like metric g on M [20] let $\nu \in E_1^{q,0}$ be the corresponding transverse volume element and $\chi = *\nu$ the characteristic form [21]. With the notation of 1.3, let ν_m and χ_m be their corresponding restrictions to $\pi_b^{-1}(U_m)$ for each U_m, and let $\mathcal{F}_m = \mathcal{F}_{U_m}$. The metric g induces canonically a Riemannian metric on W. By fixing an orientation of each U_m we obtain therefore a volume form ω_m on U_m.

On $A(M)$ we consider the trigradation defined by the orthogonal decomposition $TM = T\mathcal{F} + Q_0 + Q_1$, where $Q_0 = (T\mathcal{F})^\perp \cap T\overline{\mathcal{F}}$ and $Q_1 = (T\overline{\mathcal{F}})^\perp$; *i.e.*

$$A^{s,t,v}(M) \equiv \Gamma(\Lambda^v T^*\mathcal{F} \otimes \Lambda^t Q_0^* \otimes \Lambda^s Q_1^*).$$

Then the bigradation of $A(M)$ obtained as in Section 1 with $Q = Q_0 + Q_1$ is given by

$$A^{u,v}(M) = \sum_{s+t=u} A^{s,t,v}(M). \tag{2.1}$$

Both gradations can be restricted to forms on any open subset of M.

LEMMA 1 [7]. — *For each U_m there exists a unique $\lambda_m \in A^{0,q_0,0}(\pi_b^{-1}(U_m))$ such that $\nu_m = \pi_b^*\omega_m \wedge \lambda_m$.*

Proof. — This follows because ω_m and λ_m are nowhere zero, and the wedge product defines an isomorphism

$$A^{q_1,0,0}(\pi_b^{-1}(U_m)) \otimes A^{0,q_0,0}(\pi_b^{-1}(U_m)) \xrightarrow{\cong} A^{q,0}(\pi_b^{-1}(U_m))$$

where the three spaces are formed by smooth sections of line bundles. □

LEMMA 2. — *λ_m is a basic form; i.e. , $\lambda_m \in E_1^{q_0,0}(\mathcal{F}_m)$.*

Proof. — Clearly the interior product $i_X\lambda_m = 0$ for all $X \in \mathcal{X}(\mathcal{F}_m)$. Since ν_m and $\pi_b^*\omega_m$ are basic forms we also have

$$\pi_b^*\omega_m \wedge \theta_X\lambda_m = 0$$

for all $X \in \mathcal{X}(\mathcal{F}_m)$, where θ_X is the corresponding Lie derivative.

Let $Y_1,\ldots,Y_{q_1}$ be an orthonormal frame of U_m, and let $\widetilde{Y}_1,\ldots,\widetilde{Y}_{q_1} \in \Gamma Q_1$ be the corresponding liftings. Clearly each $\widetilde{Y}_j$ is an infinitesimal transformation of $\mathcal{F}_m$. Then, for $\widetilde{Y} = \widetilde{Y}_1 \wedge \ldots \wedge \widetilde{Y}_{q_1}$ we have

$$\begin{aligned}
0 &= i_{\widetilde{Y}}(\pi_b^*\omega_m \wedge \theta_X\lambda_m)\\
&= (i_{\widetilde{Y}}\pi_b^*\omega_m) \wedge \theta_X\lambda_m \pm \pi_b^*\omega_m \wedge i_{\widetilde{Y}}\theta_X\lambda_m\\
&= \theta_X\lambda_m \pm \pi_b^*\omega_m \wedge (\theta_X i_{\widetilde{Y}}\lambda_m - i_{\theta_X\widetilde{Y}}\lambda_m)\\
&= \theta_X\lambda_m.
\end{aligned}$$

(The third equality is given by (7.5) of Vol. III of [10]. And the fourth equality is true because $\lambda_m \in A^{0,q_0,0}(\pi_b^{-1}(U_m))$, $\widetilde{Y}_j \in \Gamma Q_i$, and

$$\theta_X\widetilde{Y} = \sum_{j=1}^{q_1} \widetilde{Y}_1 \wedge \ldots \wedge [X,\widetilde{Y}_j] \wedge \ldots \wedge \widetilde{Y}_{q_1}$$

where $[X,\widetilde{Y}_j] \in \mathcal{X}(\mathcal{F}_m)$.)

Thus $i_X\lambda_m = \theta_X\lambda_m = 0$ for all $X \in \mathcal{X}(\mathcal{F}_m)$, which means that λ_m is basic. □

For each U_m we can assume that there exists a diffeomorphism of triviality of π_b,

$$h_m : (\pi_b^{-1}(U_m),\mathcal{F}_m) \to (U_m \times M_0, U_m \times \mathcal{F}_0),$$

as in 1.3. Let $\mathrm{pr}_{m,1}$ and $\mathrm{pr}_{m,2}$ denote the canonical projections of $U_m \times M_0$ on U_m and M_0 respectively. $\mathcal{F}_0$ can be oriented so that each h_m preserves the orientations of the foliations. Thus we also obtain an orientation of M_0 since $\mathcal{F}_0$ is transversally oriented.

Choose a normalized bundle-like metric for $\mathcal{F}_0$ on M_0. Let $\nu_0 \in E_1^{q_0,0}(\mathcal{F}_0)$ be the corresponding transverse volume element, χ_0 be the corresponding characteristic form, and $\psi_m = h_m^* \mathrm{pr}_{m,2}^* \nu_0 \in E_1^{q_0,0}(\mathcal{F}_m)$.

h_m induces an isomorphism of differential spaces

$$\mathcal{E}_1(\mathcal{F}_m) \xrightarrow{\cong} \mathcal{E}_1(U_m \times \mathcal{F}_0) \cong A(U_m) \otimes \mathcal{E}_1(\mathcal{F}_0), \tag{2.2}$$

where the last isomorphism is defined by $\mathrm{pr}_{m,1}^* : A(U_m) \to A(U_m \times M_0)$ and the wedge product. In particular,

$$\mathcal{E}_1^{0,p}(\mathcal{F}_m) \xrightarrow{\cong} C^\infty(U_m) \otimes \mathcal{E}_1^{0,p}(\mathcal{F}_0) \cong C^\infty(U_m) \tag{2.3}$$

because $E_2^{q,0}(\mathcal{F}) \neq 0$ implies that the structural Lie algebra is unimodular (Proposition 1), so $\mathcal{E}_1^{0,p}(\mathcal{F}_0) \cong \mathbf{R}$ by the results indicated in 1.4.

LEMMA 3. — *The isomorphism (2.3) is given by* $\zeta \mapsto F_{m,\zeta}$, *where*

$$F_{m,\zeta}(y) = \int_{\pi_b^{-1}(y)} \psi_m \wedge \alpha \,, \qquad \text{if } \zeta = \overline{[\alpha]}_1 \text{ for } \alpha \in A^{0,p}(\pi_b^{-1}(U_m)).$$

Proof. — Let ξ_0 be the generator of $\mathcal{E}_1^{0,p}(\mathcal{F}_0)$ defined by $\mathcal{X}_0$. Then the isomorphisms

$$C^\infty(U_m) \otimes \mathcal{E}_1^{0,p}(\mathcal{F}_0) \cong \mathcal{E}_1^{0,p}(U_m \times \mathcal{F}_0)\,, \qquad C^\infty(U_m) \otimes \mathcal{E}_1^{0,p}(\mathcal{F}_0) \cong C^\infty(U_m)$$

are given respectively by

$$f \otimes \xi_0 \mapsto \overline{[(\mathrm{pr}_{m,1}^* f) \cdot (\mathrm{pr}_{m,2}^* \mathcal{X}_0)]_1}\,, \qquad f \otimes \xi_0 \mapsto f\,.$$

Therefore, since

$$\int_{\{y\} \times M_0} (\mathrm{pr}_{m,2}^* \nu_0) \wedge ((\mathrm{pr}_{m,1}^* f) \cdot (\mathrm{pr}_{m,2}^* \mathcal{X}_0)) = f(y) \cdot \int_{M_0} \nu_0 \wedge \mathcal{X}_0 = f(y),$$

the result follows. □

Consider also the map which assigns to each $\zeta \in \mathcal{E}_1^{0,p}(\mathcal{F})$ the function $G_{m,\zeta} \in C^\infty(U_m)$ defined by setting

$$G_{m,\zeta}(y) = \int_{\pi_b^{-1}(y)} \lambda_m \wedge \alpha \,, \quad \text{if } \zeta = \overline{[\alpha]}_1 \text{ for } \alpha \in A^{0,p}(M).$$

If $\zeta = \overline{[\alpha]}_1 = \overline{[\beta]}_1$ for $\alpha, \beta \in A^{0,p}(M)$ then

$$\lambda_m \wedge (\alpha - \beta) \in \overline{d_{0,1}(A^{0,p-1}(\pi_b^{-1}(U_m))}$$

by (1.1) and because λ_m is a basic form (Lemma 2). Hence, since $\deg(\lambda_m \wedge (\alpha - \beta)) = \dim(\pi_b^{-1}(y))$ we have

$$(\lambda_m \wedge (\alpha - \beta))_{|\pi_b^{-1}(y)} \in \overline{d(A(\pi_b^{-1}(y))} = d(A(\pi_b^{-1}(y)),$$

from which it follows that $G_{m,\zeta}$ is well defined.

λ_m and ψ_m are basic forms which can be thought as nowhere zero sections of the line-bundle $\Lambda^{q_0} Q_0^*$ over $\pi_b^{-1}(U_m)$. Thus there exists a unique nowhere zero function $f_m \in C^\infty(U_m)$ such that $\psi_m = (\pi_b^* f_m) \cdot \lambda_m$. The following property can be easily checked.

LEMMA 4. — *For all $\zeta \in \mathcal{E}_1^{0,p}(\mathcal{F})$ we have $F_{m,\zeta} = f_m \cdot G_{m,\zeta}$.*

On $U_m \cap U_{m'}$ we have $\omega_m = \pm\omega_{m'}$, so $\lambda_m = \pm\lambda_{m'}$ on $\pi_b^{-1}(U_m \cap U_{m'})$, obtaining

$$G_{m,\zeta} = \pm G_{m',\zeta} \tag{2.4}$$

on $U_m \cap U_{m'}$. Hence we can define the continuous function $\overline{G}_\zeta$ on W by setting

$$\overline{G}_\zeta(y) = |G_{m,\zeta}(y)| \quad \text{if } y \in U_m.$$

Now let $\xi \in \mathcal{E}_1^{0,p}(\mathcal{F})$ be the element defined by the characteristic form $\mathcal{X}$. Since $E_2^{q,0}(\mathcal{F}) \neq 0$ we have $\mathcal{E}_2^{0,p}(\mathcal{F}) \neq 0$ (by (1.3)). Choose some nonzero element $\eta \in \mathcal{E}_2^{0,p}(\mathcal{F}) \subset \mathcal{E}_1^{0,p}(\mathcal{F})$ defined by some $\alpha \in A^{0,p}(M)$.

LEMMA 5. — *$\overline{G}_\zeta$ and $\overline{G}_\eta$ are nowhere zero (so they are C^∞).*

Proof. — $\mathcal{X}_{|\pi_b^{-1}(y)}$ defines a generator of $\mathcal{E}_2^{0,p}(\mathcal{F}_{|\pi_b^{-1}(y)})$, and if $y \in U_m$, λ_m defines a generator of $E_2^{q_0,0}(\mathcal{F}_{|\pi_b^{-1}(y)})$. Therefore, since the isomorphisms of (1.3) are induced by the de Rham duality map, we obtain

$$\overline{G}_\xi(y) = \left| \int_{\pi_b^{-1}(y)} \lambda_m \wedge \mathcal{X} \right| \neq 0.$$

By Lemma 4, to prove that $\overline{G}_\eta$ is nowhere zero it is enough to prove that every $F_{m,\eta}$ is nowhere zero. And since η is closed in $\mathcal{E}_1^{0,p}(\mathcal{F})$, all the functions $F_{m,\eta}$ are constant. On the other hand, for some index m_0 we have $F_{m_0,\eta} \neq 0$ (by the injectivity of the map r_1 of (1.7)). Therefore, since W is connected it is enough to prove that if $U_m \cap U_{m'} \neq \emptyset$, then $F_{m,\eta} \neq 0$ implies $F_{m',\eta} \neq 0$. But this follows because on $U_m \cap U_{m'}$ we have $F_{m,\eta} = \pm(f_m/f_{m'}) \cdot F_{m',\eta}$ by Lemma 4 and (2.4). □

Let $\mathcal{X}' = (\pi_b^*\overline{G}_\eta / \pi_b^*\overline{G}_\xi) \cdot \mathcal{X}$, which is the characteristic form defined by a new bundle-like metric inducing the same transverse Riemannian structure. Let $\xi' = [\mathcal{X}']_1 \in \mathcal{E}_1^{0,p}(\mathcal{F})$. Then for each index m it is easy to check that

$$F_{m,\xi'} = \pm F_{m,\eta}, \tag{2.5}$$

which are constant functions. It follows that $\xi' \in \mathcal{E}_2^{0,p}(\mathcal{F})$ by the injectivity of r_1 in (1.7), and because the map (2.2) is an isomorphism of differential spaces. This finishes the proof of Proposition 2 because $\mathcal{X}'$ is obviously positive along the leaves.

3. Characterization of tautness.

PROPOSITION 3. — *Let $\mathcal{F}$ be an oriented Riemannian foliation on a compact oriented manifold M. Then $\mathcal{F}$ is taut if and only if $E_2^{q,0} \neq 0$ and the homomorphism $\mathcal{E}_2^{0,p} \to H^{1,p}(\overline{O}_1)$ of (1.2) is zero.*

Proof. — If $\mathcal{F}$ is taut there exists a p–form $\mathcal{X}$ on M, positive along the leaves and defining an element in $E_2^{0,p}$. Then the transverse volume element ν corresponding to any transverse Riemannian structure defines a nonzero element in $E_2^{q,0}$ because $[\nu \wedge \mathcal{X}]_2 \neq 0$ in $E_2^{q,p}$ (since $\nu \wedge \mathcal{X}$ is a volume form on M). Moreover, $\overline{[\mathcal{X}]}_2$ is a generator of $\mathcal{E}_2^{0,p}$, hence, by the exactness of (1.2) it follows that the connecting homomorphism $\mathcal{E}_2^{0,p} \to H^{1,p}(\overline{O}_1)$ is zero.

Reciprocally, if $E_2^{q,0} \neq 0$ then there exists an element $\xi \in \mathcal{E}_2^{0,p}$ defined by some form $\xi \in A^{0,p}(M)$ which is positive along the leaves (by Proposition 2). If the connecting homomorphism $\mathcal{E}_2^{0,p} \to H^{1,p}(\overline{O}_1)$ is zero, then there exists an element $\eta \in E_2^{0,p}$ which is mapped canonically to ξ (by the exactness of (1.2)). Choose $\alpha \in A^{0,p}(M)$ such that $\eta = [\alpha]_2$. Then we have by (1.1)

$$\mathcal{X} \in \alpha + \overline{d_{0,1}(A^{0,p-1}(M))}.$$

Since $\mathcal{X}$ is positive along the leaves, we can take some form $\beta \in \alpha + d_{0,1}(A^{0,p-1}(M))$ close enough to $\mathcal{X}$ so that β is also positive along the leaves. Then β also defines η, and $\mathcal{F}$ is taut by the criterion of Rummler-Sullivan. □

COROLLARY 1. — *Under the same hypotheses, the Tautness Conjecture is true for $\mathcal{F}$ if and only if the homomorphism $\mathcal{E}_2^{0,p} \to H^{1,p}(\overline{O}_1)$ of (1.2) is zero.*

From the proofs of Proposition 2 and Proposition 3, and using arguments of [21], [24], we obtain the following consequence (*cf.* Corollary 4 of Theorem 4.1 in [11]).

COROLLARY 2. — *Let the hypotheses be as above and assume that $\mathcal{F}$ is taut. Then for any Riemannian metric g on the vector-bundle $T\mathcal{F}$ there exists a nowhere zero basic function f such that $f \cdot g$ is in the $C^\infty-$closure, $\overline{\mathcal{T}}$, of the set $\mathcal{T}$ of restrictions to $T\mathcal{F}$ of bundle-like metrics on M for which all the leaves are minimal. If $\mathcal{F}$ has dense leaves, then g itself is in $\overline{\mathcal{T}}$.*

4. Foliations of codimension less or equal than two.

The arguments of this section show how the topology of the spectral sequence (E_i, d_i) can imply geometrical properties of the foliation.

PROPOSITION 4. — *For Riemannian foliations of codimension $q \leq 2$ on compact manifolds we have $E_2 \cong \mathcal{E}_2$ canonically; i.e. , $H(\overline{O}_1) = 0$.*

Proof. — By standard arguments we can assume that M and $\mathcal{F}$ are oriented. We will compare E_1, E_2, or E_3 with E_∞, which is Hausdorff.

For $q = 0$ we have $E_1 = E_\infty$, so $\overline{O}_1 = 0$.

For $q = 1$ we have $E_2 = E_\infty$, thus the result follows by (1.4) and the exactness of (1.2).

For $q = 2$ we have $E_3 = E_\infty$, $E_2^{2,\cdot} \equiv \mathcal{E}_2^{2,\cdot}$ (by (1.4)), $E_2^{1,\cdot} = E_\infty^{1,\cdot}$, and d_2 can be considered as $d_2 : E_2^{0,v} \to E_2^{2,v-1}$ for each integer v. Thus the closure $\overline{O}_2$ of the trivial subspace of E_2 is contained in $E_2^{0,\cdot}$. Then, since d_2 is continuous and $E_2^{2,\cdot}$ is Hausdorff, we have $d_2(\overline{O}_2) = 0$, which implies $\overline{O}_2 \subset E_3^{0,\cdot}$. So $\overline{O}_2 = 0$ because E_3 is Hausdorff, and thus E_2 is also Hausdorff. Therefore, by the exactness of (1.2) we have $H^{0,\cdot}(\overline{O}_1) = 0$, and the result follows by (1.3) and (1.4). □

COROLLARY 1. — *If $\mathcal{F}$ is an oriented Riemannian foliation of codimension $q \leq 2$ on a compact oriented manifold, then $\mathcal{F}$ is taut if and only if $E_2^{q,0} \neq 0$.*

Combining this result with the solution of the Tautness Conjecture for Riemannian flows [19] we obtain (*cf.* [5]) :

COROLLARY 2. — *Let M be a compact oriented manifold of dimension ≤ 4. Then for any oriented Riemannian foliation $\mathcal{F}$ on M, $\mathcal{F}$ is taut if and only if $E_2^{q,0} \neq 0$.*

5. Examples.

In this section we will give some examples of foliations verifying the hypotheses of Proposition 4.

Many properties can be extended from Lie foliations with dense leaves to Riemannian foliations by using the structure theorems of P. Molino. For instance, the canonical map $E_2 \to \mathcal{E}_2$ is an isomorphism for a Riemannian foliation if and only if it is an isomorphism for the corresponding Lie foliation with dense leaves [2], [3]. Thus, from Proposition 4 we obtain $E_2 \cong \mathcal{E}_2$ for a Riemannian foliation on a compact manifold if the structural Lie algebra $\mathfrak{g}$ is of dimension ≤ 2.

The case where $\mathfrak{g}$ is trivial corresponds to Riemannian foliations with compact leaves [2], [18]. The only 1−dimensional Lie algebra is abelian, and there are two non-isomorphic Lie algebras of dimension two : the abelian one, and the solvable Lie algebra with two generators, X_1 and X_2, verifying $[X_1, X_2] = X_2$.

One can construct examples of homogeneous Lie $\mathfrak{g}$−foliations with dense leaves when $\mathfrak{g}$ is nilpotent by using Malcev's theory [9]. So, when $\mathfrak{g}$ is an abelian Lie algebra of dimension one or two, those foliations verify $E_2 \cong \mathcal{E}_2$ and are taut.

When $\mathfrak{g}$ is the non-abelian Lie algebra of dimension two, we have the following example of a Lie $\mathfrak{g}$−foliation given by A. Haefliger. In this case $\mathfrak{g}$ is isomorphic to the Lie algebra of the Lie group GA of affine orientation preserving bijections of $\mathbf{R}$. Let k be a totally real number field of degree n over $\mathbf{Q}$, and let $i : k \to \mathbf{R}$ be an imbedding such that $i(u') > 0$ for all the conjugates u' of any unit u of the ring of integers of k with $i(u) > 0$. Then, using k and the above imbedding, one can construct a Lie group H of dimension $2n - 1$, a discrete uniform subgroup $\Gamma \subset H$, and a surjective homomorphism $D : H \to GA$ whose restriction to Γ is injective [9]. We have $E_2 \cong \mathcal{E}_2$ for the corresponding homogeneous foliation on H/Γ. Moreover the leaves of this Lie $\mathfrak{g}$−foliation are dense for $n \geq 3$, so it is not taut in this case because $\mathfrak{g}$ is not unimodular.

Finally, the following example is due to E. Ghys [9]. The 6−dimensional semisimple Lie groupe $H = PSL(2, \mathbf{R}) \times PSL(2, \mathbf{R})$ admits a uniform discrete subgroup Γ by a theorem of A. Borel. We may assume that Γ has a dense projection into each of the factors of H, obtaining a homogeneous foliation with dense leaves of codimension three. We may also assume that

Γ is torsion free, which implies that the action of Γ on

$$(SO(2)\backslash PSL(2,\mathbf{R})) \times (SO(2)\backslash PSL(2,\mathbf{R}))$$

is proper and without fixed points. So, on the 4–dimensional quotient manifold we obtain examples of transversally hyperbolic foliations with dense leaves of codimension two. For these foliations we also have $E_2 \cong \mathcal{E}_2$.

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