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COHOMOLOGY OF G/P FOR CLASSICAL COMPLEX LIE SUPERGROUPS G AND CHARACTERS OF SOME ATYPICAL G-MODULES (*)

by I. PENKOV (**) and V. SERGANOVA

en hommage respectueux
à J.-L. Koszul.

Introduction.

The objective of the present paper is to investigate the cohomology of certain irreducible G°-linearized sheaves $\mathcal{L}$ on G°/P, G° being the identity component of a classical Lie supergroup and P being a parabolic subsupergroup. In particular we prove two mutually dependent conjectures of [12]. The first one asserted that under certain restriction on P in case $G = \mathbf{P}(m)$, see [12], the single nonzero cohomology group of $\mathcal{L}$ is irreducible. Below we prove a more general result without any restriction on P. The second conjecture asserted an explicit character formula for a natural class of irreducible finite dimensional G°-modules and was a direct consequence of the first one. Both results have been known when $P = B$ is a Borel subsupergroup. The investigation of the cohomology of G°/B in [12] (see also [13], Chapter 4) was based on an essential generalization of Demazure's method of proving Bott's vanishing theorem, [2]. In particular, developing the notion of a simple « super-reflection », a vanishing result extending Bott's (vanishing) theorem to

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G°/B was proven. In the case of a reductive Lie group G' Bott's theorem on G'/B' leads easily to a Bott- type theorem on G'/P' for an arbitary parabolic P', however if G' is a Lie supergroup, the latter is not the fact. Therefore the question about the cohomology of G°/P remained open in [12]. The result of the present work is not a complete generalization of the main theorem of [12] to G°/P because (exept in the case $G = Q$) we use a rougher vanishing result than in [12]. Nevertheless we introduce a generalization of the simple « superreflection » suitable for any P (Steps A 2 and B 3 of the proof of Theorem 4 below). A serious motivation for investigating the cohomology of G°/P was to obtain a generalization of the typical character formula. In this way Theorem 4 below serves at the same time as a geometric version of the well known Bernstein-Gelfand-Gelfand method (this method applied to Lie superalgebras see for instance in [4] or [11]), allowing to prove an explicit character formula for a reasonable class of irreducible finite dimensional atypical representations.

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1. Preliminaries and notation.

1.1. *Algebraic Preliminaries.*

The result of this paper is a natural extension of some results in the book [13]. In particular all preliminaries needed are presented in detail in [13]. Since however [13] has not yet appeared, we do not restrict ourselves to fixing notation but recall some basic preliminary facts in the hope that this will make our work more readable.

The ground field in this paper is $\mathbb{C}$. The *(super)dimension of a* $\mathbb{Z}_2$-graded linear space $T = T_0 \oplus T_1$ will be denoted by $\dim T := = \dim T_0 + \dim T_1 \cdot \varepsilon$, ε being an odd formal variable with $\varepsilon^2 = 1$. The same notation will be used for the (super)dimension of supermanifolds. Furthermore for any $\mathbb{Z}_2$-graded space T we define the $\mathbb{Z}_2$-graded space ΠT as « T with opposite parity », i.e. $(\Pi T)_0 = T_1$, $(\Pi T)_1 = T_0$.

By definition $\mathrm{GL}(m+n\varepsilon)$ is the *general linear complex Lie supergroup* of rank $m + n\varepsilon$, $\mathrm{OSP}(m+n\varepsilon)$ is the *orthosymplectic Lie supergroup* (i.e. the Lie subsupergroup of $\mathrm{GL}(m+n\varepsilon)$ leaving invariant an even nondegenerate (super) symmetric bilinear form on $\mathbb{C}^{m+n\varepsilon}$; in this case necessarily $n=2k$), $Q(m)$ is the Lie subsupergroup of $\mathrm{GL}(m+n\varepsilon)$ leaving invariant a Π-symmetry (i.e. an odd automorphism γ with $\gamma^2=\mathrm{id}$) of $\mathbb{C}^{m+m\varepsilon}$, and $\mathbf{P}(m)$ is the Lie subsupergroup of $\mathrm{GL}(m+m\varepsilon)$ leaving an odd nondegenerate (super)antisymmetric bilinear form on $\mathbb{C}^{m+m\varepsilon}$, [13], [13′]; $\mathrm{gl}(m+n\varepsilon)$, $\mathrm{osp}(m+n\varepsilon)$, $q(m)$, $\mathbf{p}(m)$ denote respectively the Lie superalgebras of $\mathrm{GL}(m+n\varepsilon)$, $\mathrm{OSP}(m+n\varepsilon)$, $Q(m)$, $\mathbf{P}(m)$:

Throughout the whole paper G will denote one of the above listed Lie supergroups and $\mathfrak{g} := \mathrm{Lie}\, G$; specifying certain concrete series, we will write fo short just $G = \mathrm{GL}, \mathrm{OSP}$, etc., $\mathfrak{g} = \mathrm{gl}, \mathrm{osp}$, etc. By the very definition, G and $\mathfrak{g}$ act on $\mathbb{C}^{m+n\varepsilon}$ (where $m=n$ for $G=Q(m)$, $\mathbf{P}(m)$), and $\mathbb{C}^{m+n\varepsilon}$ will be called the *standard representation* respectively G and $\mathfrak{g}$. A *Borel subsupergroup* (respectively *subsuperalgebra*) of G (respectively $\mathfrak{g}$) is the identity component of the stabilizer (respectively simply the stabilizer) of a $\mathbb{Z}_2$-graded flag of maximal length in the standard representation, which is supposed to be isotropic, see [13], [13′], or [7], in case $G = \mathrm{OSP}, \mathbf{P}(\mathfrak{g}=\mathrm{osp},\mathbf{p})$, and Π-symmetric for $G = Q\,(\mathfrak{g}=q)$. *Parabolic subsupergroups (subsuperalgebras)* are identity components of stabilizers (simply stabilizers), of isotropic for $G=\mathrm{OSP}$, $\mathbf{P}$ and Π-symmetric for $G=Q$, flags of arbitrary length in the standard representation. A *Cartan subsuperalgebra* of $\mathfrak{g}$ is the centralizer of a Cartan subalgebra of $\mathfrak{g}_0$. A *Cartan subsupergroup of* G is a connected Lie subsupergroup $H \hookrightarrow G$, such that $\mathrm{Lie}\, H$ is a Cartan subsuperalgebra of $\mathfrak{g}$. For $\mathfrak{g} \neq q$ it turns out that Cartan subsuperalgebras of $\mathfrak{g}$ are simply Cartan subalgebras of $\mathfrak{g}_0$, and for $\mathfrak{g} = q(m)$ if $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \subset \mathfrak{g}$ is a Cartan subsuperalgebra, then $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, $\dim \mathfrak{h}_1 = (\dim \mathfrak{h}_0)\cdot\varepsilon$, ans $\mathfrak{h}_0$ belongs to the centre of $\mathfrak{h}$.

Fix now a Cartan subsuperalgebra $\mathfrak{h} \subset \mathfrak{g}$. If V is any finite dimensional $\mathfrak{h}_0$-semisimple $\mathfrak{g}$-module, $V = \bigoplus_{\zeta \in \mathfrak{h}_0^*} V_\zeta$, set

$$\mathrm{supp}\, V := \{\zeta \in \mathfrak{h}_0^* \mid V_\zeta \neq 0\},$$

$$\mathrm{ch}\, V := \sum_{\zeta \in \mathrm{supp}\, V} \dim V_\zeta \cdot e^\zeta$$

(each V_ζ is a $\mathbb{Z}_2$-graded space and dim denotes here superdimension). Considering $\mathfrak{g}$ itself as the adjoint module, put $\Delta := \operatorname{supp}\mathfrak{g}\backslash\{0\}$. Δ is the set of *roots of* $\mathfrak{g}$ (with respect to $\mathfrak{h}$), and for $\mathfrak{g} \neq q$ each *root space* $\mathfrak{g}_\zeta$ has dimension either 1 or ε. Therefore $\Delta = \Delta_0 \cup \Delta_1$, where $\Delta_{\substack{0\\1}} := \{\zeta \in \Delta \,|\dim \mathfrak{g}_\zeta = {}^{1}_{\varepsilon}\}$; the elements of Δ_0 are by definition the *even roots of* $\mathfrak{g}$, and the elements of Δ_1 – the *odd roots of* $\mathfrak{g}$. For $\mathfrak{g} = q$ each root space has dimension $1 + \varepsilon$. Assuming however that $\Delta = \Delta_0 \cup \Delta_1$, where $\Delta_{\substack{0\\1}}$ are two identical copies of the usual root system of type A_{m-1}, and considering Δ_0 as even and Δ_1 as the odd roots (assuming in this way $\dim \mathfrak{g}_\alpha = 1$ for $\alpha \in \Delta_0$ and $\dim \mathfrak{g}_\alpha = \varepsilon$ for $\alpha \in \Delta_1$), we can unify this case with all other ones. Furthermore we set

$$\check{\Delta} := \{\alpha \in \Delta \,| -\alpha \notin \Delta\},$$

$$\hat{\Delta} := \{\alpha \in \Delta\backslash\check{\Delta} \,| \frac{\alpha}{2} \notin \Delta, 2\alpha \notin \Delta\}, \hat{\Delta}_{\substack{0\\1}} = \hat{\Delta} \cap \Delta_{\substack{0\\1}}.$$

$\check{\Delta} = \varnothing$ for $\mathfrak{g} \neq \mathbf{p}$, and $\hat{\Delta} = \Delta$ for $\mathfrak{g} \neq \mathbf{p}, \mathfrak{g} \neq \operatorname{osp}(m+n\varepsilon)$ with $m = 2l + 1$.

The Lie superalgebras $\mathfrak{gl}(m+n\varepsilon)$ and $\operatorname{osp}(m+n\varepsilon)$ admit an invariant nondegenerate even (super)symmetric bilinear form. The form which it induces on $\mathfrak{h}^*(=\mathfrak{h}_{\bar{0}}^*)$ will be denoted by $(\ ,\ )$. $q(m)$ does not admit an even invariant nondegenerate form (in particular its Killing form, [3], [13], [13′], is identically zero), however it admits a nondegenerate $\mathfrak{g}$-invariant pairing $\mathfrak{g} \times \Pi\mathfrak{g} \to \mathbb{C}$ (or equivalently an odd nondegenerate $\mathfrak{g}$-invariant bilinear form). The induced pairing $\mathfrak{h}_{\bar{0}}^* \times \Pi\mathfrak{h}_{\bar{0}}^* \to \mathbb{C}$ will be also denoted by $(\ ,\ )$. It is easy to show that $\mathbf{p}(m)$ admits no invariant (even or odd) nondegenerate form.

The case $\mathfrak{g} = q$ has one more essential peculiarity concerning roots. Each root α defines here up to a constant a pseudoroot $\underline{\alpha}$ as an element of $\Pi\mathfrak{h}_{\bar{0}}^*\backslash\{0\}$, such that

$$[\underline{h}, g_\alpha] = \underline{\alpha}(\underline{h}) . \underline{g}_\alpha$$

for any $\underline{h} \in \mathfrak{h}_1$ and any $g_\alpha \in (\mathfrak{g}_\alpha)_1$, where $\underline{g}_\alpha \in (\mathfrak{g}_\alpha)_0$ depends only g_α. Denoting the set of pseudoroots by $\underline{\Delta}$, and fixing for each α a representative $\underline{\alpha} \in \underline{\Delta}$, we obtain a well defined map

$$\Delta \to \underline{\Delta}, \qquad \alpha \mapsto \underline{\alpha}.$$

The *weights* of an arbitrary $\mathfrak{g}$ considered are by definition the elements of $\mathfrak{h}_{\bar{0}}^*$. By $\bar{C}^+ \subset \mathfrak{h}_{\bar{0}}^*$ we denote the chamber of all dominant (with respect to a fixed Borel subalgebra $\mathfrak{b}_0$ of $\mathfrak{g}_0$, $\mathfrak{b}_0 \supset \mathfrak{h}_0$) weights, which are $\mathfrak{g}_{0,ss}$-integral, $\mathfrak{g}_{0,ss} \subset \mathfrak{g}_0$ denoting the semisimple part of $\mathfrak{g}_0$. $\chi \in \bar{C}^+$ iff the irreducible $\mathfrak{g}_0$-module with highest weight χ (with respect of $\mathfrak{b}_0$) is of finite dimension. C^+ will denote the subset of $\bar{C}^+$ consisting of all strictly dominant weights.

If $\alpha \in \Delta_1$, a weight χ is *α-regular* iff

– $(\chi,\alpha) \neq 0$ for $\mathfrak{g} = \mathrm{gl}$, osp;
– $(\chi,\underline{\alpha}) \neq 0$ for $\mathfrak{g} = q$.

Furthermore χ is *typical* iff χ is α-regular for any $\alpha \in \hat{\Delta}_1$, and χ is *regular* iff

– $(\chi,\alpha) \neq 0$ $\forall \alpha \in \Delta$ for $\mathfrak{g} = \mathrm{gl}$, osp;
– χ is typical and $(\chi,\alpha)_{\mathrm{red}} \neq 0$ $\forall \alpha \in \Delta_0$ for $\mathfrak{g} = q$ (1).

The notions of α-regularity, typicality, and regularity for $\mathfrak{g} = \mathbf{p}$ will be defined in 2.2.

Let us discuss briefly some basic properties of roots and Borel subsuperalgebras. Our considerations will concern actually also the corresponding Borel subsupergroups since Borel subsupergroups and subsuperalgebras are in 1-1 correspondence. Throughout the whole paper we will assume automatically that all Borel or parabolic subsuperalgebras considered contain $\mathfrak{h}$. First of all, set for any $\mathfrak{p} \subset \mathfrak{g}$ ($\mathfrak{p}$ being a parabolic subsuperalgebra with $\mathfrak{p} \supset \mathfrak{h}$)

$$\Delta(\mathfrak{p}) := \{\gamma \in \Delta \mid \mathfrak{g}_\gamma \cap \mathfrak{p} \neq \varnothing\}, \quad \Delta_{\substack{0\\1}}(\mathfrak{p}) := \Delta(\mathfrak{p}) \cap \Delta_{\substack{0\\1}},$$
$$\tilde{\Delta}(\mathfrak{p}) := \{\gamma \in \Delta(\mathfrak{p}) \mid -\gamma \notin \Delta \backslash \Delta(\mathfrak{p})\}, \quad \tilde{\Delta}_{\substack{0\\1}}(\mathfrak{p}) := \tilde{\Delta}(\mathfrak{p}) \cap \Delta_{\substack{0\\1}}.$$

If $\mathfrak{p} = \mathfrak{b}$, we put

$$\Delta^+(\mathfrak{b}) := \Delta(\mathfrak{b}), \quad \Delta^-(\mathfrak{b}) := \Delta \backslash \Delta^+(\mathfrak{b}),$$
$$\Delta^\pm_{\substack{0\\1}}(\mathfrak{b}) := \Delta^\pm(\mathfrak{b}) \cap \Delta_{\substack{0\\1}}.$$

The Borel subsuperalgebra $\mathfrak{b}^{\mathrm{op}}$ is characterized by

$$\mathfrak{h} \subset \mathfrak{b}^{\mathrm{op}}, \quad \Delta^+(\mathfrak{b}^{\mathrm{op}}) = \Delta^-(\mathfrak{b}),$$

and is called the *Borel subsuperalgebra opposite to* $\mathfrak{b}$ ($\mathfrak{b}^{\mathrm{op}}$ is always well defined but, as one sees immediately, for $\mathfrak{g} = \mathbf{p}$ $\Delta^+(\mathfrak{b}^{\mathrm{op}})$ does not necessarily coincide with $-\Delta^+(\mathfrak{b})$). Furthermore any $\mathfrak{b}$ induces a partial

(1) Here and below $(\ ,\ )_{\mathrm{red}}$ denotes the bilinear form on $\mathfrak{h}_0^*$ induced by the Killing form of $\mathfrak{g}_0$.

ordering $\underset{\mathfrak{b}}{\leqslant}$ on $\mathfrak{h}_{\bar{0}}^*$:

$$\lambda \underset{\mathfrak{b}}{\leqslant} \mu \Leftrightarrow \lambda = \mu + \sum_{\gamma \in \Delta^-(\mathfrak{b})} \gamma$$

for λ, $\mu \in \mathfrak{h}_{\bar{0}}^*$. Note also that $\mathfrak{b}$ is determined uniquely not only by $\Delta^+(\mathfrak{b})$, but also by the set of its simple roots (as usual, $\alpha \in \Delta^+(\mathfrak{b})$ is a *simple root of* $\mathfrak{b}$ if an equality $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Delta^+(\mathfrak{b})$ is impossible).

Below we will need the following « superanalogues » of the half sum of positive roots in the usual case. By definition

$$\check{\rho}_{\mathfrak{b}} := \frac{1}{2} \cdot \left(\sum_{\gamma \in \Delta_{\bar{0}}^+(\mathfrak{b})} \gamma + \sum_{\gamma \in \Delta_{\bar{1}}^-(\mathfrak{b})} \gamma \right),$$

$$\rho_{\mathfrak{b}} := \check{\rho} - \frac{1}{2} \cdot \left(\sum_{\gamma \in \check{\Delta} \cap \Delta_{\bar{1}}^-(\mathfrak{b})} \gamma \right),$$

$\rho_0 := \frac{1}{2} \cdot \left(\sum_{\gamma \in \Delta_{\bar{0}}^+(\mathfrak{b})} \gamma \right)$ for any $\mathfrak{b} \subset \mathfrak{g}$; for a parabolic subsuperalgebra $\mathfrak{p}$ we put

$$\rho_{\mathfrak{p}} := (\rho_{\mathfrak{p}})_0 + (\rho_{\mathfrak{p}})_1, \ (\rho_{\mathfrak{p}})_0 := \frac{1}{2} \cdot \left(\sum_{\gamma \in \Delta_0(\mathfrak{p})} \gamma \right),$$

$$(\rho_{\mathfrak{p}})_1 := \frac{1}{2} \cdot \left(\sum_{\gamma \in \Delta_1 \setminus (\Delta_1(\mathfrak{p}) \cup \check{\Delta})} \gamma \right).$$

W_0 will denote the Weyl group of $\mathfrak{g}_0$ and $l(\cdot)$ will be the length function on W_0.W_0 acts on $\mathfrak{h}_{\bar{0}}^*$ and on the sets of parabolic and Borel subsuperalgebras containing $\mathfrak{h}$. Furthermore it is essential to recall that any two Borel subsuperalgebras $\mathfrak{b}$, $\mathfrak{b}' \subset \mathfrak{g}$ with $\mathfrak{b}_0 = \mathfrak{b}_0'$ and $\operatorname{ht} \mathfrak{b} \geqslant \operatorname{ht} \mathfrak{b}'$ (where the *height* of a Borel subsuperalgebra, [13], [13′], is defined in the following way: $\operatorname{ht} \mathfrak{b} := 0$ for any $\mathfrak{b}$ in case $\mathfrak{g} \neq \mathbf{p}$ and $\operatorname{ht} \mathfrak{b} := m - \#(\Delta_{\bar{1}}^+(\mathfrak{b}) \cap \check{\Delta})$ in case $\mathfrak{g} = \mathbf{p}$) can be connected by a chain of odd reflections and inclusions. We say that two Borel subusperalgebras $\mathfrak{b}$, $\mathfrak{b}' \subset \mathfrak{g}$ *are connected by an odd reflection* iff $\mathfrak{b}_0 = \mathfrak{b}_0'$ and there exists a simple odd root $\alpha' \in \hat{\Delta}$ of $\mathfrak{b}'$, such that $\Delta_{\bar{1}}^+(\mathfrak{b}) \cap \Delta_{\bar{1}}^+(\mathfrak{b}') = \Delta^+(\mathfrak{b}') \setminus \{\alpha'\} = \Delta^+(\mathfrak{b}) \setminus \{-\alpha'\}$; one has an *inclusion* of Borel subsuperalgebras iff simply $\mathfrak{b} \subset \mathfrak{b}'$. The first is possible for $\mathfrak{g} \neq q$ and the second is possible only for $\mathfrak{g} = \mathbf{p}$. In this way for all $\mathfrak{b}$, $\mathfrak{b}'$ with $\mathfrak{b}_0 = \mathfrak{b}_0'$ and $\operatorname{ht} \mathfrak{b} \geqslant \operatorname{ht} \mathfrak{b}'$ there exists a sequence of Borel subsuperalgebras

$$\mathfrak{b} = \mathfrak{b}^1, \mathfrak{b}^2, \ldots, \mathfrak{b}^{k-1}, \mathfrak{b}^k = \mathfrak{b}',$$

such that $\mathfrak{b}^i$ and $\mathfrak{b}^{i+1}$ are connected either by an odd reflection or by an inclusion $\mathfrak{b}^i \subset \mathfrak{b}^{i+1}$. Furthermore the above sequence is determined by the corresponding sequence of roots

$$(2) \qquad \breve{\alpha}_1, \breve{\alpha}_2, \ldots, \breve{\alpha}_{k-1}$$

where $\breve{\alpha}_i \in \Delta_1^+(\mathfrak{b}^{i+1})$, $\breve{\alpha}_i \notin \Delta_1^+(\mathfrak{b}^i)$ (iff $\breve{\alpha}_i \notin \breve{\Delta}$, $-\breve{\alpha}_i \in \Delta_1^+(\mathfrak{b}^i)$) ; (it is easy to show that if $\mathfrak{g}=\mathbf{p}$ and one has an inclusion $\mathfrak{b} \subset \mathfrak{b}'$, then $\mathfrak{b}_0 = \mathfrak{b}_0'$ and $\Delta_1^+(\mathfrak{b}') = \Delta_1^+(\mathfrak{b}) \cup \{\breve{\alpha}\}$ for some odd root $\breve{\alpha} \in \breve{\Delta}$).

1.2. *Highest weight modules and their central characters.*

Let $\mathfrak{b} \subset \mathfrak{g}$ be a fixed Borel subsuperalgebra of $\mathfrak{g}$. We have an isomorphism $\mathfrak{b} \simeq \mathfrak{h} \oplus \mathfrak{n}$, $\mathfrak{h}$ being a Cartan subsuperalgebra and $\mathfrak{n}$ being a nilpotent subsuperalgebra. One shows straightforwardly that $\mathfrak{n}$ acts trivially on any finite dimensional irreducible $\mathfrak{b}$-module, and thus, as in the usual case of Lie algebras, the above isomorphism induces a 1-1 correspondence between irreducible $\mathfrak{b}$-modules of finite dimension and irreducible $\mathfrak{h}$-modules of finite dimension. In this way (since for $\mathfrak{g}\neq q$ $\mathfrak{h}=\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$) for $\mathfrak{g} \neq q$ an irreducible $\mathfrak{b}$-module is just 1- or ε-dimensional and is determined up to isomorphism by the weight, by which $\mathfrak{h}_0$ acts on it, and by its dimension. Such a $\mathfrak{b}$-module will be denoted below by λ^δ, where$\lambda \in \mathfrak{h}_0^*$ and $\delta \in \mathbb{Z}_2(\dim \lambda^\delta = \varepsilon^\delta)$. It turns out moreover that the same notation makes sense also for irreducible $\mathfrak{b}$-modules for $\mathfrak{g} = q$. These modules are discussed in detail in [11] and [13], [13′], and the important point is that $\mathfrak{h}_0$ acts on each such module by a unique weight and that the module is determined up to Π by this weight. However in the case $\mathfrak{g} = q(m)$ $\dim \lambda^\delta = k + k\varepsilon$, where k depends on m and λ; for a generic $\lambda \in \mathfrak{h}_0^*$ $k = 2^{[(m-1)/2]}$, [13], [13′].

Let now $V_\mathfrak{b}(\lambda^\delta)$ be by definition the irreducible $\mathfrak{g}$-module, such that λ^δ is its $\mathfrak{b}$-submodule. In other words λ^δ is the highest weight space of $V_\mathfrak{b}(\lambda^\delta)$ with respect to $\mathfrak{b}$. $V_\mathfrak{b}(\lambda^\delta)$ is not necessarily finite dimensional, but any irreducible $\mathfrak{g}$-module of finite dimension is isomorphic to $V_\mathfrak{b}(\lambda^\delta)$ for some λ^δ. $V_\mathfrak{b}(\lambda^\delta)$ is G°-integrable, where G° denotes the identity component of G, iff $\dim V_\mathfrak{b}(\lambda^\delta) < \infty$ and λ^δ is B-integrable. Furthermore it is essential to note that the inclusion $\lambda \in \bar{C}^+$, or equivalently the finite dimensionality of the irreducible $\mathfrak{b}_0$-module with highest weight λ (with respect to $\mathfrak{b}_0$), is (necessary but) not sufficient for the finite dimensionality of $V_\mathfrak{b}(\lambda^\delta)$ for an arbitrary $\mathfrak{b}$.

The rule of change of the highest vector of an irreducible $\mathfrak{g}$-module under an odd reflection (and an inclusion) of Borel subsuperalgebras will be essential for us. One has

LEMMA 1. — *If* $\mathfrak{b}'$ *is a Borel subsuperalgebra of* $\mathfrak{g}$ *with* $\mathfrak{b}'_0 = \mathfrak{b}_0$, *and*

a) $\mathfrak{b}$ *and* $\mathfrak{b}'$ *are connected by an odd reflection, for any* $\eta \in \mathfrak{h}_0^*$, $\varkappa \in \mathbb{Z}_2$ *one has an isomorphism of* $\mathfrak{g}$-*modules*

$$V_{\mathfrak{b}}(\eta^{\varkappa}) = \begin{cases} V_{\mathfrak{b}'}((\eta+\gamma)^{\varkappa+1}) & \text{in case } \eta + \check{\rho}_{\mathfrak{b}} \text{ is } \gamma\text{-regular} \\ V_{\mathfrak{b}'}(\eta^{\varkappa}) & \text{in case } \eta + \check{\rho}_{\mathfrak{b}} \text{ is not } \gamma\text{-regular,} \end{cases}$$

where $\gamma \in \Delta_{\bar{1}}^{+}(\mathfrak{b}') \cap \Delta_{\bar{1}}^{-}(\mathfrak{b})$ (*and, as already said, the notion of* γ-*regularity for* $\mathfrak{g} = \mathbf{p}$ *is defined in* 2.2); *furthermore if* $v \neq 0$ *is a highest vector of* $V_{\mathfrak{b}}(\eta^{\varkappa})$ *with respect to* $\mathfrak{b}$, then a highest vector of $V_{\mathfrak{b}}(\eta^{\varkappa})$ *with respect to* $\mathfrak{b}'$ *is*

— $g_{\gamma}v$ *in case* $\eta + \check{\rho}_{\mathfrak{b}}$ *is* γ-*regular*,

— v *in case* $\eta + \check{\rho}_{\mathfrak{b}}$ *is not* γ-*regular*,

g_{γ} *being a generator of the root space* $\mathfrak{g}_{\gamma}$;

b) $\mathfrak{b} \subset \mathfrak{b}'$, *for any* $\eta \in \mathfrak{h}_0^*$, $\varkappa \in \mathbb{Z}_2$ *one has*

$$V_{\mathfrak{b}}(\eta^{\varkappa}) = V_{\mathfrak{b}'}(\eta^{\varkappa}),$$

and the highest vectors of $V_{\mathfrak{b}}(\eta^{\varkappa})$ *with respect to* $\mathfrak{b}$ *and* $\mathfrak{b}'$ *coincide.*

Comment on the proof. — b) is obvious and the proof of a) is a nondifficult computation; see [13], [13′], 2.5, Theorem 3. □

Lemma 1 implies the important

COROLLARY 1. — *Let*

$$\mathfrak{b} = \mathfrak{b}^1, \quad \mathfrak{b}^2, \ldots, \mathfrak{b}^{k-1}, \quad \mathfrak{b}^k = \mathfrak{b}'$$

be a chain of odd reflections and inclusions, connecting two given Borel subsuperalgebras $\mathfrak{b}$, $\mathfrak{b}' \subset \mathfrak{g}$ (*with* $\mathfrak{b}_0 = \mathfrak{b}'_0$). *Then for an arbitrary* $\lambda \in \mathfrak{h}_0^*$ one has

$$V_{\mathfrak{b}}(\lambda^{\delta}) \simeq V_{\mathfrak{b}}(\lambda'^{\delta'}),$$

where

$$\lambda' := \lambda + \sum_{j_r \neq i_1(\lambda), \ldots, i_{l'}(\lambda)} \check{\alpha}_{j_r},$$

$$\delta' := \delta + (k - l' - 1) \bmod 2$$

and

$$\check{\alpha}_{i_1(\lambda)}, \ldots, \check{\alpha}_{i_{l'}(\lambda)}$$

is the maximal subsequence of (2), *such that* $\check{\alpha}_{i_t} = \check{\alpha}_{i_t(\lambda)} \notin \check{\Delta}$ *and*

$$\lambda + \check{\rho}_{\mathfrak{b}\, i_t} - \check{\alpha}_{i_1} - \check{\alpha}_{i_2} - \ldots - \check{\alpha}_{i_{t-1}}$$

(where $\check{\alpha}_0 := 0$) *is not* $\check{\alpha}_{i_t}$*-regular for any* $t = 1, \ldots, l'$. *Furthermore if* $v \neq 0$ *is a highest vector of* $V_{\mathfrak{b}}(\lambda^{\delta})$ *with respect to* $\mathfrak{b}$, *then*

$$v' = \left(\prod_{j_r \neq i_1(\lambda), \ldots, i_{l'}(\lambda)} g\check{\alpha}_{j_r} \right) \cdot v$$

is a highest vector of $V_{\mathfrak{b}}(\lambda^{\delta})$ *with respect to* $\mathfrak{b}'$. □

Consider the supercentre Z of the enveloping algebra $U(\mathfrak{g})$. It is well known that Z acts on any irreducible $\mathfrak{g}$-module by a central character, i.e. by a homomorphism $Z \to \mathbb{C}$, see for instance [13], [13']. The main tool in studing the centre itself and its characters is the following

THEOREM 1 (A. Sergeev, V. Kac). — *Let* $\mathfrak{g} \neq \mathbf{p}$. *There exists an (Harish-Chandra) injective homomorphism*

$$HC_{\mathfrak{g}} : Z \to S^{\cdot}(\mathfrak{h}_0)$$

($S^{\cdot}$ denoting symmetric algebra), such that

$$\operatorname{im} HC_{\mathfrak{g}} = \left\{ \begin{matrix} f \in S^{\cdot}(\mathfrak{h}_0)^{W_0} \mid \alpha \in \hat{\Delta}_1, \ \lambda \in \mathfrak{h}_0^*, \text{ and } (\lambda, \alpha) = 0 \\ (\textit{resp. } (\lambda, \underline{\alpha}) = 0 \textit{ for } \mathfrak{g} = q) \Rightarrow f(\lambda + t\alpha) = f(\lambda) \ \forall t \in \mathbb{C} \end{matrix} \right\}.$$

The proof see in [10] (some earlier references are [14] and [6]).

Denote now by $\theta^{\lambda + \rho_{\mathfrak{b}}}$ the (super)central character of $V_{\mathfrak{b}}(\lambda^{\delta})$. As it is shown for instance in [13], [13'], this notation is well defined, i.e. $\theta^{\lambda + \rho_{\mathfrak{b}}}$ depends only on $\lambda + \rho_{\mathfrak{b}}$ (for a fixed $\mathfrak{h}$).

LEMMA 2. — *Let* $\mathfrak{g} \neq \mathbf{p}$. *The following conditions on* $\chi \in \mathfrak{h}_0^*$ *are equivalent :*

— χ *is typical;*
— $\theta^{\chi} = \theta^{\eta}$ *(for some* $\eta \in \mathfrak{h}_0^*$) *implies* $\eta = w(\chi)$ *for a certain* $w \in W_0$;
— *the homomorphism* $\chi_W : S^{\cdot}(\mathfrak{h}_0)^{W_0} \to \mathbb{C}$, *induced by* χ, *is the unique extension of* $\theta^{\chi} \circ HC_{\mathfrak{g}}$ *to a character of* $S(\mathfrak{h}_0)^{W_0}$.

A natural proof is based on Theorem 1, [13], [13′] (however V.Kac has used the result already in [4] and [5], before Theorem 1 was known). □

Next we have

THEOREM 2. — *Let* $\mathfrak{b} \subset \mathfrak{g}$ *be a Borel subusperalgebra. Then if* $\dim V_{\mathfrak{b}}(\lambda^{\delta}) < \infty$ *and* $\lambda + \rho_{\mathfrak{b}}$ *is typical,*

$$(3)\quad \mathrm{ch}\, V_{\mathfrak{b}}(\lambda^{\delta}) = \frac{\dim \lambda^{\delta}}{\sum_{w' \in W_0} \operatorname{sgn} w' e^{w'(\rho_0)}} \cdot \left(\sum_{w' \in W_0} \operatorname{sgn} w' . w'(e^{\lambda+\rho_0} \prod_{\gamma \in \Delta_{\bar{1}}^{-}(\mathfrak{b}) \cap (\Delta_{\bar{1}}^{-}(\mathfrak{b}) \cap \check{\Delta})} (1+\varepsilon e^{\gamma})) \right) \cdot$$

□

(3) has been proven by V. Kac in 1977 for $\mathfrak{g} = \mathfrak{gl}$, $\mathfrak{osp}$, [4], [5], and later respectively by D. Leites for $\mathfrak{g} = \mathbf{p}$ and by the first author for $\mathfrak{g} = q$, [11]. There have been several attempts to generalize (3) (in particular [1], [8], [9], [15], [16], [17]) to the case of an arbitrary finite dimensional $\mathfrak{g}$-module, but although certain effects are already known (see [13] for a description of the situation), this is still a challenge. In § 2 we extend (4) to a natural class of atypical modules, which are « typical modulo a parabolic subsuperalgebra ».

A final remark concerning central characters of $\mathfrak{g}$-modules is that if

$$(4)\qquad 0 \to E' \to E \to E'' \to 0$$

is an exact sequence of $\mathfrak{g}$-modules, and E' and E'' admit central characters θ' and θ'', then $\theta' \neq \theta''$ implies the splitting of (4) (i.e. in particular the existence of an isomorphism $E \simeq E' \oplus E''$). The proof is almost obvious, see [13], [13′]. A consequence we need is, that is V is a finite dimensional indecomposable $\mathfrak{g}$-module, then the central characters of all its irreducible composition factors coincide.

1.3. *Supergeometric preliminaries.*

We fix the supergeometric objects with which we deal below. For more details consult [13], [12], and [7]. If G is any of the Lie supergroups considered in the present paper (G° is the identity component of G) and $P \hookrightarrow G^{\circ}$ is a parabolic subuspergroup, the universal categorical quotient (in the category or superschemes) G°/P exists and is a connected (complex algebraic) supermanifold. This follows from Manin's construction of the supermanifolds of G-flags, [7], because one shows immediately that the supermanifold of G-flags (i.e. of isotropic flags in case $G = \mathrm{OSP}$, $\mathbf{P}$, and of Π-symmetric flags in case $G = Q$, see [13] or [7]) in the standard representation of type equal to the type of the P-invariant G-flag is nothing but G°/P. Iff $P = B$, i.e. iff P is a Borel subsupergroup, $G^\circ/P = G^\circ/B$ is a supermanifold of G-flags of maximal length.

When P' runs over all parabolic subsupergroups of G G°/P' form a finite oriented graph with vertices G°/P' themselves and edges — all submersions $G^\circ/P' \to G^\circ/P''$ induced by inclusions $P' \hookrightarrow P'' \hookrightarrow G$. Let us consider in particular the subgraph $\{G^\circ/B'\}$ with vertices G°/B' for all possible B'. If $G \neq \mathbf{P}$, this graph is fully disjoint, i.e. has no nontrivial edges (edges different from the ones induced by the identities $B' = B'$), and if $G = G^\circ = Q$, $\{G/B'\}$ consists of a single vertex because all Borel subsupergroups of Q are conjugated.

For $G = \mathbf{P}(G = G^\circ$ but) the structure of $\{G/B'\}$ is less trivial; $\{G/B'\}$ looks in this case like this

$$
\begin{array}{c|cccccc}
m & G/B_{ad} & & & & \\
 & \downarrow & & & & \\
m-1 & G/B' & \cdots & G/B'' & & \\
 & & & \downarrow & & \\
\cdot & & & \cdots & & \\
\cdot & & & \cdots & & \\
\cdot & & & \cdots & & \\
 & & & \downarrow & & \\
1 & & & G/B''' & \cdots & G/B^{(iv)} \\
 & & & & & \downarrow \\
0 & & & & & G/B_d ,
\end{array}
$$

i.e. it has $m + 1$ « levels » parametrized by the height of the Borel subsupergroup, all « levels » are fully disjoint subgraphs, « levels » 0 and m consist each of one homogeneous superspace (the Borel subsupergroups B_d and B_{ad} are described explicitly in 2.2.), and any two neighbouring « levels » are connected by at least one edge.

Fix now $P \hookrightarrow G$ and let $\mathscr{v}$ be a finite dimensional P-module. Then one defines as usual (see [13], Chapter 3) the induced G°-linearized $\mathcal{O}_{G^\circ/P}$-module $\tilde{\mathscr{v}}$. (The geometrical fiber of $\tilde{\mathscr{v}}$ at the geometrical point $P \in (G^\circ/P)_{\mathrm{red}}$ is nothing but the P-module $\mathscr{v}$ itself.) In particular for any integral weight η, such that the B-module structure on η^δ extends to a P-module structure, $\tilde{\eta}^\varkappa$ is a well defined G°-linearized $\mathcal{O}_{G^\circ/P}$-module of rank equal rk $\eta^\varkappa$. An explicit description of $\tilde{\eta}^\varkappa$ in terms of the tautological flag on G°/P see in [13] or [12]. It is also essential to note that one can extend the definition of $\tilde{\eta}^\varkappa$ to the case of a $\mathfrak{g}_{0,ss}$-integral weight ($\mathfrak{g}_{0,ss}$ being the semisimple part of $\mathfrak{g}_0$), such that $\eta^\varkappa$ admits a Lie $P = \mathfrak{p}$-module structure, [13], [12]. In the latter case $\tilde{\eta}^\varkappa$ in only a $\mathfrak{g}$-linearized $\mathcal{O}_{G^\circ/P}$-module.

Now we will introduce another parametrization of the sheaves $\tilde{\eta}^\varkappa$, which will be used below. Define first for any $\mathfrak{g}_{0,ss}$-integral weight η and any Borel subsuperalgebra $\mathfrak{b} \subset \mathfrak{g}$ the $\mathfrak{g}_{0,ss}$-integral weight $\eta_l(\mathfrak{b})$ in the following way. Consider the Borel subsuperalgebra $w_m(\mathfrak{b})$, where $w_m \in W_0$ is the element of maximal length, and connect it with $\mathfrak{b}^{\mathrm{op}}$ by a chain of odd reflections and inclusions (this is possible since $\Delta_0(w_m(\mathfrak{b})) = \Delta_0(\mathfrak{b}^{\mathrm{op}}) = -\Delta_0(\mathfrak{b})$, 1.1; note also for $G = Q$ simply $w_m(\mathfrak{b}) = \mathfrak{b}^{\mathrm{op}}$!). Let this chain be

$$w_m(\mathfrak{b}) = \mathfrak{b}^1, \quad \mathfrak{b}^2, \dots, \mathfrak{b}^{k-1}, \quad \mathfrak{b}^k = \mathfrak{b}^{\mathrm{op}}.$$

Then if $\lambda := w_m(\eta)$, set

$$\eta_l(\mathfrak{b}) := \lambda',$$

λ' being defined by the procedure of Corollary 1. Invariantly $\eta_l(\mathfrak{b})$ is determined uniquely from the isomorphism

$$V_{w_m(\mathfrak{b})}(w_m(\eta)^\delta) = V_{\mathfrak{b}^{\mathrm{op}}}(\eta_l(\mathfrak{b})^{\delta'})$$

for some $\delta' \in \mathbb{Z}_2$ (for $G = Q$ $\eta_l(\mathfrak{b}) = w_m(\eta)$). The notation $\eta_l(\mathfrak{b})$ is motivated by the fact that if $V_\mathfrak{b}(\eta^\delta) < \infty$, then $\eta_l(\mathfrak{b})$ is simply the *lowest weight of* $V_\mathfrak{b}(\eta^\delta)$ *with respect to* $\mathfrak{b}$ (the latter being nothing but the highest weight of $V_\mathfrak{b}(\eta^\delta)$ with respect to $\mathfrak{b}^{\mathrm{op}}$).

Now we set

$$\mathcal{O}_{G^\circ/B}(\lambda^\delta) := \widetilde{\lambda_l(\mathfrak{b})}^{\delta'}$$

for any $\mathfrak{g}_{0,ss}$-integral weight λ. Furthermore let us introduce the sets

$$\mathfrak{h}^*_\mathfrak{p} := \left\{ \lambda \in \mathfrak{h}^*_0 \,\middle|\, \begin{array}{l} \lambda \text{ is } \mathfrak{g}_{0,ss}\text{-integral and the } \mathfrak{b}\text{-module} \\ \lambda_l(\mathfrak{b})^{\delta'} \text{ admits a } \mathfrak{p}\text{-module structure} \end{array} \right\},$$

$$C^+_\mathfrak{p}(\bar{C}^+_\mathfrak{p}) := C^+ \cap \mathfrak{h}^*_\mathfrak{p}(\bar{C}^+ \cap \mathfrak{h}^*_\mathfrak{p}).$$

By the very definition of $\mathfrak{h}_{\mathfrak{p}}^*$, if $\lambda \in \mathfrak{h}_{\mathfrak{p}}^*$, we can push down $\mathcal{O}_{G^\circ/B}(\lambda^\delta)$ to G°/P via the canonical submersion

$$\pi : G^\circ/B \to G^\circ/P$$

and obtain in this way a well defined $\mathfrak{g}$-linearized $\mathcal{O}_{G^\circ/P}$-module $\mathcal{O}_{G^\circ/P}(\lambda^\delta)$ of rank equal to rk $\mathcal{O}_{G^\circ/B}(\lambda^\delta)$.

Let us discuss now briefly the cohomological properties of $\mathcal{O}_{G_\circ/P}(\lambda^\delta)$ (where possibly $P = B$). First of all $\mathcal{O}_{G^\circ/P}(\lambda^\delta)$ has a standard filtration with composition factors of the form

$$S^k(N^*_{(G^\circ/P)_{\text{red}}/(G^\circ/P)}) \otimes (\mathcal{O}_{G^\circ/P}(\lambda^\delta))_{\text{red}} \ (^2),$$

where

$$(\mathcal{O}_{G^\circ/P}(\lambda^\delta))_{\text{red}} \simeq \mathbb{C}^{\dim \lambda_l(\mathfrak{b})^{\delta'}} \otimes \mathcal{O}_{(G^\circ/P)_{\text{red}}}(w_m(\lambda_l(\mathfrak{b}))^{\delta'}).$$

(We assume $\mathcal{O}_{(G^\circ/P)_{\text{red}}}(\eta^\varkappa)$ being defined in the same manner as $\mathcal{O}_{(G^\circ/P)}(\eta^\varkappa)$. However, since there are no odd reflections and inclusions between Borel subgroups of G_{red}, we have simply

$$\mathcal{O}_{(G^\circ/P)_{\text{red}}}(\eta^\varkappa) := \widetilde{w_m(\eta)}^\varkappa, \quad \text{rk}\ \mathcal{O}_{(G^\circ/P)_{\text{red}}}(\eta^\varkappa) = \varepsilon^\varkappa$$

for any G considered). Furthermore one shows easily that for each k, $k = 0, \ldots, (\text{rk}\ N^*_{(G^\circ/P_{\text{red}})/(G^\circ/P)})/\varepsilon$, the pull-back

$$\pi^*_{\text{red}}(S^k(N^*_{(G^\circ/P)_{\text{red}}/(G^\circ/P)}) \otimes (\mathcal{O}_{G^\circ/P}(\lambda^\delta))_{\text{red}})$$

admits a filtration with composition factors

$$\mathcal{O}_{(G^\circ/B)_{\text{red}}}\left(w_m(\lambda_l(\mathfrak{b}) - \sum_{\gamma_i \in I} \gamma_i)^{\delta' + (k) \bmod 2}\right),$$

$I = (\gamma_1, \ldots, \gamma_k)$ being an arbitrary k-tuple of different elements of $\Delta_{\bar{1}}^-(\mathfrak{b}) \backslash (\Delta_{\bar{1}}^-(\mathfrak{b}) \cap \tilde{\Delta}_1(\mathfrak{p}))$ (each composition factor having multiplicity $\dim \lambda_l(\mathfrak{b})^{\delta'}$). But it is a consequence of Bott's theorem, [2], that the cohomology of any $\mathcal{O}_{(G^\circ/P)_{\text{red}}}$-module $\mathcal{L}'$ coincides with the cohomology of $\pi^*_{\text{red}}\mathcal{L}'$, and the cohomology of the line bundles $\mathcal{O}_{(G^\circ/B)_{\text{red}}}\left(w_m(\lambda_l(\mathfrak{b}) - \sum_{\gamma_i \in I} \gamma_i)^{\delta' + (k) \bmod 2}\right)$ is well known. All this enables us

(²) If X is a supermanifold and $i_X : X_{\text{red}} \hookrightarrow X$ is the canonical closed immersion, $N^*_{X_{\text{red}}/X}$ denotes the conormal bundle of X_{red} in X, and $F_{\text{red}} := i_X^* F$ for any $\mathcal{O}_X$-module F.

to establish some first basic facts about the cohomology of $\mathcal{O}_{G^\circ/P}(\lambda^\delta)$. Denote by Ech $\mathcal{L}$ the Euler characteristic of formal characters of cohomologies of a $\mathfrak{g}$-linearized $\mathcal{O}_{G^\circ/P}$-module $\mathcal{L}$. (Note that Ech $\mathcal{L}$ depends only on the $\mathfrak{g}_0$-structure on $\mathcal{L}$, i.e. Ech $\mathcal{L}$ is well defined actually for any $\mathfrak{g}_0$-linearized $\mathcal{O}_{G^\circ/P}$-module $\mathcal{L}$.) Combining the above considerations with Bott's theorem and Frobenius duality (see [13], Chapter 4) one proves

PROPOSITION 1. — *For any* $\lambda \in \mathfrak{h}_\mathfrak{p}^*$

a) $\Gamma(\mathcal{O}_{G^\circ/P}(\lambda^\delta))$ *is an indecomposable* $\mathfrak{g}$-*module;*

b) *the conditions* $\Gamma(\mathcal{O}_{G^\circ/P}(\lambda^\delta)) \neq 0$ *and* $\dim V_\mathfrak{b}(\lambda^\delta) < \infty$ *for* $\mathfrak{b} \subset \mathfrak{p}$ *are equivalent, and moreover if they are satisfied, one has a canonical injection of* $\mathfrak{g}$-*modules*

$$\Theta_\lambda : V_\mathfrak{b}(\lambda^\delta) \hookrightarrow \Gamma(\mathcal{O}_{G^\circ/P}(\lambda^\delta));$$

c) *one has*

$$(5)\quad \text{Ech}\, \mathcal{O}_{G^\circ/P}(\lambda^\delta) = \frac{\dim \lambda_l(\mathfrak{b})^{\delta'}}{\sum_{w' \in W_0} \text{sgn}\, w' e^{w'(-\rho_0)}} \cdot \left(\sum_{w' \in W_0} \text{sgn}\, w'.w'(e^{\lambda_l(\mathfrak{b})-\rho_0} \prod_{\gamma \in \Delta_{\bar{1}}^-(\mathfrak{b}) \setminus (\Delta_{\bar{1}}^-(\mathfrak{b}) \cap \tilde{\Delta}_1(\mathfrak{p}))} (1+\varepsilon e^{-\gamma})) \right).$$

d) *For a generic* λ *of* $\mathfrak{h}_\mathfrak{p}^*$, *i.e. if* λ *belongs to a certain Zariski-open subset of* $\mathfrak{h}_\mathfrak{p}^*$,

$$H^i(\mathcal{O}_{G^\circ/P}(\lambda^\delta)) = 0 \qquad \text{for} \qquad i \neq l(w),$$

where $w \in W_0$ *is defined by the requirement that* $w(\lambda_l(\mathfrak{b}) - \rho_0)$ *is antidominant, i.e.* $-w(\lambda_l(\mathfrak{b}) - \rho_0) \in \bar{C}^+$. *(For a generic* λ *of* $\mathfrak{h}_\mathfrak{p}^*$ *this determines* w *uniquely.)* □

The first essential fact of « super Borel-Weil-Bott theory » is

THEOREM 3. — *Let* $B \hookrightarrow G$ *be a Borel subsupergroup, such that* $\check{\Delta} \subset \Delta(\mathfrak{p})$ *for* $G = \mathbf{P}$. *If* $\lambda + \rho_\mathfrak{p}$ *is typical, and in case* $G = \mathbf{P}$ *also regular, see* 2.2, *then*

$$H^i(\mathcal{O}_{G^\circ/B}(\lambda^\delta)) = 0 \qquad \text{for} \qquad i \neq l(w),$$

where $-\ w(\lambda_l(\mathfrak{b})-\rho_{\mathfrak{p}})\in C^+$, *and one has*

$$H^{l(w)}(\mathcal{O}_{G^o/B}(\lambda^\delta)) = V_{\mathfrak{b}^{op}}((w(\lambda_l(\mathfrak{b})-\rho_{\mathfrak{b}})+\rho_{\mathfrak{b}})^{\delta(w)}) = V_{\mathfrak{b}}((w'(\lambda+\rho_{\mathfrak{b}})-\rho_{\mathfrak{b}})^{\delta(w')}),$$

where by definition $w'(\lambda+\rho_{\mathfrak{b}})\in C^+$ *and* $\delta(w)$, $\delta(w')\in\mathbb{Z}_2$.

The theorem is proved in detail in [13], (and in [12] ([3])), where one can find also a generalization of the result for an arbitrary Borel subsupergroup of $\mathbf{P}$. □

Let us complete this section by a remark. First of all (using Corollary 1) one shows straightforwardly that for a generic λ of $\bar{C}^+_{\mathfrak{p}}$ and any $\mathfrak{b}\subset\mathfrak{p}$ $\lambda_l(\mathfrak{b}) = w_m(\lambda+(\rho_{\mathfrak{p}})_1) + (\rho_{\mathfrak{p}})_1$. Second, one checks that setting

$$\Delta_1(\mathfrak{b})_{\mathfrak{p}} := \{\gamma\in\Delta_{\bar{1}}^-(\mathfrak{b})\,|\,\gamma\notin w_m(\tilde{\Delta}(\mathfrak{p}))\}$$

and

$$\lambda_l(\mathfrak{b}) = w_m(\lambda+(\rho_{\mathfrak{p}})_1)+(\rho_{\mathfrak{p}})_1,$$

we have

$$(6)\quad \frac{\dim\lambda_l(\mathfrak{b})^{\delta'}}{\sum_{w'\in W_0}\operatorname{sgn} w' e^{w'(-\rho_0)}} \cdot\left(\sum_{w'\in W_0}\operatorname{sgn} w'.w'(e^{\lambda_l(\mathfrak{b})-\rho_0}\prod_{\gamma\in\Delta_{\bar{1}}^-(\mathfrak{b})\setminus(\Delta_{\bar{1}}^-(\mathfrak{b})\cap\tilde{\Delta}_1(\mathfrak{p})\cap\check{\Delta})}(1+\varepsilon e^{-\gamma}))\right)$$

$$= \frac{\dim\lambda^\delta}{\sum_{w'\in W_0}\operatorname{sgn} w' e^{w'(\rho_0)}}\left(\sum_{w'\in W_0}\operatorname{sgn} w'.w'(e^{\lambda+\rho_0}.\prod_{\gamma\in\Delta_1(\mathfrak{b})_{\mathfrak{p}}\setminus(\Delta_1(\mathfrak{b})_{\mathfrak{p}}\cap\check{\Delta})}(1+\varepsilon e^{\gamma}))\right).$$

Therefore Proposition 1 implies that for $\mathfrak{g}\neq\mathbf{p}$ (and again for a generic λ of $\bar{C}^+_{\mathfrak{p}}$) $\operatorname{ch} V_{\mathfrak{b}}(\lambda^\delta) = (\operatorname{ch}\operatorname{im}\Theta_\lambda) \leqslant$ expression (6), because in this case $\check{\Delta} = \varnothing$ and (6) is nothing but the right hand side of (5). It turns out however that the latter inequality is valid also for $\mathfrak{g} = \mathbf{p}$, and in the subsequent section we prove that for a generic $\lambda\in\bar{C}^+_{\mathfrak{p}}$ this is an equality.

([3]) Unfortunately the result of [12] which concerns the case $G = \mathbf{P}$ needs a modification; compare Theorem 2 in [12] with Theorem 4 below.

2. The result: cohomology of $\mathcal{O}_{G^\circ/P}(\lambda^\delta)$ for a generic λ of $\mathfrak{h}_\mathfrak{p}^*$ and a character formula.

Fix now a Borel subgroup B_{red} of G_{red}. Then for any parabolic subsupergroup P of G with $P_{\mathrm{red}} \hookrightarrow B_{\mathrm{red}}$ and for any $w' \in W$ $\mathrm{P}_{w'}$ will denote a parabolic subsuperogroup of G, such that $\mathfrak{p}_{w'} = \mathrm{Lie}\, P_{w'}$ is a minimal parabolic subsuperalgebra of $\mathfrak{g}$ with

$$\mathrm{Lie}\, B_{\mathrm{red}} \subset \mathfrak{p}_{w'}, \qquad \Delta(\mathfrak{p}_{w'}) \supset \{w'(\alpha) \mid \alpha \in \Delta(\mathfrak{p}),\ -\alpha \in \Delta(\mathfrak{p})\}.$$

($P_{w'}$ is not necessarily uniquely defined; for instance for $P = B$ $B_{w'}$ is an arbitrary Borel subsupergroup containing B_{red}). The main result of our paper is

THEOREM 4. — *Assume the pair $B \hookrightarrow P$ being fixed. If λ is a generic element of $\mathfrak{h}_\mathfrak{p}^*$, $\mu_l := w(\lambda_l(\mathfrak{b}) - \rho_\mathfrak{p}) + \rho_{\mathfrak{p}_w}$ (for some $\mathfrak{p}_w$), where $-w(\lambda_l(\mathfrak{b}) - \rho_0) \in \bar{C}^+$, and $\mathfrak{b}_w$ is a Borel subsuperalgebra of $\mathfrak{p}_w$ with $\mathfrak{h} \subset \mathfrak{b}_w$, then the single nonzero cohomology group $H^{l(w)}(\mathcal{O}_{G^\circ/P}(\lambda^\delta))$ of $(\mathcal{O}_{G^\circ/P}(\lambda^\delta)$ (4);*

— *is isomorphic to $V_{\mathfrak{b}_w}(\mu^\varkappa) \simeq V_{\mathfrak{b}_w^{op}}(\mu_l^{\varkappa'})$, where by definition $\mu_l(\mathfrak{b}_w) = \mu_l$ and $\varkappa$, $\varkappa' \in Z_2$ if $G = \mathrm{GL}$, OSP, Q; in the special case $G = Q$ a sufficient condition for this to be satisfied is $(\lambda,\alpha)_{\mathrm{red}} \neq 0 \neq (\lambda,\underline{\alpha})$ $\forall \alpha \in \Delta_{\bar{1}}^-(\mathfrak{b})_\mathfrak{p}$;*

— *admits a $\mathfrak{g}$-filtration with composition factors $V_{\mathfrak{b}_w^{op}}\big(\mu_l + \sum_i 2l_{k_i} w(\varepsilon_{k_i})\big)^{\varkappa' + \left(\sum_i l_{k_i}\right) \bmod 2}$ for a certain $\varkappa' \in \mathbb{Z}_2$, and for all possible $l_{k_i} \in \{0,1\}$, where $-2\varepsilon_{k_i}$ runs over the set $(\check{\Delta} \cap \Delta_{\bar{1}}^-(\mathfrak{b})) \setminus (\check{\Delta} \cap \Delta_{\bar{1}}^-(\mathfrak{b}) \cap \tilde{\Delta}_1(\mathfrak{p}))$ if $G = \mathbf{P}$. In particular for $\check{\Delta} \subset \underline{\Delta}(\mathfrak{p})$ $H^{l(w)}(\mathcal{O}_{G^\circ/P}(\lambda^\delta)) = V_{\mathfrak{b}_w^{op}}(\mu_l^{\varkappa'}) = V_{\mathfrak{b}_w}(\mu^\varkappa)$, where as in* a) $\mu_l(\mathfrak{b}_w) = \mu_l$).

Although we have given a separate formulation of Theorem 2 for $G = \mathbf{P}$, it is clear that the claim for the case $G = \mathbf{P}$ is simply more general than the claim for $G = \mathrm{GL}$, OSP, Q; indeed replacing in the claim for $\mathbf{P}$ $\mathfrak{g}$ by gl, osp, or q, we obtain the corresponding statement just because $\check{\Delta} = \varnothing$ in each of these cases. Unfortunately we are not able to give a unified proof of both claims. The reason is that the technique of central characters of $U(\mathfrak{g})$-modules is well developed still only for $\mathfrak{g} \neq \mathbf{p}$. But since, as we shall see below, $\mathbf{p}$ is in some sense simpler than osp or q, we are able to prove Theorem 2 for $\mathfrak{g} = \mathbf{p}$ not

(4) For simplicity we assume already that $H^i(\mathcal{O}_{G^\circ/P}(\lambda^\delta)) = 0$ $\forall i \neq l(w)$ by Proposition 1, d).

using a description of the centre of $U(\mathfrak{g})$ and of its characters. Therefore our plan of proof is the following: first, using the results of 1.2, we will establish Theorem 2 for $G \neq \mathbf{P}$, and then we will present a longer but very explicit proof for $G = \mathbf{P}$.

Proof of Theorem 2. A: $G = \mathrm{GL}$, OSP, Q.

*Step A*1. Assume $w = \mathrm{id}$. Then $\mathfrak{b}_w = \mathfrak{b}$ and one has the injection $\Theta_\lambda : V_{\mathfrak{b}}(\lambda^\delta) \hookrightarrow \Gamma(\mathcal{O}_{G^\circ/P}(\lambda^\delta))$. We claim now that in this case Θ_λ is an isomorphism for a generic λ of $\mathfrak{h}_{\mathfrak{p}}^*$, i.e. for λ belonging to a Zariski-open set in $\mathfrak{h}_{\mathfrak{p}}^*$. Indeed, if $V_{\mathfrak{b}}(\zeta^\omega) \hookrightarrow \operatorname{coker}\Theta_\lambda$, then

$$(7) \qquad \zeta_l(\mathfrak{b}) = \lambda_l(\mathfrak{b}) - \sum_{\gamma_i \in \Delta_{\bar{1}}^-(\mathfrak{b}) \setminus (\Delta_{\bar{1}}^-(\mathfrak{b}) \cap \tilde{\Delta}_1(\mathfrak{p}))} \varkappa_i \gamma_i\,(^5), \qquad \varkappa_i \in \{0,1\},$$

because the considerations of 1.3 imply that $\Gamma(\mathcal{O}_{G^\circ/P}(\lambda^\delta))$ has a $\mathfrak{g}_0$-module filtration with irreducible composition factors of the form

$$\Gamma(\mathcal{O}_{(G^\circ/B)_{\mathrm{red}}}((w_m(\lambda_l(\mathfrak{b}) - \sum_{\gamma_i \in I} \gamma_i))^{\delta' + (\# I) \bmod 2})) = V_{\mathfrak{b}_0^{\mathrm{op}}}((\lambda_l(\mathfrak{b}) - \sum_{\gamma_i \in I} \gamma_i)^{\delta' + (\# I) \bmod 2})$$

(the latter equality follows from Bott's theorem, [2]), and therefore (7) is satisfied for the weight of any $\mathfrak{b}^{\mathrm{op}}$-singular vector in $\Gamma(\mathcal{O}_{G^\circ/P}(\lambda^\delta))$. Moreover by Proposition 1, a) $\Gamma(\mathcal{O}_{G^\circ/P}(\lambda^\delta))$ is indecomposable, the central characters of all its irreducible composition factors coincide, 1.2, and thus

$$(8) \qquad \theta^{\lambda + \rho_{\mathfrak{b}}} = \theta^{\zeta + \rho_{\mathfrak{b}}}.$$

Now we will show that, for a generic λ of $\mathfrak{h}_{\mathfrak{p}}^*$, (8) together with (7) implies $\zeta = \lambda$, i.e. $\operatorname{coker}\Theta_\lambda = 0$. A precise assumption on λ, under which this is true, is

$$(9) \qquad (\lambda + \rho_{\mathfrak{b}}, \alpha) \neq 0 \quad \forall \alpha \in \Delta_1(\mathfrak{b})_{\mathfrak{p}} \cap \hat{\Delta} \quad \text{in case } G = \mathrm{GL},\ \mathrm{OSP};$$
$$(\lambda, \underline{\alpha}) \neq 0 \quad \forall \alpha \in \Delta_1(\mathfrak{b})_{\mathfrak{p}} \quad \text{in case } G = Q.$$

Note first that $\theta^{\lambda + \rho_{\mathfrak{b}}} = \theta^{\lambda_l(\mathfrak{b}) + \rho_{\mathfrak{b}^{\mathrm{op}}}}$ because $\lambda_l(\mathfrak{b})$ is the lowest weight of $V_{\mathfrak{b}}(\lambda^\delta)$, i.e. $V_{\mathfrak{b}}(\lambda^\delta) = V_{\mathfrak{b}^{\mathrm{op}}}(\lambda_l(\mathfrak{b})^{\delta'})$. Since $\lambda_l(\mathfrak{b})^{\delta'}$ is a $\mathfrak{p}$-module, $\theta^{\lambda_l(\mathfrak{b}) + \rho_{\mathfrak{b}^{\mathrm{op}}}}$ factors through the natural surjection $Z \to Z_{U(\mathfrak{g}_{\mathfrak{p}})}$, where $\mathfrak{g}_{\mathfrak{p}}$ is the

(5) In the cases considered $\Delta^-(\mathfrak{b}) = -\Delta^+(\mathfrak{b})$ for any $\mathfrak{b}$, thus in (7) one can replace all three minus signs by plus signs.

classical Lie superalgebra, defined by the requirements:

(i) $\mathfrak{h}(\mathfrak{g}_\mathfrak{p})^* \simeq \mathfrak{h}^*/\{$linear envelope of $\tilde{\Delta}(\mathfrak{p})$ for $\mathfrak{g} = \mathfrak{gl}$, $\mathfrak{osp}$ and of $\tilde{\Delta}(\mathfrak{p}) \oplus \{\underline{\alpha} \mid \alpha \in \tilde{\Delta}(\mathfrak{p})\}$ for $\mathfrak{g} = \mathfrak{q}\}$;

(ii) $\Delta(\mathfrak{g}_\mathfrak{p})$ is the image of Δ in $\mathfrak{h}(\mathfrak{g}_\mathfrak{p})_{\bar{0}}^*$.

Denote now by η' the image of $\eta \in \mathfrak{h}_{\bar{0}}^*$ in $\mathfrak{h}(\mathfrak{g}_\mathfrak{p})_{\bar{0}}^*$. Then (9) implies that the characters of $Z_{U(\mathfrak{g}_\mathfrak{p})}$ $\theta^{(\lambda_l(\mathfrak{b})+\rho_{\mathfrak{b}^{op}})'}$ and $\theta^{(\zeta_l(\mathfrak{b})+\rho_{\mathfrak{b}^{op}})'}$ coincide. Next we note that (10) is equivalent to

$$(10) \quad (\lambda_l(\mathfrak{b})+\rho_{\mathfrak{b}^{op}},\alpha) \neq 0 \quad \forall\alpha \in \tilde{\Delta}_1(\mathfrak{p}) \cap \hat{\Delta} \text{ in case } G = GL,\ OSP,$$
$$(\lambda,\underline{\alpha}) \neq 0 \ \forall\alpha \in \tilde{\Delta}_1(\mathfrak{p}) \text{ in case } G = Q.$$

But (10) implies now that $\theta^{(\lambda_l(\mathfrak{b})+\rho_{\mathfrak{b}^{op}})'}$ is a typical character of $Z_{U(\mathfrak{g}_\mathfrak{p})}$, i.e. that $\theta^{(\lambda_l(\mathfrak{b})+\rho_{\mathfrak{b}^{op}})'} \circ HC_{\mathfrak{g}_\mathfrak{p}}$ admits a unique extension to a character of $S^\cdot(\mathfrak{h}(\mathfrak{g}_\mathfrak{p})^{W_0(\mathfrak{g}_\mathfrak{p})})$. Therefore applying Lemma 2 in 1.2 to $\mathfrak{g}_\mathfrak{p}$, we obtain

$$(11) \qquad \lambda_l(\mathfrak{b})' = \zeta_l(\mathfrak{b})',$$

since $(\lambda_l(\mathfrak{b})+\rho_{\mathfrak{b}^{op}})'$ and $(\zeta_l(\mathfrak{b})+\rho_{\mathfrak{b}^{op}})'$ are both dominant with respect to the image of $\mathfrak{b}_{\bar{0}}^{op}$ in $\mathfrak{g}_\mathfrak{p}$. Furthermore (11) and (7) give immediately $\lambda_l(\mathfrak{b}) = \zeta_l(\mathfrak{b})$, or equivalently $\lambda = \zeta$, which is a contradiction. In this way coker $\Theta_\lambda = 0$ and Θ_λ is an isomorphism in the situation considered.

Step A2. Let now λ be a generic element of $\mathfrak{h}_\mathfrak{p}^*$. Then if $-w(\lambda_l(\mathfrak{b})-\rho_0) \in \bar{C}^+$, $H^{l(w)}(\mathcal{O}_{G^\circ/P}(\lambda^\delta))$ is the unique nonzero cohomology group of $\mathcal{O}_{G^\circ/P}(\lambda^\delta)$ and

$$(12) \quad (-1)^{l(w)} \operatorname{Ech} \mathcal{O}_{G^\circ/P} = \operatorname{ch} H^{l(w)}(\mathcal{O}_{G^\circ/P}(\lambda^\delta))$$
$$= (-1)^{l(w)} \cdot \frac{\dim \lambda_l(\mathfrak{b})^{\delta'}}{\sum_{w' \in W_0} \operatorname{sgn} w'\, e^{w'(-\rho_0)}} \cdot \Bigg(\sum_{w' \in W_0} \operatorname{sgn} w' .\, w'\Big(e^{\lambda_l(\mathfrak{b})-\rho_0} \prod_{\gamma \in \Delta_{\bar{1}}^-(\mathfrak{b}) \setminus (\Delta_{\bar{1}}^-(\mathfrak{b}) \cap \tilde{\Delta}_1(\mathfrak{p}))} (1+\varepsilon e^{-\gamma})\Big)\Bigg).$$

The essential point is to note now that $\mu \in \bar{C}^+_{\mathfrak{p}_w}$ (where $\mu_l(\mathfrak{b}_w) = \mu_l$) and $\mathcal{O}_{G^\circ/P_w}(\mu^\times)$ is a $\mathfrak{g}$-linearized sheaf on G°/P_w with

$$H^i(\mathcal{O}_{G^\circ/P_w}(\mu^\times)) = 0, \qquad \forall i > 0$$

for a generic λ. Moreover by Step 1 we have

$$(13) \qquad \Gamma(\mathcal{O}_{G^\circ/P_w}(\mu^\times)) = V_{\mathfrak{b}_w}(\mu^\times) = V_{\mathfrak{b}_w^{op}}(\mu_l^{\times\prime}).$$

This observation proves the claim of the Theorem since one checks immediately that in the case considered

$$\operatorname{ch} H^{l(w)}(\mathcal{O}_{G^\circ/P}(\lambda^\delta)) = \operatorname{ch} \Gamma(\mathcal{O}_{G^\circ/P_w}(\mu^\varkappa))$$

for a certain $\varkappa \in \mathbb{Z}_2$, and an irreducible $\mathfrak{g}$-module is determined up to isomorphism (or up to Π for $G=Q$) by its formal character. It remains to note only that for $G = Q$ the condition $(\lambda,\alpha)_{\text{red}} \neq 0$ for $\lambda \in \mathfrak{h}_{\mathfrak{p}}^*$, $\alpha \in \Delta_1(\mathfrak{b})_{\mathfrak{p}}$, implies $H^j(\mathcal{O}_{G^\circ/P}(\lambda^\delta)) = 0$ $\forall j \neq l(w)$ and moreover also $H^i(\mathcal{O}_{G^\circ/Pw}(\mu^\varkappa)) = 0$, $\forall i > 0$. This can be checked straighforwardly and we leave the details to the reader. Since furthermore condition (9) is sufficient for establishing (13), our claim is proved also for $G = Q$.

2.2. *The case* $G = \mathbf{P}$.

Before turning to the proof for the remaining series $G = \mathbf{P}$, we will study in more detail the structure of $\mathfrak{g} = \mathbf{p}$. The crucial property needed below is that $\mathfrak{g}$ admits a $\mathbb{Z}$-gradation of the form $\mathfrak{g} = \mathfrak{g}_{(-1)} \oplus \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}$, where $\mathfrak{g}_{(0)} = \mathfrak{g}_0$ and $[\mathfrak{g}_{(1)}, \mathfrak{g}_{(1)}] = [\mathfrak{g}_{(-1)}, \mathfrak{g}_{(-1)}] = 0$. Indeed, in matrix form $\mathfrak{g} = \mathbf{p}(m)$ can be represented as the set of $(m+m) \times (m+m)$ block matrices

$$\Lambda = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

with (usual supercommutator, see [3], [5], or [13], [13′], and) $D = -A^t$, $B = -B^t$, $C = C^t$, and setting $\mathfrak{g}_{(1)} = \{\Lambda \in \mathfrak{g} \mid A=D=B=0\}$, $\mathfrak{g}_{(-1)} = \{\Lambda \in \mathfrak{g} \mid A=D=C=0\}$, one checks immediately that we obtain a $\mathbb{Z}$-gradation of $\mathfrak{g}$ with the desired properties. (A $\mathbb{Z}$-gradation with the same properties admit also $\tilde{\mathfrak{g}} = \mathrm{gl}(m+n\varepsilon)$ and $\tilde{\mathfrak{g}} = \mathrm{osp}(2+2k\varepsilon)$. Even more in the latter cases the situation is simpler because $\dim \tilde{\mathfrak{g}}_{(-1)} = \dim \tilde{\mathfrak{g}}_{(1)}$, while for the above introduced $\mathbb{Z}$-gradation of $\mathfrak{g} = \mathbf{p}(m)$ $\dim \mathfrak{g}_{(1)} - \dim \mathfrak{g}_{(-1)} = m \cdot \varepsilon$. (The effect is due of course to the fact that $\check{\Delta} = \varnothing$ for $\mathrm{gl}(m+n\varepsilon)$, $\mathrm{osp}(2+2k\varepsilon)$, but $\check{\Delta} \neq \varnothing$ for $\mathfrak{g} = \mathbf{p}$). The approach applied below to $\mathfrak{g} = \mathbf{p}$ can be applied also in a very similar and essentially simpler manner also to $\mathrm{gl}(m+n\varepsilon)$, $\mathrm{osp}(2+2k\varepsilon)$. This leads to an alternative proof of Theorem 2 for $\mathrm{gl}(m+n\varepsilon)$ and $\mathrm{osp}(2+2k\varepsilon)$, which is more direct in the sense that it does not use a description of the centre of the enveloping algebra).

Choosing furthermore $\mathfrak{h} = \mathfrak{h}_0 \subset \mathfrak{g}$ to be the sub(super)algebra of diagonal matrices, introducing standard coordinates $\lambda_{(i)}$ in $\mathfrak{h}^*$ (as in [13], [13′]), and setting

$$\varepsilon_i := (0,\dots,0,1,0,\dots,0)\in\mathfrak{h}^*, \qquad i=1,\dots,m,$$

we have

$$\mathfrak{h}^* = \left\{\sum_i \lambda_{(i)}\varepsilon_i \mid \lambda_{(i)}\in\mathbb{C}\right\},$$

$$\Delta = \{\varepsilon_i-\varepsilon_j,\ \pm(\varepsilon_i+\varepsilon_j), i\neq j, -2\varepsilon_i\},$$

$$\check{\Delta} = \{-2\varepsilon_i\},\ \hat{\Delta} = \Delta\backslash\check{\Delta}.$$

The conditions

$$\mathfrak{b}_d \supset \mathfrak{h}, \qquad \Delta(\mathfrak{b}_d) = \{\varepsilon_i-\varepsilon_j \text{ for } i<j, -\varepsilon_i-\varepsilon_j\},$$
$$\mathfrak{b}_{ad} \supset \mathfrak{h}, \qquad \Delta(\mathfrak{b}_{ad}) = \{\varepsilon_i-\varepsilon_j, \text{ for } i<j, \varepsilon_i+\varepsilon_j \text{ for } i\neq j\}$$

determine uniquely respectively the *distinguished* and *antidistinguished Borel subsuperalgebra* containing $\mathfrak{h}$. (The supergroups B_d and B_{ad} are defined setting Lie $B_d := \mathfrak{b}_d$, Lie $B_{ad} := \mathfrak{b}_{ad}$). The simple roots of $\mathfrak{b}_d$ and $\mathfrak{b}_{ad}$ are respectively:

$$\mathfrak{b}_d: \{-2\varepsilon_1, \varepsilon_1-\varepsilon_2,\dots,\varepsilon_{m-1}-\varepsilon_m\},$$
$$\mathfrak{b}_{ad}: \{\varepsilon_1-\varepsilon_2,\dots,\varepsilon_{m-1}-\varepsilon_m, \varepsilon_{m-1}+\varepsilon_m\};$$

furthermore we put for short $\rho_d := \rho_{\mathfrak{b}_d}$, $\rho_{ad} := \rho_{\mathfrak{b}_{ad}}$.

Set now $\mathfrak{b} := \mathfrak{b}_d^{\mathrm{op}}$, $\mathfrak{b}' := \mathfrak{b}_{ad}^{\mathrm{op}}$ and fix sequence (2) from 1.1 to be the sequence

$$-2\varepsilon_1,\ -\varepsilon_1-\varepsilon_2,\ -2\varepsilon_2,\ -\varepsilon_1-\varepsilon_3,\ -\varepsilon_2-\varepsilon_3,\ -2\varepsilon_3,\ \dots,\ \varepsilon_{m-1}-\varepsilon_m,\ -2\varepsilon_m$$

(the roots $-\varepsilon_i-\varepsilon_j$, $i\neq j$, correspond to odd reflections and the roots $-2\varepsilon_i$ correspond to inclusions). Let furthermore

$$(14) \qquad \alpha_1, \alpha_2, \dots, \alpha_l, \qquad l = k-1-\#\check{\Delta}$$

be the subsequence of (2) (in the concrete case when $\mathfrak{b}=\mathfrak{b}_d^{\mathrm{op}}$ and $\mathfrak{b}'=\mathfrak{b}_{ad}^{\mathrm{op}}$) consisting of elements of $\Delta_1^+(\mathfrak{b}_d)\backslash\check{\Delta}$.

If $\chi\in C^+$, we define the subsequence of $(\chi, \mathfrak{b}_d^{\mathrm{op}})$-*marked* roots of (14) as the maximal subsequence $\beta_1,\dots,\beta_r$ of (14), for which

$$(\chi-\beta_1-\dots-\beta_{s-1})_{(i_s)} = (\chi-\beta_1-\dots-\beta_{s-1})_{(j_s)} + 1 \text{ (}^6\text{)}$$

(6) For any weight $\eta\in\mathfrak{h}^*$ $\eta_{(t)}$ denotes here and below its t-th standard coordinate.

for any $s = 1, \ldots, r$, where $\beta_s = -\varepsilon_{i} - \varepsilon_{j_s}$, $i_s > j_s$ (and $\beta_0 := 0$). Denote by M_χ the set $\{\beta_1, \ldots, \beta_r\}$, by $\beta_1^\circ, \ldots, \beta_p^\circ$ – the subsequence of (14) consisting of non – (χ, β_d^{op})-marked roots $(r+p=l)$, and by A_χ – the set $\{\beta_1^\circ, \ldots, \beta_p^\circ\}$.

More generally, if $\mathfrak{b} \subset \mathfrak{g} = \mathbf{p}(m)$ is a Borel subsuperalgebra with $\mathfrak{b}_0 = (\mathfrak{b}_d^{op})_0$, and $\chi \in \bar{C}^+$, we will say that $\beta \in \hat{\Delta}_1^-(\mathfrak{b}) \cap \hat{\Delta}$ is *$(\chi, \mathfrak{b})$-marked* iff there exists a sequence $\beta_1, \ldots, \beta_t$, $\beta_i \in \Delta_1^-(\beta) \cap \check{\Delta}$, with $\beta = \beta_t$, and such that for any $l = 0, \ldots, t-1$

$$\left(\chi - \sum_1^l \beta_i\right)_{(k_{l+1})} = \left(\chi - \sum_1^l \beta_i\right)_{(j_{l+1})} + 1,$$

where ($\beta_0 = 0$ and) $\beta_{l+1} = \pm(\varepsilon_{k_{l+1}} + \varepsilon_{j_{l+1}})$, $k_{l+1} > j_{l+1}$. The set of $(\chi, \mathfrak{b})$-marked roots (in this more general sense) will be denoted by $\mathbb{M}_\chi^{\mathfrak{b}}$ and the set $(\Delta_1^-(\mathfrak{b}) \cap \hat{\Delta}) \backslash \mathbb{M}_\chi^{\mathfrak{b}}$-by $\mathbb{A}_\chi^{\mathfrak{b}}$.

In the special case when $\chi \in C^+$ and $\mathfrak{b} = \mathfrak{b}_d$ or $\mathfrak{b} = \mathfrak{b}_{ad}$ one can compute all $(\chi, \mathfrak{b})$-marked roots using only a fixed sequence of roots. Indeed one proves straightforwardly (by induction on m)

LEMMA 3. – *For any $\chi \in C^+$*

$$\mathbb{M}_\chi^{\mathfrak{b}} = \begin{cases} M_\chi & \text{for} \quad \mathfrak{b} = \mathfrak{b}_d^{op} \\ M'_\chi & \text{for} \quad \mathfrak{b} = \mathfrak{b}_{ad}^{op}, \end{cases}$$

where one defines M'_χ in the same way as M_χ, replacing (14) *by the sequence*

(14') $\quad \alpha'_1 := -\alpha_l, \alpha'_2 := -\alpha_{l-1}, \ldots, \alpha'_l := -\alpha_1.$ □

$(\chi, \mathfrak{b})$-marked roots are closely related to the procedure of Corollary 1. In particular one checks immediately (and this will be essential below) that if $\mathfrak{b} = \mathfrak{b}_d^{op}$ and $\mathfrak{b}' = \mathfrak{b}_{ad}^{op}$, then Corollary 1 claims nothing but the existence of an isomorphism $V_{\mathfrak{b}}(\xi^\varkappa) \simeq V_{\mathfrak{b}'}\left(\left(\xi + \sum_{\eta \in A_{\xi-\rho_d}}^{n} \eta\right)^{\varkappa + (\# A_{\xi} - \rho_d) \bmod 2}\right)$ for any ξ with $\dim V_{\mathfrak{b}}(\xi^\varkappa) < \infty$.

It remains to define the notions of α-regularity and typicality for the case $\mathfrak{g} = \mathbf{p}$: a weight χ of $\mathbf{p}$ is called:

– *α-regular* for $\alpha \in \hat{\Delta}_1 = \Delta_1 \backslash \check{\Delta}$ iff

$$\chi_{(s)} \neq \chi_{(s')}, \quad \text{where} \quad \alpha = \pm(\varepsilon_s + \varepsilon_{s'});$$

— *typical* iff

$$\chi_{(s)} \neq \chi_{(s')} + 1 \quad \forall s, s', 1 \leqslant s, s' \leqslant m;$$

(for $\chi \in C^+$ typicality is equivalent to the condition $\mathbb{M}^{\mathfrak{b}}_{\chi} = \varnothing$);

— *regular* iff

$$\chi_{(s)} \neq \chi_{(s')}, \ \chi_{(s)} \neq \chi_{(s')} + 1, \ \chi_{(s)} \neq \chi_{(s')} + 2, \ \chi_{(s)} \neq \chi_{(s')} + 3 \qquad \forall s, s', 1 \leqslant s, s' \leqslant m.$$

Now we are able to continue the

Proof of Theorem 2. B: $G = \mathbf{P}$. — Without loss of generality we can assume in the rest of the proof that $B_{\text{red}} = (B_d)_{\text{red}}$. Furthermore if $\mathfrak{b} = \operatorname{Lie} B$ and $\dim V_{\mathfrak{b}^{\text{op}}}(\mu^{\varkappa''}) < \infty$ for some $\mu \in \mathfrak{h}_0^*$, we define the weights μ_d and μ_{ad} from the isomorphisms $V_{\mathfrak{b}^{\text{op}}}(\mu^{\varkappa}) \simeq V_{\mathfrak{b}_d^{\text{op}}}(\mu_d^{\varkappa''}) \simeq V_{\mathfrak{b}_{ad}^{\text{op}}}(\mu_{ad}^{\varkappa'''})$.

Step $B1$: Let $w = \text{id}$ and $B = B_d(\mathfrak{b} = \mathfrak{b}_d)$. Using a different argument, we will show that Θ_λ is an isomorphism for a generic $\lambda \in \bar{C}_{\mathfrak{p}}^+$ also in this situation.

First of all one checks immediately that $\lambda \in \bar{C}_{\mathfrak{p}}^+$ implies here $-A_{\lambda_d - \rho_d} = -\mathbb{A}^{\mathfrak{b}^{\text{op}}}_{\lambda_l(\mathfrak{b}) + \rho_{\mathfrak{b}^{\text{op}}}} \subset \Delta_1^-(\mathfrak{b}) \backslash (\Delta_1^-(\mathfrak{b}) \cap \tilde{\Delta}_1(\mathfrak{p}) \cap \hat{\Delta})$, and for a generic λ of $\bar{C}_{\mathfrak{p}}^+$ we have obviously $-A_{\lambda_d - \rho_d} = \Delta_1^-(\mathfrak{b}) \backslash (\Delta_1^-(\mathfrak{b}) \cap \tilde{\Delta}_1(\mathfrak{p}) \cap \check{\Delta})$. In order to prove that Θ_λ is an isomorphism it suffices to show that each irreducible composition factor $V_{\mathfrak{b}_0^{\text{op}}}((\lambda_l(\mathfrak{b}) - \sum_{\lambda_i \in I} \lambda_i)^{\delta' + (\# I) \bmod 2})$ of $\Gamma(\mathcal{O}_{G/P}(\lambda^\delta))$ is a $\mathfrak{g}_0$-component of $\operatorname{im} \Theta_\lambda$, i.e. to establish the inequality

$$\text{(15)} \qquad \operatorname{ch} V_{\mathfrak{b}}(\lambda^\delta) \geqslant \operatorname{ch} \Gamma(\mathcal{O}_{G/P}(\lambda^\delta)) = \text{expression (6)}.$$

Assume now $V_{\mathfrak{b}}(\zeta^\omega) \hookrightarrow \operatorname{coker} \Theta_\lambda$. Then (7) is satisfied also in our case. Furthermore for a generic λ of $\bar{C}_{\mathfrak{p}}^+$ $-A_{\zeta_d - \rho_d} \subset \Delta_1^-(\mathfrak{b}) \backslash (\Delta_1^-(\mathfrak{b}) \cap \tilde{\Delta}_1(\mathfrak{p}) \cap \hat{\Delta})$ for any $\zeta \in \bar{C}^+$ satisfying (7), because for a fixed $\beta \in \hat{\Delta}_1$ the condition $\beta \notin -A_{\zeta_d - \rho_d}$ is an open condition on λ. But (as we have already noted) Corollary 1 gives now $\zeta_{ad} := \zeta_d + \sum_{\eta \in A_{\zeta_d - \rho_d}} \eta$, and $A_{\zeta_d - \rho_d} \supset A_{\lambda_d - \rho_d}$ together with (7) implies

$$\text{(16)} \qquad \zeta_{ad} \underset{\mathfrak{b}_{ad}}{\leqslant} \lambda_{ad},$$

which is impossible because the multiplicity of $e^{\lambda_{ad}}$ in $\operatorname{ch} \Gamma(\mathcal{O}_{G^\circ/P}(\lambda^\delta))$

is equal $\varepsilon^{\delta'''}$ and for each $\xi \in \operatorname{supp} \Gamma(\mathcal{O}_{G^\circ/P}(\lambda^\delta))$, $\xi \neq \lambda_{ad}$, one has

$$\xi \underset{\mathfrak{b}_{ad}}{>} \lambda_{ad}.$$

This contradiction implies coker $\Theta_\lambda = 0$, i.e. the claim of the Theorem in the case considered.

Step B2: Let now $w = \mathrm{id}$, but $B \hookrightarrow P$ be an arbitrary pair (with $B_{\mathrm{red}} = (B_d)_{\mathrm{red}}$). By Proposition 1, c)

$$(17)\quad \operatorname{ch} \Gamma(\mathcal{O}_{G/P}(\lambda^\delta)) = \frac{\varepsilon^{\delta'}}{\sum_{w' \in W_0} \operatorname{sgn} w' e^{w'(-\rho_0)}} \cdot \left(\sum_{w' \in W_0} \operatorname{sgn} w' . w'(e^{\lambda_l(\mathfrak{b}) - \rho_0} . \prod_{\gamma \in \Delta_{\bar 1}^-(\mathfrak{b}) \setminus (\Delta_{\bar 1}^-(\mathfrak{b}) \cap \tilde\Delta_1(\mathfrak{p}))} (1 + \varepsilon e^{-\gamma})) \right),$$

and in this case the right hand side of (17) is equal to

$$\sum_{l_{k_i}} \left(\frac{\varepsilon^{\delta'}}{\sum_{w' \in W_0} \operatorname{sgn} w' e^{w'(-\rho_0)}} \cdot \sum_{w' \in W_0} \operatorname{sgn} w' . w' \left((e^{\lambda_l(\mathfrak{b}) - \rho_0 + \sum_i 2 l_{k_i} \varepsilon_{k_i}} \prod_{\gamma \in \Delta_{\bar 1}^-(\mathfrak{b}) \setminus (\Delta_{\bar 1}^-(\mathfrak{b}) \cap \tilde\Delta_1(\mathfrak{p}) \cap \check\Delta)} (1 + \varepsilon e^{-\gamma})) \right) \right),$$

where

$$(\check\Delta \cap \Delta_{\bar 1}^-(\mathfrak{b})) \setminus (\check\Delta \cap \Delta_{\bar 1}^-(\mathfrak{b}) \cap \tilde\Delta_1(\mathfrak{p})) = \{-2\varepsilon_{k_1}, \dots, -2\varepsilon_{k_t}\},\ k_1 < \dots < k_t$$

and $\sum_{l_{k_i}}$ denotes the sum over all possible $l_{k_i} \in \{0,1\}$, $i = 1, \dots, t$. Next we observe that $\Gamma(\mathcal{O}_{G/P}(\lambda^\delta))$ is endowed with a natural $\mathfrak{g}$-filtration, among the composition factors of which one finds $\mathfrak{b}^{\mathrm{op}}$-highest weight modules with highest weights running over $\{\lambda_l(\mathfrak{b}) + \sum_i 2 l_{k_i} . \varepsilon_{k_i}\}$ for all possible $l_{k_i} \in \{0,1\}$. Indeed im Θ_λ is already the first factor of our filtration. Considering furthermore $\Gamma(\mathcal{O}_{G/P}(\lambda^\delta))/\mathrm{im}\, \Theta_\lambda$, we see that for a generic $\lambda \in \bar C_{\mathfrak{p}}^+$ $\lambda_l(\mathfrak{b}) + 2\varepsilon_{k_1} \in \operatorname{supp} \Gamma(\mathcal{O}_{G/P}(\lambda^\delta))/\mathrm{im}\, \Theta_\lambda$ (since evidently for a generic λ of $\bar C_{\mathfrak{p}}^+$ $\lambda_l(\mathfrak{b}) + 2\varepsilon_{k_1} \in \operatorname{supp} \Gamma(\mathcal{O}_{G/P}(\lambda^\delta))$ but $\lambda_l(\mathfrak{b}) + 2\varepsilon_{k_1} \notin \operatorname{supp}(\mathrm{im}\, \Theta_\lambda = V_{\mathfrak{b}}(\lambda^\delta))$), and that $(\lambda_l(\mathfrak{b}) + 2\varepsilon_{k_1})^{\delta'+1}$ is a $\mathfrak{b}^{\mathrm{op}}$-submodule of $\Gamma(\mathcal{O}_{G/P}(\lambda^\delta))/\mathrm{im}\, \Theta_\lambda$ (because $\lambda_l(\mathfrak{b}) + 2\varepsilon_{k_1}$ is maximal in $\operatorname{supp} \Gamma(\mathcal{O}_{G/P}(\lambda^\delta))/\mathrm{im}\, \Theta_\lambda$ with respect to the partial ordering $\underset{\mathfrak{b}^{\mathrm{op}}}{\geqslant}$). Denote by $\hat V_{\mathfrak{b}^{\mathrm{op}}}((\lambda_l(\mathfrak{b}) + 2\varepsilon_{k_1})^{\delta'+1})$ the highest weight submodule of

$\Gamma(\mathcal{O}_{G/P}(\lambda^\delta))/\mathrm{im}\,\Theta_\lambda$ generated by $(\lambda_l(\mathfrak{b})+2\varepsilon_{k_1})^{\delta'+1}$. Ordering now the weights $\lambda_l(\mathfrak{b})+\sum_i 2l_{k_i}.\varepsilon_i$ in the way

$$\lambda_l(\mathfrak{b})+2\varepsilon_{k_1}, \quad \lambda_l(\mathfrak{b})+2\varepsilon_{k_2}, \ldots, \lambda_l(\mathfrak{b})+2\varepsilon_{k_t},$$
$$\lambda_l(\mathfrak{b})+2\varepsilon_{k_1}+2\varepsilon_{k_2}, \ldots, \lambda_l(\mathfrak{b})+\sum_{i=1}^{t} 2\varepsilon_{k_i},$$

and continuing the above process, i.e. passing to the module $(\Gamma(\mathcal{O}_{G/P}(\lambda^\delta))/\mathrm{im}\,\Theta_\lambda)/\hat{V}_{\mathfrak{b}^{\mathrm{op}}}((\lambda_l(\mathfrak{b})+2\varepsilon_{k_1})^{\delta'+1})$ and so on, one obtains the desired filtration.

Our further observation is that again for a generic λ of $\bar{C}^+_{\mathfrak{p}}$

$$-\mathbb{A}^{\mathfrak{b}^{\mathrm{op}}}_{\lambda_l(\mathfrak{b})+\rho_{\mathfrak{b}^{\mathrm{op}}}} \subset \Delta_1^-(\mathfrak{b})\backslash(\Delta_1^-(\mathfrak{b})\cap\tilde{\Delta}_1(\mathfrak{p})),$$

and even more that in our case

$$\Delta_1^-(\mathfrak{b})\backslash(\Delta_1^-(\mathfrak{b})\cap\tilde{\Delta}_1(\mathfrak{p})\cap\check{\Delta}_1) = -\mathbb{A}^{\mathfrak{b}^{\mathrm{op}}}_{\lambda_l(\mathfrak{b})+\sum_i 2l_{k_i}\varepsilon_{k_i}+\rho_{\mathfrak{b}^{\mathrm{op}}}}$$

for (a generic λ of $\bar{C}^+_{\mathfrak{p}}$ and) arbitrary $l_{k_i}\in\{0,1\}$. Moreover (using Corollary 1) one checks immediately that $\lambda_d+\sum_i 2l_{k_i}\varepsilon_{k_i}$ is the highest weight with respect to $\mathfrak{b}_d^{\mathrm{op}}$ of $V_{\mathfrak{b}^{\mathrm{op}}}((\lambda_l(\mathfrak{b})+\sum_i 2l_{k_i}\varepsilon_{k_i})^{\delta''+\left(\sum_i l_{k_i}\right)\mathrm{mod}\,2})$ for arbitrary $l_{k_i}\in\{0,1\}$. Now the argument from Step $B1$ (applied consecutively to the composition factors of the filtration) implies here

$$\mathrm{ch}\, V_{\mathfrak{b}_d^{\mathrm{op}}}((\lambda_d+\sum_i 2l_{k_i}\varepsilon_{k_i})^{\delta''+\left(\sum_i l_{k_i}\right)\mathrm{mod}\,2}) \geqslant \frac{\varepsilon^{\delta''+\left(\sum_i 2l_{k_i}\right)\mathrm{mod}\,2}}{\sum_{w'\in W_0}\mathrm{sgn}\, w' e^{w'(-\rho_0)}}$$
$$\cdot\left(\sum_{w'\in W_0}\mathrm{sgn}\, w'.w'\left(e^{\lambda_d-\rho_0+\sum_i 2l_{k_i}\varepsilon_{k_i}}\prod_{\gamma\in A_{\lambda_d}+\sum_i 2l_{k_i}.\varepsilon_{k_i}-\rho_d}(1+\varepsilon e^{\gamma})\right)\right)$$

for a generic λ of $\bar{C}^+_{\mathfrak{p}}$ and any l_{k_i}, or equivalently

$$\mathrm{ch}\, V_{\mathfrak{b}^{\mathrm{op}}}((\lambda_l(\mathfrak{b})+\sum_i 2l_{k_i}\varepsilon_{k_i})^{\delta'+\left(\sum_i l_{k_i}\right)\mathrm{mod}\,2}) \geqslant \frac{\varepsilon^{\delta''+\left(\sum_i l_{k_i}\right)\mathrm{mod}\,2}}{\sum_{w'\in W_0}\mathrm{sgn}\, w' e^{w'(-\rho_0)}}$$
$$\cdot\left(\sum_{w'\in W_0}\mathrm{sgn}\, w'.w'\left(e^{\lambda_l(\mathfrak{b})-\rho_0+\sum_i l_{k_i}\varepsilon_{k_i}}\prod_{\gamma\in-\mathbb{A}_l^{\mathfrak{b}^{\mathrm{op}}}(\mathfrak{b})+\sum_i 2l_{k_i}\varepsilon_{k_i}+\rho_{\mathfrak{b}^{\mathrm{op}}}=\Delta_1^-(\mathfrak{b})\backslash(\Delta_1^-(\mathfrak{b})\cap\Delta_1^-(\mathfrak{p})\cap\check{\Delta})}(1+\varepsilon e^{-\gamma})\right)\right).$$

Therefore

$$\operatorname{ch}\Gamma(\mathcal{O}_{G/P}(\lambda^{\delta}))$$

$$\geqslant \sum_{l_{k_i}} \frac{\varepsilon^{\delta'+\left(\sum_i l_{k_i}\right) \bmod 2}}{\sum_{w'\in W_0} \operatorname{sgn} w' e^{w'(-\rho_0)}} \left(\sum_{w'\in W_0} \operatorname{sgn} w' . w' \left(e^{\lambda_l(\mathfrak{b})-\rho_0+\sum_i l_{k_i}\varepsilon_{k_i}} \prod_{\gamma\in\Delta_{\bar 1}(\mathfrak{b})\setminus(\Delta_{\bar 1}(\mathfrak{b})\cap\bar\Delta_1(\mathfrak{p})\cap\check\Delta)} (1+\varepsilon e^{-\gamma})\right)\right).$$

But the already constructed filtration on $\Gamma(\mathcal{O}_{G/P}(\lambda^{\delta}))$ gives

$$\operatorname{ch}\Gamma(\mathcal{O}_{G/P}(\lambda^{\delta})) \leqslant \sum_{l_{k_i}} \operatorname{ch} V_{\mathfrak{b}^{op}}((\lambda_l(\mathfrak{b})+\sum_i 2l_{k_i}\varepsilon_{k_i})^{\delta'+\left(\sum_i l_{k_i}\right)\bmod 2}),$$

which implies

$$\operatorname{ch}\hat{V}_{\mathfrak{b}^{op}}((\lambda_l(\mathfrak{b})+\sum_i 2l_{k_i}\varepsilon_{k_i})^{\delta'+\left(\sum_i l_{k_i}\right)\bmod 2})$$

$$= \operatorname{ch} V_{\mathfrak{b}^{op}}((\lambda_l(\mathfrak{b})+\sum_i 2l_{k_i}\varepsilon_{k_i})^{\delta'+\left(\sum_i l_{k_i}\right)\bmod 2})$$

for all $l_{k_i}\in\{0,1\}$, and even more that the composition factors of our filtration are precisely the $\mathfrak{g}$-modules

$$V_{\mathfrak{b}^{op}}((\lambda_l(\mathfrak{b})+\sum_i 2l_{k_i}\varepsilon_{k_i})^{\delta'+\left(\sum_i l_{k_i}\right)\bmod 2})$$

for arbitrary $l_{k_i}\in\{0,1\}$. In this way the proof of our Theorem is completed for the case $w=\mathrm{id}$.

Step B3: Let now λ be a generic element of $\mathfrak{h}_{\mathfrak{p}}^*$ (and $B \hookrightarrow P$ be an arbitrary pair with $B_{\mathrm{red}}=(B_d)_{\mathrm{red}}$. Then exactly as in Step A2, $H^i(\mathcal{O}_{G/P}(\lambda^{\delta}))=0$ for $i\neq l(w)$, $(-1)^{l(w)}\operatorname{Ech}\mathcal{O}_{G/P}(\lambda^{\delta})$ is given by formula (12), and $\mathcal{O}_{G/P_w}(\mu^{\varkappa})$ is a well defined (and here even more invertible !) $\mathfrak{g}$-linearized sheaf on G°/P_w with $H^i(\mathcal{O}_{G/P_w}(\mu^{\varkappa}))=0$ $\forall i>0$. Furthermore Step B2 implies that in our case $\Gamma(\mathcal{O}_{G/P_w}(\mu^{\varkappa}))$ admits a $\mathfrak{g}$-filtration with composition factors

$$V_{\mathfrak{b}_w^{op}}((\mu_l+\sum_i 2l_{k_i}\varepsilon_{k_i})^{\varkappa'+\left(\sum_i l_{k_i}\right)\bmod 2})$$

for all possible $l_{k_i} \in \{0,1\}$, where $-2\varepsilon_{k_i}$ runs over $(\check{\Delta}\cap\Delta_{\bar{1}}^-(\mathfrak{b}_w))\backslash(\check{\Delta}\cap\Delta_{\bar{1}}^-(\mathfrak{b}_w)\cap\tilde{\Delta}_1(\mathfrak{p}_w))$. However, in order to complete the proof, we need here an extra argument. More precisely, using the explicit form of the right hand side of (12) one constructs similarly to Step B2 a natural filtration on $H^{l(w)}(\mathcal{O}_{G/P}(\lambda^\delta))$, the composition factors of which are quotients of the Verma modules

$$\tilde{V}_{\mathfrak{b}_w^{\mathrm{op}}}((\mu_l + \sum_i 2l_{k_i}.w(\varepsilon_{k_i}))^{\varkappa'+\left(\sum_i l_{k_i}\right)\bmod 2}) \; (^8)$$

for all possible $l_{k_i} \in \{0,1\}$, where $-2\varepsilon_{k_i}$ runs now over $(\check{\Delta}\cap\Delta_{\bar{1}}^-(\mathfrak{b}))\backslash(\check{\Delta}\cap\Delta_{\bar{1}}^-(\mathfrak{b})\cap\tilde{\Delta}_1(\mathfrak{p}))$ and $\varkappa \in \mathbb{Z}_2$. (Indeed one checks immediately that $\mu_l^{\varkappa'}$ is a $\mathfrak{b}_w^{\mathrm{op}}$-submodule of $H^{l(w)}(\mathcal{O}_{G/P}(\lambda^\delta))$ for a certain $\varkappa' \in \mathbb{Z}_2$, then that $(\mu_l + 2\varepsilon_{k_j})^{\varkappa'+1}$ for $j = \min\{j_0 | w(\varepsilon_{k_{i_0}}) = \varepsilon_{k_{j_0}}, -2\varepsilon_{k_i} \in (\Delta_{\bar{1}}^-(\mathfrak{b})\cap\check{\Delta})\backslash(\Delta_{\bar{1}}^-(\mathfrak{b})\cap\tilde{\Delta}_1(\mathfrak{p})\cap\check{\Delta}\}$, $i = 1, \ldots, t$, is a $\mathfrak{b}_w^{\mathrm{op}}$-submodule of the quotient $H^{l(w)}(\mathcal{O}_{G/P}(\lambda^\delta))/(\mathfrak{g}$-module generated by $\mu_l^{\varkappa'})$, etc., i.e. one carries out the same procedure as in Step B2). The crucial observation in that now the filtration of $\Gamma(\mathcal{O}_{G/P_w}(\mu^\varkappa))$ implies for a generic $\eta \in \bar{C}^+_{\mathfrak{p}_w}$

$$\operatorname{ch} V_{\mathfrak{b}_w^{\mathrm{op}}}((\eta_l + \sum_i 2l_{k_i} w(\varepsilon_{k_i}))^{\varkappa'+\left(\sum_i l_{k_i}\right)\bmod 2}) = \frac{\varepsilon^{\varkappa'+\left(\sum_i l_{k_i}\right)\bmod 2}}{\sum_{w'\in W_0} \operatorname{sgn} w' e^{w'(-\rho_0)}} \cdot \left(\sum_{w'\in W_0} \operatorname{sgn} w'.w'(e^{\eta_l + \sum_i 2l_{k_i} w(\varepsilon_{k_i}) - \rho_0} \prod_{\gamma\in\Delta_{\bar{1}}^-(\mathfrak{b}_w)\backslash(\Delta_{\bar{1}}^-(\mathfrak{b}_w)\cap\tilde{\Delta}_1(\mathfrak{p}_w)\cap\check{\Delta})} (1+\varepsilon e^{-\gamma}))\right).$$

Therefore we obtain that the composition factors of $H^{l(w)}(\mathcal{O}_{G/P}(\lambda^\delta))$ coincide with

$$V_{\mathfrak{b}_w^{\mathrm{op}}}((\mu_l + \sum_i 2l_{k_i}.(\varepsilon_{k_i}))^{\varkappa'+\left(\sum_i l_{k_i}\right)\bmod 2})$$

for all possible l_{k_i}, which completes the proof of the Theorem. □

Theorem 4 implies

(8) For any $\eta \in \mathfrak{h}_{\bar{0}}^*$ and any η one defines the Verma module $\tilde{V}_{\mathfrak{b}}(\eta^\delta)$ setting $\tilde{V}_{\mathfrak{b}}(\eta^\delta) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \eta^\delta$.

COROLLARY 2. – *Let* $\varkappa$ = gl, osp, $\mathfrak{g}$, $\mathbf{p}$, *and* $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p}$ *be fixed. Then for a generic* λ *of* $\bar{C}_{\mathfrak{p}}^{+}$ (*i.e. for* λ *belonging to a Zariski-open subset of* $\bar{C}_{\mathfrak{p}}^{+}$) *one has*

$$(18)\quad \operatorname{ch} V_{\mathfrak{b}}(\lambda^{\delta}) = \frac{\dim \lambda^{\delta}}{\sum_{w' \in W_0} \operatorname{sgn} w' e^{w'(\rho_0)}} \cdot \left(\sum_{w' \in W_0} \operatorname{sgn} w' . w'\Big(e^{\lambda+\rho_0} . \prod_{\gamma \in \Delta_1(\mathfrak{b})_{\mathfrak{p}} \setminus (\Delta_1(\mathfrak{b})_{\mathfrak{p}} \cap \check{\Delta})} (1+\varepsilon e^{\gamma})\Big) \right).$$

In the special case when $\mathfrak{g} = q$ a sufficient condition (18) to be satisfied is $(\lambda,\alpha)_{\text{red}} \neq 0 \neq (\lambda,\underline{\alpha})\ \forall \alpha \in \Delta_1(\mathfrak{b})_{\mathfrak{p}}$.

Proof. – By Proposition 1, c) $H^i(\mathcal{O}_{G^\circ/P}(\lambda^\delta)) = 0$ for $i > 0$ implies that $\operatorname{ch} \Gamma(\mathcal{O}_{G^\circ/P}(\lambda^\delta))$ equals to the right hand side of (5). Furthermore if $\mathfrak{g} \neq \mathbf{p}$, Theorem 4 gives $\Gamma(\mathcal{O}_{G^\circ/P}(\lambda^\delta)) = V_{\mathfrak{b}}(\lambda^\delta)$ for a generic $\lambda \in \bar{C}_{\mathfrak{p}}^{+}$, and (since the condition $H^i(\mathcal{O}_{G^\circ/P}(\lambda^\delta))=0$ for $i>0$ is also an open condition on $\lambda \in \bar{C}_{\mathfrak{p}}^{+}$) we obtain in this way that (for a generic $\lambda \in \bar{C}_{\mathfrak{p}}^{+}$) $\operatorname{ch} V_{\mathfrak{b}}(\lambda^\delta)$ is given by the right hand side of (5). (According to Theorem 4, for $\mathfrak{g}=q$ the condition $(\lambda,\alpha)_{\text{red}} \neq 0 \neq (\lambda,\underline{\alpha})\ \forall \alpha \in \Delta_1(\mathfrak{b})_{\mathfrak{p}}$, together with $\lambda \in \bar{C}_{\mathfrak{p}}^{+}$, implies $\Gamma(\mathcal{O}_{G^\circ/P}(\lambda^\delta)) \simeq V_{\mathfrak{b}}(\lambda^\delta)$, $H^i(\mathcal{O}_{G^\circ/P}(\lambda^\delta))=0$ for $i>0$, and therefore it is also sufficient for the equality $\operatorname{ch} V_{\mathfrak{b}}(\lambda^\delta)=$ right hand side of (5)). But for $\mathfrak{g} \neq \mathbf{p}\ \check{\Delta} = \varnothing$ and, according to the remark at the end of § 1, the right hand side of (5) is equal to the right hand side of (6), i.e. to the right hand of (18). This proves our claim for $\mathfrak{g} \neq \mathbf{p}$. If now $\mathfrak{g} = \mathbf{p}$, the above argument remains valid in the special case when $\mathfrak{b}_d \subset \mathfrak{p}$. However in Step B2 of the proof of Theorem 4 we have already generalized practically the same argument to an arbitrary pair $\mathfrak{b} \subset \mathfrak{p}$. Indeed the inequality

$$\operatorname{ch} V_{\mathfrak{b}^{op}}\Big((\lambda_l(\mathfrak{b}) + \sum_i 2l_{k_i}\varepsilon_{k_i})^{\delta' + \left(\sum_i l_{k_i}\right) \bmod 2}\Big) \geqslant \frac{\varepsilon^{\delta' + \left(\sum_i l_{k_i}\right) \bmod 2}}{\sum_{w' \in W_0} \operatorname{sgn} w' e^{w'(-\rho_0)}} \cdot \left(\sum_{w' \in W_0} \operatorname{sgn} w' . w'\Big(e^{\lambda_l(\mathfrak{b}) - \rho_0 + \sum_i 2l_{k_i}\varepsilon_{k_i}} . \prod_{\gamma \in \Delta_{\bar{1}}(\mathfrak{b}) \setminus (\Delta_{\bar{1}}(\mathfrak{b}) \cap \tilde{\Delta}_1(\mathfrak{p}) \cap \check{\Delta})} (1+\varepsilon e^{-\gamma})\Big) \right)$$

from Step B2 together with Proposition 1, the result of Step B2, and the remark at the end of § 1 imply the claim of Corollary 2 also for $\mathfrak{g} = \mathbf{p}$. □

Corollary 2 is obviously an extension of Theorem 2 to modules, which are « typical modulo $\mathfrak{p}$ ». The precise formulation of this typicality condition could naturally be the requirement (9), however in the proof of (18) we use a slightly stronger condition, encoded in the requirement that $H^i(\mathcal{O}_{G^\circ/P}(\lambda^\delta)) = 0 \; \forall i > 0$. The situation is most simple for $G = Q$, where the condition $\dim V_{\mathfrak{b}}(\lambda^\delta) < \infty$ for $\lambda \in \bar{C}_{\mathfrak{p}}^+$ (which is equivalent to $(\lambda, \alpha)_{\text{red}} \neq 0 \;\; \forall \alpha \in \Delta_{\bar{1}}^-(\mathfrak{b})_{\mathfrak{p}}$) implies automatically $H^i(\mathcal{O}_{G^\circ/P}(\lambda^\delta)) = 0$ $\forall i > 0$. Therefore if $\mathfrak{g} = q$, the result of Corollary 2 is precisely an extension of the result of [11] to weights « typical modulo $\mathfrak{p}$ » (formula (18) being in this special case nothing but a formula obtained first by Sergeev in the special situation of irreducible submodules of the tensor algebra of the standard representation, [15]).

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