ANNALES DE L'INSTITUT FOURIER

MARTIN MARKL

On the rational homotopy Lie algebra of spaces with finite dimensional rational cohomology and homotopy

Annales de l'institut Fourier, tome 39, nº 1 (1989), p. 193-206

<http://www.numdam.org/item?id=AIF_1989__39_1_193_0>

Ann. Inst. Fourier, Grenoble
39, 1 (1989), 193-206

ON THE RATIONAL HOMOTOPY LIE ALGEBRA OF SPACES WITH FINITE DIMENSIONAL RATIONAL COHOMOLOGY AND HOMOTOPY

by Martin MARKL

Introduction.

A path connected topological space S is said to have type F , if

$$\dim(H^*(S;Q)) < \infty \quad \text{and} \quad \dim(\pi^*_\psi(S)) < \infty ,$$

where $\pi^*_\psi(S)$ denotes the ψ-homotopy of the space S [12; p. 61]. If S is simply connected, the previous condition is, of course, equivalent with

$$\dim(H^*(S;Q)) < \infty \quad \text{and} \quad \dim(\pi_*(S)\otimes Q) < \infty \qquad \text{(see [2]).}$$

Spaces of type F were studied by many authors, see for example [2], [3], [4] and [5]. J. Friedlander and S. Halperin gave in [2] the characterization of all rational graded vector spaces V_* , for which there exists a space S of type F with $V_* \cong \pi_*(S)\otimes Q$ in the category of graded spaces.

Suppose that S is simply connected and denote by ΩS the loop space of S . The Samelson product induces on $\pi_*(\Omega S)\otimes Q \cong \pi_{*+1}(S)\otimes Q$ the structure of a graded Lie algebra over rationals which is called the (rational) homotopy Lie algebra of the space S [8; p.210]. It is natural to ask how to characterize all graded rational Lie algebras Π_* for which there exists a simply connected space S of type F with $\Pi_* \cong \pi_*(\Omega S)\otimes Q$ in the category of graded Lie algebras. Unfortunately, this problem seems to have

---

Key-words : Rational homotopy algebra – Space of type F – Minimal model.
A.M.S. Classification : 55P62 – 55Q15.

no reasonable solution (see [5; p.114]). On the other hand, this question leads to the study of the set $f\mathcal{L}(W)$ of all graded Lie algebra structures on a given graded vector space W , that are the homotopy Lie algebras of spaces of type F . This set forms a subset of the algebraic variety $\mathcal{L}(W)$ of all graded Lie algebra structures on W (see § 2). We prove, roughly speaking, that there are (under suitable assumptions) only three possibilities :

- $f\mathcal{L}(W) = \emptyset$, *i.e.* no graded Lie algebra structure on W can be realized by the homotopy Lie algebra of a simply connected space of type F ,
- $f\mathcal{L}(W)$ is a proper, nonempty and Zariski-open subset of $\mathcal{L}(W)$,
- $f\mathcal{L}(W) = \mathcal{L}(W)$, *i.e.* every graded Lie algebra structure on W can be realized by the homotopy Lie algebra of a simply connected space of type F .

We also show that these cases are characterized by the combinatorial condition, similar to the "strong arithmetic condition" of [2; p.117].

1. Preliminaries.

In this paper we adopt the terminology of [12] and [3]. A minimal algebra $(\Lambda M, D)$ is said to be pure, if $D(M^{\text{even}}) = 0$ and $D(M^{\text{odd}}) \subset \Lambda M^{\text{even}}$ [3; p.179]. For a minimal algebra $(\Lambda M, d)$ we define the differential d_p by

$d_p(M^{\text{even}}) = 0\,,\ \ d_p(M^{\text{odd}}) \subset \Lambda M^{\text{even}}$ and $(d - d_p)(M^{\text{odd}}) \subset \Lambda^+ M^{\text{odd}} \cdot \Lambda M.$

The differential d_p is called the pure modification of d . If the dimension of the vector space M is finite, then

$$\dim(H^*(\Lambda M, d)) < \infty \ \text{ if and only if } \ \dim(H^*(\Lambda M, d_p)) < \infty \tag{1.1}$$

by [3; Proposition 1]. Let C^* be the cochain functor from the category of differential graded Lie algebras to the category of differential graded commutative algebras, $C^* : LDG \to ADGC$ [12; I.1]. It relates the minimal model $(\Lambda M, d)$ of a simply connected space S and its homotopy Lie algebra Π_* by :

$$C^*((\Pi_*, \partial = 0)) \cong (\Lambda M, d_2)\ , \tag{1.2}$$

where d_2 denotes the quadratic part of the differential d [12; p.88].

Let V be a (positively) graded finite dimensional rational vector space and let $x_1, \ldots, x_r, y_1, \ldots, y_q$ be a homogeneous basis, $\deg(x_i) = 2a_i$,

$\deg(y_j) = 2b_j - 1$, $1 \le i \le r$, $1 \le j \le q$. The integers $b_1, \dots, b_q; a_1, \dots, a_r$ will be called, according to [2], the exponents of the graded space V .

Let [;] be a graded Lie algebra product (bracket) on a graded vector space W [12; 0.4]. Denote by sW the suspension of W , *i.e.* the graded vector space defined by $(sW)_p = W_{p-1}$. If we write $C^*((W, [\ ,\ ], \partial = 0)) = (\Lambda V, d)$ then, by definition, the differential d is quadratic and

$$V = (sW)^*(= \mathrm{Hom}(sW, Q)) .$$

Choose a basis $x_1, \dots, x_r, y_1, \dots, y_q$ of V as above and let $b_1, \dots, b_q, a_1, \dots, a_r$ be the exponents of the space V . Clearly, the pure modification d_p of the differential d is characterized by a sequence $g_1, \dots, g_q$ of quadratic polynomials from $Q[x_1, \dots, x_r]$, $g_j = d_p(y_j) \in \Lambda(x_1, \dots, x_r) = Q[x_1, \dots, x_r]$, $1 \le j \le q$. Using [2; Theorem 3] we can easily deduce the following observation (the proof is given in § 4).

Observation. — Suppose that $(W, [\ ;\ ])$ is the homotopy Lie algebra of a simply connected space of type F . Then the following condition must be satisfied (compare with the definition before [2; Theorem 1]) :

for every subsequence A^* of $(a_1, \dots, a_r)$ of length s $(1 \le s \le r)$ there exist at least s elements b_j of $(b_1, \dots, b_q)$ of the form $b_j = \sum_{a_i \in A^*} \gamma_{ij} a_i$, where γ_{ij} are non-negative integers and

- either $\sum_{a_i \in A^*} \gamma_{ij} \ge 3$,
- or $\sum_{a_i \in A^*} \gamma_{ij} = 2$ and each quadratic monomial $\prod_{a_i \in A^*} (x_i)^{\gamma_{ij}}$ occurs

in the polynomial g_j .

2. Results.

Let V be a finite dimensional rational graded vector space and $b_1, \dots, b_q, a_1, \dots, a_r$ its exponents. We shall always assume that $a_i > 0$ and $b_j > 1$, $1 \le i \le r$, $1 \le j \le q$. Denote by W the desuspension $s^{-1}V^*$, *i.e.* the graded space defined by $(s^{-1}V^*)_p = V^*_{p+1}$. Clearly $2b_1 - 2, \dots, 2b_q - 2, 2a_1 - 1, \dots, 2a_r - 1$ are the degrees of a homogeneous basis of W .

Let $\mathcal{L}(W)$ be the system of all graded Lie algebra structures on W . Systems of such a type will be considered as (not necessarily irreducible) affine algebraic varieties (= closed algebraic sets) over Q in the same sense as, for example, in [7]. Similarly, let $\mathcal{L}_p(W)$ denote the variety of all graded Lie algebra products on W satisfying the following "purity" condition :

$$\text{(2.1)} \qquad \begin{array}{l} \text{if } x \text{ and } y \text{ are homogeneous and } [x;y] \neq 0 \text{ , then} \\ \deg(x) \text{ and } \deg(y) \text{ are both odd.} \end{array}$$

This condition means nothing else than the purity of $C^*((W,[;],\partial=0))$. Finally, denote by $f\mathcal{L}(W)$ (resp. $f\mathcal{L}_p(W)$) the system of all graded Lie algebra structures (resp. graded Lie algebra structures satisfying (2.1)) on W which can be realized by the homotopy Lie algebra of a simply connected space of type F .

Write for simplicity $B = (b_1, \ldots, b_q)$ and $A = (a_1, \ldots, a_r)$. In the situation described above we denote, for a positive integer k , by "AC_k" the following condition :

for every subsequence A^* of A of length s $(1 \leq s \leq r)$ there exist at least s elements b_j of B of the form

$$b_j = \sum_{a_i \in A^*} \gamma_{ij} a_i \ ,$$

where γ_{ij} are non-negative integers and $\sum_{a_i \in A^*} \gamma_{ij} \geq k$.

Remark. — The condition "AC_2" is precisely the "strong arithmetic condition" introduced in [2], hence the simply connected case of Theorem 1 in [2] reads in the terminology introduced above as follows :

the condition "AC_2" is satisfied if and only if $f\mathcal{L}(W) \neq \emptyset$.

Moreover, it easily follows from (1.1) that $f\mathcal{L}(W) \neq \emptyset$ if and only if $f\mathcal{L}_p(W) \neq \emptyset$ (see also the following paragraphs). Notice also that the Jacobi identity in graded Lie algebras satisfying (2.1) is trivial, hence $\mathcal{L}_p(W)$ is in fact isomorphic with the affine space Q^d for suitable d . Therefore each Zariski-open subset of $\mathcal{L}_p(W)$ is dense.

THEOREM 1. — *There are only three possibilities :*

- *First case : $f\mathcal{L}_p(W)$ is empty*

- *Second case : $f\mathcal{L}_p(W)$ is a nonempty, Zariski-open (and hence dense) subset of $\mathcal{L}_p(W)$, but $f\mathcal{L}_p(W) \neq \mathcal{L}_p(W)$*

- *Third case : $f\mathcal{L}_p(W) = \mathcal{L}_p(W)$.*

These cases are characterized as follows :

- *First case is equivalent with "non AC_2"*
- *Second case is equivalent with "AC_2 et non AC_3"*
- *Third case is equivalent with "AC_3" .*

This theorem is proved in § 4. Note that the conditions "AC_k" are easily verifiable. From the previous theorem and the Observation we easily obtain :

COROLLARY 2. — *If the condition "AC_3" is satisfied, then each pure (= satisfying (2.1)) Lie algebra product on W can be realized by the homotopy Lie algebra of a simply connected space of type F . If the condition "AC_3" is not satisfied, then no simply connected space of type F has the homotopy Lie algebra isomorphic with the algebra $(W, [\ ;\ ] = 0)$.*

Let us denote by $\mathcal{M}(V)$ (resp. $\mathcal{M}_p(V)$) the affine variety of all minimal (resp. pure minimal) algebras of the form $(\Lambda V, d)$. We can define the map $F : \mathcal{M}(V) \to \mathcal{L}(W)$ by $F((\Lambda V, d)) = (W, [\ ,\ ])$, where the algebra $(W, [\ ;\ ])$ is characterized by $C^*((W, [\ ;\ ], \partial = 0)) = (\Lambda V, d_2)$. The restriction gives the map $F_p : \mathcal{M}_p(V) \to \mathcal{L}_p(W)$. Define the map $p : \mathcal{L}(W) \to \mathcal{L}_p(W)$ by $p((W, [\ ;\ ])) = (W, [\ ;\ ]_p)$, where $[x; y]_p = [x; y]$ for deg(x) and deg(y) odd and $[x; y]_p = 0$ otherwise, $x, y \in W$ are homogeneous elements. Finally, we denote by $P : \mathcal{M}(V) \to \mathcal{M}_p(V)$ the map $P((\Lambda V, d)) = (\Lambda V, d_p)$ (d_p is defined in § 1). Our maps form the following commutative diagram :

$$\begin{array}{ccc} \mathcal{L}(W) & \xleftarrow{F} & \mathcal{M}(V) \\ {\scriptstyle p}\downarrow & & {\scriptstyle P}\downarrow \\ \mathcal{L}_p(W) & \xleftarrow{F_p} & \mathcal{M}_p(V) \end{array}$$

THEOREM 3. — *Let $4.\min\{2a_i, 2b_j - 1;\ 1 \le i \le r, 1 \le j \le q\} > \max\{2a_i, 2b_j - 1;\ 1 \le i \le r,\ 1 \le j \le q\} + 2$ or, more generally, let the canonical map from $\mathcal{M}(V)$ to the pullback of the diagram*

$$\begin{array}{ccc} \mathcal{L}(W) & & \\ {\scriptstyle p}\downarrow & & \\ \mathcal{L}_p(W) & \xleftarrow{F_p} & \mathcal{M}_p(V) \end{array}$$

be an epimorphism. Then the classification given in Theorem 1 is valid also for $f\mathcal{L}(W)$ in $\mathcal{L}(W)$.

The previous theorem contains the following interesting information.

COROLLARY 4. — *Suppose that the condition "AC_3" is satisfied and that* $4.\min\{\deg(v)\ ;\ v \in V$ *is homogeneous* $\} > \max\{\deg(v)\ ;\ v \in V$ *is homogeneous* $\}+2$. *Then each Lie algebra structure on the vector space* W *can be realized by the homotopy Lie algebra of a simply connected space of type* F .

THEOREM 5. — *Let the variety* $\mathcal{M}(V)$ *be irreducible. Then the condition "AC_2" is satisfied if and only if the set* $f\mathcal{L}(W)$ *is dense in* $\mathcal{L}(W)$.

Of course, if the condition "AC_2" is not satisfied, then the set $f\mathcal{L}(W)$ is empty (see the remark before Theorem 1). Our theorems are proved in § 4. We give the example showing the necessity of the irreducibility assumption in the last one.

Let V be the space homogeneously generated by the set $\{y_1, y_2, y_3, x\}$, $\deg(y_1) = 3$, $\deg(y_2) = 11$, $\deg(y_3) = 13$ and $\deg(x) = 4$. Then clearly $\mathcal{M}(V) \cong \{(a,b) \in Q^2; ab = 0\}$ and this set is reducible. It is easy to see that $\mathcal{L}(W) \cong Q$ and that $f\mathcal{L}(W) =$ Point, although the condition "AC_3" (and hence also "AC_2") is satisfied. It is interesting to compare this with the situation of Theorem 1, where "AC_3" implies $f\mathcal{L}_p(W) = \mathcal{L}_p(W)$. We see that the couples $(\mathcal{L}(W), f\mathcal{L}(W))$ and $(\mathcal{L}_p(W), f\mathcal{L}_p(W))$ have, in general, quite different properties.

On the other hand, there are interesting examples when Theorem 5 is applicable. For example, if V is the graded space based by the set $\{y_1, y_2, y_2', y_3, x\}$, $\deg(y_1) = 3$, $\deg(y_2) = \deg(y_2') = 11$, $\deg(y_3) = 13$ and $\deg(x) = 4$, then clearly $\mathcal{M}(V) \cong \{(a,b,c,d) \in Q^4; ac + bd = 0\}$ which can be shown to be irreducible. By Theorem 5, $f\mathcal{L}(W)$ is dense in $\mathcal{L}(W) = Q^2$ (it can be shown even that $f\mathcal{L}(W) = \mathcal{L}(W)$).

3. Main lemma.

In this paragraph we deduce the lemma, which forms the basis tool for proving our theorems. We adopt the usual terminology of [6], [9] and [10]. All objects are considered over an arbitrary (not necessary algebraically closed) field k of characteristic zero. Let $x_1, \dots, x_r, a_1, \dots, a_s$ be graded indeterminates, $\deg(x_i) > 0$, $\deg(a_j) = 0$ for $1 \le i \le r$, $1 \le j \le s$. We shall denote for brevity $x = (x_1, \dots, x_r)$ and $a = (a_1, \dots, a_s)$. For example, the graded polynomial ring $k[x_1, \dots, x_r, a_1, \dots, a_s]$ will be denoted simply by $k[x,a]$. Let A be the affine space with "coordinates" $a_1, \dots, a_s$:

$$A = \{(a_1, \dots, a_s);\ a_j \in k,\ 1 \le j \le s\} \cong k^s \ .$$

For a point $\alpha \in A$ and an ideal $I \subset k[x,a]$ let I_α be the ideal in $k[x]$ defined by

$$I_\alpha = \{f(x,\alpha); f(x,a) \in I\} \ .$$

Finally, for a subset $X \subset A$ write

$$X^I = \{\alpha \in X; \dim_k(k[x]/I_\alpha) < \infty\} \ .$$

The main result of this paragraph reads as follows :

MAIN LEMMA. — *Suppose that the ideal I is homogeneous (i.e. generated by a set of homogeneous elements, see [10; chap. VII]) in the graded ring $k[x,a]$. Then*

$$A^I = \{\alpha \in A; \dim_k(k[x]/I_\alpha) < \infty\}$$

is a (possibly empty) Zariski-open subset of A .

It can be easily shown that the lemma is not valid without the homogeneity assumption. Also the assumption $\deg(x_i) > 0$, $\deg(a_j) = 0$, $1 \le i \le r$, $1 \le j \le s$, is necessary.

Fix an algebraic closure $\overline{k}$ of the field k . The inclusion $k \subset \overline{k}$ defines the natural injection $k[x,a] \hookrightarrow \overline{k}[x,a]$ and we can clearly consider all objects over $\overline{k}$; I generates the ideal $\overline{I} \subset \overline{k}[x,a]$ and the "$\overline{k}$–version" of A is :

$$\overline{A} = \{(a_1, \dots, a_s); a_j \in \overline{k}, 1 \le j \le s\} \cong \overline{k}^s \ .$$

Then again $A \subset \overline{A}$. We can easily verify that for each $\alpha \in A$:

$$\dim_k(k[x]/I_\alpha) < \infty \text{ if and only if } \dim_{\overline{k}}(\overline{k}[x]/\overline{I}_\alpha) < \infty \ ,$$

hence $A^I = \overline{A}^{\overline{I}} \cap A$. Because $A \cap U$ is clearly Zariski-open (over k) in A for each Zariski-open (over $\overline{k}$) subset U of $\overline{A}$, it is sufficient to prove the lemma under the assumption that *k is algebraically closed.* First step towards the proof of Main Lemma is the following proposition.

PROPOSITION 1. — *For each Zariski-closed subset F of the affine space A either $F^I = \emptyset$ or F^I contains a nonempty subset, Zariski-open in F .*

Proof of the proposition. — Because clearly $(F_1 \cup F_2)^I = F_1^I \cup F_2^I$, we can always suppose that the set F is irreducible, hence the ideal

$$J = \{f \in k[a]; f(\alpha) = 0 \text{ for each } \alpha \in F\}$$

is prime. Denote by B the affine space

$$B = \{(x_1, \ldots, x_r, a_1, \ldots, a_s); x_i, a_j \in k, 1 \le i \le r, 1 \le j \le s\}$$

and let $P : B \to A$ be the natural projection. As usually, for an ideal K of a polynomial ring, denote by $Z(K)$ the zero set of K in the corresponding affine space [6; I.1]. We know that [2; Remark 1.9] :

(3.1) $\dim_k(k[x]/I_\alpha) < \infty$ if and only if the set $Z(I_\alpha)$ is finite.

Denote $M = Z(I) \cap P^{-1}(F)$. Because clearly $Z(I_\alpha) = Z(I) \cap P^{-1}(\alpha)$, we obtain easily from (3.1) that

(3.2) $$F^I = \{\alpha \in F; P^{-1}(\alpha) \cap M \text{ is finite}\}.$$

The ideal J can be considered as a subset of $k[x, a]$ and it makes sense to denote by D the ideal generated by I and J in $k[x, a]$. Note that $M = Z(D)$. If we decompose the algebraic set M into the union of irreducible components, $M = M_1 \cup \ldots \cup M_m$, then

$$Q_i = \{f \in k[x, a]; f(\xi) = 0 \text{ for each } \xi \in M_i\}$$

are the associated primes of the ideal D , $1 \le i \le m$. Similarly as above we obtain

(3.3) $$F^{Q_i} = \{\alpha \in F; P^{-1}(\alpha) \cap M_i \text{ is finite}\} , \ 1 \le i \le m ,$$

hence it is clear from the description (3.2) of the set F^I that

$$F^I = \bigcap_{1 \le i \le m} F^{Q_i} .$$

The set F is supposed to be irreducible, hence every nonempty Zariski-open subset of F is dense in F and it is clearly sufficient to prove that for each i , $1 \le i \le m$,

(3.4) either $F^{Q_i} = \emptyset$ or F^{Q_i} contains a nonempty subset, Zariski-open in F .

Fix i , $1 \le i \le m$. Because the ideals I and J are homogeneous, the ideal $D = (I, J)$ is homogeneous, too. By [10; p.154] each associated prime Q_i of D is also homogeneous, hence Q_i is generated by a system of the form

$$g_1(x, a), \ldots, g_u(x, a), h_1(a), \ldots, h_v(a) ,$$

where $g_t \in k[x, a]$ are homogeneous of positive degrees and $h_j \in k[a]$ are homogeneous of degree zero, $1 \le t \le u$, $1 \le j \le v$ (because $\deg(x_k) > 0$, no x_k can occur in a polynomial of degree zero, $1 \le k \le r$). This observation is the *key point* of our proof.

Denote by H the ideal generated in $k[a]$ by the polynomials $h_1, \ldots, h_v$. We claim that $P(M_i) = Z(H)$. Indeed, because the polynomials $g_1, \ldots, g_u$ have positive degrees, they are zero on elements of the form $(0, \alpha)$ for each $\alpha \in A$. Consequently, $(0, \alpha) \in Z(Q_i) = M_i$ provided $\alpha \in Z(H)$. Because $\alpha = P(0, \alpha)$, we see that $Z(H) \subset P(M_i)$. On the other hand, if $(\xi, \alpha) \in M_i = Z(Q_i)$ then clearly $h_j(\alpha) = 0$ for each j , $1 \leq j \leq v$, and $\alpha = P(\xi, \alpha) \in Z(H)$, which proves the inclusion $P(M_i) \subset Z(H)$.

By definition, $P(M_i) \subset F$ and we distinguish the following two cases :

A. $P(M_i) \subsetneqq F$. In this case, the set $U_i = F \setminus Z(H)$ is nonempty and Zariski-open in F . Because $P^{-1}(\alpha) \cap M_i = \emptyset$ for each $\alpha \in U_i$, $U_i \subset F^{Q_i}$ by (3.3) and the condition (3.4) is satisfied.

B. $P(M_i) = F$. Denote $F' = \{(0, \alpha); \alpha \in F\}$. Clearly $F' \subset M_i$, hence $\dim(F) = \dim(F') \leq \dim(M_i)$. The restriction $P|M_i$ defines the map $\pi : M_i \to F$, which is epic by our assumption. Again we distinguish two cases :

B.1. $\dim(M_i) > \dim(F)$. By the definition of the dimension, the set $\pi^{-1}(\alpha)$ is finite if and only if $\dim(\pi^{-1}(\alpha)) = 0$. The theorem [11; I.6. Theorem 7] (compare also [1; AG 10.1]) says that the set

$$F^{Q_i} = \{\alpha \in F; \dim(\pi^{-1}(\alpha)) = 0\}$$

is empty and (3.4) is valid.

B.2. $\dim(M_i) = \dim(F)$. Because $F' \subset M_i$ and $\dim(F') = \dim(M_i)$, from the irreducibility of the set M_i we see that $F' = M_i$, hence $\pi^{-1}(\alpha) = \{(0, \alpha)\}$. We have $F^{Q_i} = F$ and (3.4) is again satisfied. Our proposition is proved.

Proof of Main Lemma. — Suppose we have constructed a sequence $A_1 \supsetneqq A_2 \supsetneqq \cdots \supsetneqq A_k$, $k \geq 1$, of closed subsets of A with the property $(A \setminus A_k) \subset A^I$. If $A_k^I = \emptyset$ then $A^I = (A \setminus A_k)$ is open. In the opposite case there exists, by Proposition 1, a nonempty open subset $U_k \subset A_k$ with $U_k \subset A_k^I$. In this case we define $A_{k+1} = (A_k \setminus U_k)$. The set A_{k+1} is closed, $A_k \supsetneqq A_{k+1}$ and $(A \setminus A_{k+1}) \subset A^I$. Since the topological space A is Noetherian [6; 1.4.7], this procedure gives rise to a closed $A_m \subset A$ with $(A \setminus A_m) = A^I$. The lemma is proved.

4. Remaining proofs.

In this paragraph we prove the theorems of § 2. We adopt the notation introduced in previous paragraphs.

Let $f\mathcal{M}_p(V)$ denote the subset of $\mathcal{M}_p(V)$ consisting of all pure minimal algebras having finite dimensional cohomology. It is not hard to deduce from (1.1) that $f\mathcal{L}_p(W) = F_p(f\mathcal{M}_p(V))$. The algebras belonging to $\mathcal{M}_p(V)$ are of the form

$$(\Lambda(x_1,\ldots,x_r,y_1,\ldots,y_q),d)\ ,\quad \deg(x_i)=2a_i\ ,\quad \deg(y_j)=2b_j-1\ ,$$

with $d(x_i) = 0$ and $d(y_j) \in \Lambda(x_1,\ldots,x_r) = Q[x_1,\ldots,x_r]$ for $1 \le i \le r$, $1 \le j \le q$. Thus each element of $\mathcal{M}_p(V)$ is characterized by a sequence $f_1,\ldots,f_q$ of polynomials, $f_j = d(y_j) \in Q[x_1,\ldots,x_r]$, $1 \le j \le q$. Our minimal algebra clearly belongs to $f\mathcal{M}_p(V)$ if and only if

$$\dim_Q(Q[x_1,\ldots,x_r]/(f_1,\ldots,f_r)) < \infty\ ,\quad \text{see also [2].}$$

PROPOSITION 2.

a) *"$f\mathcal{M}_p(V) = \emptyset$" is equivalent with "non AC_2",*

b) *"$f\mathcal{M}_p(V)$ is a nonempty subset, Zariski-open in $\mathcal{M}_p(V)$" is equivalent with "AC_2",*

c) *"$F_p(f\mathcal{M}_p(V)) = \mathcal{L}_p(W)$" is equivalent with "$AC_3$".*

Proof of a). — This equivalence is in fact the main result of [2]; see also the note before Theorem 1.

Proof of b). — For each j , $1 \le j \le q$, denote by Φ_j the family of all at least quadratic (*i.e.* of length ≥ 2) monomials $\sigma \in Q[x_1,\ldots,x_r]$ with $\deg(\sigma) = 2b_j$. Write $\Phi_j = \{\sigma_1^j,\ldots,\sigma_{k_j}^j\}$ and denote

$$f_j(x,a^j) = f_j(x_1,\ldots,x_r,a_1^j,\ldots,a_{k_j}^j) = \sum_{1\le s\le k_j} a_s^j\sigma_s^j\ ,\quad 1\le j\le q\ .$$

Then $\mathcal{M}_p(V)$ is isomorphic to the affine space A with the "coordinates" $a_1^1,\ldots,a_{k_1}^1,\ldots,a_1^q,\ldots,a_{k_q}^q$ in the evident sense. If we put $\deg(a_s^j) = 0$ for $1 \le j \le q$, $1 \le s \le k_s$, then $I = (f_1,\ldots,f_q)$ is a homogeneous ideal in the graded polynomial ring $Q[x_1,\ldots,x_r,a_1^1,\ldots,a_{k_1}^1,\ldots,a_1^q,\ldots,a_{k_q}^q]$. Applying Main Lemma to this situation we see that the set A^I , which is clearly isomorphic with $f\mathcal{M}_p(V)$, is Zariski-open in $A \cong \mathcal{M}_p(V)$. Combining this with a) we obtain the requisite equivalence.

Proof of c). — The set $\mathcal{L}_p(W)$ can be identified with the subset of $\mathcal{M}_p(V)$ consisting of all minimal algebras with pure quadratic differential in the natural way. Under this identification F_p acts as taking the quadratic part and "$F_p(f\mathcal{M}_p(V)) = \mathcal{L}_p(W)$" means that for each pure quadratic differential δ on ΛV there exists a pure minimal algebra $(\Lambda V, d) \in f\mathcal{M}_p(V)$ such that the quadratic part d_2 of the differential d is equal to δ. Especially the equation $F_p(f\mathcal{M}_p(V)) = \mathcal{L}_p(W)$ implies the existence of $(\Lambda V, d) \in f\mathcal{M}_p(V)$ with trivial quadratic part. Then "AC_3" must be satisfied by Observation in § 1.

On the other hand, let "AC_3" be satisfied and let ψ_j be, similarly as in the proof of b), the set of all at least cubic (= of length ≥ 3) monomials $\mu \in Q[x_1, \ldots, x_r]$ with $\deg(\mu) = 2b_j$, $1 \leq j \leq q$. The families $\psi_1, \ldots, \psi_q$ satisfy the condition P.C. of [2; p.119] and there is a sequence $f_1, \ldots, f_q \in Q[x_1, \ldots, x_r]$ of polynomials such that each f_j is a linear combination of monomials from ψ_j and

$$\dim_Q(Q[x_1, \ldots, x_r]/(f_1, \ldots, f_q)) < \infty \qquad [2\ ;\ \text{Theorem 3}].$$

By the definition of ψ_j all the polynomials $f_1, \ldots, f_q$ have zero quadratic part.

Now, let $(\Lambda V, \delta)$ be a pure minimal algebra with quadratic differential and denote $g_j = \delta(y_j) \in Q[x_1, \ldots, x_r]$, $1 \leq j \leq q$. Then the pure differential d , defined for each sequence $\alpha_1, \ldots, \alpha_q$ of nonzero rationals by

$$d(y_j) = (\alpha_j)^{-1} \cdot f_j + g_j \ , \quad 1 \leq j \leq q \ ,$$

has the quadratic part equal to δ . By the following lemma we can find the rationals $\alpha_1, \ldots, \alpha_q$ such that $(\Lambda V, d) \in f\mathcal{M}_p(V)$ which completes our proof.

LEMMA. — *Let $f_1, \ldots, f_q, g_1, \ldots, g_q \in Q[x_1, \ldots, x_r]$ be homogeneous elements and let $\dim_Q(Q[x_1, \ldots, x_r]/(f_1, \ldots, f_q)) < \infty$. Then there exists a sequence $\alpha_1, \ldots, \alpha_q$ of nonzero rational numbers such that*

$$\dim_Q(Q[x_1, \ldots, x_r]/((\alpha_1)^{-1}f_1 + g_1, \ldots, (\alpha_q)^{-1}f_q + g_q)) < \infty \ .$$

Proof of the lemma. — For $1 \leq i \leq q$ define $h_i(x, a) = f_i(x) + a_i g_i(x)$. If we define $\deg(a_i) = 0$ for $1 \leq i \leq q$, then $h_1, \ldots, h_q$ are homogeneous elements of the polynomial ring $k[x_1, \ldots, x_r, a_1, \ldots, a_q]$; let us denote by I the ideal $(h_1, \ldots, h_q)$. If we abbreviate by A the affine space $A = \{(a_1, \ldots, a_q)\ ;\ a_i \in Q\ ,\ 1 \leq i \leq q\}$, the set A^I is Zariski-open in A by

Main Lemma. By our assumption, $\dim_Q(k[x_1,\ldots,x_r]/(f_1,\ldots,f_q)) < \infty$, hence $(0,\ldots,0) \in A^I$ and A^I is nonempty. Clearly there exists a point $(\alpha_1,\ldots,\alpha_q) \in A^I$ having all coordinates different from zero. Because

$$(f_1 + \alpha_1 g_1, \ldots, f_q + \alpha_q g_q) = ((\alpha_1)^{-1} f_1 + g_1, \ldots, (\alpha_q)^{-1} f_q + g_q) ,$$

our point $(\alpha_1,\ldots,\alpha_q)$ has the requisite properties.

Proof of Theorem 1. — As we remarked in the proof of Proposition 2, the affine space $\mathcal{L}_p(W)$ can be identified with an affine subspace of the affine space $\mathcal{M}_p(V)$, under this identification $F_p : \mathcal{M}_p(V) \to \mathcal{L}_p(W)$ is simply the canonical projection, hence an open epimorphism. Theorem 1 now follows from the classification given in Proposition 2.

Proof of Theorem 3. — We easily deduce from (1.1) that $f\mathcal{L}(W) = FP^{-1}(f\mathcal{M}_p(V))$. Taking the space $\{(x,y) \in \mathcal{L}(W) \times \mathcal{M}_p(V); p(x) = F_p(y)\}$ as the pullback of the diagram we see that if the canonical map from $\mathcal{M}(V)$ to the pullback is epic, then $f\mathcal{L}(W) = p^{-1}(f\mathcal{L}_p(W))$. The theorem now follows from Theorem 1 and from the evident fact that $p : \mathcal{L}(W) \to \mathcal{L}_p(W)$ is a continuous epimorphism.

For $p > 0$ the set $\Lambda^p V = \{v_1 \wedge \ldots \wedge v_p; v_1,\ldots,v_p \in V\}$ forms a vector subspace of ΛV and $\bigoplus_{p\geq 0} \Lambda^p V \cong \Lambda V$ (we put $\Lambda^0 V = Q$). Let $q_p : \Lambda V \to \Lambda^p V$ be the projection. For a linear endomorphism G of ΛV and $i \geq 2$ denote by $G_i : \Lambda V \to \Lambda V$ the linear map defined by $G_i|\Lambda^p V = q_{p+i-1} \circ G$. Finally, for $j \geq 1$ denote $G_{>j} = \sum_{i>j} G_i$.

The canonical map from $\mathcal{M}(V)$ to the pullback is clearly epic if and only if for each pure minimal differential d on ΛV and for each quadratic differential D on ΛV whose pure modification D_p is equal to the quadratic part d_2 of d there exists a differential δ on ΛV whose pure modification is equal to d and whose quadratic part is equal to D .

Let D and d be as above. Define the derivation δ by $\delta = D + d_{>2}$.

Then clearly $\delta^2 = D^2 + (\delta^2)_{>3} = (\delta^2)_{>3}$ and it is not hard to verify that under the assumption

$$\begin{aligned} 4.\ &\min\{\deg(v); v \in V \text{ is homogeneous}\} \\ &> \max\{\deg(v); v \in V \text{ is homogeneous}\} + 2 \end{aligned}$$

is always $(\delta^2)_{>3} = 0$, consequently δ is a differential satisfying $\delta_p = d$ and $\delta_2 = D$.

Proof of Theorem 5. — Recall that $f\mathcal{L}(W) = FP^{-1}(f\mathcal{M}_p(V))$ (see the proof of Theorem 3). The map $P : \mathcal{M}(V) \to \mathcal{M}_p(V)$ is continuous and epic and the set $P^{-1}(U)$ is, because of the irreducibility of $\mathcal{M}(V)$, dense for each nonempty open subset $U \subset \mathcal{M}_p(V)$. The map $F : \mathcal{M}(V) \to \mathcal{L}(W)$ is also continuous and epic and the rest follows from Proposition 2.

Proof of Observation. — Let Ω_j be, for $1 \le j \le q$, the system of all monomials $\omega \in Q[x_1, \dots, x_r]$ with $\deg(\omega) = 2b_j$, such that

- either ω is at least cubic (= of length ≥ 3),
- or ω is quadratic and it occurs in the polynomial g_j .

Suppose that there exists $(\Lambda V, D) \in f\mathcal{M}(V)$ with $C^*((W,[\ ;\ ],\partial=0))$ $= (\Lambda V, D_2)$. Then each polynomial $f_j = D_p(y_j)$ must be clearly a rational linear combination of elements of Ω_j , $1 \le j \le q$. Being $(W, [\ ;\ ])$ the homotopy Lie algebra of a space of type F , by [2; Theorem 3] the systems $\Omega_1, \dots, \Omega_q$ must satisfy the condition P.C. of [2; p. 119]. But P.C. for $\Omega_1, \dots, \Omega_q$ is clearly equivalent with the condition given in Observation.

I would like to take this opportunity to thank Professor J.-C. Thomas for his helpful advice. Also conversations with my friend Honza Nekovář were helpful in my thinking about this paper.

BIBLIOGRAPHY

[1] A. BOREL, Linear algebraic groups, W.A. Benjamin, New-York, 1969.

[2] J.-B. FRIEDLANDER, S. HALPERIN, An arithmetic characterization of the rational homotopy groups of certain spaces, Inv. Math., 53 (1979), 117–133.

[3] S. HALPERIN, Finiteness in the minimal models of Sullivan, Trans. Amer. Math. Soc., 230 (1977), 173–199.

[4] S. HALPERIN, Spaces whose rational homology and ψ–homotopy are both finite dimensional, Astérisque, 113–114 (1984), 198–205.

[5] S. HALPERIN, The structure of $\pi_*(\Omega S)$, Astérisque, 113–114, 109–117.

[6] R. HARTSHORNE, Algebraic geometry, Springer, 1977.

[7] J.-M. LEMAIRE, F. SIGRIST, Dénombrement de types d'homotopie rationnelle, C.R. Acad. Paris, Sér. A, 287 (1978), 109–112.

[8] D. QUILLEN, Rational homotopy theory, Ann. Math., 90 (1969), 205–295.

[9] P. SAMUEL, O. ZARISKI, Commutative algebra, Vol. I, Princeton N.J., Van Nostrand, 1958.

[10] P. SAMUEL, O. ZARISKI, Commutative algebra, Vol. II, Princeton N.J., Van Nostrand, 1960.

[11] I.-R. SHAFAREVICH, Osnovy algebraicheskoj geometrii, Moskva, 1972.

[12] D. TANRÉ, Homotopie rationnelle : Modèles de Chen, Quillen, Sullivan, Lecture Notes in Math. 1025, Springer, 1983.

Manuscrit reçu le 28 octobre 1986,
révisé le 26 janvier 1988.

Martin MARKL,
Matematický Ústav ČSAV
Žitná 25
115 67 Praha 1 (Czechoslovakia).